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UNIQUENESS OF POSITIVE SOLUTIONS FOR A CLASS OF ODE'S WITH DIRICHLET BOUNDARY CONDITIONS

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Abstract. We study the uniqueness of positive solutions of the boundary-value problem

$$u'' + a(t)u' + f(t,u) = 0, \quad t \in (0,b)$$
$$u(0) = 0, u(b) = 0\,,$$

where $0 < b < \infty$, $a \in C^1[0,\infty)$ and $f \in C^1([0,\infty)\times[0,\infty),[0,\infty))$ satisfy suitable conditions. The proof of our main result is based on the shooting method and the Sturm comparison theorem.

1. Introduction

In this paper, we consider the uniqueness of positive solutions for the problem

$$u'' + a(t)u' + f(t,u) = 0, \quad t \in (0,b) \tag{1.1}$$
$$u(0) = 0, \quad u(b) = 0\,, \tag{1.2}$$

where $0 < b < \infty$, $a \in C^1[0,\infty)$ and $f \in C^1([0,\infty)\times[0,\infty),[0,\infty))$ satisfy suitable conditions.

By a *positive solution* of (1.1)-(1.2), we understand a function $u(t)$ which is positive in $t \in (0,b)$ and satisfies the differential equation (1.1) and the boundary conditions (1.2).

To study the uniqueness problem of (1.1)-(1.2), we will use the shooting method and the Sturm comparison theorem. Some of the ideas used in this paper are motivated by Lynn Erbe and Moxun Tang [2, 3] and Fu and Lin [4].

This paper is organized as follows. In section 2, we state the main result (see Theorem 2.1 below) and give some examples to illustrate the applicability of our results. In section 3, we show some preliminary results. Finally we prove the main result in Section 4.

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2. Main Result

To investigate the uniqueness of (1.1)-(1.2), we introduce the initial-value problem

$$u'' + a(t)u' + f(t,u) = 0 \tag{2.1}$$

$$u(0) = 0, \quad u'(0) = \alpha > 0. \tag{2.2}$$

For a given number $\alpha > 0$, we know from the assumptions $a \in C^1[0,\infty)$ and $f \in C^1([0,\infty)\times[0,\infty),[0,\infty))$ that (2.1)-(2.2) has a unique solution $u(t,\alpha)$ defined on $(0,T_\alpha)$, where T_α is either $+\infty$ or a positive number such that u can not be further continued to the right of T_α. If $\alpha > 0$, then $u(0,\alpha) = 0,\ u'(0,\alpha) = \alpha > 0$. Therefore, there exists a positive number $\epsilon \in (0,T_\alpha)$ such that

$$u(t,\alpha) > 0, \quad t \in (0,\ \epsilon).$$

When $u(t,\alpha)$ vanishes at some $t_0 \in (0,T_\alpha)$, we define $b(\alpha)$ to be the first zero of $u(t,\alpha)$ in $(0,T_\alpha)$. More precisely, $b(\alpha)$ is a function of α which has the properties

$$u(t,\alpha) > 0, \quad t \in (0,b(\alpha)); \quad u(b(\alpha),\alpha) = 0.$$

If $u(t,\alpha)$ is positive in $(0,T_\alpha)$, then we define $b(\alpha) = T_\alpha$. Denote

$$N := \{\alpha : \alpha > 0,\ b(\alpha) < T_\alpha\}.$$

It is obviously that problem (1.1)-(1.2) has no positive solution if N is an empty set. Hence, we suppose $N \neq \emptyset$.

Now, for any given $b > 0$, if we can prove there exists at most one $\alpha \in N$ such that $b = b(\alpha)$, then we conclude the uniqueness of positive solutions of the problem (1.1)-(1.2).

