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SOLUTIONS OF FOURTH-ORDER PARTIAL DIFFERENTIAL EQUATIONS IN A NOISE REMOVAL MODEL

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Abstract. In this paper, we discuss the existence and uniqueness of weak solutions for a fourth-order partial differential equation stemmed from image processing for noise removal. We also present some numerical tests for high order filters.

1. Introduction

We study the fourth-order initial-boundary value problem

$$\frac{\partial u}{\partial t}+\frac{\partial^2}{\partial x^2}\Phi'\big(\frac{\partial^2 u}{\partial x^2}\big)=0 \quad (x,t)\in Q_T, \tag{1.1}$$

$$u(0,t)=u(1,t)=u'(0,t)=u'(1,t)=0 \quad t\in(0,T), \tag{1.2}$$

$$u(x,0)=u_0(x) \quad x\in I, \tag{1.3}$$

where $I=(0,1)$, $Q_T=I\times(0,T)$ and $\Phi:\mathbb{R}\to\mathbb{R}^+$ is an N function; i.e. $\Phi(\cdot)$ is even, continuous, convex with $\Phi>0$ for $t>0$,

$$\lim_{t\to 0}\frac{\Phi(t)}{t}\to 0 \quad \text{and} \quad \lim_{t\to\pm\infty}\frac{\Phi(t)}{|t|}\to+\infty. \tag{1.4}$$

Here we assume that Φ satisfies the Δ_2-condition:

$$\Phi(2\xi)\le K\Phi(\xi), \quad |\xi|\ge R, \tag{1.5}$$

where $K>2$ and R are two positive constants.

In recent years, many nonlinear PDEs are proposed to deal with the trade-off between noise removal and edge preservation. Among them, the fourth-order parabolic PDEs have drawn great interest [4, 7, 11, 12, 18, 19, 20]. Since they seek to minimize a cost functional which is an increasing function of the absolute value of the Laplacian of the image intensity function, they could decrease the staircasing property which may be undesirable under some circumstances [4, 15]. In general, the forms of fourth-order PDEs are analogous with the second order ones. For example, in [18], You and Kaveh proposed equation

$$u_t=-\Delta(g(\Delta u)\Delta u),$$

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where $g(s) = 1/(1+s^2)$, which is analogous with the Perona-Malik model [13]. In [12], Lysaker *et al* used the equation

$$u_t = -\Delta\big(\frac{\Delta u}{|\Delta u|}\big),$$

which is similar to TV model [14]. In [8], Didas used the equation (1.1) with $\Phi(x) = 2\lambda\left(\sqrt{\lambda^2 - x^2} - \lambda\right)$, where $\lambda > 0$ and it is the Charbonnier filter [3].

Our model includes a class of more general equations [3, 8], e.g. $\Phi(s) = \frac{1}{p}|s|^p$, $p > 1$. When $p = 2$, a linear filter could be obtained. While this filter has very strong isotropic smoothing properties and does not preserve edges very well. One should then decrease p in order to preserve the edges as much as possible, that is to say fast diffusion is desired. There are some other functions which satisfy the conditions (1.4) and (1.5), for example:

$$\Phi(s) = |s|\ln(1+|s|),$$

and

$$\Phi(s) = |s|L_k(|s|),$$

where $L_i(s) = \ln(1 + L_{i-1}(s))$ $(i = 1, 2, \dots, k)$ and $L_0(s) = \ln(1+|s|)$, see [9, 16].

Although the effectiveness of fourth order diffusion equations for noise removal has been proposed in [4, 6, 7, 11, 12, 18], very little has been known about theoretical analysis. We refer to [17], Chapter 4 for a nonlinear equation with double-degeneracy, [10] for traveling wave solutions in one dimension, [5] for the existence and uniqueness of (1.1) for $\Phi'(s) = \arctan(s)$, [11] for the existence of a fourth order PDE by variational methods and [19] for a generalized thin film equation.

It is worth while mentioning that the initial data is chosen by the original image generally. We take the zero boundary value conditions for convenience, which corresponds to padding the boundary of the image with black.

The plan of the paper is the following. In Section 2, we state some preliminaries and the main theorem. Section 3 is devoted to the proofs of our main results and Section 4 deals with some numerical experiments using finite difference methods by an explicit scheme.

