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EXISTENCE OF POSITIVE SOLUTIONS FOR THE SYMMETRY THREE-POINT BOUNDARY-VALUE PROBLEM

QIAOZHEN MA

ABSTRACT. In this paper, we show the existence of single and multiple positive solutions for the symmetry three-point boundary value problem under suitable conditions by using classical fixed point theorem in cones.

1. Introduction

Since Gupta [3] studied three-point boundary value problems for the nonlinear ordinary differential equation, many classical results have been obtained by using Leray-Schauder continuation theorem, nonlinear alternatives of Leray-Schauder and coincidence degree theory. For more information, we refer the reader to [1, 3, 6, 7] and reference therein. The study of multi-point boundary-value problems for linear second-order differential equations was initiated by II'in and Moiseev [4]. While the multi-point boundary value problem arise in the different areas of applied mathematics and physics. For instance, many problems in the theory of elastic stability can be handled as a multi-point problem [8]. Therefore, it's necessary to continue to extend and investigate.

Ma [6], by using fixed-point index theorems and Leray-Schauder degree and upper and lower solutions, considered the multiplicity of positive solutions of the problem

$$u'' + \lambda h(t)f(u) = 0, \quad t \in (0,1), \tag{1.1}$$

$$u(0) = 0, \quad u(1) = \alpha u(\eta), \tag{1.2}$$

where $0 < \eta < 1$, $0 < \alpha < 1/\eta$, assuming that $f \in C([0,\infty),[0,\infty))$, $h \in C([0,1),[0,\infty))$, and f is superlinear. In the present paper, we study the existence of single and multiple positive solutions to nonlinear symmetry three-point boundary value problem

$$u'' + \lambda a(t)f(u) = 0, \quad t \in (0,1), \tag{1.3}$$

$$u(0) = \beta u(\eta), \quad u(1) = \alpha u(\eta). \tag{1.4}$$

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Where $\lambda > 0$ is a positive parameter, $\alpha > 0$, $\beta > 0$, $0 < \eta < 1$.

Clearly, problem (1.3)-(1.4) is more generic than (1.1)-(1.2), that is to say, our problem is (1.1)-(1.2) for $\beta = 0$. Moreover, (1.3)-(1.4) is transformed immediately into the classical Dirichlet problem for $\alpha = \beta = 0$. And when $\beta = 0$, $\alpha = 1$, $\eta \to 1$ problem (1.3)-(1.4) is changed into the mixed boundary value problem. In addition, our results will be obtained under conditions that do not require f to be either superlinear or sublinear. In short, our problem gives a frame to these problems under more generic conditions. We make the following assumptions.

(i) $a \in C([0,1],[0,+\infty))$ and there exists $x_0 \in [0,1]$ such that $a(x_0) > 0$.

(ii) $f \in C([0,+\infty),[0,+\infty))$ and there exist nonnegative constants in the extended reals, f_0, f_∞, such that

$$f_0 = \lim_{u\to 0+} \frac{f(u)}{u}, \quad f_\infty = \lim_{u\to\infty} \frac{f(u)}{u}.$$

(iii) $f(0) > 0$, for $t \in [0,1]$.

Remark 1.1. It is easy to see that if (iii) holds, then there exist two constants $a, b \in (0,\infty)$, such that $0 < f(u) \le b$, for $u \in [0,a]$.

The key tool in our approach is the following Krasnoselskii's fixed point theorem in a cone.

Theorem 1.2 ([2]). *Let E be a Banach space and $K \subset E$ be a cone in E. Suppose that Ω_1, Ω_2 are bounded open subset of K with $0 \in \Omega_1, \bar{\Omega}_1 \subset \Omega_2$, and $A : K \to K$ is a completely continuous operator such that either*

$$\|Aw\| \le \|w\|, \quad w \in \partial\Omega_1, \quad \|Aw\| \ge \|w\|, \quad w \in \partial\Omega_2, \quad \text{or}$$
$$\|Aw\| \ge \|w\|, \quad w \in \partial\Omega_1, \quad \|Aw\| \le \|w\|, \quad w \in \partial\Omega_2.$$

Then A has a fixed point in $\bar{\Omega}_2 \setminus \Omega_1$.

