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EXPONENTIAL STABILITY AND BLOW-UP FOR ABSTRACT NONLINEAR SYSTEM WITH SOURCE TERMS

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Abstract. In this article we consider an abstract nonlinear system with nonlinear source terms. We prove the exponential stability by the energy method. Also under suitable conditions on the initial values, we show that the nonlinear source terms are able to guarantee the blow-up of the solutions by convex method.

1. Introduction

Let $A : D(A) \to L^2(\Omega)$ be a self-adjoint positive definite operator, $D(A) \subset L^2(\Omega)$ is a dense and compact embedding where Ω is a open bounded subset of $\mathbb{R}^n$ $(n \geq 1)$. We consider the system

$$\begin{gathered}
u_{tt} + A^2u + M(\|A^{\alpha/2}u\|_2^2 + \|A^{\alpha/2}v\|_2^2)A^\alpha u + N(\|A^{\beta/2}u\|_2^2)A^\beta u_t = f(u,v), \\
\text{in } \Omega \times (0,\infty), \\
v_{tt} + A^2v + M(\|A^{\alpha/2}u\|_2^2 + \|A^{\alpha/2}v\|_2^2)A^\alpha v + N(\|A^{\beta/2}v\|_2^2)A^\beta v_t = g(u,v), \\
\text{in } \Omega \times (0,\infty),
\end{gathered} \tag{1.1}$$

with the initial value conditions

$$\begin{gathered}
u(x,0) = u_0, v(x,0) = v_0, \quad \text{in } \Omega, \\
u_t(x,0) = u_1, v_t(x,0) = v_1, \quad \text{in } \Omega,
\end{gathered} \tag{1.2}$$

where $0 < \beta \leq \alpha \leq 1$, M and N are continuous functions. The functions f and g model the interior dissipations in the equations.

Many authors have studied the nonlinear wave equation

$$u_{tt} - M\left(\|\nabla u\|_2^2\right)\Delta u = 0. \tag{1.3}$$

This model was proposed by Kirchhoff [6] in one dimensional in which $M(s)$ is a linear function, and describe the transversal vibration of a string. Then many authors devote themselves to local existence and global existence result of system (1.3), see [1, 2, 3, 8]. Mizumachi [9] added linear damping on (1.3) and obtained the decay estimates for the solutions.

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Ikehata [5] considered the local solvability for abstract equation

$$u_{tt} - M\big(\|A^{1/2}u\|^2\big)Au + \delta u_t = f(u),$$

where A is a positive definite and self-adjoint operator in Hilbert space $(X, \|\cdot\|)$, $f : D(A^{1/2}) \to X$ is a nonlinear operator, and $M(s)$ is a C^1 function satisfying

$$M(s) \geq m_0 > 0.$$

He obtained the existence of strong solution without compactness hypothesis.

Rivera [11] studied the equation with damping

$$u_{tt} + M\left(\|A^{1/2}u\|^2\right)Au + Au_t = 0,$$

and proved that if the initial value $(u_0, u_1) \in D(A) \times X$, then the corresponding solution of the system satisfies

$$u \in C^2\big([0,T]; D(A^k)\big), \quad \forall k \in N.$$

Lazo [7] studied the nonlinear wave equation

$$u_{tt} + M\big(\|A^{1/2}u\|^2\big)Au + N\left(\|A^{\alpha}u\|^2\right)A^{\alpha}u_t = f.$$

He proved the existence of global solutions in a Hilbert space by using Galerkin's method.

Wu [12] considered the following nonlinear viscoelastic wave equations of Kirchhoff type with the nonlinear damping and source terms,

$$\begin{gathered}
u_{tt} - M\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right)\Delta u + \int_0^t g_1(t-s)\Delta u(s)ds + |u_t|^{p-1}u_t = f_1(u,v),\\
v_{tt} - M\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right)\Delta v + \int_0^t g_2(t-s)\Delta v(s)ds + |v_t|^{q-1}v_t = f_2(u,v).
\end{gathered}$$

He proved that, with the initial date in the stable set and for a wider class of relaxation functions, the decay rate of the system depends on the exponents of the damping terms by using Nakao's method. Conversely, for certain initial date in the unstable set, he obtained the blow-up result when the initial energy is nonnegative.

