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CURVATURE BLOW-UP FOR THE PERIODIC CH-MCH-NOVIKOV EQUATION

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Abstract. We study the CH-mCH-Novikov equation with cubic nonlinearity, which is derived by an asymptotic method from the classical shallow water theory. This model can be related to three different important shallow water equations: CH equation, mCH equation and Novikov equation. We show the curvature blow-up of the CH-mCH-Novikov equation by the method of characteristics and conserved quantities to the Riccati-type differential inequality.

1. Introduction

We consider the periodic equation with cubic nonlinearity which is an asymptotic model from the classical shallow water theory, called the CH-mCH-Novikov equation

$$\begin{gathered} m_t + k_1(2u_x m + u m_x) + k_2((u^2 - u_x^2)m)_x + k_3(u^2 m_x + 3uu_x m) = 0, \\ t > 0,\ x \in \mathbb{S}, \\ u(0,x) = u_0(x), \quad x \in \mathbb{S}, \end{gathered} \tag{1.1}$$

where $m = u - u_{xx}$, and k_i $(i = 1,2,3)$ are constants.

We know that there are two important dimensionless parameters in water-wave theory: amplitude parameter $\varepsilon = a/h_0$ and shallowness parameter $\mu = h_0^2/\lambda^2$, where h_0 is the mean depth of water, a and λ are the typical amplitude and wavelength of the waves, respectively. When we say the shallow-water (or long-wave), it means there is a presumption of small depth (compared with wavelength), i.e. $\mu \ll 1$. Whereas there are at least two cases for the amplitude parameter $\varepsilon = a/h_0$: Boussinesq scaling (weakly nonlinear regime): $\mu \ll 1$, $\varepsilon = O(\mu)$; and the Camassa-Holm (CH) scaling (moderately nonlinear regime): $\mu \ll 1$, $\varepsilon = O(\sqrt{\mu})$.

The following equation is derived for the scaled surface elevation by using $\mu \ll 1$ and $\varepsilon = O(\mu^{2/5})$ [3]:

$$\begin{aligned} &m_t + u_x - \frac{\mu}{4}u_{xxx} + \frac{\varepsilon}{2}(2u_x m + u m_x) + \frac{c_1\varepsilon^2}{4}((u^2 - \beta\mu u_x^2)m)_x \\ &- \frac{c_2\varepsilon^2}{4}(u^2 m_x + 3uu_x m) = 0 + O(\varepsilon^5, \mu^2), \end{aligned}$$

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where $m = u - \beta\mu u_{xx}$. By scaling $u(t,x) \to \frac{1}{2}\varepsilon u(\sqrt{\beta\mu}t, \sqrt{\beta\mu}x)$, we obtain

$$m_t + u_x - \frac{3}{5}u_{xxx} + 2u_x m + u m_x + c_2((u^2 - u_x^2)m)_x + c_3(u^2 m_x + 3uu_x m) = 0.$$

Equation (1.1) is the general case of the above equation, related to the three integrable systems: Camassa-Holm equation, modified Camassa-Holm equation and Novikov equation.

When $k_1 = 1,\ k_2 = 0$ and $k_3 = 0$, (1.1) reduces to

$$m_t + u m_x + 2m u_x = 0, \quad m = u - u_{xx}. \tag{1.2}$$

This model (1.2) is derived by using the CH scaling $\mu \ll 1,\ \varepsilon = O(\sqrt{\mu})$, which is proposed as a model to describe the uni-directional propagation of shallow water waves over a flat bottom [2, 9]. It also models the propagation of axially symmetric waves in hyperelastic rods [7, 14]. The CH equation is completely integrable for a large class of initial data, for which it can be solved by the inverse scattering method [4]. In contrast to the KdV equation, the CH equation has three remarkable distinctive properties [23, 24]. First, although CH is completely integrable, it can describe wave breaking phenomena. The second is the existence of peakons. Indeed, the CH equation has the single peakon [2] and the multi-peakon solutions [12]. It is significant that the peakons are orbitally stable: the shape is stable under small perturbations [6, 15]. These peakons capture a feature of the waves of greatest height for the free-boundary incompressible Euler equations [20]. The last one is the variety of interesting geometric formulations of the CH equation [5, 8, 14, 16].

When $k_1 = 0,\ k_2 = 1$ and $k_3 = 0$, (1.1) reduces to

$$m_t + ((u^2 - u_x^2)m)_x = 0. \tag{1.3}$$

The mCH equation (1.3) is derived by applying the method of tri-Hamiltonian duality to the bi-Hamiltonian representation of the modified Korteweg-deVries (mKdV) equation [8, 19]. The equation is formally integrable and can be rewritten as the bi-Hamiltonian form and the Lax pair [19]. Moreover, the mCH equation exhibits new features, including wave breaking and blow up criteria that do not appear in the original CH equation [10]. On the other hand, since the mCH equation also arises from an intrinsic (arc-length preserving) invariant planar curve flow in Euclidean geometry [10], it can be regarded as a Euclidean-invariant counterpart to the KdV equation from the viewpoint of curve flows in Klein geometries [5, 17].

When $k_1 = 0$, $k_2 = 0$ and $k_3 = 1$, (1.1) becomes to the Novikov equation [21, 22]:

$$m_t + u^2 m_x + 3uu_x m = 0.$$

It is known that the Novikov equation is integrable with Lax pair [18]. A matrix Lax pair representation to the Novikov equation was provided by Hone and Wang [13]. With that representation it can be shown that the Novikov equation is related to a negative flow in the Sawada-Kotera hierarchy. It is also noticed that the Novikov equation admits a bi-Hamiltonian structure [13]. Hone, Lundmark and Szmigielski [11] obtained multi-peakons of the Novikov equation explicitly by using the inverse scattering approach.

The motivation of this study comes from the curvature blow-up for cubic nonlinear models. We know that the curvature blow-up phenomena is be found in the mCH equation [10] and the generalized modified Camassa-Holm(gmCH) equation [1]. This leads to a natural question of understanding how the interaction between these cubic nonlinearities would affect the singularity formation mechanism. In this paper, the key ingredients are that we will choose initial data such that m_0 does not change sign. This eliminates fast local oscillation of solutions, i.e. $u \pm u_x \geq 0$.

The remainder of the paper is organized as follows. In Section 2, we present some preliminary results. In Section 3, we are devoted to the precise blow-up scenario about the CH-mCH-Novikov equation. In Section 4, the curvature blow-up data are illustrated.