2. Preliminaries and Main Result

In the following sections we always assume $\Phi(\cdot)$ is a function satisfied the condition (1.4) and (1.5). Then the N-function $\Psi(\cdot)$ which conjugates to $\Phi(\cdot)$ is defined by

$$\Psi(s) = \sup_{t\in\mathbb{R}}\{t\cdot s - \Phi(t)\}.$$

We have the following Young's inequality,

$$s\cdot t \le \Phi(s) + \Psi(t).$$

For all $|s| > R$, we get (see [1, 16])

$$\Phi(s) \le \Phi'(s)s \le (K-1)\Phi(s) \tag{2.1}$$

and

$$0 \le \Psi(\Phi'(s)) = \Phi'(s)s - \Phi(s) \le (K-2)\Phi(s). \tag{2.2}$$

For any $s, t \in \mathbb{R}$, we have (see [1, 16])

$$(\Phi'(s) - \Phi'(t))\cdot(s-t) \ge 0. \tag{2.3}$$

Lemma 2.1 ([1, 16]). *If Ψ conjugates to Φ, then there exist positive numbers $p > 1$, $R > 0$, $R' > 0$, $K_1 > 0$ and $K_2 > 0$ such that for all $s, t \in \mathbb{R}$,*

$$\Phi(s) \le K_1 |s|^p, \quad |s| \ge R, \tag{2.4}$$

$$\Psi(t) \ge K_2 |t|^{p'}, \quad |t| \ge R', \quad p' = \frac{p}{p-1}. \tag{2.5}$$

Lemma 2.2 ([2, 16]). *Suppose $\{f_j\} \subset L^1(I; \mathbb{R})$ satisfies that*

$$\int_I \Phi(f_j) dx \le C,$$

where C is a positive constant. Then there exist a subsequence $\{f_{m_j}\} \subset \{f_j\}$ and a function $f \in L^1(I; \mathbb{R})$ such that

$$f_{m_j} \rightharpoonup f \quad \text{weakly in } L^1(I, \mathbb{R}) \text{ as } j \to \infty$$

with

$$\int_I \Phi(f) dx \le \liminf_{j \to \infty} \int_I \Phi(f_{m_j}) dx \le C.$$

Now we define the weak solution of problem (1.1)–(1.3).

Definition 2.3. Let T be a fixed positive constant. A function $u : Q_T \to \mathbb{R}$ is called a weak solution of the problem (1.1)–(1.3), if the following conditions are fulfilled:

(1) $u \in C([0, T]; L^2(I)) \cap L^\infty(0, T; W_0^{2,1}(I))$ and $\iint_{Q_T} \Phi\big(\frac{\partial^2 u}{\partial x^2}\big)\, dx\, dt < +\infty$.
(2) For any $\varphi \in C_0^\infty(Q_T)$,

$$\iint_{Q_T} \big\{ - u \frac{\partial \varphi}{\partial t} + \Phi'\big(\frac{\partial^2 u}{\partial x^2}\big) \frac{\partial^2 \varphi}{\partial x^2} \big\} dx\, dt = 0.$$

(3) $u(x, 0) = u_0(x)$ in $L^2(I)$.

We state our main result as follows.

Theorem 2.4. *Let $u_0 \in L^2(I)$ with $\int_I \Phi(\frac{\partial^2 u_0}{\partial x^2})\, dx \le C$ and compatibility conditions on $\{0, 1\} \times \{t = 0\}$. Then problem (1.1)–(1.3) admits one and only one weak solution.*

3. Proof of the Main Theorem

We use the time discrete method to construct an approximate solution. Divide the interval $(0, T)$ into N equal segments and denote $h = T/N$. Consider the problem:

$$\frac{1}{h} (u_{k+1} - u_k) + \frac{d^2}{dx^2} \Phi'\big(\frac{d^2 u_{k+1}}{dx^2}\big) = 0, \tag{3.1}$$

$$u_{k+1}(0) = u_{k+1}(1) = u'_{k+1}(0) = u'_{k+1}(1) = 0, \tag{3.2}$$

where $k = 0, 1, \dots, N-1$, and u_0 is the initial data.

Lemma 3.1. *For $u_k \in L^2(I)$, the problem (3.1)-(3.2) admits one and only one weak solution $u_{k+1} \in W_0^{2,1}(I)$, such that for any $\phi(x) \in C_0^\infty(I)$,*

$$\frac{1}{h} \int_0^1 (u_{k+1} - u_k)\, \phi dx + \int_0^1 \Phi'\big(\frac{d^2 u_{k+1}}{dx^2}\big) \frac{d^2 \phi}{dx^2} dx = 0, \tag{3.3}$$

and

$$\int_0^1 \Phi\big(\frac{d^2u_{k+1}}{dx^2}\big)dx \le C,$$

where C is a constant depended only on $\|u_k\|_{L^2(I)}$ and h.