Mu and Ma [10] considered the following nonlinear viscoelastic wave equations of Kirchhoff type with Balakrishnan-Taylor damping,

$$\begin{gathered}
u_{tt} - \Big(a + b\|\nabla u\|_2^2 + \sigma\int_\Omega \nabla u\cdot\nabla u_t\,dx\Big)\Delta u + \int_0^t g_1(t-s)\Delta u(s)ds = f_1(u,v),\\
v_{tt} - \Big(a + b\|\nabla v\|_2^2 + \sigma\int_\Omega \nabla v\cdot\nabla v_t\,dx\Big)\Delta v + \int_0^t g_2(t-s)\Delta v(s)ds = f_2(u,v).
\end{gathered}$$

By the modified perturbed energy technique, the authors showed that the decay rate of the system is similar to that of relaxation functions. They also proved that nonlinear source of polynomial type is able to force solutions to blow up in finite time even if stronger damping exists.

Zhang et al [14] obtained the existence of global weak solutions for the coupled system

$$\begin{gathered}
u_{tt} + M(\|A^{1/2}u\|^2 + \|A^{1/2}v\|^2)Au + N(\|A^{\alpha}u\|^2)A^{\alpha}u_t = f(x,t),\\
v_{tt} + M(\|A^{1/2}u\|^2 + \|A^{1/2}v\|^2)Av + N(\|A^{\alpha}v\|^2)A^{\alpha}v_t = g(x,t).
\end{gathered}$$

Hao et al [4] proved the well posedness of the solution for system

$$u_{tt} + A^2u + M(\|A^{\alpha/2}u\|^2 + \|A^{\alpha/2}v\|^2)A^{\alpha}u + N(\|A^{\beta/2}u\|^2)A^{\beta}u_t = f(x,t),$$

$$v_{tt} + A^2 v + M(\|A^{\alpha/2}u\|^2 + \|A^{\alpha/2}v\|^2)A^\alpha v + N(\|A^{\beta/2}v\|^2)A^\beta v_t = g(x,t),$$

by Galerkin's method. However, they did not obtain the blow up or decay property.

In this paper, under suitable conditions, we prove that the system is exponential stable when the initial value lies in the stable set and blow up when the initial energy is negative or non-negative but small. Our plan in this paper is as follows. In section 2, we present some materials and assumptions needed later. In section 3, we prove the decay result by the energy method. In section 4, by using the convex method, we prove the blow-up phenomena of solutions.

2. Preliminaries

In this section, we present some materials and assumptions needed in the rest of this paper.

We denote $\|\cdot\|_q = \|\cdot\|_{L^q(\Omega)}$, $1 \le q < \infty$, and denote $(\cdot,\cdot)$ the usual inner product of $L^2(\Omega)$. We denote $H = L^2(\Omega)$ and c to be a generic positive constant which might change from line to line.

Next we give some assumptions for system (1.1)–(1.2).

(A1) There exist constants $c_0 > 0, \gamma > 2$, $p > 1$ and a positive C^1 function $F : R^2 \to R$ such that

$$\begin{gathered}\frac{\partial F(u,v)}{\partial u} = f(u,v), \quad \frac{\partial F(u,v)}{\partial v} = g(u,v), \quad uf(u,v) + vg(u,v) = \gamma F, \\ \int_\Omega F(u,v)dx \le c_0 \left(\|u\|_{p+1}^{p+1} + \|v\|_{p+1}^{p+1}\right),\end{gathered} \tag{2.1}$$

where p satisfies the inequality

$$\|u\|_{p+1} \le c_1 \|Au\|_2, \quad \forall u \in D(A), \tag{2.2}$$

(A2) There exist positive constants m_0, n_0, such that

$$M(z) \ge m_0, \quad N(z) \ge n_0, \quad \forall z \ge 0. \tag{2.3}$$

(A3) There exists positive constants c_1, c_2, such that

$$\|Au\|_2 \ge c_2 \|A^{\frac{r}{2}}u\|_2 \ge c_3 \|u\|_2, \quad \forall u \in D(A),\ r \in (0,1]. \tag{2.4}$$