2. Preliminaries

To discuss the wave breaking phenomenon of the periodic CH-mCH-Novikov equation (1.1), we rewrite it as

$$\begin{gathered} u_t = -k_1 G * (2u_x m + u m_x) - k_2 G * ((u^2 - u_x^2)m)_x - k_3 G * (u^2 m_x + 3u u_x m), \\ t > 0,\ x \in \mathbb{S}, \\ u(0,x) = u_0(x), \quad x \in \mathbb{S}, \end{gathered} \tag{2.1}$$

where $G(x) = \frac{\cosh(x-[x]-\frac{1}{2})}{2\sinh(1/2)}$, $[x]$ represents the largest integer part of x, and $G(x)$ is the fundamental solution of $(1-\partial_x^2)^{-1}$ on the unit circle $\mathbb{S} = \mathbb{R}/\mathbb{Z}$, that is for any $x \in \mathbb{S}$.

Let $G(x) = \Lambda_1(x) + \Lambda_2(x)$, where $\Lambda_1(x) = \frac{e^{x-[x]-\frac{1}{2}}}{4\sinh(\frac{1}{2})}$ and $\Lambda_2(x) = \frac{e^{-x+[x]+\frac{1}{2}}}{4\sinh(\frac{1}{2})}$. Then $G_x(x) = \Lambda_2(x) - \Lambda_1(x)$.

Lemma 2.1. *Assume that $u_0 \in H^s(\mathbb{S}) \cap L^1(\mathbb{S})$ with $s > 5/2$. Suppose that u is the corresponding solution to* (1.1) *with the initial data u_0. Then*

$$H_0[u_0] = \int_{\mathbb{S}} (u^2 + u_x^2)\,dx = \int_{\mathbb{S}} (u_0^2 + u_{0,x}^2)\,dx. \tag{2.2}$$

Proof. We write (1.1) as

$$\begin{aligned} &u_t - u_{txx} + k_1(2u_x(u-u_{xx}) + u(u_x - u_{xxx})) + k_2((u^2-u_x^2)(u-u_{xx}))_x \\ &+ k_3(u^2(u_x - u_{xxx}) + 3uu_x(u-u_{xx})) = 0. \end{aligned} \tag{2.3}$$

Multiplying (2.3) by u and integrating by parts, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{S}} (u^2(t,x) + u_x^2(t,x))\,dx + k_1\int_{\mathbb{S}} (u_x^3 - u_x^3)\,dx \\ &- k_2 \int_{\mathbb{S}} (u^3 u_x - u^2 u_x u_{xx} - u u_x^3 + u_x^3 u_{xxx})\,dx \\ &+ k_3 \int_{\mathbb{S}} (u^3 u_x - u^3 u_{xxx} + 3u^3 u_x - 3u^2 u_x u_{xx})\,dx = 0. \end{aligned} \tag{2.4}$$

Then

$$\int_{\mathbb{S}} (u^2 + u_x^2)\,dx = \int_{\mathbb{S}} (u_0^2 + u_{0,x}^2)\,dx.$$

Then the proof of the lemma is complete. $\square$

The following inequality is often used for the wave-breaking phenomena of the periodic CH-mCH-Novikov equation.

Lemma 2.2. [24] *For every $f \in H^1(\mathbb{S})$, $\alpha \in \mathbb{R}$ we have*

$$\max_{x\in[0,1]} f^2(x) \leq \mu \int_{\mathbb{S}} (f^2 + \alpha^2 f_x^2)\,dx,$$

where

$$\mu = \frac{\cosh(\frac{1}{2\alpha})}{2\alpha\sinh(\frac{1}{2\alpha})}.$$

Moreover, μ is the minimum value. So in this sense μ is the optimal constant which is obtained by the associated Green function

$$G(x) = \frac{\cosh(\frac{x}{\alpha} - \frac{[x]}{\alpha} - \frac{1}{2\alpha})}{2\alpha\sinh(\frac{1}{2\alpha})}.$$

When $\alpha = 1$, the constant $\mu = \frac{e+1}{2(e-1)}$ is sharp.

Lemma 2.3. [25] *Let $-\frac{e+1}{e-1} \le \gamma \le \frac{e+1}{e-1}$, $\alpha \in \mathbb{R}$. Then if $u \in H^1(\mathbb{S})$ such that $u(t,1) = u(t,0)$, we obtain*

$$\begin{aligned} &(G \pm \gamma G_x) * (u^2 + \frac{1}{2}u_x^2 - \alpha u) \\ &\ge \begin{cases} \frac{1}{2}(u - \frac{\alpha}{2})^2 - \frac{\alpha^2}{4}, & |\gamma| \le 1, \\ \frac{1}{4}((e+1) - |\gamma|(e-1))(u - \frac{\alpha}{2})^2 - \frac{\alpha^2}{4}, & 1 \le |\gamma| \le \frac{e+1}{e-1}, \end{cases} \end{aligned} \tag{2.5}$$

and

$$\Lambda_{1,2} * (2u^2 + u_x^2) \ge \frac{1}{2}u^2.$$

3. Precise blow-up scenario

The local well-posedness theorem about the periodic CH-mCH-Novikov equation can be obtained from the standard argument of [3] with a slight modification.

Theorem 3.1. *Let $u_0 \in H^s(\mathbb{S}), s > 5/2$. Then there exists a time $T > 0$ such that the periodic problem* (1.1) *has a unique strong solution*

$$u \in C([0,T]; H^s(\mathbb{S})) \cap C^1([0,T]; H^{s-1}(\mathbb{S})).$$

Now we define the following characteristics associated to (1.1) as

$$\begin{gathered} q_t(t,x) = [k_1 u + k_2(u^2 - u_x^2) + k_3 u^2](t, q(t,x)), \quad x \in \mathbb{S}, t \in [0, T^*), \\ q(0,x) = x, \quad x \in \mathbb{S}. \end{gathered} \tag{3.1}$$

Then we can easily satisfy the following proposition.

Proposition 3.2. *Suppose that $u_0 \in H^s(\mathbb{S})$ with $s > 5/2$ and $T > 0$ be the maximal existence time of the strong solution u to the initial value problem* (3.1)*. Then* (3.1) *has a unique solution $q \in C^1([0,T) \times \mathbb{S})$ such that $q(t,\cdot)$ is an increasing diffeomorphism of $\mathbb{S}$ with*

$$q_x(t,x) = \exp(\int_0^t (k_1 u_x + 2k_2 m u_x + 2k_3 u u_x)(s, q(s,x)) ds), \tag{3.2}$$

for $(t,x) \in [0,T) \times \mathbb{S}$. Moreover, for all $(t,x) \in [0,T) \times \mathbb{S}$ there holds

$$m(t, q(t,x)) = m_0(x) \exp(-\int_0^t (2k_1 u_x + 2k_2 m u_x + 3k_3 u u_x)(s, q(s,x))\, dx), \tag{3.3}$$

where $m_0(x) = m(0,x)$.