We denote the *variation* of $u(t,\alpha)$ by

$$\phi(t,\alpha) = \partial u(t,\alpha)/\partial\alpha, \quad t \in [0,T_\alpha).$$

Then $\phi(t,\alpha)$ satisfies

$$\phi'' + a(t)\phi' + f_u(t,u)\phi = 0, \quad t \in [0,T_\alpha) \tag{2.3}$$

$$\phi(0) = 0, \quad \phi'(0) = 1, \tag{2.4}$$

where the notation $f_u(t,u)$ denotes $\partial f(t,u)/\partial u$. Let L be the linear operator

$$L(\phi) = \phi'' + a(t)\phi' + f_u(t,u)\phi, \quad t \in [0,T_\alpha) \tag{2.5}$$

and

$$G_h(t) = u(t,\alpha) + \frac{h-1}{2}\frac{v(t)}{v'(t)}u'(t,\alpha), \quad t \in [0,T_\alpha) \tag{2.6}$$

where

$$v(t) = \int_0^t \exp\big(-\int_0^\tau a(s)ds\big)d\tau$$

and accordingly

$$v'(t) = \exp\big(-\int_0^t a(s)ds\big).$$

It is easy to check that

$$v''(t) + a(t)v'(t) = 0, \quad t \in [0,T_\alpha).$$

A different function $G_h(t)$ has been used by Erbe and Tang [2]. However, the function $G_h(t)$ defined by (2.6) is first introduced here.

Differentiating $G_h(t)$ with respect to t, we get

$$\begin{aligned}
G_h'(t) &= u'(t,\alpha) + \frac{h-1}{2}(\frac{v(t)}{v'(t)})'u'(t,\alpha) + \frac{h-1}{2}\frac{v(t)}{v'(t)}u''(t,\alpha)\\
&= u'(t,\alpha) + \frac{h-1}{2}(\frac{v(t)}{v'(t)})'u'(t,\alpha)\\
&\quad + \frac{h-1}{2}\frac{v(t)}{v'(t)}(-f(t,u(t,\alpha)) - a(t)u'(t,\alpha))\\
&= \big(1 + \frac{h-1}{2}(\frac{v(t)}{v'(t)})' - \frac{h-1}{2}a(t)\frac{v(t)}{v'(t)}\big)u'(t,\alpha) - \frac{h-1}{2}\frac{v(t)}{v'(t)}f(t,u(t,\alpha))
\end{aligned}$$

and

$$\begin{aligned}
G_h''(t) &= [\frac{h-1}{2}(\frac{v}{v'})'' - \frac{h-1}{2}a'(t)\frac{v}{v'} - \frac{h-1}{2}a(t)(\frac{v}{v'})']u'\\
&\quad + (1 + \frac{h-1}{2}(\frac{v}{v'})' - \frac{h-1}{2}a(t)\frac{v}{v'})(-f(t,u) - a(t)u')\\
&\quad - \frac{h-1}{2}(\frac{v}{v'})'f(t,u) - \frac{h-1}{2}\frac{v}{v'}f_u(t,u)u' - \frac{h-1}{2}\frac{v}{v'}f_t(t,u)\\
&= [-a(t) - \frac{h-1}{2}a(t)(\frac{v}{v'})' + a^2(t)\frac{h-1}{2}\frac{v}{v'}]u'\\
&\quad - [1 + (h-1)(\frac{v}{v'})' - \frac{h-1}{2}a(t)\frac{v}{v'}]f(t,u) - \frac{h-1}{2}\frac{v}{v'}f_u(t,u)u'\\
&\quad - \frac{h-1}{2}\frac{v}{v'}f_u(t,u).
\end{aligned}$$

where $f_t(t,u)$ denotes $\partial f(t,u)/\partial t$. So we have

$$\begin{aligned}
L(G_h(t)) &= G_h''(t) + a(t)G_h'(t) + f_u(t,u)G_h(t)\\
&= G_h''(t) + a(t)(1 + \frac{h-1}{2}(\frac{v}{v'})' - \frac{h-1}{2}a(t)\frac{v}{v'})u'\\
&\quad - \frac{h-1}{2}\frac{v}{v'}a(t)f(t,u) + f_u(t,u)u + \frac{h-1}{2}\frac{v}{v'}f_u(t,u)u'\\
&= f_u(t,u)u - [1 + (h-1)(\frac{v}{v'})']f(t,u) - \frac{h-1}{2}\frac{v}{v'}f_u(t,u).
\end{aligned}$$

Denote $L_h(t) = L(t,u,\alpha) = L(G_h(t))$, then

$$L_h(t) = f_u(t,u)u - [1 + (h-1)(\frac{v}{v'})']f(t,u) - \frac{h-1}{2}\frac{v}{v'}f_u(t,u).$$

Note that when $h = 1$, we simply have $G_1(t) = u(t,\alpha)$, and $L_1(t) = uf_u(t,u) - f(t,u)$.