We denote the eigenvalues of the self-adjoint positive definite operator A by $\{\lambda_j\}_{j\in N_+}$. Thus we have $0 < \lambda_1 < \lambda_2 < \dots < \lambda_n < \dots$, and $\lambda_n \to +\infty$ $(n \to +\infty)$. The corresponding eigenvector series is $\{\omega_j\}_{j\in N_+}$. Let $D(A^s) = \{u \in D(A^{1/2}) : A^s u \in H\}$, and

$$\begin{gathered}(u,v)_{D(A^s)} = (A^s u, A^s u) = \sum_{j=1}^{+\infty} \lambda_j^{2s} (u,\omega_j)(v,\omega_j), \quad \forall u,v \in D(A^s), \\ \|u\|_{D(A^s)}^2 = (u,u)_{D(A^s)} = \sum_{j=1}^{+\infty} \lambda_j^{2s} (u,\omega_j)^2,\ \forall u \in D(A^s).\end{gathered}$$

Especially, $H = D(A^0)$, and we denote $V = D(A^{\frac{\alpha}{2}})$.

Now, we state the following well posedness of the solution of system (1.1)-(1.2) which can be derived by Galerkin's method just as in [4].

Lemma 2.1. *Assume that* (A1)–(A3) *hold. If* $(u_0, u_1), (v_0, v_1) \in V \cap D(A^\beta) \times H$*, then system* (1.1)-(1.2) *exists only one weak solution* $(u, v) = (u(x, t), v(x, t))$ *satisfying*

$$(u, v) \in L^\infty(0, T; D(A^\beta)) \cap L^2(0, T; D(A^{\frac{\alpha+\beta}{2}})),$$
$$(u_t, v_t) \in L^\infty(0, T; H) \cap L^2(0, T; D(A^{\beta/2})).$$

We let

$$\hat{M}(z) = \int_0^z M(s)ds, \quad \hat{N}(z) = \int_0^z N(s)ds,$$

and define the energy functional

$$E(t) = \frac{1}{2}\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) + J(t), \tag{2.5}$$

where

$$J(t) = \frac{1}{2}w^2(t) - \int_\Omega F(u, v)dx,$$
$$w(t) = \Big(\|Au\|_2^2 + \|Av\|_2^2 + \hat{M}(\|A^{\alpha/2}u\|_2^2 + \|A^{\alpha/2}v\|_2^2)\Big)^{1/2}. \tag{2.6}$$

By a simple calculation, we obtain

$$E'(t) = -N(\|A^{\beta/2}u\|_2^2)\|A^{\beta/2}u_t\|_2^2 - N(\|A^{\beta/2}v\|_2^2)\|A^{\beta/2}v_t\|_2^2. \tag{2.7}$$

From (2.7) it follows that the energy $E(t)$ is non-increasing. Employing (2.1), (2.2) and (2.6), we conclude that

$$\begin{aligned}
\int_\Omega F(u, v)dx &\le c_0\Big(\|u\|_{p+1}^{p+1} + \|v\|_{p+1}^{p+1}\Big) \\
&\le c_0 c_3^{p+1}\left(\|Au\|_2^{p+1} + \|Av\|_2^{p+1}\right) \\
&\le 2c_0 c_3^{p+1} w^{p+1}(t) := \frac{\eta}{p+1} w^{p+1}(t),
\end{aligned} \tag{2.8}$$

where $\eta = 2(p+1)c_0 c_3^{p+1}$ is a positive constant.