Proof. From (1.1) it suffices to derive

$$m_t + (k_1 u + k_2(u^2 - u_x^2) + k_3 u^2) m_x = -(2k_1 u_x + 2k_2 u_x m + 3k_3 u u_x) m.$$

Then, using the characteristics (3.1), we obtain (3.3). $\square$

Remark 3.3. *Suppose $u_0 \in H^s(\mathbb{S})$ with $s > 5/2$. Let $T > 0$ be the maximal existence time of the strong solution u to the corresponding initial value problem* (1.1)*. If $m_0(x) > 0$ for all $x \in \mathbb{S}$, then $m(t,x) > 0$ for all $(t,x) \in [0,T) \times \mathbb{S}$. Moreover, we have $u \pm u_x \geq 0$.*

Similar to the other CH-type equation, (1.1) can be reformulated into a nonlocal transport form (2.1). We can get the following criterion lemma 3.4. The proof follows a similar idea as in [3], and hence we omit it.

Lemma 3.4. *Let $u_0 \in H^s(\mathbb{S}), s > 5/2$ and u be the solution of* (1.1)*. Assume that $T^* > 0$ is the maximum time of existence. Then*

$$T^* < \infty \Rightarrow \int_0^{T^*} \|k_1 u_x(\tau) + k_2 m u_x(\tau) + 2k_3 u u_x(\tau)\|_{L^\infty} d\tau = \infty. \tag{3.4}$$

Remark 3.5. The blow-up criterion (3.4) implies that the lifespan T^* does not depend on the regularity index s of the initial data u_0.

Moreover, we prove the following accurate wave-breaking criteria.

Lemma 3.6. *Suppose that $u_0 \in H^s(\mathbb{S}), s > 5/2$. The corresponding solution u to the periodic problem* (1.1) *blows up in finite time $T^* > 0$ if and only if*

$$\lim_{t\to T^*} \inf_{x\in\mathbb{S}} \{k_1 u_x(t,x) + k_2 m(t,x) u_x(t,x) + 2k_3 u(t,x) u_x(t,x)\} = -\infty. \tag{3.5}$$

Proof. In view of Remark 3.5, it suffices to consider the case $s = 3$. Suppose that if $k_1 u_x(t,x) + k_2 m(t,x) u_x(t,x) + 2k_3 u(t,x) u_x(t,x)$ is bounded from below on $[0,T^*) \times \mathbb{S}$, and there exists a constant $M > 0$ such that

$$k_1 u_x(t,x) + k_2 m(t,x) u_x(t,x) + 2k_3 u(t,x) u_x(t,x) \geq -M, \quad [0,T^*) \times \mathbb{S}. \tag{3.6}$$

Multiplying (1.1) by m and integrating over $\mathbb{S}$, and then integration by parts, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{S}} m^2\,dx + \frac{3k_1}{2}\int_{\mathbb{S}} m^2 u_x\,dx + \int_{\mathbb{S}} (k_2 u_x m + 2k_3 u u_x) m^2\,dx = 0. \tag{3.7}$$

The initial condition implies that $m_0 \in H^{s-2} \subset L^q$ for any $2 \leq q \leq \infty$. Similarly we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{S}} m_x^2\,dx + k_1\int_{\mathbb{S}} (2u_x m + u m_x)_x m_x\,dx + k_2\int_{\mathbb{S}} ((u^2 - u_x^2)m)_{xx} m_x\,dx \\ &+ k_3\int_{\mathbb{S}} (u^2 m_x + 3u u_x m)_x m_x\,dx = 0. \end{aligned} \tag{3.8}$$

Integrating by parts yields

$$k_1\int_{\mathbb{S}} (2u_x m + u m_x)_x m_x\,dx = -k_1\int_{\mathbb{S}} u_x m^2\,dx + \frac{5k_1}{2}\int_{\mathbb{S}} u_x m_x^2\,dx, \tag{3.9}$$

$$k_2\int_{\mathbb{S}} ((u^2 - u_x^2)m)_{xx} m_x\,dx = \int_{\mathbb{S}} (5k_2 u_x m) m_x^2\,dx - \int_{\mathbb{S}} (\frac{2}{3}k_2 u_x m) m^2\,dx, \tag{3.10}$$

$$\begin{aligned} &k_3\int_{\mathbb{S}} (u^2 m_x + 3u u_x m)_x m_x\,dx \\ &= \int_{\mathbb{S}} (4k_3 u u_x) m_x^2\,dx - \int_{\mathbb{S}} (6k_3 u u_x) m^2\,dx + \int_{\mathbb{S}} 8k_3 u m_x m^2\,dx. \end{aligned} \tag{3.11}$$

Plugging (3.9)-(3.11) into (3.8), we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{S}} m_x^2\,dx - k_1\int_{\mathbb{S}} u_x m^2\,dx + \frac{5k_1}{2}\int_{\mathbb{S}} u_x m_x^2\,dx \\ &+ \int_{\mathbb{S}} (5k_2 u_x m + 4k_3 u u_x) m_x^2\,dx - \int_{\mathbb{S}} (\frac{2}{3}k_2 u_x m + 6k_3 u u_x) m^2\,dx \\ &+ \int_{\mathbb{S}} 8k_3 u m_x m^2\,dx = 0. \end{aligned} \tag{3.12}$$

So from this, (3.7), and (3.12), we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{S}} (m^2 + m_x^2)\,dx \\ &= -\int_{\mathbb{S}} (k_1 u_x + k_2 u_x m + 2k_3 u u_x) m^2\,dx - \int_{\mathbb{S}} (\frac{5}{2}k_1 u_x + 5k_2 u_x m + 4k_3 u u_x) m_x^2\,dx \\ &\quad - \int_{\mathbb{S}} (\frac{2}{3}k_2 u_x m + 6k_3 u u_x) m^2\,dx + \int_{\mathbb{S}} 8k_3 u m_x m^2\,dx \\ &= -\int_{\mathbb{S}} (k_1 u_x + k_2 u_x m + 2k_3 u u_x) m^2\,dx - \int_{\mathbb{S}} (5k_1 u_x + 5k_2 u_x m + 10k_3 u u_x) m_x^2\,dx \\ &\quad + \frac{5}{2}k_1\int_{\mathbb{S}} u_x m_x^2\,dx + 6k_3\int_{\mathbb{S}} u u_x m_x^2\,dx - \int_{\mathbb{S}} (\frac{2}{3}k_2 u_x m + 6k_3 u u_x) m^2\,dx \\ &\quad + \int_{\mathbb{S}} 8k_3 u m_x m^2\,dx. \end{aligned}$$