We will use the following assumptions:

(A1) $f(t,0) \equiv 0$, and $uf_u(t,u) > f(t,u) > 0$ for all $t > 0$, $u > 0$
(A2) $a(t) \le 0$, $a(t)v(t) + v'(t) \ge 0$, for all $t \ge 0$
(A3) If $\alpha > 0$ and $h \ge 1$, and there exists a $\widetilde{t} \in (0, b(\alpha))$, such that $L_h(\widetilde{t},\alpha) \ge 0$, then $L_h(t,\alpha) \ge 0$ for all $t \in [\widetilde{t},\ b(\alpha))$
(A4) $(v(t)/v'(t))'f(t,u) + (v(t)/2v'(t))f_u(t,u) > 0$, for all $t > 0$, $u > 0$.

The main result of this paper is as follows.

Theorem 2.1. *Assume (A1)-(A4) hold. Then* (1.1)-(1.2) *has at most one positive solution.*

Example 2.2. Let $f(t,u)=u^p$ $(p>1)$, $a(t)\equiv -1$. Then

$$a(t)v(t)+v'(t)=1>0,\quad L_h(t)=u^p[p-1-(h-1)e^{-t}].$$

Obviously, (A1), (A2) are satisfied. Since e^{-t} is strictly decreasing for $t>0$, (A3) is satisfied. Meanwhile,

$$(\frac{v}{v'})'f(t,u)+\frac{v}{2v'}f_u(t,u)=e^{-t}u^p>0,\quad t>0,\ u>0.$$

(A4) is also satisfied.

Example 2.3. Let $f(t,u)=t^lu^p$ $(l>0,\ p>1)$ and $a(t)\equiv 0$. Then

$$a(t)v(t)+v'(t)=1>0,\quad L_h(t)=t^lu^p[p-h-(h-1)l/2]$$

and

$$(\frac{v}{v'})'f(t,u)+\frac{v}{2v'}f_u(t,u)=t^lu^p(1+\frac{l}{2t^2})>0,\quad t>0,\ u>0.$$

Obviously, (A1)-(A4) are satisfied.

3. Preliminary Results

Lemma 3.1. *Suppose $f(t,0)\equiv 0$ for all $t\ge 0$, and*

$$\phi(b(\alpha),\alpha)\neq 0,\quad \alpha\in N. \tag{3.1}$$

Then one of the following cases must occur

(i) *N is an open interval*

(ii) *$N=(0,j_1)\cup(j_2,\infty)$ with $0<j_1<j_2<+\infty$. Moreover, $b'(\alpha)>0$ for all $(0,j_1)$; $b'(\alpha)<0$ for all (j_2,∞).*

Proof. From the definition of $b(\alpha)$, we have that $u'(b(\alpha),\alpha)\le 0$ and for all $\alpha\in N$,

$$u(b(\alpha),\alpha)=0. \tag{3.2}$$

If $u'(b(\alpha),\alpha)=0$, then (3.2) with the assumption $f(t,0)\equiv 0$ for $t\ge 0$ imply

$$u(t,\alpha)\equiv 0,\quad t>0$$

However this contradicts the fact $u'(0,\alpha)=\alpha>0$. Therefore, we must have

$$u'(b(\alpha),\alpha)<0. \tag{3.3}$$

By the Implicit Function Theorem, $b(\alpha)$ is well-defined as a function of α in N and $b(\alpha)\in C^1(N)$. Furthermore, it follows from (3.3) that N is an open set.

Differentiating both sides of the identity (3.2) with respect to α, we obtain

$$u'(b(\alpha),\alpha)b'(\alpha)+\phi(b(\alpha),\alpha)=0. \tag{3.4}$$

Combining this with (3.1), it follows that $b'(\alpha)\neq 0$.

We note that if $\bar\alpha\in(0,\infty)\setminus N$ with $\{\alpha_n\}\subset N$ and $\alpha_n\to\bar\alpha$ as $n\to\infty$, then

$$b(\alpha_n)\to+\infty.$$

Otherwise, on the contrary , we may suppose that $b(\alpha_n)\to t_1$ as $n\to\infty$. Let $n\to\infty$, we have from $u(b(\alpha_n),\alpha_n)=0$ that $u(t_1,\bar\alpha)=0$. However this contradicts $\bar\alpha\notin N$.