Proof. We investigate the functional defined on $W_0^{2,1}(I)$ by

$$E(v) = \frac{1}{2h}\int_0^1 (v-u_k)^2 dx + \int_0^1 \Phi\big(\frac{d^2v}{dx^2}\big)dx.$$

We choose $v=0$, then

$$0 \le \inf_{v\in W_0^{2,1}(I)} E(v) \le E(0) = \frac{1}{2h}\int_0^1 u_k^2 dx.$$

By lemma 2.2, we can extract a minimizing sequence $\{v_n\}_{n=1}^\infty \subset W_0^{2,1}(I)$ such that

$$E(v_n) \to \inf_{v\in W_0^{2,1}(I)} E(v), \quad \text{as } n\to\infty,$$

and

$$\int_0^1 |v_n|^2 dx + \int_0^1 \Phi\big(\frac{d^2v_n}{dx^2}\big)dx \le C.$$

By (1.4) and Lemma 2.2, we may find a subsequence $\{v_{n_j}\}_{j=1}^\infty \subset \{v_n\}_{n=1}^\infty$ and a function u_{k+1}, such that $v_{n_j} \rightharpoonup u_{k+1}$ weakly in $W_0^{2,1}(I)$ and

$$\int_0^1 \Phi\big(\frac{d^2u_{k+1}}{dx^2}\big) \le C.$$

Since $\Phi(s)$ is convex and by relaxation, we have that u_{k+1} is a weak solution of the problem (3.1)–(3.2).

Assume u_{k+1} and v_{k+1} are both solutions of the problem (3.1)–(3.2). Then for every $\phi(x)\in C_0^\infty(I)$, we have

$$\frac{1}{h}\int_0^1 (u_{k+1}-v_{k+1})\phi dx + \int_0^1 \Big(\Phi'\big(\frac{d^2u_{k+1}}{dx^2}\big) - \Phi'\big(\frac{d^2v_{k+1}}{dx^2}\big)\Big)\frac{d^2\phi}{dx^2}dx = 0.$$

By (2.2) and the approximation argument, we could take $\phi(x) = u_{k+1}-v_{k+1}$ as the test function. We get

$$\begin{aligned}&\frac{1}{h}\int_0^1 (u_{k+1}-v_{k+1})^2\, dx\\ &+ \int_0^1 \Big(\Phi'\big(\frac{d^2u_{k+1}}{dx^2}\big) - \Phi'\big(\frac{d^2v_{k+1}}{dx^2}\big)\Big)\Big(\frac{d^2u_{k+1}}{dx^2} - \frac{d^2v_{k+1}}{dx^2}\Big)dx = 0.\end{aligned}$$

By (2.3), the two terms on the left hand side are both nonnegative. We get $u_{k+1} = v_{k+1}$ a.e. in I. Then the proof is complete. □

Let $\chi^{h,j}(t)$ be the indicator function of $[h(j-1), hj)$. We construct an approximate function by

$$u^h(x,t) = \sum_{j=1}^N \chi^{h,j}(t)u_{j-1}(x) \quad \text{with} \quad u^h(x,0) = u_0(x).$$

Lemma 3.2. *For the weak solution u_{k+1} of the problem (3.1)–(3.2), the following estimates hold*

$$h\sum_{k=0}^{N-1}\int_0^1 \Phi'\left(\frac{d^2u_{k+1}}{dx^2}\right)\frac{d^2u_{k+1}}{dx^2}dx \le C, \tag{3.4}$$

$$\sup_{0<t<T}\int_0^1 \Phi\left(\frac{\partial^2 u^h}{\partial x^2}\right)dx \le C, \tag{3.5}$$

where C is a constant independent of h.