Moreover,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{S}} (m^2 + m_x^2)\,dx \\ &\le 5\int_{\mathbb{S}} M(m^2 + m_x^2)\,dx + \frac{5}{2}k_1\int_{\mathbb{S}} u_x m_x^2\,dx + 6k_3\int_{\mathbb{S}} u u_x m_x^2\,dx \\ &\quad - \int_{\mathbb{S}} (\frac{2}{3}k_2 u_x m + 6k_3 u u_x) m^2\,dx + \int_{\mathbb{S}} 8k_3 u m_x m^2\,dx \\ &\le 5\int_{\mathbb{S}} M(m^2 + m_x^2)\,dx + \frac{5}{2}|k_1|\|u_x\|_{L^\infty}\|m\|_{H^1}^2 + 6|k_3|\|u\|_{L^\infty}\|u_x\|_{L^\infty}\|m\|_{H^1}^2 \\ &\quad + \frac{2|k_2| + 8|k_3|}{3}\|u\|_{H^1}^2\|m\|_{H^1}^2. \end{aligned}$$

Note that from Lemmas 2.1, 2.2 and 2.3, we have

$$\frac{1}{2}\frac{d}{dt}\|m\|_{H^1}^2 \le (5M + \frac{5}{2}|k_1|\sqrt{\mu H_0} + 6|k_3|\mu H_0 + \frac{2|k_2| + 8|k_3|}{3}H_0)\|m\|_{H^1}^2$$

Solving the inequality, it follows that

$$\|m(t)\|_{H^1}^2 \le e^{2(5M + \frac{5}{2}|k_1|\sqrt{\mu H_0} + 6|k_3|\mu H_0 + \frac{2|k_2| + 8|k_3|}{3}H_0)t}\|m_0\|_{H^1}^2,$$

for $t \in [0, T^*)$. Then Theorem 3.1 ensures that the solution does not blow-up in finite time. On the other hand, if

$$\lim_{t\to T^*}\{\inf_{x\in\mathbb{S}}(k_1 u_x(t,x) + k_2 m(t,x) u_x(t,x) + 2k_3 u(t,x) u_x(t,x))\} = -\infty,$$

then either u_x or m blows up in finite time. $\square$

Now, we present the dynamics of a few important quantities along the characteristics $q(t,x_0)$. Where $'$ denotes the derivative $\partial_t+(k_1u+k_2(u^2-u_x^2)+k_3u^2)\partial_x$ along the characteristics.

Lemma 3.7. *Let $u_0\in H^s(\mathbb{S})$ with $s>5/2$ and $\widehat{u}'(t)=u'(t,q(t,x_0))$, $\widehat{u_x}'(t)=u_x'(t,q(t,x_0))$, $\widehat{m}'(t)=m'(t,q(t,x_0))$, and $\widehat{M}'(t)=(mu_x)(t,q(t,x_0))$. Then $\widehat{u}'(t),\widehat{u_x}'(t),\widehat{m}'(t),\widehat{M}'(t)$ satisfy the following integro-differential equations*

$$\begin{aligned}
\widehat{u}'(t)&=-\frac{2}{3}k_2\widehat{u_x}^3+(\frac{k_2}{3}+\frac{k_3}{2})[\Lambda_1*(u-u_x)^3-\Lambda_2*(u+u_x)^3]\\
&\quad-k_1[\Lambda_2*(u^2+\frac{1}{2}u_x^2)-\Lambda_1*(u^2+\frac{1}{2}u_x^2)],\\
\widehat{u_x}'(t)&=k_1(\widehat{u}^2-\frac{1}{2}\widehat{u_x}^2)+k_2(\frac{1}{3}\widehat{u}^3-\widehat{u}\widehat{u_x}^2)+\frac{k_3\widehat{u}}{2}(\widehat{u}^2-\widehat{u_x}^2)\\
&\quad-(\frac{k_2}{3}+\frac{k_3}{2})[\Lambda_1*(u-u_x)^3+\Lambda_2*(u+u_x)^3]\\
&\quad-k_1[\Lambda_1*(u^2+\frac{1}{2}u_x^2)+\Lambda_2*(u^2+\frac{1}{2}u_x^2)],\\
\widehat{m}'(t)&=-(2k_1\widehat{u_x}+2k_2\widehat{u_x}\widehat{m}+3k_3\widehat{u}\widehat{u_x})\widehat{m},\\
\widehat{M}'(t)&=-2k_2\widehat{M}^2+k_1\widehat{m}(\widehat{u}^2-\frac{5}{2}\widehat{u_x}^2)+\frac{\widehat{m}\widehat{u}}{6}[(2k_2+3k_3)\widehat{u}^2\\
&\quad-(6k_2+21k_3)\widehat{u_x}^2]-(\frac{k_2}{3}+\frac{k_3}{2})\widehat{m}[\Lambda_1*(u-u_x)^3+\Lambda_2*(u+u_x)^3]\\
&\quad-k_1\widehat{m}[\Lambda_1*(u^2+\frac{1}{2}u_x^2)+\Lambda_2*(u^2+\frac{1}{2}u_x^2)].
\end{aligned}$$

Proof. In view of (1.1), we can obtain

$$\widehat{m}_t+(k_1\widehat{u}+k_2(\widehat{u}^2-\widehat{u_x}^2)+k_3\widehat{u}^2)\widehat{m_x}=-(2k_1\widehat{u_x}+2k_2\widehat{u_x}\widehat{m}+3k_3\widehat{u}\widehat{u_x})\widehat{m},\tag{3.13}$$

which is the equation about $\widehat{m}'(t)$. By (3.13), we obtain

$$\widehat{u_t}=-k_1G*(2u_xm+um_x)-k_2G*[(u^2-u_x^2)m]_x-k_3G*(u^2m+3uu_xm).\tag{3.14}$$

By a direct calculation, we have

$$\begin{aligned}
G*(2u_xm+um_x)&=uu_x+\Lambda_1*(u^2+\frac{1}{2}u_x^2)-\Lambda_2*(u^2+\frac{1}{2}u_x^2),\\
G*[(u^2-u_x^2)m]_x&=(u^2-u_x^2)u_x+\frac{2}{3}u_x^3-\frac{1}{3}[\Lambda_1*(u-u_x)^3-\Lambda_2*(u+u_x)^3],\\
G*(u^2m_x+3uu_xm)&=u^2u_x-\frac{1}{2}[\Lambda_1*(u-u_x)^3-\Lambda_2*(u+u_x)^3].
\end{aligned}$$

Therefore, plugging the above three equations into (3.14) leads to $\widehat{u}'(t)$.