If N be not an open interval, and let $J_1=(j_0,j_1)$ and $J_2=(j_2,j_3)$ be two distinct components of N with $0<j_1<j_2<\infty$. Then

$$\lim_{\alpha\to j_1^-}b(\alpha)=\lim_{\alpha\to j_2^+}b(\alpha)=+\infty$$

Since $b(\alpha)$ is strictly monotonic in each component of N, we conclude that $b'(\alpha) > 0$ in J_1, and $b'(\alpha) < 0$ in J_2. Meanwhile

$$\lim_{\alpha\to j_0^+} b(\alpha) < +\infty, \quad \lim_{\alpha\to j_3^-} b(\alpha) < +\infty$$

It follows that $j_0 = 0$ and $j_3 = +\infty$. Therefore $N = (0, j_1) \cup (j_2, \infty)$ with $b'(\alpha) > 0$ in $(0, j_1)$, and $b'(\alpha) < 0$ in (j_2, ∞). The proof is completed. □

Lemma 3.2. *Let* $\alpha \in N$*, and let* $f(t, u)$ *satisfy (A1). Then* $\phi(t, \alpha)$ *has at least one zero in* $(0, b(\alpha))$.

Proof. Note that $L(\phi) = 0$, i.e.,

$$\phi'' + a(t)\phi' + f_u(t, u)\phi = 0 \tag{3.5}$$

Meanwhile,

$$G_h''(t) + a(t)G_h'(t) + f_u(t, u)G_h(t) = L_h(t) \tag{3.6}$$

Multiply both sides of (3.6) by $\exp\big(\int_0^t a(s)ds\big)\phi(t, \alpha)$, and multiply both sides of (3.5) by $\exp\big(\int_0^t a(s)ds\big)G_h(t)$, then subtract the resulting identities, we have

$$\big[\exp\big(\int_0^t a(s)ds\big)(G_h'\phi - G_h\phi')\big]' = \exp\big(\int_0^t a(s)ds\big)\phi(t, \alpha)L_h(t). \tag{3.7}$$

Set $h = 1$ in (3.7), we get

$$\big[\exp\big(\int_0^t a(s)ds\big)(u'\phi - u\phi')\big]' = \exp\big(\int_0^t a(s)ds\big)\phi(t, \alpha)(f_u(t, u)u - f(t, u)) \tag{3.8}$$

Suppose on the contrary that $\phi(t, \alpha)$ does not vanish in $(0, b(\alpha))$. Then we know from (2.4) that

$$\phi(t, \alpha) > 0, \quad t \in (0, b(\alpha)).$$

This implies that the right hand side of (3.8) is positive in $(0, b(\alpha))$, and accordingly

$$\big[\exp\big(\int_0^t a(s)ds\big)(u'\phi - u\phi')\big]' > 0, \quad t \in (0, b(\alpha)). \tag{3.9}$$

Since

$$\exp\big(\int_0^t a(s)ds\big)(u'\phi - u\phi')\Big|_{t=0} = 0$$

we have form (3.9) that

$$\exp\big(\int_0^t a(s)ds\big)(u'\phi - u\phi')\Big|_{t=b(\alpha)} > 0. \tag{3.10}$$

On the other hand

$$\exp\big(\int_0^t a(s)ds\big)(u'\phi - u\phi')\big|_{t=b(\alpha)} = \exp\big(\int_0^{b(\alpha)} a(s)ds\big)u'(b(\alpha), \alpha)\phi(b(\alpha), \alpha) \le 0$$

which contradicts (3.10). Therefore, $\phi(t, \alpha)$ has at least one zero in $(0, b(\alpha))$. □

4. Proof of Theorem 2.1

We note that for any $\alpha \in N$, (3.4) holds. If we can show that

$$\phi(b(\alpha), \alpha) < 0, \quad \alpha \in N \tag{4.1}$$

then from (4.1), (3.3) and (3.4), it follows that $b'(\alpha) < 0$. Combining this with Lemma 3.1, we conclude that N is an open interval and $b'(\alpha) < 0$ for all $\alpha \in N$. So $b(\alpha)$ is a strictly decreasing function in N. Thus, for any given $b > 0$, there is at most one $\alpha \in N$ such that $b(\alpha) = b$. If such α exists exactly, then $u(t, \alpha)$, which is the unique solution of the initial value problem (2.1)-(2.2), must be the positive solution of boundary value problem (1.1)-(1.2). Combining this with the uniqueness of solution of the initial value problem, we know that the positive solution of (1.1)-(1.2) is unique.