Proof. Noticing that $C_0^\infty(I)$ is dense in $W_0^{2,1}(I)$, we may choose $\phi(x)\in W_0^{2,1}(I)$ as the test function in (3.3). Let $\phi(x)=u_{k+1}$ in (3.3). Then

$$\frac{1}{h}\int_0^1 (u_{k+1}-u_k)u_{k+1}dx + \int_0^1 \Phi'(\frac{d^2u_{k+1}}{dx^2})\frac{d^2u_{k+1}}{dx^2}dx = 0.$$

So we have

$$\frac{1}{h}\int_0^1 u_{k+1}^2dx + \int_0^1 \Phi'(\frac{d^2u_{k+1}}{dx^2})\frac{d^2u_{k+1}}{dx^2}dx \le \frac{1}{2h}\int_0^1 (u_{k+1}^2+u_k^2)dx;$$

i.e.,

$$\frac{1}{2}\int_0^1 u_{k+1}^2dx + h\int_0^1 \Phi'(\frac{d^2u_{k+1}}{dx^2})\frac{d^2u_{k+1}}{dx^2}dx \le \frac{1}{2}\int_0^1 u_k^2dx. \tag{3.6}$$

Summing up (3.6) for k from 0 to $N-1$, we have

$$\frac{1}{2}\int_0^1 u_N^2dx + h\sum_{k=0}^{N-1}\int_0^1 \Phi'(\frac{d^2u_{k+1}}{dx^2})\frac{d^2u_{k+1}}{dx^2}dx \le \frac{1}{2}\int_0^1 u_0^2dx.$$

Then (3.4) is obtained.

Letting $\phi(x)=u_{k+1}-u_k$ in (3.3), we obtain

$$\frac{1}{h}\int_0^1 (u_{k+1}-u_k)^2dx + \int_0^1 \Phi'(\frac{d^2u_{k+1}}{dx^2})(\frac{d^2u_{k+1}}{dx^2}-\frac{d^2u_k}{dx^2})dx = 0.$$

Since the first term of above equality is nonnegative, by Young's inequality and (2.2), we have

$$\begin{aligned}
&\int_0^1 \Phi'(\frac{d^2u_{k+1}}{dx^2})\frac{d^2u_{k+1}}{dx^2}dx\\
&\le \int_0^1 \Phi'(\frac{d^2u_{k+1}}{dx^2})\frac{d^2u_k}{dx^2}dx\\
&\le \int \Psi\Big(\Phi'(\frac{d^2u_{k+1}}{dx^2})\Big)dx + \int_0^1 \Phi(\frac{d^2u_k}{dx^2})dx\\
&= \int_0^1 \Phi'(\frac{d^2u_{k+1}}{dx^2})\frac{d^2u_{k+1}}{dx^2}dx - \int_0^1 \Phi(\frac{d^2u_{k+1}}{dx^2})dx + \int_0^1 \Phi(\frac{d^2u_k}{dx^2})dx.
\end{aligned}$$

Thus

$$\int_0^1 \Phi(\frac{d^2u_{k+1}}{dx^2})dx \le \int_0^1 \Phi(\frac{d^2u_k}{dx^2})dx.$$

For any m with $1\le m\le N-1$, summing up the above inequality for k from 0 to $m-1$, we have

$$\int_0^1 \Phi(\frac{d^2u_m}{dx^2})dx \le \int_0^1 \Phi(\frac{d^2u_0}{dx^2})dx.$$

So we get (3.5) and the proof is complete. □

Lemma 3.3. *For the weak solution u_{k+1} of (3.1)–(3.2), we have*

$$-Ch \le \int_0^1 |u_{k+1}|^2 - |u_k|^2 dx \le 0, \tag{3.7}$$

where C is a positive constant independent of h.

Proof. The second inequality of (3.7) is obvious by (3.6). Choosing $\phi(x) = u_k$ in (3.3), we have

$$\frac{1}{h}\int_0^1 (u_{k+1} - u_k)u_k dx + \int_0^1 \Phi'\big(\frac{d^2 u_{k+1}}{dx^2}\big)\frac{d^2 u_k}{dx^2} dx = 0.$$

So by Young' inequality and inequality (2.2), we have

$$\begin{aligned}
\frac{1}{h}\int_0^1 (u_k - u_{k+1})u_k dx &\le \int_0^1 \Phi'\big(\frac{d^2 u_{k+1}}{dx^2}\big)\frac{d^2 u_k}{dx^2} dx \\
&\le \int_0^1 \Psi\Big(\Phi'\big(\frac{d^2 u_{k+1}}{dx^2}\big)\Big)dx + \int_0^1 \Phi\big(\frac{d^2 u_k}{dx^2}\big)dx \\
&\le (K-2)\int_0^1 \Phi\big(\frac{d^2 u_{k+1}}{dx^2}\big)dx + \int_0^1 \Phi\big(\frac{d^2 u_k}{dx^2}\big)dx.
\end{aligned}$$