Differentiating (3.14) with respect to x, we obtain

$$\widehat{u}_{xt}=-k_1G*(2u_xm+um_x)_x-k_2G*[(u^2-u_x^2)m]_{xx}-k_3G*(u^2m+3uu_xm)_x.\tag{3.15}$$

By the same method and calculating the following three items

$$\begin{aligned}
&G*(2u_xm+um_x)_x\\
&=uu_{xx}+u^2-\frac{1}{2}u_x^2+[\Lambda_1*(u^2+\frac{1}{2}u_x^2)+\Lambda_2*(u^2+\frac{1}{2}u_x^2)],
\end{aligned}\tag{3.16}$$

$$\begin{aligned}
&G*[(u^2-u_x^2)m]_{xx}\\
&=(u^2-u_x^2)u_{xx}+(\frac{1}{3}u^3-uu_x^2)-\frac{1}{3}[\Lambda_1*(u-u_x)^3+\Lambda_2*(u+u_x)^3],
\end{aligned}\tag{3.17}$$

$$
\begin{aligned}
&G*(u^2m_x+3uu_xm)_x\\
&=u^2u_{xx}+\frac{u}{2}(u^2-u_x^2)-\frac{1}{2}[\Lambda_1*(u-u_x)^3+\Lambda_2*(u+u_x)^3],
\end{aligned}
\tag{3.18}
$$

Note that $\widehat{u_x}'(t)$ can be obtained from (3.15)-(3.18). Moreover,

$$
\begin{aligned}
&\widehat{M}'(t)\\
&=(\widehat{m}\widehat{u_x})'(t)=\widehat{m}'(t)\widehat{u_x}(t)+\widehat{m}(t)\widehat{u_x}'(t)\\
&=-2k_2\widehat{M}^2+k_1\widehat{m}(\widehat{u}^2-\frac{5}{2}\widehat{u_x}^2)+\frac{\widehat{m}\widehat{u}}{6}[(2k_2+3k_3)\widehat{u}^2-(6k_2+21k_3)\widehat{u_x}^2]\\
&\quad-(\frac{k_2}{3}+\frac{k_3}{2})\widehat{m}[\Lambda_1*(u-u_x)^3+\Lambda_2*(u+u_x)^3]\\
&\quad-k_1\widehat{m}[\Lambda_1*(u^2+\frac{1}{2}u_x^2)+\Lambda_2*(u^2+\frac{1}{2}u_x^2)].
\end{aligned}
\tag{3.19}
$$

Thus, the proof is complete. □

4. Curvature blow-up

From the blow-up criterion Lemma 3.6, we know that the conservation law $H_0[u_0]$ indicates two possible scenarios for the formation of singularity, namely, the wave-breaking ($u_x\to\infty$) or curvature blow-up ($u_{xx}\to\infty$). Now we prove that the wave-breaking phenomena of the CH-mCH-Novikov equation (1.1) is the curvature blow-up ($u_{xx}\to\infty$).

Theorem 4.1. *Suppose that $k_1<0$, $k_2<0$, and (1)*

$$
\begin{gathered}
k_1^2\le-\frac{16}{21}k_2,\\
k_1^2+\frac{2}{3}k_2\le k_3\le-\frac{2}{21}k_2,
\end{gathered}
$$

or (2)

$$
\begin{gathered}
k_1^2>-\frac{16}{21}k_2,\\
\max\{k_1^2+\frac{2}{3}k_2,-\frac{2}{5}k_2\}<k_3<k_1^2-\frac{2}{5}k_2,
\end{gathered}
$$

where $m_0\in H^s(\mathbb{S})$ for $s>1/2$ and $m_0>0$. Assume that there exists some point $x_0\in\mathbb{S}$ such that $m_0(x_0)>0$ and

$$
u_{0,x}(x_0)\ge\Big(\frac{2k_2+3k_3-3k_1^2}{4k_2}\Big)^{1/2}u_0(x_0).
\tag{4.1}
$$

Then the solution $u(t,x)$ blows up in finite time $T^*\le-\frac{1}{2k_2m_0(x_0)u_{0,x}(x_0)}$.

Proof. From (3.19), we have the equation

$$
\begin{aligned}
&\widehat{M}'(t)\\
&=-2k_2\widehat{M}^2+k_1\widehat{m}(\widehat{u}^2-\frac{5}{2}\widehat{u_x}^2)+\frac{\widehat{m}\widehat{u}}{6}[(2k_2+3k_3)\widehat{u}^2-(6k_2+21k_3)\widehat{u_x}^2]\\
&\quad-(\frac{k_2}{3}+\frac{k_3}{2})\widehat{m}[\Lambda_1*(u-u_x)^3+\Lambda_2*(u+u_x)^3]\\
&\quad-k_1\widehat{m}[\Lambda_1*(u^2+\frac{1}{2}u_x^2)+\Lambda_2*(u^2+\frac{1}{2}u_x^2)].
\end{aligned}
\tag{4.2}
$$

According to (3.3), we know that $\widehat{m}$, $\widehat{u} > 0$ when $m_0(x_0) > 0$. To obtain a Riccati-type inequality from (4.2), we assume that

$$k_1 < 0,\ k_2 < 0,\ \frac{k_2}{3} + \frac{k_3}{2} < 0 \ \Leftrightarrow\ k_3 < -\frac{2}{3}k_2. \tag{4.3}$$

Therefore,

$$\begin{aligned}
&\widehat{M}'(t)\\
&\geq -2k_2\widehat{M}^2 + k_1\widehat{m}(\widehat{u}^2 - \frac{5}{2}\widehat{u_x}^2) + \frac{\widehat{m}\widehat{u}}{6}[(2k_2+3k_3)\widehat{u}^2 - (6k_2+21k_3)\widehat{u_x}^2] - \frac{1}{2}k_1\widehat{m}\widehat{u}^2\\
&= -2k_2\widehat{M}^2 + \frac{k_1}{2}\widehat{m}(\widehat{u}^2 - 5\widehat{u_x}^2) + \frac{\widehat{m}\widehat{u}}{6}[(2k_2+3k_3)\widehat{u}^2 - (6k_2+21k_3)\widehat{u_x}^2].
\end{aligned} \tag{4.4}$$

Suppose that

$$\begin{gathered}
1 - 5\frac{\widehat{u_x}^2}{\widehat{u}^2} \leq 0 \Rightarrow \frac{\widehat{u_x}^2}{\widehat{u}^2} \geq \frac{1}{5},\\
(2k_2+3k_3) - (6k_2+21k_3)\frac{\widehat{u_x}^2}{\widehat{u}^2} \geq 0 \Rightarrow \frac{\widehat{u_x}^2}{\widehat{u}^2} \geq \frac{2k_2+3k_3}{6k_2+21k_3},
\end{gathered} \tag{4.5}$$

where $6k_2 + 21k_3 \leq 0 \Leftrightarrow k_3 \leq -\frac{2}{7}k_2$. Comparing with the values of $\frac{1}{5}$ and $\frac{2k_2+3k_3}{6k_2+21k_3}$, we discuss it in two cases.