3. Exponential decay result

In this section, we prove a decay result for system (1.1)-(1.2). For this purpose, we define the potential well

$$W = \Big\{(u, v) \in D(A) \times D(A) : I(t) = w^2(t) - \gamma \int_\Omega F(u, v)dx > 0\Big\} \cup (0, 0).$$

Lemma 3.1. *Let* (u, v) *be the solution of system* (1.1)-(1.2) *and assume that* (A1)–(A3) *hold. If* $(u_0, v_0) \in W$*, and*

$$\zeta = \frac{\eta}{p+1}\Big(\frac{2\gamma}{\gamma - 2}E(0)\Big)^{\frac{p-1}{2}} < \frac{1}{\gamma},$$

then

$$(u(t), v(t)) \in W, \quad \forall t \ge 0.$$

Proof. If $(u_0, v_0) \in W$, from the definition of W, we obtain $I(0) > 0$. By the continuity of $I(t)$, there exists $T^* \in (0, \infty)$, such that for $t \in [0, T^*]$, $I(t) \geq 0$. Then we have

$$J(t) = \frac{\gamma - 2}{2\gamma} w^2(t) + \frac{1}{\gamma} I(t) \geq \frac{\gamma - 2}{2\gamma} w^2(t), \quad t \in [0, T^*],$$

thus we obtain

$$w^2(t) \leq \frac{2\gamma}{\gamma - 2} J(t) \leq \frac{2\gamma}{\gamma - 2} E(t) \leq \frac{2\gamma}{\gamma - 2} E(0), \quad t \in [0, T^*].$$

Combining this and (2.8), we obtain

$$\int_\Omega F(u, v) dx \leq \zeta w^2(t), \quad t \in [0, T^*].$$

Then by the assumption on ζ, we have

$$(u(t), v(t)) \in W, \quad t \in [0, T^*].$$

Repeating the process, T^* extends increasingly. □

Lemma 3.2. *Let (u, v) be the solution of system* (1.1)-(1.2)*, We assume that* (A1)–(A3) *hold and the function $M(z)$ satisfies*

$$\hat{M}(z) \leq M(z) z, \quad z \geq 0. \tag{3.1}$$

Then the functional

$$F(t) = (u, u_t) + (v, v_t) + \frac{1}{2} \hat{N}(\|A^{\beta/2} u\|_2^2) + \frac{1}{2} \hat{N}(\|A^{\beta/2} v\|_2^2)$$

satisfies

$$F'(t) \leq -I(t) + \|u_t\|_2^2 + \|v_t\|_2^2. \tag{3.2}$$

Proof. Differentiating $F(t)$ and by (1.1), we have

$$\begin{aligned} F'(t) &= \|u_t\|_2^2 + \|v_t\|_2^2 - M(\|A^{\alpha/2} u\|_2^2 + \|A^{\alpha/2} v\|_2^2)(\|A^{\alpha/2} u\|_2^2 + \|A^{\alpha/2} v\|_2^2) \\ &\quad - \|Au\|_2^2 - \|Av\|_2^2 + \gamma \int_\Omega F(u, v) dx. \end{aligned}$$

By (3.1) and the definition of $I(t)$, we obtain (3.2). □

We define the Lyapunov functional

$$L(t) = mE(t) + F(t), \tag{3.3}$$

in which m is a big positive constant to be determined later.