By (3.5) of Lemma 3.2, we have

$$\int_0^1 u_k^2 dx - \int_0^1 u_{k+1}u_k dx \le Ch.$$

Therefore,

$$\int_0^1 u_k^2 dx \le Ch + \int_0^1 u_{k+1}u_k dx \le Ch + \frac{1}{2}\int_0^1 u_k^2 dx + \frac{1}{2}\int_0^1 u_{k+1}^2 dx.$$

Thus, we obtain that

$$\frac{1}{2}\int_0^1 u_k^2 dx \le Ch + \frac{1}{2}\int_0^1 u_{k+1}^2 dx.$$

So the proof of this lemma is complete. □

Corollary 3.4.

$$\int_0^1 |u^h|^2 dx \le \int_0^1 |u_0|^2 dx.$$

Proof of Theorem 2.4. Let

$$\xi_h = \Phi'\big(\frac{\partial^2 u^h}{\partial x^2}\big) \quad \text{and} \quad \Delta^h u^h = u_{k+1} - u_k.$$

By (3.3) we see that

$$\iint_{Q_T} \Big(\frac{1}{h}\Delta^h u^h \varphi + \xi_h \frac{\partial^2 \varphi}{\partial x^2}\Big)\, dx\, dt = 0, \tag{3.8}$$

for any $\varphi \in C_0^\infty(Q_T)$.

By Lemma 2.2, Lemma 3.2 and Corollary 3.4, we can draw a subsequence $\{u^h\}$, denoted still by $\{u^h\}$, such that

$$u^h \rightharpoonup u \quad \text{weakly * in } L^\infty(0,T,W_0^{2,1}(I)),$$
$$\iint_{Q_T} \Phi\big(\frac{\partial^2 u}{\partial x^2}\big)dx\,dt \le C,$$
$$u^h \rightharpoonup u \quad \text{weakly * in } L^\infty(0,T,L^2(I)).$$

By (2.2),

$$\iint_{Q_T} \Psi\left(\xi_h\right) dx\,dt \le \iint_{Q_T} (K-2)\Phi\big(\frac{\partial u^h}{\partial x^2}\big)dx\,dt \le C.$$

And by lemma 2.1,

$$\iint_{Q_T} |\xi_h|^{p'} dx\,dt \le C,$$

for some $p' > 1$. Thus, we may extract a subsequence from ξ_h, denoted still by ξ_h, such that

$$\xi_h \rightharpoonup \xi \quad \text{weakly in } L^{p'}(\Omega).$$

Since $\Psi(s)$ is a convex function, we obtain

$$\iint_{Q_T} \Psi(\xi)dx\,dt \le \liminf_{h\to 0} \iint_{Q_T} \Psi(\xi_h)dx\,dt \le C.$$

Using Young's inequality again, we have

$$\iint_{Q_T} \big|\xi \cdot \frac{\partial^2 u}{\partial x^2}\big|dx\,dt \le \iint_{Q_T} \Psi(\xi) + \Phi\big(\frac{\partial^2 u}{\partial x^2}\big)dx\,dt \le C.$$

By the discrete equation (3.8), we see that

$$\frac{1}{h}\Delta^h u^h \quad \text{is bounded in } L^\infty(0,T;W^{-2,1}(I))$$

and

$$\frac{1}{h}\Delta^h u^h \rightharpoonup \frac{\partial u}{\partial t} \quad \text{weakly * in } L^\infty(0,T;W^{-2,1}(I)).$$

Letting $h \to 0$ in (3.8), we have in the sense of distributions

$$\frac{\partial u}{\partial t} + \frac{\partial^2 \xi}{\partial x^2} = 0. \tag{3.9}$$

Now we will prove $\xi = \Phi'\big(\frac{\partial^2 u}{\partial x^2}\big)$. Denote

$$f_h(t) = \frac{t-kh}{2h}\Big(\int_0^1 |u_{k+1}|^2 dx - \int_0^1 |u_k|^2 dx\Big) + \frac{1}{2}\int_0^1 u_k^2 dx,$$

where $kh < t \le (k+1)h$, $k = 0,1,2,\dots,N-1$. By (3.7), we have

$$\frac{1}{2}\int_0^1 |u_k|^2 dx - Ch \le f_h(t) \le \frac{1}{2}\int_0^1 |u_k|^2 dx,$$
$$-C \le f_h'(t) \le 0.$$