(1) When

$$\frac{1}{5} \leq \frac{2k_2+3k_3}{6k_2+21k_3} \Leftrightarrow \frac{2}{3}k_2 \leq k_3 \leq -\frac{2}{7}k_2, \tag{4.6}$$

we have

$$\frac{\widehat{u_x}^2}{\widehat{u}^2} \geq \frac{2k_2+3k_3}{6k_2+21k_3}. \tag{4.7}$$

In particular, a finite-time blow-up of $\widehat{M}$ is realized if the ration $|\frac{u_x}{u}|$ stays reasonable big along the characteristics. If $k_1 < 0$, using Lemma 2.3 and $u_x \to +\infty$ we have

$$\begin{aligned}
(\frac{\widehat{u_x}}{\widehat{u}})' &= \frac{1}{\widehat{u}^2}[k_1\widehat{u}(\widehat{u}^2 - \frac{1}{2}\widehat{u_x}^2) - k_1(\widehat{u}+\widehat{u_x})\Lambda_1 * (u^2 + \frac{1}{2}u_x^2)\\
&\quad - k_1(\widehat{u}-\widehat{u_x})\Lambda_2 * (u^2 + \frac{1}{2}u_x^2)] + \frac{\widehat{u}^2 - \widehat{u_x}^2}{\widehat{u}^2}[(\frac{k_2}{3} + \frac{k_3}{2})\widehat{u}^2 - \frac{2k_2}{3}\widehat{u_x}^2]\\
&\quad - \frac{2k_2+3k_3}{6\widehat{u}^2}[(\widehat{u}+\widehat{u_x})\Lambda_1 * (u-u_x)^3 + (\widehat{u}-\widehat{u_x})\Lambda_2 * (u+u_x)^3]\\
&= \frac{1}{\widehat{u}^2}[k_1\widehat{u}(\widehat{u}^2 - \frac{1}{2}\widehat{u_x}^2) - k_1\widehat{u}G * (u^2 + \frac{1}{2}u_x^2) + k_1\widehat{u_x}G_x * (u^2 + \frac{1}{2}u_x^2)]\\
&\quad + \frac{\widehat{u}^2 - \widehat{u_x}^2}{\widehat{u}^2}[(\frac{k_2}{3} + \frac{k_3}{2})\widehat{u}^2 - \frac{2k_2}{3}\widehat{u_x}^2]\\
&\quad - \frac{2k_2+3k_3}{6\widehat{u}^2}[(\widehat{u}+\widehat{u_x})\Lambda_1 * (u-u_x)^3 + (\widehat{u}-\widehat{u_x})\Lambda_2 * (u+u_x)^3].
\end{aligned}$$

Moreover,

$$\begin{aligned}
(\frac{\widehat{u_x}}{\widehat{u}})' &\geq \frac{1}{\widehat{u}^2}[k_1\widehat{u}(\widehat{u}^2-\frac{1}{2}\widehat{u_x}^2)-k_1|\widehat{u_x}|(G-G_x)*(u^2+\frac{1}{2}u_x^2)] \\
&\quad +\frac{\widehat{u}^2-\widehat{u_x}^2}{\widehat{u}^2}[(\frac{k_2}{3}+\frac{k_3}{2})\widehat{u}^2-\frac{2k_2}{3}\widehat{u_x}^2] \\
&\geq \frac{\widehat{u}^2-\widehat{u_x}^2}{\widehat{u}^2}[k_1\widehat{u}+(\frac{k_2}{3}+\frac{k_3}{2})\widehat{u}^2-\frac{2k_2}{3}\widehat{u_x}^2] \\
&= \frac{\widehat{u}^2-\widehat{u_x}^2}{\widehat{u}^2}[-\frac{1}{2}+(-\frac{k_1^2}{2}+\frac{k_2}{3}+\frac{k_3}{2})\widehat{u}^2-\frac{2k_2}{3}\widehat{u_x}^2],
\end{aligned} \tag{4.8}$$

where

$$-\frac{1}{2}+(-\frac{k_1^2}{2}+\frac{k_2}{3}+\frac{k_3}{2})\widehat{u}^2-\frac{2k_2}{3}\widehat{u_x}^2 \geq 0 \Leftrightarrow \frac{\widehat{u_x}^2}{\widehat{u}^2} \geq \frac{2k_2+3k_3-3k_1^2}{4k_2}. \tag{4.9}$$

We know that $|u_x| \leq u$ from Remark 3.3. Then

$$0 \leq \frac{2k_2+3k_3-3k_1^2}{4k_2} \leq 1 \ \Leftrightarrow\ \frac{2}{3}k_2+k_3 \leq k_1^2 \leq k_3-\frac{2}{3}k_2. \tag{4.10}$$

From (4.8) and (4.9), we have chosen the initial data so that

$$(\frac{\widehat{u_x}}{\widehat{u}})(0) \geq \sqrt{\frac{2k_2+3k_3-3k_1^2}{4k_2}}. \tag{4.11}$$

Thus, $\frac{\widehat{u_x}}{\widehat{u}}$ increases initially. Moreover,

$$(\frac{\widehat{u_x}}{\widehat{u}})(t) \geq (\frac{\widehat{u_x}}{\widehat{u}})(0) \geq \sqrt{\frac{2k_2+3k_3-3k_1^2}{4k_2}}. \tag{4.12}$$

Then we have

$$\frac{\widehat{u_x}^2}{\widehat{u}^2} \geq \frac{2k_2+3k_3-3k_1^2}{4k_2} \geq \frac{2k_2+3k_3}{4k_2} \geq \frac{2k_2+3k_3}{6k_2+21k_3}, \tag{4.13}$$

where

$$2k_2+21k_3 \leq 0 \ \Leftrightarrow\ k_3 \leq -\frac{2}{21}k_2. \tag{4.14}$$

From (4.3), (4.6), (4.10), (4.14), we have $k_1^2 \leq -\frac{16}{21}k_2, k_1^2+\frac{2}{3}k_2 \leq k_3 \leq -\frac{2}{21}k_2$. Plugging this into (4.4) it yields that $\widehat{M}'(t) \geq -2k_2\widehat{M}^2$, and that $\widehat{M}(t)$ blows up in finite time with an estimate of the blow-up time T^* as

$$T^* \leq -\frac{1}{2k_2\widehat{M(0)}} = -\frac{1}{2k_2m_0(x_0)u_{0,x}(x_0)}.$$

(2) When

$$\frac{1}{5} > \frac{2k_2+3k_3}{6k_2+21k_3} \Leftrightarrow k_3 > -\frac{2}{7}k_2 \quad or \quad k_3 < -\frac{2}{3}k_2. \tag{4.15}$$