Proof of Theorem 2.1. We need only to prove (4.1). To this end, we divide the proof into six steps.
Step 1. We show that there exists unique $c(\alpha) \in (0, b(\alpha))$ such that

$$u'(c(\alpha), \alpha) = 0 \tag{4.2}$$

and

$$u'(t, \alpha) > 0 \text{ on } [0, c(\alpha)); \quad u'(t, \alpha) < 0 \text{ on } (c(\alpha), b(\alpha)] \tag{4.3}$$

In fact, if $\tau \in (0, b(\alpha))$ such that $u'(\tau, \alpha) = 0$. Then from (2.1) and (A1), we have that

$$u''(\tau) = f(\tau, u(\tau, \alpha)) < 0$$

which means that τ is a local maximum of $u(t, \alpha)$. Combining this with the fact $u(0, \alpha) = u(b(\alpha), \alpha) = 0$, it concludes that there exists unique $c(\alpha) \in (0, b(\alpha))$ such that (4.2) and (4.3) hold.
Step 2. We show that

$$\phi(t, \alpha) > 0, \quad t \in (0, c(\alpha)]. \tag{4.4}$$

From the facts that $u(t, \alpha) > 0$ and $u'(t, \alpha) > 0$ on $(0, c(\alpha))$, we have that

$$G_h(t) > 0, \quad t \in (0, c(\alpha)]$$

whenever $h \geq 1$. Since

$$\begin{aligned} L_h(c(\alpha)) &= \big[f_u(t,u)u - \big(1 + (h-1)(\frac{v}{v'})'\big)f(t,u) - \frac{h-1}{2}\frac{v}{v'}f_u(t,u)\big]_{t=c(\alpha)} \\ &= \big[f_u(t,u)u - \big(1 - (\frac{v}{v'})'\big)f(t,u) + \frac{v}{2v'}f_u(t,u) \\ &\quad - h\ \big((\frac{v}{v'})'f(t,u) + \frac{v}{2v'}f_u(t,u)\big)\big]_{t=c(\alpha)} \end{aligned}$$

Combing it with (A4), we can choose $\bar{h} \in (1, \infty)$ so large such that

$$L_{\bar{h}}(c(\alpha)) < 0$$

which together with (A3) implies that

$$L_{\bar{h}}(t) < 0, \quad t \in (0, c(\alpha)]. \tag{4.5}$$

Suppose on the contrary that (4.4) is not true, and let t_2 be the first zero of $\phi(t, \alpha)$ in $(0, c(\alpha)]$. Then

$$\phi(t, \alpha) > 0 \text{ on } t \in (0, t_2), \quad \phi(t_2, \alpha) = 0. \tag{4.6}$$

Note that $\phi(t,\alpha)$ and $G_{\bar h}(t)$ satisfy

$$\big[\exp\big(\int_0^t a(s)ds\big)(G'_{\bar h}\phi - G_{\bar h}\phi')\big]' = \exp\big(\int_0^t a(s)ds\big)\phi(t,\alpha)L_{\bar h}(t)$$

which together with (4.5), (4.6) imply

$$\big[\exp\big(\int_0^t a(s)ds\big)(G'_{\bar h}\phi - G_{\bar h}\phi')\big]' < 0, \quad t\in(0,t_2) \tag{4.7}$$

Since $G_{\bar h}(0) = u(0,\alpha) = 0$, it follows that

$$\big[\exp\big(\int_0^t a(s)ds\big)(G'_{\bar h}\phi - G_{\bar h}\phi')\big]_{t=0} = 0$$

which and (4.7) yield

$$\big[\exp\big(\int_0^t a(s)ds\big)(G'_{\bar h}\phi - G_{\bar h}\phi')\big]_{t=t_2} < 0. \tag{4.8}$$

