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ON ASYMPTOTIC BEHAVIOUR OF OSCILLATORY SOLUTIONS FOR FOURTH ORDER DIFFERENTIAL EQUATIONS

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Abstract. We establish sufficient conditions for the linear differential equations of fourth order

$$(r(t)y'''(t))' = a(t)y(t) + b(t)y'(t) + c(t)y''(t) + f(t)$$

so that all oscillatory solutions of the equation satisfy

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} y'(t) = \lim_{t\to\infty} y''(t) = \lim_{t\to\infty} r(t)y'''(t) = 0,$$

where $r : [0, \infty) \to (0, \infty), a, b, c$ and $f : [0, \infty) \to R$ are continuous functions. A suitable Green's function and its estimates are used in this paper.

1. Introduction

Bainov and Dimitrova [2] proved the following result (See also Theorem 3.1.1 with $n = 4$ in [1]).

Theorem 1.1. *Assume*

$$\lim_{t\to\infty} \int_{t_0}^{t} \frac{1}{r_1(s_1)} \int_{t_0}^{s_1} \frac{1}{r_2(s_2)} \int_{t_0}^{s_2} \frac{1}{r_3(s_3)}\, ds_3\, ds_2\, ds_1 < \infty, \tag{1.1}$$

$$\int_{t_0}^{\infty} |a(t)|\, dt < \infty, \tag{1.2}$$

$$\int_{t_0}^{\infty} |f(t)|\, dt < \infty, \tag{1.3}$$

Then all solutions of

$$(r_3(t)(r_2(t)(r_1(t)y'(t))')')' + a(t)F((Ay)(t)) = f(t) \tag{1.4}$$

are bounded and all oscillatory solutions of (1.4) *tend to zero as* $t \to \infty$, *where* $F \in C(R, R)$ *and* $F(u)$ *is a bounded function on* $R, r_i \in C^{4-i}([t_0, \infty); (0, \infty)), 1 \le i \le 3$, $a, f \in C([t_0, \infty); R)$ *and* A *is an operator with certain properties.*

The motivation of the present work has come from Theorem 1.1. Since $F(u) = u$ is not bounded, then Theorem 1.1 cannot be applied to its corresponding linear equation. Our purpose is to show that under the conditions of Theorem 1.1, every

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oscillatory solution of the considered equation along with their first and second order derivatives tend to zero as $t \to \infty$. In fact, we consider the more general fourth order linear differential equations of the form

$$(r(t)y'''(t))' = a(t)y(t) + b(t)y'(t) + c(t)y''(t) + f(t) \tag{1.5}$$

where $r : [0,\infty) \to (0,\infty), a, b, c$ and $f : [0,\infty) \to R$ are continuous functions. We shall show that all oscillatory solutions of (1.5) along with their first and second order derivatives tend to zero as $t \to \infty$.

Asymptotic behaviour of oscillatory solutions of second order differential equations have been studied by many authors, see ([6, 7, 8, 9]). For higher order differential equations, one may see the paper due to Chen and Yeh [3] and the recent work due to Padhi [5] and the references cited therein. The monograph due to [1] gives a survey on the asymptotic decay of oscillatory solutions of differential equations. In [5], Padhi considered a more general forced differential equation where he obtained a new sufficient condition under which all oscillatory solutions of the equation tend to zero as $t \to \infty$. The result improve all earlier existing results. In a recent note [4], Padhi studied the asymptotic behaviour of oscillatory solutions of third order linear differential equations. It seems that asymptotic behaviour of oscillatory solutions of fourth order differential equations of the form (1.5) has not been studied in the literature. Motivated by the result in [4], this work pays an attention for the asymptotic behaviour of oscillatory solutions of the equations of the form (1.5). The technique used in the work is the help of a Green's function and its estimates. This technique was used by Padhi [4]. The sufficient conditions given in this paper may be treated as a different set of condition given in [5]. Sufficient conditions for oscillations of equations of the form (1.5) with $b(t) = 0$ and $c(t) = 0$ are given in [1].

The work is organized as follows: Section 1 is introductory where as the main result of the paper is given in Section 2 and an open problem is left to the reader.

We note that a solution of the above mentioned equations is said to be oscillatory if it has arbitrarily large zeros.

2. Main Results

The main result of the paper is the following.