Then from (4.5), we have

$$\frac{\widehat{u_x}^2}{\widehat{u^2}} > \frac{1}{5}. \tag{4.16}$$

From (4.9), we have

$$\frac{\widehat{u}_x^2}{\widehat{u}^2} \geq \frac{2k_2+3k_3-3k_1^2}{4k_2} > \frac{1}{5} \ \Leftrightarrow\ k_1^2 > k_3+\frac{2}{5}k_2, k_3 > -\frac{2}{5}k_2, \tag{4.17}$$

where we know that $|u_x| \le u$ from Remark 3.3, so

$$0 \le \frac{2k_2+3k_3-3k_1^2}{4k_2} \le 1 \ \Leftrightarrow\ \frac{2}{3}k_2 + k_3 \le k_1^2 \le k_3 - \frac{2}{3}k_2.$$

So, we have

$$k_3 + \frac{2}{5}k_2 < k_1^2 \le k_3 - \frac{2}{3}k_2, \quad k_3 > \frac{2}{3}k_2. \tag{4.18}$$

From (4.8) and (4.9), we have chosen the initial data so that

$$(\widehat{\frac{u_x}{u}})(0) \ge \sqrt{\frac{2k_2+3k_3-3k_1^2}{4k_2}}. \tag{4.19}$$

Thus, $\widehat{\frac{u_x}{u}}$ increases initially. Moreover,

$$(\widehat{\frac{u_x}{u}})(t) \ge (\widehat{\frac{u_x}{u}})(0) \ge \sqrt{\frac{2k_2+3k_3-3k_1^2}{4k_2}}. \tag{4.20}$$

From (4.3), (4.17), (4.18), we obtain $\max\{k_1^2 + \frac{2}{3}k_2, -\frac{2}{5}k_2\} < k_3 < k_1^2 - \frac{2}{5}k_2$. Plugging this into (4.4) it yields that $\widehat{M}'(t) \ge -2k_2\widehat{M}^2$, and that $\widehat{M}(t)$ blows up in finite time with an estimate of the blow-up time T^* as

$$T^* \le -\frac{1}{2k_2\widehat{M(0)}} = -\frac{1}{2k_2 m_0(x_0)u_{0,x}(x_0)}. \qquad \square$$

Remark 4.2. (1) From Lemma 3.6, we obtain that the true blow-up quantity is $k_2 m u_x$. In Theorem 4.1, when $k_2 < 0$, we seek data that lead to $mu_x \to +\infty$. So that we consider the case when $k_2 > 0$, it leads to $mu_x \to -\infty$

(2) The blow-up time T^* is only related to the parameter k_2. We know that the mCH equation plays a dominant role in blow-up phenomena when the CH equation, the mCH equation and the Novikov equation act simultaneously.

Using a similar argument as above, we prove the following corollary when $k_2 > 0$.

Corollary 4.3. *Suppose that $k_1 > 0$, $k_2 > 0$, $-\frac{2}{15}k_2 - k_3 \le k_1^2 < \frac{2}{3}k_2 - k_3$ and $-\frac{2}{27}k_2 < k_3 \le \frac{2}{3}k_2$. Let $m_0 \in H^s(\mathbb{S})$ for $s > 1/2$ and $m_0 > 0$. Assume that there exists some point $x_0 \in \mathbb{S}$ such that $m_0(x_0) > 0$ and*

$$u_{0,x}(x_0) \le -\sqrt{\frac{2k_2+3k_3+3k_1^2}{4k_2}}\,u_0(x_0). \tag{4.21}$$

Then the solution $u(t,x)$ blows up in finite time with an estimate of the blow-up time T^* as $T^* \le -\frac{1}{2k_2 m_0(x_0)u_{0,x}(x_0)}$.

Proof. Now we look for $\widehat{M} \to -\infty$. We recall the equation

$$\begin{aligned}
\widehat{M}'(t) &= -2k_2\widehat{M}^2 + k_1\widehat{m}\widehat{u}^2\Big(1 - \frac{5}{2}\frac{\widehat{u_x}^2}{\widehat{u}^2}\Big) + \frac{\widehat{m}\widehat{u}^3}{6}\Big[(2k_2+3k_3) - (6k_2+21k_3)\frac{\widehat{u_x}^2}{\widehat{u}^2}\Big] \\
&\quad - \Big(\frac{k_2}{3} + \frac{k_3}{2}\Big)m[\Lambda_1 * (u-u_x)^3 + \Lambda_2 * (u+u_x)^3] \\
&\quad - k_1 m\big(\Lambda_1 * [u^2(1 + \frac{1}{2}\frac{u_x^2}{u^2})] + \Lambda_2 * [u^2(1+\frac{1}{2}\frac{u_x^2}{u^2})]\big).
\end{aligned}$$

We assume that

$$k_1 > 0,\ k_2 > 0,\ \frac{k_2}{3} + \frac{k_3}{2} > 0 \ \Leftrightarrow\ k_3 > -\frac{2}{3}k_2. \tag{4.22}$$

To obtain a Riccati-type inequality we have

$$\begin{gathered}
1-\frac{5\widehat{u_x}^2}{2\widehat{u}^2}\le 0 \Leftrightarrow \frac{\widehat{u_x}^2}{\widehat{u}^2}\ge \frac{2}{5},\\
(2k_2+3k_3)-(6k_2+21k_3)\frac{\widehat{u_x}^2}{\widehat{u}^2}\le 0 \Leftrightarrow \frac{\widehat{u_x}^2}{\widehat{u}^2}\ge \frac{2k_2+3k_3}{6k_2+21k_3},
\end{gathered} \tag{4.23}$$

where $6k_2+21k_3\ge 0 \Leftrightarrow k_3\ge -\frac{2}{7}k_2$. Now we discuss the following two cases:

(1) When $\frac{2k_2+3k_3}{6k_2+21k_3}\ge\frac{2}{5}\Leftrightarrow -\frac{2k_2}{7}\le k_3\le -\frac{2k_2}{27}$, and $\frac{2k_2+3k_3}{6k_2+21k_3}\le 1\Leftrightarrow -\frac{2k_2}{7}\le k_3\le -\frac{2k_2}{9}$. Then

$$-\frac{2k_2}{7}\le k_3\le -\frac{2k_2}{9}. \tag{4.24}$$

We have

$$\frac{\widehat{u_x}^2}{\widehat{u}^2}\ge \frac{2k_2+3k_3}{6k_2+21k_3}. \tag{4.25}$$