Theorem 3.3. *If the assumptions of Lemma 3.1 and* (3.1) *hold, and $N(z) \in L^\infty(0, \infty)$, then there exist two positive constants ω and κ, such that*

$$E(t) \leq \kappa e^{-\omega t}, \quad t \geq 0. \tag{3.4}$$

Proof. From (2.4), (2.5), (3.3), Young's inequality, Lemma 3.1, and combining with the condition $N(z) \in L^\infty(0,\infty)$, we have

$$\begin{aligned}
&L(t) - \frac{m}{2}E(t) = F(t) + \frac{m}{2}E(t) \\
&\geq -\frac{1}{2}\left(\|u\|_2^2 + \|v\|_2^2 + \|u_t\|_2^2 + \|v_t\|_2^2\right) - c\|A^{\beta/2}u\|_2^2 - c\|A^{\beta/2}v\|_2^2 + \frac{m}{2}E(t) \\
&\geq -c\left(\|Au\|_2^2 + \|Av\|_2^2 + \|u_t\|_2^2 + \|v_t\|_2^2\right) + \frac{m}{2}E(t) \\
&= \Big(\frac{m}{4} - c\Big)\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) + \Big(\frac{m(\gamma-2)}{4\gamma} - c\Big)\left(\|Au\|_2^2 + \|Av\|_2^2\right) + \frac{m}{2\gamma}I(t) \\
&\quad + \frac{m(\gamma-2)}{4\gamma}\hat{M}(\|A^{\alpha/2}u\|_2^2 + \|A^{\alpha/2}v\|_2^2) \\
&\geq \Big(\frac{m}{4} - c\Big)\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) + \Big(\frac{m(\gamma-2)}{4\gamma} - c\Big)\left(\|Au\|_2^2 + \|Av\|_2^2\right).
\end{aligned} \tag{3.5}$$

On the other hand, by similar calculation, we obtain

$$\begin{aligned}
&2mE(t) - L(t) = mE(t) - F(t) \\
&\geq \Big(\frac{m}{2} - c\Big)\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) + \Big(\frac{m(\gamma-2)}{2\gamma} - c\Big)\left(\|Au\|_2^2 + \|Av\|_2^2\right).
\end{aligned} \tag{3.6}$$

We choose N large enough, such that

$$L(t) - \frac{N}{2}E(t) \geq 0, \quad 2NE(t) - L(t) \geq 0,$$

thus we obtain

$$L(t) \sim E(t). \tag{3.7}$$

From Lemma 3.1, we have a constant $\eta_1 \in (0,1)$, such that

$$\gamma\int_\Omega F(u,v)dx \leq (1-\eta_1)w^2(t).$$

Hence we have

$$I(t) \geq \eta_1 w^2(t).$$

Then we arrive at

$$\begin{aligned}
E(t) &= \frac{1}{2}\left(||u_t||_2^2 + ||v_t||_2^2\right) + \frac{\gamma-2}{2\gamma}w^2(t) + \frac{1}{\gamma}I(t) \\
&\leq \frac{1}{2}\left(||u_t||_2^2 + ||v_t||_2^2\right) + \eta_2 I(t),
\end{aligned} \tag{3.8}$$

where $\eta_2 = \frac{\gamma-2}{2\gamma\eta_1} + \frac{1}{\gamma}$.

Differentiating $L(t)$ and by (2.3), (2.4), (2.7), (3.2), we obtain

$$L'(t) \leq -(cN-1)(\|u_t\|_2^2 + \|v_t\|_2^2) - I(t).$$

Let N large enough, such that $cN-1>0$ and (3.7) holds, exploiting (3.8), we have

$$L'(t) \leq -cE(t).$$

Because of (3.7), we have some constant $\omega > 0$ such that

$$L'(t) \leq -\omega L(t). \tag{3.9}$$

Integrating (3.9), we have $L(t) \leq ce^{-\omega t}$. This completes the proof. $\square$

4. Blow-up result

Let
$$G(\lambda)=\frac{1}{2}\lambda^2-\frac{\eta}{p+1}\lambda^{p+1},\quad \lambda>0.$$
By calculation, we can get that $E_1:=G(\lambda_1)=\frac{p-1}{2(p+1)}\lambda_1^2$ is the maximum value of the function $G(\lambda)$, here $\lambda_1=\eta^{-\frac{1}{p-1}}$.