2. Preliminary Lemmas

Lemma 2.1 ([5]). *Let $\beta \ne \frac{1-\alpha\eta}{1-\eta}$. Then, for $y \in C[0,1]$, boundary-value problem*

$$u'' + y(t) = 0, \quad t \in (0,1), \tag{2.1}$$

$$u(0) = \beta u(\eta), \quad u(1) = \alpha u(\eta). \tag{2.2}$$

has a unique solution

$$\begin{aligned} u(t) = &-\int_0^t (t-s)y(s)ds + \frac{(\beta-\alpha)t - \beta}{(1-\alpha\eta) - \beta(1-\eta)} \int_0^\eta (\eta - s)y(s)ds \\ &+ \frac{(1-\beta)t + \beta\eta}{(1-\alpha\eta) - \beta(1-\eta)} \int_0^1 (1-s)y(s)ds. \end{aligned}$$

Lemma 2.2 ([5]). *Let $0 < \alpha < 1/\eta$, $0 < \beta < \frac{1-\alpha\eta}{1-\eta}$. Then, for $y \in C[0,1]$, and $y \ge 0$, the unique solution of problem (2.1)-(2.2) satisfies*

$$u(t) \ge 0, \ t \in [0,1].$$

Lemma 2.3 ([5]). *Let $0 < \alpha < \frac{1}{\eta}$, $0 < \beta < \frac{1-\alpha\eta}{1-\eta}$. Then, for $y \in C[0,1]$, and $y \ge 0$, the unique solution of problem (2.1)-(2.2) satisfy*

$$\min_{t\in[0,1]} u(t) \ge \gamma \|u\|,$$

where

$$\gamma = \min\{\frac{\alpha(1-\eta)}{1-\alpha\eta}, \alpha\eta, \beta\eta, \beta(1-\eta)\}.$$

Note that $u = u(t)$ is a solution of (1.3)-(1.4), if and only if

$$\begin{aligned} u(t) =&\lambda[-\int_0^t (t-s)a(s)f(u(s))ds + \frac{(\beta-\alpha)t-\beta}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^\eta (\eta-s)a(s)f(u(s))ds \\ &+ \frac{(1-\beta)t+\beta\eta}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^1 (1-s)a(s)f(u(s))ds] := A_\lambda u(t). \end{aligned} \tag{2.3}$$

Define a cone K in the Banach space $C[0,1]$,

$$K = \{u : u \in C[0,1],\ u \geq 0,\ \min_{t\in[0,1]} u(t) \geq \gamma\|u\|\}.$$

By Lemmas 2.2 and 2.3, we know that $A_\lambda K \subset K$ and it is not hard to verify that $A_\lambda : K \to K$ is a completely continuous.

3. Main Results

Throughout this paper, we shall use the following notation

$$A = \frac{1+\beta(1+\eta)}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^1 (1-s)a(s)ds, \quad B = \frac{\beta(1-\eta)}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^\eta sa(s)ds.$$

Here and below we assume that $\alpha\eta < 1$.

Theorem 3.1. *Suppose that (i)-(ii) hold. Then we have*

1. *If $Af_0 < \gamma Bf_\infty$, then for each $\lambda \in (\frac{1}{\gamma Bf_\infty}, \frac{1}{Af_0})$, the problem* (1.3)-(1.4) *has at least one positive solution.*
2. *If $f_0 = 0$ and $f_\infty = \infty$, then for any $\lambda \in (0, \infty)$, the problem* (1.3)-(1.4) *has at least one positive solution.*
3. *If $f_\infty = \infty$, $0 < f_0 < \infty$, then for each $\lambda \in (0, \frac{1}{Af_0})$, the problem* (1.3)-(1.4) *has at least one positive solution.*
4. *If $f_0 = 0$, $0 < f_\infty < \infty$, then for each $\lambda \in (\frac{1}{\gamma Bf_\infty}, \infty)$, the problem* (1.3)-(1.4) *has at least one positive solution.*

Proof. Since the proof of (2)-(4) is similar to the proof of (1), we only prove (1). Let $\lambda \in (\frac{1}{\gamma Bf_\infty}, \frac{1}{Af_0})$, and choose $\varepsilon > 0$ such that

$$\frac{1}{\gamma B(f_\infty-\varepsilon)} \leq \lambda \leq \frac{1}{A(f_0+\varepsilon)}. \tag{3.1}$$

By the definition of f_0, there exists $H_1 > 0$ such that $f(x) \le (f_0+\varepsilon)x$ for $x \in [0, H_1]$. Let $u \in K$ with $\|u\| = H_1$, by (2.3) and (3.1), we conclude that

$$\begin{aligned}
A_\lambda u(t) &\le \frac{\lambda\beta t}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^\eta (\eta-s)a(s)f(u(s))ds \\
&\quad + \frac{\lambda(t+\beta\eta)}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^1 (1-s)a(s)f(u(s))ds \\
&\le \frac{\lambda\beta}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^1 (1-s)a(s)f(u(s))ds \\
&\quad + \frac{\lambda(1+\beta\eta)}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^1 (1-s)a(s)f(u(s))ds \\
&= \frac{\lambda(1+\beta+\beta\eta)}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^1 (1-s)a(s)f(u(s))ds \\
&\le \frac{\lambda(1+\beta+\beta\eta)}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^1 (1-s)a(s)(f_0+\varepsilon)u(s)ds \\
&\le \lambda A(f_0+\varepsilon)\|u\| \le \|u\|.
\end{aligned} \tag{3.2}$$