According to the Ascoli-Arzela theorem, there exists a function $f(t) \in C([0,T])$, such that

$$\lim_{h\to 0} f_h(t) = \frac{1}{2}\lim_{h\to 0}\int_0^1 |u^h|^2 dx = f(t) \quad \text{uniformly for } t \in [0,T].$$

It follows from (3.6) that

$$\frac{1}{2}\int_0^1 |u^h|^2dx + \iint_{Q_T} \Phi'\big(\frac{\partial^2 u^h}{\partial x^2}\big)\frac{\partial^2 u^h}{\partial x^2}dx\,dt \le \frac{1}{2}\int_0^1 |u_0|^2dx.$$

Letting $h \to 0$ in the above inequality we have

$$\begin{aligned}
&\liminf_{h\to 0} \iint_{Q_T} \Phi'\left(\frac{\partial^2 u^h}{\partial x^2}\right)\frac{\partial^2 u^h}{\partial x^2}dx\,dt\\
&\le f(0) - f(T)\\
&= \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\int_0^{T-\varepsilon} (f(t) - f(t+\varepsilon))dt\\
&= \lim_{\varepsilon\to 0^+}\lim_{h\to 0} \frac{1}{2\varepsilon}\int_0^{T-\varepsilon}\int_0^1 (|u^h(x,t)|^2 - |u^h(x,t+\varepsilon)|^2)dx\,dt\\
&\le \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\int_0^{T-\varepsilon}\int_0^1 (u(x,t) - u(x,t+\varepsilon))\cdot u\,dx\,dt\\
&\le -\int_0^T \langle\frac{\partial u}{\partial t}, u\rangle dt,
\end{aligned}$$

where $\langle\cdot\rangle$ denotes the dual product of the function in $W^{-2,1}(I)$ and $W_0^{2,1}(I)$. So we have

$$\liminf_{h\to 0} \iint_{Q_T} \Phi'\big(\frac{\partial^2 u^h}{\partial x^2}\big)\frac{\partial^2 u^h}{\partial x^2}dx\,dt \le \iint_{Q_T} \xi\frac{\partial^2 u}{\partial x^2}dx\,dt. \tag{3.10}$$

Define $F[u] = \int_0^1 \Phi\big(\frac{\partial^2 u}{\partial x^2}\big)dx$ and choose a function $g \in L^\infty(0,T;W_0^{2,1}(I))$ with $\iint_{Q_T} \Phi\big(\frac{\partial^2 g}{\partial x^2}\big)\,dx\,dt < +\infty$. Because $\Phi(s)$ is convex, we have

$$\iint_{Q_T} \Phi\big(\frac{\partial^2 g}{\partial x^2}\big)dx\,dt - \iint_{Q_T} \Phi\big(\frac{\partial^2 u^h}{\partial x^2}\big)dx\,dt \ge \iint_{Q_T} \Phi'\big(\frac{\partial^2 u^h}{\partial x^2}\big)\frac{\partial^2 (g-u^h)}{\partial x^2}dx\,dt.$$

Letting $h \to 0$ and by (3.10), we get

$$\iint_{Q_T} \Phi\big(\frac{\partial^2 g}{\partial x^2}\big)dx\,dt - \iint_{Q_T} \Phi\big(\frac{\partial^2 u}{\partial x^2}\big)dx\,dt \ge \iint_{Q_T} \xi\cdot\frac{\partial^2 (g-u)}{\partial x^2}dx\,dt.$$

Replacing g by $\varepsilon g + u$, we see that

$$\frac{1}{\varepsilon}(F[u+\varepsilon g] - F[u]) \ge \iint_{Q_T} \xi\cdot\frac{\partial^2 g}{\partial x^2}dx\,dt$$

and

$$\iint_{Q_T} \frac{\delta F[u]}{\delta u} g dx\,dt = \iint_{Q_T} \Phi'\big(\frac{\partial^2 u}{\partial x^2}\big)\frac{\partial^2 g}{\partial x^2}dx\,dt \ge \iint_{Q_T} \xi\cdot\frac{\partial^2 g}{\partial x^2}dx\,dt.$$

Due to the arbitrariness of g, we get that $\xi = \Phi'\big(\frac{\partial^2 u}{\partial x^2}\big)$. By (3.9), u is the weak solution of the problem (1.1)–(1.3).