Lemma 4.1. *Let (u,v) be the solution of system (1.1)-(1.2). We assume that (A1),(A2) hold, $w(0)>\lambda_1$ and $0<E(0)<E_1$, then there exists λ_2, such that*
$$w(t)\geq\lambda_2>\lambda_1,\quad t\geq 0,$$
and
$$\int_\Omega F(u,v)dx\geq\frac{\eta}{p+1}\lambda_2^{p+1}.$$

Lemma 4.2 ([15])**.** *Suppose that there is a positive, twice-differential function $Y(t)$ satisfies the inequality*
$$Y''(t)Y(t)-\varsigma\left(Y'(t)\right)^2\geq 0,\quad t\geq 0,$$
where the constant $\varsigma>1$, then there is a $t^*<\frac{Y(0)}{(\varsigma-1)Y'(0)}$ such that $Y(t)\to\infty$ as $t\to t^*$.

Theorem 4.3. *Let (u,v) be the solution of system (1.1)-(1.2). We assume that (A1), (A2) hold and*
$$\hat{M}(z)\geq M(z)z,\quad \hat{N}(z)\geq N(z)z. \tag{4.1}$$
If anyone of the following conditions is satisfied:

- (i) $E(0)<0$;
- (ii) $E(0)=0$, $2(u_0,u_1)+2(v_0,v_1)>0$;
- (iii) $0<E(0)<\varrho E_1$,*where* $\varrho=\min\{1,\frac{p+1}{(\gamma-1)(p-1)}(\gamma-2-\frac{p-1}{p+1})\}$ *and* $\gamma\geq 3$,

then system (1.1)-(1.2) blows up in finite time.

Proof. We prove this theorem by contradiction. Assume that the solution (u,v) is global. Then we can define, for sufficiently large $T>0$,
$$\begin{aligned}\Phi(t)&=\|u\|_2^2+\|v\|_2^2+\int_0^t\hat{N}(\|A^{\beta/2}u(t-s)\|_2^2)ds+\int_0^t\hat{N}(\|A^{\beta/2}v(t-s)\|_2^2)ds\\&\quad+(T-t)\left[\hat{N}(\|A^{\beta/2}u_0\|_2^2)|+\hat{N}(\|A^{\beta/2}v_0\|_2^2)\right]+k_0(t+t_0)^2,\quad t\in[0,T],\end{aligned}$$
where k_0, $t_0\geq 0$ are constants to be determined later.

Differentiating $\Phi(t)$, we have
$$\begin{aligned}\Phi'(t)&=2(u,u_t)+2(v,v_t)+2\int_0^t N(\|A^{\beta/2}u(s)\|_2^2)\left(A^{\beta/2}u_t(s),A^{\beta/2}u(s)\right)ds\\&\quad+2\int_0^t N(\|A^{\beta/2}v(s)\|_2^2)\left(A^{\beta/2}v_t(s),A^{\beta/2}v(s)\right)ds+2k_0(t+t_0).\end{aligned}$$
Taking the derivation of $\Phi'(t)$, we obtain
$$\begin{aligned}\Phi''(t)&=2\|u_t\|_2^2+2\|v_t\|_2^2+2\gamma\int_\Omega F(u,v)dx-2\|Au\|_2^2-2\|Av\|_2^2\\&\quad-2M(\|A^{\alpha/2}u\|_2^2+\|A^{\alpha/2}v\|_2^2)(\|A^{\alpha/2}u\|_2^2+\|A^{\alpha/2}v\|_2^2)+2k_0.\end{aligned}$$

In the following, we deal with $\Phi''(t)$ in different situations.