As a result, $\|A_\lambda u\| \le \|u\|$. Let $\Omega_1 = \{u \in K : \|u\| < H_1\}$, then

$$\|A_\lambda u\| \le \|u\|, \quad \text{for } u \in K \cap \partial\Omega_1. \tag{3.3}$$

Again thanks to the definition of f_∞, there exists $\hat{H}_2 > 0$ such that $f(x) \ge (f_\infty - \varepsilon)x$, for every $x \in [\hat{H}_2, \infty)$. Denote $H_2 = \max\{2H_1, \frac{\hat{H}_2}{\gamma}\}$, $\Omega_2 = \{u \in K : \|u\| < H_2\}$.

If $u \in K$ with $\|u\| = H_2$, then $\min_{t\in[0,1]} u(t) \ge \gamma\|u\| \ge \hat{H}_2$. It leads to

$$\begin{aligned}
A_\lambda u(0) &= -\frac{\lambda\beta}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^\eta (\eta-s)a(s)f(u(s))ds \\
&\quad + \frac{\lambda\beta\eta}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^1 (1-s)a(s)f(u(s))ds \\
&\ge -\frac{\lambda\beta}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^\eta (\eta-s)a(s)f(u(s))ds \\
&\quad + \frac{\lambda\beta\eta}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^\eta (1-s)a(s)f(u(s))ds \\
&= \frac{\lambda\beta(1-\eta)}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^\eta sa(s)f(u(s))ds \\
&\ge \frac{\lambda\beta(1-\eta)}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^\eta sa(s)(f_\infty-\varepsilon)u(s)ds \\
&\ge \lambda\gamma B(f_\infty-\varepsilon)\|u\| \ge \|u\|.
\end{aligned} \tag{3.4}$$

Consequently, $\|A_\lambda u\| \ge \|u\|$ for $u \in K \cap \partial\Omega_2$.

Thus, according to the first condition of Theorem 1.2, A_λ has a fixed point $u(t)$ with $H_1 \le \|u\| \le H_2$ in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$. □

Theorem 3.2. *Suppose that (i)-(ii) hold. Then we have*

(1) *If $Af_\infty < \gamma Bf_0$, then for each $\lambda \in (\frac{1}{\gamma Bf_0}, \frac{1}{Af_\infty})$, the problem* (1.3)-(1.4) *has at least one positive solution.*

(2) *If* $f_0 = \infty$ *and* $f_\infty = 0$*, then for any* $\lambda \in (0, \infty)$*, the problem* (1.3)-(1.4) *has at least one positive solution.*
(3) *If* $f_\infty = \infty$*,* $0 < f_0 < \infty$*, then for each* $\lambda \in (0, \frac{1}{Af_\infty})$*, the problem* (1.3)-(1.4) *has at least one positive solution.*
(4) *If* $f_0 = 0$*,* $0 < f_\infty < \infty$*, then for each* $\lambda \in (\frac{1}{\gamma B f_0}, \infty)$*, the problem* (1.3)-(1.4) *has at least one positive solution.*

Proof. Since the proof of (2)-(4) is similar to the proof of (1), we only prove (1). Let $\lambda \in (\frac{1}{\gamma B f_0}, \frac{1}{Af_\infty})$, and choose $\varepsilon > 0$ such that

$$\frac{1}{\gamma B(f_0 - \varepsilon)} \le \lambda \le \frac{1}{A(f_\infty + \varepsilon)}. \tag{3.5}$$

By the definition of f_0, there exists $H_3 > 0$ such that $f(x) \ge (f_0 - \varepsilon)x$ for $x \in [0, H_3]$. Let $u \in K$ with $\|u\| = H_3$ such that $\min_{t\in[0,1]} u(t) \ge \gamma\|u\|$. Similar to the estimates of (3.4), we obtain

$$\begin{aligned} A_\lambda u(0) &\ge \frac{\lambda\beta(1-\eta)}{(1-\alpha\eta) - \beta(1-\eta)} \int_0^\eta sa(s)f(u(s))ds \\ &\ge \frac{\lambda\beta(1-\eta)}{(1-\alpha\eta) - \beta(1-\eta)} \int_0^\eta sa(s)(f_0 - \varepsilon)u(s)ds \\ &\ge \lambda\gamma B(f_0 - \varepsilon)\|u\| \ge \|u\|. \end{aligned} \tag{3.6}$$