In particular, a finite-time blow-up of $\widehat{M}$ can be realized if the ration $|\frac{u_x}{u}|$ stays reasonable big along the characteristics. We have

$$\begin{aligned}
(\frac{\widehat{u_x}}{\widehat{u}})' &= \frac{1}{\widehat{u^2}}[k_1\widehat{u}(\widehat{u}^2-\frac{1}{2}\widehat{u_x}^2)-k_1(\widehat{u}+\widehat{u_x})\Lambda_1*(u^2+\frac{1}{2}u_x^2)\\
&\quad -k_1(\widehat{u}-\widehat{u_x})\Lambda_2*(u^2+\frac{1}{2}u_x^2)]+\frac{\widehat{u}^2-\widehat{u_x}^2}{\widehat{u}^2}[(\frac{k_2}{3}+\frac{k_3}{2})\widehat{u}^2-\frac{2k_2}{3}\widehat{u_x}^2]\\
&\quad -\frac{2k_2+3k_3}{6\widehat{u}^2}[(\widehat{u}+\widehat{u_x})\Lambda_1*(u-u_x)^3+(\widehat{u}-\widehat{u_x})\Lambda_2*(u+u_x)^3]\\
&\le \frac{1}{\widehat{u}^2}[k_1\widehat{u}(\widehat{u}^2-\frac{1}{2}\widehat{u_x}^2)-\frac{1}{4}k_1\widehat{u}^2(\widehat{u}+\widehat{u_x})-\frac{1}{4}k_1\widehat{u}^2(\widehat{u}-\widehat{u_x})]\\
&\quad +\frac{\widehat{u}^2-\widehat{u_x}^2}{\widehat{u}^2}[(\frac{k_2}{3}+\frac{k_3}{2})\widehat{u}^2-\frac{2k_2}{3}\widehat{u_x}^2].
\end{aligned}$$

So we have

$$\begin{aligned}
(\frac{\widehat{u_x}}{\widehat{u}})' &= \frac{1}{\widehat{u}^2}[\frac{k_1}{2}\widehat{u}(\widehat{u}^2-\widehat{u_x}^2)]+\frac{\widehat{u}^2-\widehat{u_x}^2}{\widehat{u}^2}[(\frac{k_2}{3}+\frac{k_3}{2})\widehat{u}^2-\frac{2k_2}{3}\widehat{u_x}^2]\\
&\le \frac{\widehat{u}^2-\widehat{u_x}^2}{\widehat{u}^2}[\frac{1}{8}+\frac{k_1^2}{2}\widehat{u}^2+(\frac{k_2}{3}+\frac{k_3}{2})\widehat{u}^2-\frac{2k_2}{3}\widehat{u_x}^2]\\
&= \frac{\widehat{u}^2-\widehat{u_x}^2}{\widehat{u}^2}[\frac{1}{8}+(\frac{k_1^2}{2}+\frac{k_2}{3}+\frac{k_3}{2})\widehat{u}^2-\frac{2k_2}{3}\widehat{u_x}^2],
\end{aligned} \tag{4.26}$$

where

$$\begin{aligned}
&\frac{1}{8}+(\frac{k_1^2}{2}+\frac{k_2}{3}+\frac{k_3}{2})\widehat{u}^2-\frac{2k_2}{3}\widehat{u_x}^2<0\\
&\Leftrightarrow \frac{2k_2}{3}\frac{\widehat{u_x}^2}{\widehat{u}^2}\ge \frac{k_1^2}{2}+\frac{k_2}{3}+\frac{k_3}{2}+\frac{1}{8\widehat{u}^2}\\
&\Leftrightarrow \frac{2k_2}{3}\frac{\widehat{u_x}^2}{\widehat{u}^2}\ge \frac{k_1^2}{2}+\frac{k_2}{3}+\frac{k_3}{2}+\frac{1}{8\mu E_0}\\
&\Leftrightarrow \frac{\widehat{u_x}^2}{\widehat{u}^2}\ge \frac{3}{16\mu E_0k_2}+\frac{2k_2+3k_3}{4k_2}+\frac{3k_1^2}{4k_2}\\
&\Leftrightarrow \frac{\widehat{u_x}^2}{\widehat{u}^2}\ge \frac{2k_2+3k_3+3k_1^2}{4k_2}.
\end{aligned}$$

We know that $|u_x| \le u$, so

$$\frac{2k_2+3k_3+3k_1^2}{4k_2} \le 1 \Leftrightarrow k_1^2 \le \frac{2}{3}k_2 - k_3 \quad \text{and} \quad \frac{2}{3}k_2 - k_3 > 0 \Leftrightarrow k_3 < \frac{2}{3}k_2. \quad (4.27)$$

We have chosen the initial data so that

$$(\frac{\widehat{u_x}}{\widehat{u}})(0) \ge \sqrt{\frac{2k_2+3k_3+3k_1^2}{4k_2}}. \quad (4.28)$$

We need $\frac{\widehat{u_x}^2}{\widehat{u}^2} > \frac{2k_2+3k_3+3k_1^2}{4k_2} \ge \frac{2k_2+3k_3}{4k_2} \ge \frac{2k_2+3k_3}{6k_2+21k_3} \ge \frac{2}{5}$. Therefore,

$$2k_2 + 21k_3 \ge 0 \Leftrightarrow k_3 \ge -\frac{2}{21}k_2. \quad (4.29)$$

Clearly, (4.29) contradicts (4.24), so Case 1 can not happen.

(2) When $\frac{2k_2+3k_3}{6k_2+21k_3} < \frac{2}{5} \Leftrightarrow k_3 < -\frac{2k_2}{7}$ or $k_3 > -\frac{2k_2}{27}$ and $6k_2+21k_3 > 0 \Leftrightarrow k_3 > -\frac{2}{7}k_2$, we have

$$\frac{\widehat{u_x}^2}{\widehat{u}^2} \ge \frac{2}{5}.$$

With the same method, we have chosen the initial data so that

$$(\frac{\widehat{u_x}}{\widehat{u}})(0) \ge \sqrt{\frac{2k_2+3k_3+3k_1^2}{4k_2}}. \quad (4.30)$$

We need $\frac{\widehat{u_x}^2}{\widehat{u}^2} > \frac{2k_2+3k_3+3k_1^2}{4k_2}$. Then $\frac{\widehat{u_x}}{\widehat{u}}$ decreases initially. So, we have

$$\frac{\widehat{u_x}^2}{\widehat{u}^2} > \frac{2k_2+3k_3+3k_1^2}{4k_2} \ge \frac{2}{5} \Leftrightarrow k_1^2 \ge -\frac{2}{15}k_2 - k_3. \quad (4.31)$$

From (4.22), (4.26), (4.30) and (4.31), we have $-\frac{2}{15}k_2 - k_3 \le k_1^2 < \frac{2}{3}k_2 - k_3$, $-\frac{2}{27}k_2 < k_3 < \frac{2}{3}k_2$. Then we obtain the desired Riccati inequality for $\widehat{M}$

$$\widehat{M}'(t) \le -2k_2\widehat{M}^2,$$

which implies that $\widehat{M}(t) \to -\infty$ as $t \to T^*$, where $T^* \le -\frac{1}{2k_2 m_0(x_0) u_{0,x}(x_0)}$. □

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