Theorem 2.1. *Let* (1.2) *and* (1.3) *hold. Further suppose that*

$$\int_0^\infty \frac{t^2}{r(t)}\,dt < \infty, \quad \int_0^\infty |b(t)|\,dt < \infty, \quad \int_0^\infty |c(t)|\,dt < \infty. \tag{2.1}$$

Then every oscillatory solution of the equation (1.5) *satisfies*

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} y'(t) = \lim_{t\to\infty} y''(t) = \lim_{t\to\infty} r(t)y'''(t) = 0. \tag{2.2}$$

Proof. Let $\{t_k\}_{k=1}^\infty$, $1 < t_k < t_{k+1}$, $(k = 1, 2, 3, \dots)$ be such that $y(t_k) = 0$. Then for each natural k, there exists $t'''_{k+1} \in (t_{k+1}, t_{k+3})$ such that $y'''(t'''_{k+1}) = 0$. Hence (1.5) implies

$$y'''(t) = \frac{1}{r(t)} \int_{t'''_{k+1}}^t [a(s)y(s) + b(s)y'(s) + c(s)y''(s) + f(s)]\,ds. \tag{2.3}$$

We can find t'_{k+1} and $t''_{k+1}(t_{k+1} < t'_{k+1} < t''_{k+1})$ such that $y(t_{k+1}) = 0, y'(t'_{k+1}) = 0$ and $y''(t''_{k+1}) = 0$. We note that $t'''_{k+1} \le t''_{k+1}$.

If now we set

$$\rho_{ik} = \max\{|y^{(i)}(t)| : t_{k+1} \le t \le t''_{k+1}\}, \quad i = 0,1,2,$$
$$\rho_{3k} = \max\{r(t)|y'''(t)| : t_{k+1} \le t \le t''_{k+1}\},$$
$$\epsilon_k = \int_{t_k}^{t''_{k+1}} \big(\frac{t^2}{r(t)} + |a(t)| + |b(t)| + |c(t)| + |f(t)|\big)\, dt,$$

then it follows that

$$|y'''(t)| \le \frac{1}{r(t)}\epsilon_k(\rho_{0k} + \rho_{1k} + \rho_{2k} + 1), \quad t_{k+1} \le t \le t''_{k+1}, \tag{2.4}$$
$$\rho_{3k} \le \epsilon_k(\rho_{0k} + \rho_{1k} + \rho_{2k} + 1), \; t_{k+1} \le t \le t''_{k+1}. \tag{2.5}$$

On the other hand, by conditions (1.2),(1.3) and (2.1), we have

$$\lim_{k\to\infty} \epsilon_k = 0. \tag{2.6}$$

Therefore, without any loss of generality, it can be assumed that

$$\epsilon_k < \frac{1}{\sqrt{7}}, \quad (k = 1,2,3,\dots). \tag{2.7}$$

By Green's formula, for each natural k, we have

$$\begin{aligned} y(t) &= \int_{t_{k+1}}^{t''_{k+1}} G_k(t,s) y'''(s)\, ds, \\ y'(t) &= \int_{t_{k+1}}^{t''_{k+1}} \frac{\delta G_k(t,s)}{\delta s} y'''(s)\, ds, \\ y''(t) &= \int_{t_{k+1}}^{t''_{k+1}} \frac{\delta^2 G_k(t,s)}{\delta s^2} y'''(s)\, ds \end{aligned} \tag{2.8}$$

where

$$G_k(t,s) = \begin{cases} s \in [t_{k+1}, t'_{k+1}] : \begin{cases} \frac{(t-t_k)}{2}(2s - t - t_k), & t \le s \\ \frac{(s-t_{k+1})^2}{2}, & s \le t. \end{cases} \\ s \in [t'_{k+1}, t''_{k+1}] : \begin{cases} \frac{(t-t_{k+1})}{2}(2t'_{k+1} - t - t_k), & t \le s \\ \frac{(t-s)^2}{2} + \frac{(t-t_{k+1})}{2}(2t'_{k+1} - t - t_{k+1}), & s \le t \end{cases} \end{cases}$$

is the Green's function for $y'''(t) = 0, y(t_{k+1}) = 0,\; y'(t'_{k+1}) = 0,\; y''(t''_{k+1}) = 0$. Moreover,

$$|G_k(t,s)| < \frac{3s^2}{2}, \quad |\frac{\delta G_k(t,s)}{\delta t}| < s, \quad |\frac{\delta^2 G_k(t,s)}{\delta t^2}| < 1,$$

for $t_{k+1} \le s \le t''_{k+1}$. By these estimates and inequalities (2.4) and (2.8), we have