Cases (i) **and** (ii)**:** By (2.5), (2.7), (4.1) and $\gamma > 2$ we have

$$
\begin{aligned}
&\Phi''(t)\\
&= 2\gamma \int_0^t N(\|A^{\beta/2}u(s)\|_2^2)\|A^{\beta/2}u_t(s)\|_2^2 + N(\|A^{\beta/2}v(s)\|_2^2)\|A^{\beta/2}v_t(s)\|_2^2 ds\\
&\quad + 2\gamma\,[E(t) - E(0)] + \Phi''(t)\\
&\geq (\gamma-2)\Big[\|Au\|_2^2 + \|Av\|_2^2 + M(\|A^{\alpha/2}u\|_2^2 + \|A^{\alpha/2}v\|_2^2)(\|A^{\alpha/2}u\|_2^2 + \|A^{\alpha/2}v\|_2^2)\Big]\\
&\quad + 2\gamma \int_0^t N(\|A^{\beta/2}u(s)\|_2^2)\|A^{\beta/2}u_t(s)\|_2^2 + N(\|A^{\beta/2}v(s)\|_2^2)\|A^{\beta/2}v_t(s)\|_2^2 ds\\
&\quad + (\gamma+2)\,\big[\|u_t\|_2^2 + \|v_t\|_2^2\big] - 2\gamma E(0) + 2k_0\\
&\geq (\gamma+2)\Big[\int_0^t N(\|A^{\beta/2}u(s)\|_2^2)\|A^{\beta/2}u_t(s)\|_2^2 + N(\|A^{\beta/2}v(s)\|_2^2)\|A^{\beta/2}v_t(s)\|_2^2 ds\\
&\quad + \|u_t\|_2^2 + \|v_t\|_2^2 + k_0\Big] - \gamma\,[k_0 + 2E(0)]\,.
\end{aligned}
$$

Let

$$
\begin{gathered}
P = \|u\|_2^2, \quad Q = \|v\|_2^2, \quad \tilde{P} = \|u_t\|_2^2, \quad \tilde{Q} = \|v_t\|_2^2,\\
R = \int_0^t N(\|A^{\beta/2}u(s)\|_2^2)\|A^{\beta/2}u(s)\|_2^2 ds,\\
S = \int_0^t N(\|A^{\beta/2}v(s)\|_2^2)\|A^{\beta/2}v(s)\|_2^2 ds,\\
\tilde{R} = \int_0^t N(\|A^{\beta/2}u(s)\|_2^2)\|A^{\beta/2}u_t(s)\|_2^2 ds,\\
\tilde{S} = \int_0^t N(\|A^{\beta/2}v(s)\|_2^2)\|A^{\beta/2}v_t(s)\|_2^2 ds.
\end{gathered}
$$

We select $0 < k_0 < -2E(0)$ in Case (i) and $k_0 = 0$ in Case (ii), then by the inequality

$$
\begin{aligned}
&\int_0^t N(\|u(s)\|_2^2)(u_t(s), u(s)) ds\\
&\leq \int_0^t N(\|u(s)\|_2^2)\|u_t(s)\|_2\|u(s)\|_2 ds\\
&\leq \Big(\int_0^t N(\|u(s)\|_2^2)\|u_t(s)\|_2^2 ds\Big)^{1/2}\Big(\int_0^t N(\|u(s)\|_2^2)\|u(s)\|_2^2 ds\Big)^{1/2},
\end{aligned}
$$

By using Hölder inequality and (4.1), we obtain

$$
\begin{aligned}
&\Phi''\Phi - \frac{\gamma+2}{4}(\Phi')^2\\
&\geq (\gamma+2)\,\big[P + Q + R + S + k_0(t+t_0)^2\big]\,\big[\tilde{P} + \tilde{Q} + \tilde{R} + \tilde{S} + k_0\big]\\
&\quad - (\gamma+2)[P^{1/2}\tilde{P}^{1/2} + Q^{1/2}\tilde{Q}^{1/2} + R^{1/2}\tilde{R}^{1/2} + S^{1/2}\tilde{S}^{1/2} + k_0(t+t_0)]^2 \geq 0.
\end{aligned}
$$

In Case (i), we take t_0 sufficiently large such that

$$
\Phi'(0) = 2(u_0, u_1) + 2(v_0, v_1) + 2k_0 t_0 > 0.
$$

Noticing that $\Phi(0) > 0$, by Lemma 4.2, we conclude that there exist $t^* > 0$, such that

$$\lim_{t\to t^*} \Phi(t) = \infty.$$

Since t^* is independent of T, we assume that $t^* < T$, which is contradicted the hypothesis that the solution (u, v) is global.