Hence, it follows that $\|A_\lambda u\| \ge \|u\|$. Set $\Omega_1 = \{u \in K : \|u\| < H_3\}$, we claim

$$\|A_\lambda u\| \ge \|u\|, \quad \text{for } u \in K \cap \partial\Omega_1.$$

Again in line with the definition of f_∞, there exists $\tilde{H}_4$ such that $f(x) \le (f_0 + \varepsilon)x$, for $x \in [\tilde{H}_4, \infty)$. We discuss two possible cases:.

Case 1. Suppose that f is bounded, that is, there exists a positive constant M_1 such that $f(x) \le M_1$ for all $x \in [0, \infty)$. Set $H_4 = \max\{2H_3, \lambda M_1 A\}$. If $u \in K$ with $\|u\| = H_4$, similar to (3.2), we obtain

$$\begin{aligned} A_\lambda u(t) &\le \frac{\lambda(1+\beta+\beta\eta)}{(1-\alpha\eta) - \beta(1-\eta)} \int_0^1 (1-s)a(s)f(u(s))ds \\ &\le \lambda M_1 A \le H_4 = \|u\|. \end{aligned} \tag{3.7}$$

Thus, by setting $\Omega_2 = \{u \in K : \|u\| < H_4\}$, we get

$$\|A_\lambda u\| \le \|u\|, \quad \text{for} \quad u \in K \cap \partial\Omega_2.$$

Case 2. Suppose that f is unbounded, we choose $H_4 > \max\{2H_3, \gamma^{-1}\tilde{H}_4\}$ such that $f(x) \le f(H_4)$, for $x \in [0, H_4]$. Let $u \in K$ with $\|u\| = H_4$, we have

$$\begin{aligned} A_\lambda u(t) &\le \frac{\lambda(1+\beta+\beta\eta)}{(1-\alpha\eta) - \beta(1-\eta)} \int_0^1 (1-s)a(s)f(u(s))ds \\ &\le \frac{\lambda(1+\beta+\beta\eta)}{(1-\alpha\eta) - \beta(1-\eta)} \int_0^1 (1-s)a(s)f(H_4)ds \\ &\le \lambda A(f_\infty + \varepsilon)H_4 \le \|u\|. \end{aligned} \tag{3.8}$$

Let $\Omega_2 = \{u \in K : \|u\| < H_4\}$, this yields

$$\|A_\lambda u\| \le \|u\|, \quad \text{for } u \in K \cap \partial\Omega_2.$$

As a result, from the above estimates and by Theorem 1.2, it follows that A_λ has a fixed point $u \in K \cap (\bar{\Omega}_2 \setminus \Omega_1)$. $\square$

Theorem 3.3. *Suppose that (i)-(ii) are true. In addition, assume that there exist two positive constants H_5, H_6 with $H_5 < \gamma H_6$ and $AH_6 \le BH_5$ such that*

(1) $f(x) \le \frac{H_5}{\lambda A}$, $\forall x \in [0, H_5]$,
(2) $f(x) \ge \frac{H_6}{\lambda B}$, $\forall x \in [\gamma H_6, H_6]$.

Then problem (1.3)-(1.4) *has at least one positive solution $u^* \in K$ with $H_5 \le \|u^*\| \le H_6$.*

The proof is similar to the proofs of Theorems 3.1 and 3.2, so we omit it.

Theorem 3.4. *Suppose that (i)-(iii) hold, moreover, $f_\infty = \infty$. Then there exists a positive constant Λ_1 such that problem* (1.3)-(1.4) *has at least two positive solutions for λ small enough.*

Proof. From (3) of theorems 3.1 and 3.2, we can see that (1.3)-(1.4) has a positive solution u_1 satisfying

$$\|u_1\| \ge H, \tag{3.9}$$

where H is a suitable constant for $\lambda \in (0, \mu^*)$, and $\mu^* = \min\{\frac{1}{Af_0}, \frac{1}{Af_\infty}\}$.

To find the second positive solution of (1.3)-(1.4), we set

$$f^*(u) = \begin{cases} f(u), & \text{for } u \in [0, a], \\ f(a), & \text{for } u \in [a, \infty), \end{cases} \tag{3.10}$$

then $0 < f^*(u) \le b$ for $u \in [0, \infty)$, where a, b are given in remark 1.1.