On the other hand, since $\phi'(t_2)\le 0$ and $G_{\bar h}(t_2) > 0$, we have

$$\begin{aligned}&\big[\exp\big(\int_0^t a(s)ds\big)(G'_{\bar h}\phi - G_{\bar h}\phi')\big]_{t=t_2}\\ &= \exp\big(\int_0^{t_2} a(s)ds\big)(-G_{\bar h}(t_2)\phi'(t_2)) \ge 0\end{aligned}$$

This contradicts (4.8). Therefore (4.4) holds.
Step 3. We show that if $h>1$ then $G_h(t)$ has exactly one zero τ_h in $(c(\alpha),b(\alpha))$. If $h>1$, we have from the definition of $G_h(t)$ that

$$G_h(c(\alpha)) = u(c(\alpha)) > 0, \quad G_h(b(\alpha)) < 0$$

which implies that $G_h(t)$ with $h>1$ must have zeros in $(c(\alpha),b(\alpha))$.

Next we show that $G_h(t)$ with $h>1$ has at most one zero in $(c(\alpha),b(\alpha))$. For any given $h>1$, let $G_h(t)=0$ for some $t\in(c(\alpha),b(\alpha))$. Then

$$u(t,\alpha) + \frac{h-1}{2}\frac{v(t)}{v'(t)}u'(t,\alpha) = 0$$

and consequently

$$\frac{u'(t)}{u(t)} = \frac{2}{1-h}\frac{v'(t)}{v(t)}.$$

Set

$$w_1(s) = \frac{u'(s)}{u(s)}, \quad s\in(c(\alpha),b(\alpha))$$

and

$$w_2(s) = \frac{2}{1-h}\frac{v'(s)}{v(s)}, \quad s\in(c(\alpha),b(\alpha))$$

By (A2), we have

$$\begin{gathered}w_1'(s) = \frac{-a(s)u'u - f(s,u)u - u'^2}{u^2} < 0, \quad s\in(c(\alpha),\ b(\alpha))\\ w_2'(s) = \frac{2}{1-h}\frac{-v'(a(s)v+v')}{v^2} \ge 0, \quad s\in(c(\alpha),\ b(\alpha)).\end{gathered}$$

Hence, $w_1(s)$ and $w_2(s)$ intersect at most once in $(c(\alpha),b(\alpha))$, and accordingly $G_h(t)$ $(h>1)$ has at most one zero in $(c(\alpha),b(\alpha))$.

Step 4. Let $\theta(\alpha)$ be the first zero of $\phi(t,\alpha)$ in $(c(\alpha),b(\alpha))$. We show that there exists an unique $p\in(1,\infty)$ such that the unique zero, τ_p, of $G_p(t)$ in $(c(\alpha),b(\alpha)]$ satisfying

$$\tau_p=\theta(\alpha) \tag{4.9}$$

$$G_p(t)>0 \text{ on } (0,\theta(\alpha)),\quad G_p(\theta(\alpha))=0,\quad G_p(t)<0 \text{ on } (\theta(\alpha),b(\alpha)], \tag{4.10}$$

$$\phi(t,\alpha)>0 \text{ on } (0,\theta(\alpha)),\quad \phi(\theta(\alpha),\alpha)=0. \tag{4.11}$$

Note that for $t\in(c(\alpha),b(\alpha))$, $u'(t,\alpha)<0$. From the definition of $G_h(t)$, we have that for any fixed $\tau\in(c(\alpha),b(\alpha))$, $G_h(\tau)$ is continuous and strictly decreasing with respect to h. Since $G_1(\tau)=u(\tau,\alpha)>0$ and $\lim_{h\to+\infty}G_h(\tau)=-\infty$, there must be a unique $h>1$ such that

$$G_h(\tau)=0.$$

In particular, for $\theta(\alpha)\in(c(\alpha),b(\alpha))$, there exists a unique number $p\in(1,\infty)$ such that

$$G_p(\theta(\alpha))=0$$

i.e. $\tau_p=\theta(\alpha)$.

Equations (4.10) and (4.11) can be deduced from the fact that both G_p has unique zero in $(c(\alpha),b(\alpha))$ and $\theta(\alpha)$ is the first zero of $\phi(t,\alpha)$ in $(c(\alpha),b(\alpha))$.