In Case (ii), we have $\Phi(0) > 0$ and $\Phi'(0) > 0$, then we use the same argument as Case (i).

Case (iii): By (2.5)-(2.7), (4.1), $\gamma > 3$ and Lemma 4.1, we obtain

$$\begin{aligned}
&\Phi''(t)\\
&\geq (\gamma+1)\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) + (\gamma-3)w^2(t) + 2k_0 + 2\int_\Omega F(u,v)dx - 2(\gamma-1)E(t)\\
&= (\gamma+1)\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) + (\gamma-3)w^2(t) + 2k_0 + 2\int_\Omega F(u,v)dx - 2(\gamma-1)E(0)\\
&\quad + 2(\gamma-1)\int_0^t N(\|A^{\beta/2}u(s)\|_2^2)\|A^{\beta/2}u_t(s)\|_2^2 + N(\|A^{\beta/2}v(s)\|_2^2)\|A^{\beta/2}v_t(s)\|_2^2 ds\\
&\geq \gamma\left[\tilde{P} + \tilde{Q} + \tilde{R} + \tilde{S} + k_0\right] + (\gamma-3)w(t)^2 + 2\int_\Omega F(u,v)dx\\
&\quad - 2(\gamma-1)E(0) - (\gamma-2)k_0\\
&\geq \gamma\left[\tilde{P} + \tilde{Q} + \tilde{R} + \tilde{S} + k_0\right] + (\gamma-3)\lambda_2^2 + \frac{2\eta}{p+1}\lambda_2^{p+1}\\
&- 2(\gamma-1)E(0) - (\gamma-2)k_0\\
&\geq \gamma\left[\tilde{P} + \tilde{Q} + \tilde{R} + \tilde{S} + k_0\right] + \left(\gamma - 2 - \frac{p-1}{p+1}\right)\lambda_1^2\\
&\quad - 2(\gamma-1)E(0) - (\gamma-2)k_0.
\end{aligned}$$

By denoting $C := \left(\gamma - 2 - \frac{p-1}{p+1}\right)\lambda_1^2 - 2(\gamma-1)E(0)$, we have

$$\Phi''(t) \geq \gamma\left[\tilde{P} + \tilde{Q} + \tilde{R} + \tilde{S} + k_0\right] + C - (\gamma-2)k_0.$$

Furthermore we know that $C > 0$ because of $E(0) < \varrho E_1$. By selecting

$$0 < k_0 \leq \frac{C}{\gamma - 2},$$

we obtain

$$\Phi''(t) \geq \gamma[\tilde{P} + \tilde{Q} + \tilde{R} + \tilde{S} + k_0].$$

Finally we have

$$\begin{aligned}
&\Phi''\Phi - \frac{\gamma}{4}(\Phi')^2\\
&\geq \gamma\left[P + Q + R + S + k_0(t+t_0)^2\right][\tilde{P} + \tilde{Q} + \tilde{R} + \tilde{S} + k_0]\\
&\quad - \gamma[P^{1/2}\tilde{P}^{1/2} + Q^{1/2}\tilde{Q}^{1/2} + R^{1/2}\tilde{R}^{1/2} + S^{1/2}\tilde{S}^{1/2} + k_0(t+t_0)]^2\\
&\geq 0.
\end{aligned}$$

Similarity to the Case (i), we select t_0 sufficiently large such that

$$\Phi'(0) = 2(u_0, u_1) + 2(v_0, v_1) + 2k_0t_0 > 0.$$

Noticing that $\Phi(0) > 0$, we repeat the process and conclude the desired result. $\square$

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