Now we consider the auxiliary equation

$$u'' + \lambda a(t) f^*(u) = 0, \quad t \in (0, 1) \tag{3.11}$$

with the boundary value conditions

$$u(0) = \beta u(\eta), \quad u(1) = \alpha u(\eta). \tag{3.12}$$

It is easy to check that (3.11)-(3.12) is equivalent to the fixed point equation $u = F_\lambda u$, where

$$\begin{aligned} F_\lambda u(t) := \lambda[&- \int_0^t (t-s) a(s) f^*(u(s)) ds \\ &+ \frac{(\beta - \alpha)t - \beta}{(1 - \alpha\eta) - \beta(1 - \eta)} \int_0^\eta (\eta - s) a(s) f^*(u(s)) ds \\ &+ \frac{(1 - \beta)t + \beta\eta}{(1 - \alpha\eta) - \beta(1 - \eta)} \int_0^1 (1 - s) a(s) f^*(u(s)) ds]. \end{aligned}$$

Clearly, $F_\lambda : K \to K$ is completely continuous and $F_\lambda(K) \subset K$. Set

$$H_7 = \min\{\frac{H}{2}, a\}, \tag{3.13}$$

$$\Lambda = \min\{H_7[\frac{(1 + \beta + \beta\eta)M}{(1 - \alpha\eta) - \beta(1 - \eta)} \int_0^1 (1 - s) a(s) ds]^{-1}, \mu^*\}$$

and fix $\lambda \in (0, \Lambda)$, where $M = \max\{f^*(u) : 0 \le u \le H_7\}$.

Choose $\Omega_3 = \{u \in C[0,1] : \|u\| < H_7\}$, then for $u \in K \cap \partial\Omega_3$, we have

$$\begin{aligned} F_\lambda u(t) &\le \frac{\lambda(1+\beta+\beta\eta)}{(1-\alpha\eta)-\beta(1-\eta)} \int_0^1 (1-s)a(s)f^*(u(s))ds \\ &\le \frac{\lambda M(1+\beta+\beta\eta)}{(1-\alpha\eta)-\beta(1-\eta)} \int_0^1 (1-s)a(s)ds \\ &\le H_7. \end{aligned} \tag{3.14}$$

Therefore, $\|F_\lambda u\| \le \|u\|$, for $u \in K \cap \partial\Omega_3$.

From (iii) we know that $\lim_{u\to 0+} \frac{f^*(u)}{u} = +\infty$. This means that there exists a constant H_8 ($H_8 < H_7$) such that $f^*(u) \ge \rho u$ for $u \in [0, H_8]$, where

$$\frac{\lambda\rho\beta\gamma(1-\eta)}{(1-\alpha\eta)-\beta(1-\eta)} \int_0^1 sa(s)ds \ge 1.$$

Also

$$\begin{aligned} F_\lambda u(0) &\ge \frac{\lambda\beta(1-\eta)}{(1-\alpha\eta)-\beta(1-\eta)} \int_0^\eta sa(s)f^*(u(s))ds \\ &\ge \frac{\lambda\beta(1-\eta)}{(1-\alpha\eta)-\beta(1-\eta)} \int_0^\eta sa(s)\rho u(s)ds \\ &\ge \frac{\lambda\rho\beta\gamma(1-\eta)}{(1-\alpha\eta)-\beta(1-\eta)} \int_0^\eta sa(s)ds\|u\| \ge \|u\|. \end{aligned} \tag{3.15}$$

Thus, we may let $\Omega_4 = \{u \in C[0,1] : \|u\| < H_8\}$, so that $\|F_\lambda u\| \ge \|u\|$, for $u \in K \cap \partial\Omega_4$.

By the second part of Theorem 1.2, it follows that (3.11)-(3.12) has a positive solution u_2 satisfying

$$H_8 \le \|u_2\| \le H_7. \tag{3.16}$$

Combining with (3.10), (3.13), we obtain that u_2 is also a solution of (1.3)-(1.4).

In other words, from (3.9) and (3.16) we show that (1.3)-(1.4) has two distinct positive solutions u_1 and u_2 for $\lambda \in (0, \Lambda_1)$. $\square$

Theorem 3.5. *Suppose that (i)-(iii) hold, furthermore, $f_0 = f_\infty = 0$. Then the problem* (1.3)-(1.4) *has at least two positive solutions for λ large enough.*

Proof is the same as that of Theorem 3.4, we omit it.

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Qiaozhen Ma
College of Mathematics and Information Science, Northwest Normal University, Lanzhou, Gansu, 730070, China
E-mail address: maqzh@nwnu.edu.cn