Next, we prove the uniqueness of the weak solution of the problem (1.1)–(1.3). Suppose there exist two weak solutions u and v. Using an approximation technique (see [17, 19]), for any test function $\varphi(x,t) \in C^\infty(\bar{Q}_T)$, we have

$$\iint_{Q_T} -(u-v)\frac{\partial\varphi}{\partial t}dx\,dt + \iint_{Q_T} \Big(\Phi'\big(\frac{\partial^2 u}{\partial x^2}\big) - \Phi'\big(\frac{\partial^2 v}{\partial x^2}\big)\Big)\frac{\partial^2\varphi}{\partial x^2}dx\,dt = 0.$$

Furthermore, we may take $u-v$ as a test function and then get

$$\frac{1}{2}\int_0^1 |u-v|^2(t)dx\,dt + \iint_{Q_t}\Big(\Phi'\big(\frac{\partial^2 u}{\partial x^2}\big) - \Phi'\big(\frac{\partial^2 v}{\partial x^2}\big)\Big)\Big(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 v}{\partial x^2}\Big)dx\,dt = 0,$$

where $Q_t = (0,t)\times I$. Since the two terms on the left hand side are nonnegative by inequality (2.4), we have $u = v$ a.e. in Q_T. Thus the proof is complete. □

4. Numerical experiments

After the theoretical analysis, we shall do some numerical tests of higher order filters in practice to compare our model with the other well-known models of [13, 18]. In our model, we take $\Phi(s) = |s|\ln(1+|s|)$. For convenience, we are in favor of implementation of an explicit Euler method, i.e.

$$\frac{u^{k+1}-u^k}{\Delta t} + \frac{\partial^2}{\partial x^2}\Phi'\big(\frac{\partial^2 u^k}{\partial x^2}\big) = 0, \quad (x,t)\in Q_T.$$

For each figure, we use 1 for space steps, 0.2 for time steps of figure (c) and 0.001 for time steps of figures (d) and (e). Steady state was achieved for figure (d) and figure (e) in less than 20000 iterations. In figure (c), we fixed the number of iterations to 1500.

Fig.1 (a) shows the initial signal and (b) the noisy signal. By the figures from (c) to (e), we could conclude that the second order filtering yields enhancement of edges and staircase-like structures, the fourth order filtering results tend to be piecewise linear with enhanced curvature. At the same time we could also see that the fourth order filtering is further affirmed by the almost piecewise constant derivative which is also shown in Figure 1.

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References

[1] R. A. Adams, Sobolev Space, *New York, Academic Press*, 1975.
[2] L. Ambrosio, N. Fusco and D. Pallara, Functions of bounded variation and free discontinuity problems, *Clarendon press, Oxford*, 2000.
[3] G. Aubert and L. Vese, A variational method in image recovery, *SIAM Journal on Numerical Analysis*, **34** (1997), 1948–1979.
[4] A. Chambolle and P. L. Lions, Image recovery via total variation minimization and related problems, *Numer. Math.* **76** (1997), 167–188.
[5] A. L. Bertozzi and J. B. Greer, Low-curvature image simplifiers: global regularity of smooth solutions and Laplacian limiting schemes, *Comm. Pure Appl. Math.*, **57** (2004), no. 6, 764–790.
[6] T. F. Chan and S. Esedoḡlu, Aspects of total variation regularized L^1 function approximation, *SIAM J. Appl. Math.* **65** (2005), no. 5, 1817–1837 (electronic).
[7] T. Chan, A. Marquina and P. Mulet, High-order total variation-based image restoration, *SIAM J. Sci. Comput.* **22** (2000), no. 2, 503–516.
[8] S. Didas, J. Weickert and B. Burgeth, Stability and local feature enhancement of higher order nonlinear diffusion filtering, *Pattern Recognition: 27th DAGM Symposium, Vienna, Austria*, **2005**.
[9] M. Fuchs and G. Mingione, Full $C^{1,\alpha}$-regularity for free and constrained local minimizers of elliptic variational integrals with nearly linear growth, *Manuscripta Math.*, **102** (2000), no.2, 227–250.