$$\rho_{0k} \le \frac{3}{2}\epsilon_k(\rho_{0k} + \rho_{1k} + \rho_{2k} + 1) \int_{t_{k+1}}^{t''_{k+1}} \frac{s^2}{r(s)}\, ds \le \frac{3}{2}\epsilon_k^2(\rho_{0k} + \rho_{1k} + \rho_{2k} + 1),$$
$$\rho_{1k} \le \epsilon_k(\rho_{0k} + \rho_{1k} + \rho_{2k} + 1) \int_{t_{k+1}}^{t''_{k+1}} \frac{s}{r(s)}\, ds \le \epsilon_k^2(\rho_{0k} + \rho_{1k} + \rho_{2k} + 1),$$
$$\rho_{2k} \le \epsilon_k(\rho_{0k} + \rho_{1k} + \rho_{2k} + 1) \int_{t_{k+1}}^{t''_{k+1}} \frac{1}{r(s)}\, ds \le \epsilon_k^2(\rho_{0k} + \rho_{1k} + \rho_{2k} + 1).$$

Thus

$$\begin{aligned}\rho_{0k}+\rho_{1k}+\rho_{2k} &\le \frac{7}{2}\epsilon_k^2(\rho_{0k}+\rho_{1k}+\rho_{2k})+\frac{7}{2}\epsilon_k^2\\ &\le \frac{7}{2}\cdot\frac{1}{7}(\rho_{0k}+\rho_{1k}+\rho_{2k})+\frac{7}{2}\epsilon_k^2\\ &\le \frac{1}{2}(\rho_{0k}+\rho_{1k}+\rho_{2k})+\frac{7}{2}\epsilon_k^2\end{aligned}$$

which in turn implies

$$\rho_{0k}+\rho_{1k}+\rho_{2k}\le 7\epsilon_k^2$$

and from (2.5)

$$\rho_{3k}\le \epsilon_k(\rho_{0k}+\rho_{1k}+\rho_{2k}+1)\le \epsilon_k(7\epsilon_k^2+1).$$

Since (2.6) holds, then the above inequality yields that $\rho_{ik}\to 0$ as $k\to\infty, i=0,1,2,3$. This completes the proof of the theorem. □

Example 2.2. Consider the equation

$$(t^4y'''(t))'=\frac{1}{t^2}y(t)+\frac{1}{t^2}y'(t)+\frac{1}{t^2}y''(t)+f(t),\quad t\ge 1, \tag{2.9}$$

where

$$\begin{aligned}f(t)=\frac{\sin t}{t^2}+\frac{20\cos t}{t^3}-\frac{190\sin t}{t^4}-\frac{896\cos t}{t^5}+\frac{1760\sin t}{t^6}\\ -\frac{\cos t}{t^8}-\frac{6\sin t}{t^9}-\frac{12\cos t}{t^9}+\frac{42\sin t}{t^{10}}.\end{aligned}$$

All the conditions of Theorem 2.1 are satisfied. We note that $y(t)=\frac{\sin t}{t^6}$ is an oscillatory solution of the equation (2.9) satisfying the property (2.2).

It is clear that the conclusion of Theorem 2.1 holds for the homogeneous equation

$$(r(t)y'''(t))'=a(t)y(t)+b(t)y'(t)+c(t)y''(t) \tag{2.10}$$

However, from Example 2.2, it seems that the forcing term $f(t)$ plays a crucial role in constructing the example. Thus, it would be interesting to obtain an example for the homogeneous equation (2.10) satisfying the conclusions of Theorem 2.1 under the conditions (1.2) and (2.1).

Remak 2.3. It would be interesting to obtain sufficient conditions on the coefficient functions using the above technique so that any arbitrary oscillatory solution $y(t)$ of the general n-th order linear differential equations of the form

$$(r(t)y^{(n-1)}(t))'=\sum_{i=0}^{n-2}p_i(t)y^{(i)}(t)+f(t) \tag{2.11}$$

satisfies

$$\lim_{t\to\infty}y(t)=\lim_{t\to\infty}y'(t)=\lim_{t\to\infty}y''(t)=\dots=\lim_{t\to\infty}y^{(n-2)}(t)=\lim_{t\to\infty}r(t)y^{(n-1)}(t)=0 \tag{2.12}$$

where r and f are as defined earlier and $p_i:[0,\infty)\to R\,(i=0,1,2,\dots,n-2)$.

It seems that the following sufficient conditions are needed to prove the above remark.

Open problem. Under condition (1.3) and

$$\int_0^\infty \frac{t^{n-2}}{r(t)}\,dt < \infty, \quad \int_0^\infty |p_i(t)|\,dt < \infty, \quad i = 0, 1, \dots, n-2, \tag{2.13}$$

does every oscillatory solution $y(t)$ of (2.11) satisfy property (2.12)?

A suitable Green's function and their estimates maybe needed to answer the above open problem. However, we have not found that Green's function yet.

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