Step 5. We show that there exists $t_3\in(0,\theta(\alpha)]$ such that

$$L_p(t_3)\ge 0. \tag{4.12}$$

Suppose on the contrary that $L_p(t)<0$ on $(0,\theta(\alpha)]$. Note that $\phi(t,\alpha)$ and $G_p(t)$ satisfy

$$\big[\exp\big(\int_0^t a(s)ds\big)(G_p'\phi-G_p\phi')\big]'=\phi(t,\alpha)L_p(t)\exp\big(\int_0^t a(s)ds\big). \tag{4.13}$$

Since the right-hand side of (4.13) is negative on $(0,\theta(\alpha))$,

$$\big[\exp\big(\int_0^t a(s)ds\big)(G_p'\phi-G_p\phi')\big]'<0,\quad t\in(0,\theta(\alpha))$$

This and

$$\exp\big(\int_0^t a(s)ds\big)(G_p'\phi-G_p\phi')\Big|_{t=0}=0$$

imply that

$$\exp\big(\int_0^t a(s)ds\big)(G_p'\phi-G_p\phi')\Big|_{t=\theta(\alpha)}<0 \tag{4.14}$$

On the other hand, we have from (4.9) and the definitions of p and $\theta(\alpha)$ that

$$\exp\big(\int_0^t a(s)ds\big)(G_p'\phi-G_p\phi')\Big|_{t=\theta(\alpha)}=0$$

This contradicts (4.14). Therefore, (4.12) holds for some $t_3\in(0,\theta(\alpha)]$.

Step 6. We show that $\theta(\alpha)$ is the unique zero of $\phi(t,\alpha)$ in $(c(\alpha),b(\alpha)]$. Suppose on the contrary that there exists $\tau_1\in(\theta(\alpha),b(\alpha)]$ such that

$$\phi(\tau_1,\alpha)=0,\quad \phi(t,\alpha)<0\quad \text{on } (\theta(\alpha),\tau_1)$$

and

$$\phi'(\tau_1,\alpha)>0. \tag{4.15}$$

(Note that if $\phi'(\tau_1,\alpha)=0$, then $\phi\equiv 0$. This contradicts (2.4)) We have from (A3) and (4.12) that

$$L_p(t)\ge 0,\quad t\in[\theta(\alpha),b(\alpha)).$$

Integrating both sides of (4.13) from $\theta(\alpha)$ to τ_1, we get

$$-\exp\Big(\int_0^{\tau_1}a(s)ds\Big)G_p(\tau_1)\phi'(\tau_1,\alpha)=\int_{\theta(\alpha)}^{\tau_1}\exp\Big(\int_0^{t}a(s)ds\Big)\phi(t,\alpha)L_p(t)dt. \tag{4.16}$$

This together with the fact $\phi(t,\alpha)<0$ on $(\theta(\alpha),\tau_1)$ implies that

$$-\exp\Big(\int_0^{\tau_1}a(s)ds\Big)G_p(\tau_1(\alpha))\phi'(\tau_1,\alpha)\le 0. \tag{4.17}$$

On the other hand, we have from the fact $G_p(\tau_1)<0$ and (4.15) that

$$-\exp\Big(\int_0^{\tau_1}a(s)ds\Big)G_p(\tau_1)\phi'(\tau_1,\alpha)>0.$$

This contradicts (4.17). Therefore, $\phi(t,\alpha)$ has not zero point in $(\theta(\alpha),b(\alpha)]$, and consequently $\phi(b(\alpha),\alpha)<0$. □

References

[1] C. V. Coffman; *Uniqueness of the positive radial solution on an annulus of the Dirichlet problem for* $\Delta u-u+u^3=0$, J. Differential Equations, 128(1996), 379-386.

[2] L. Erbe and M. Tang; *Uniqueness of positive radial solutions of* $\Delta u+f(|x|,u)=0$, Differential and Integral Equations, 11(1998),725-743.

[3] L. Erbe and M. Tang; *Uniqueness theorems for positive solutions of quasilinear elliptic equations in a ball*, J. Differential Equations, 138(1997), 351-379.

[4] C. C. Fu and S. S. Lin; *Uniqueness of positive radial solutions for semilinear elliptic equations on annular domains*. Nonlinear Anal., 44(6) (2001), 749–758.

[5] W. M. Ni and R. D. Nussbaum; *Uniqueness and nonuniqueness for positive radial solutions of* $\Delta u+f(u,r)=0$, Comm. Pure Appl. Math., 38(1) (1985), 67–108.

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