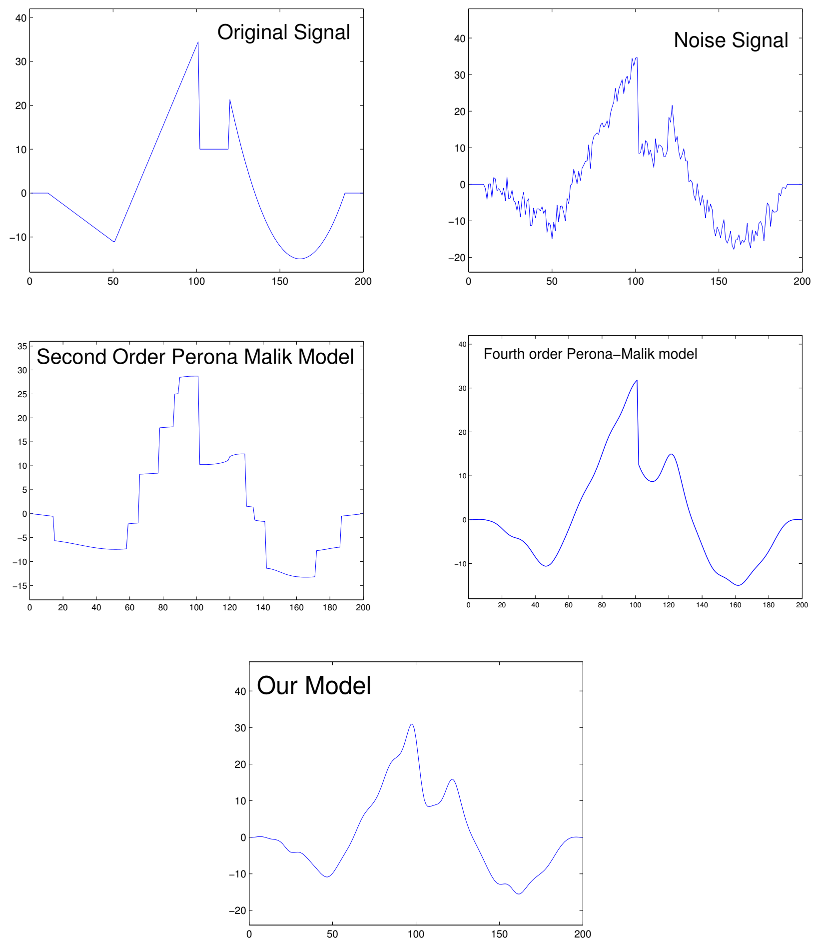

FIGURE 1. One-dimensional signal evaluation: original signal, noisy signal, and restored by second order Perona-Malik model, fourth-order Perona-Malik model and our model.

[10] J. B. Greer and A. L. Bertozzi, Traveling wave solutions of fourth order PDEs for image processing, *SIAM J. Math. Anal.* **36** (2004), no.1, 38–68 (electronic).

[11] W. Hinterberger and O. Scherzer, Variational methods on the space of functions of bounded Hessian for convexification and denoising, *Computing*, **76** (2006), no.1, 109–133.

[12] M. Lysaker, A. Lundervold and X. C. Tai, Noise removal using fourth-order partial differential equation with applications to medical magnetic resonance images in space and time, *IEEE. Transactions on image processing*, vol. **12** (2003),no.12, 1579–1590.

[13] P. Perona and J. Malik, Scale space and edge detection using anisotropic difusion, *IEEE Transactions on Pattern Analysisi and Machine Intelligence*, **12** (1990), 629-639.

[14] L. Rudin, S. Osher, and E. Fatemi, Nonlinear total variation based noise removal algorithms, *Physica D*, **60** (1992), 259–268.

[15] D. Strong and T. Chan, Edge-preserving and scale-dependent properties of total variation regularization, *Inverse Problems*, **19** (2003), 165–187.

[16] L. Wang and S. Zhou, Existence and uniqueness of weak solutions for a nonlinear parabolic equation realated to image analysis, to apper.

[17] Z. Wu, J. Zhao, J. Yin and H. Li, Nonlinear Diffusion Equations, *World Scientific*, 2001.
[18] Y. L. You and M. Kaveh, Fourth-order partial differential equations for noise removal, *IEEE Transactions on Image Processing*, vol. **9**(2000), no.10, 1723–1730.
[19] M. Xu and S. L. Zhou, Existence and uniqueness of weak solutions for a generalized thin film equation, *Nonlinear Anal.* **60** (2005), no. 4, 755–774.
[20] M. Xu and S. L. Zhou, Existence and uniqueness of weak solutions for a fourth- order nonlinear parabolic equation, *J. Math. Anal. Appl.* **325** (2007), 636–654.

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