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NON-RADIAL NORMALIZED SOLUTIONS FOR A NONLINEAR SCHRÖDINGER EQUATION

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Abstract. This article concerns the existence of multiple non-radial positive solutions of the L^2-constrained problem

$$-\Delta u - Q(\varepsilon x)|u|^{p-2}u = \lambda u, \quad \text{in } \mathbb{R}^N,$$
$$\int_{\mathbb{R}^N} |u|^2 dx = 1,$$

where $Q(x)$ is a radially symmetric function, $\varepsilon > 0$ is a small parameter, $N \geq 2$, and $p \in (2, 2 + \frac{4}{N})$ is assumed to be mass sub-critical. We are interested in the symmetry breaking of the normalized solutions and we prove the existence of multiple non-radial positive solutions as local minimizers of the energy functional.

1. Introduction

We consider nonlinear Schrödinger equations with a L^2 constraint,

$$\begin{gathered} -\Delta u - Q(\varepsilon x)|u|^{p-2}u = \lambda u, \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = 1, \end{gathered} \tag{1.1}$$

where $2 < p < 2 + \frac{4}{N}, N \geq 2$, the potential function $Q(x)$ is radially symmetric, that is $Q(|x|) = Q(x)$, and $\varepsilon > 0$ is a parameter. Many models of such type can be seen in the literature. It is especially important in theory and application to study the existence and properties of ground states and bound states solutions. The existence of solutions for nonlinear Schrödinger equation, including their properties, is of great interest in the field and has been studied extensively in the past (e.g., [1, 3, 8, 10, 11, 14, 15, 18]) and references therein. In addition, a growing number of articles considering the existence of multiple bound states of nonlinear Schrödinger equation, together with L^2 constraint, have appeared in the field; see [4, 5, 6, 7, 12, 14, 17] and references therein.

Let $d > 0$ and $\mu > 0$ be constants. First consider a case of constant potential and define

$$c(d,\mu) = \inf_{u \in H^1(\mathbb{R}^N), \|u\|_2^2 = \mu} \Big(\frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{d}{p} \int_{\mathbb{R}^N} |u|^p dx \Big). \tag{1.2}$$

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Here and after we use $\|\cdot\|_r$ to denote the L^r norm for $r \geq 1$. Cazenave-Lions [5] proved that $c(d,\mu) < 0$ and $c(d,\mu)$ are attained, up to a translation, by a radially symmetric minimizer. This minimizing feature leads to the orbital stability of the standing waves associated with the time-dependent nonlinear Schrödinger equation; see [5, 4]. Using the same method and being easier because of the compact embedding from H_r^1 into L^p for $N \geq 2$, one can prove that when $Q(x) = Q(|x|)$ is a radial function and replaces the constant d in the problem (1.2), the minimization problem is also solvable in the space of radial functions; see [17]. Here H_r^1 denotes the subspace of $H^1(\mathbb{R}^N)$ of radially symmetric functions.

A natural question is that when Q is a radial potential whether there exist non-radial solutions – symmetry breaking phenomenon. The question was studied by Yang [17] in which the authors give conditions under which a ground state solution is non-radial, i.e., symmetry breaking occurs. Then Yang [17] also gives conditions which assure the existence of multiple non-radial bound state solutions. These solutions are constructed as global minimizers of the energy functional in some symmetric subspaces. More precisely it was proved that when $N = 2$ and $N \geq 4$, (1.1) has radial and non-radial solutions. Furthermore, there exist multiple non-radial solutions [17] as $\varepsilon \to 0$. However, the global minimization scheme may not work for the case $N = 3$. The reason for such circumstances is that in the three-dimensional case, multiple bumps may concentrate at the origin or concentrate along the z-axis and run to infinity in two opposite directions, i.e., the north and south poles (See Proposition 2.3 below for more details). Also for the existence of multiple non-radial solutions the condition on Q in [17] depends on the number of solutions, which is less desirable.

This article gives a different method to settle the issues related to the above problem. We prove the existence of multiple non-radial positive solutions for all dimensions $N \geq 2$ with conditions on Q independent of the number of solutions. The conclusion is much stronger than the original one. It is worth mentioning that solutions given in this paper are found to be local minimizers, while the solutions of [17] are global minimizers in symmetric subspaces. Under the conditions in this paper we do not know whether the global minimizers exist and whether they are non-radial solutions even if they do exist, yet local minimizers do exist under our conditions.

To solve equation (1.1) we look for the critical points of the functional

$$J_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{p}\int_{\mathbb{R}^N} Q(\varepsilon x)|u|^p dx,\ u \in H^1(\mathbb{R}^N) \tag{1.3}$$

under the assumption $\|u\|_2^2 = 1$. We use the following assumption

(A1) $Q \in C(\mathbb{R}^N, \mathbb{R})$ is radially symmetric, i.e., $Q(x) = Q(|x|)$, $Q(x)$ achieves its maximum on $\{x : |x| = 1\}$, and there exist $a > 0$ and $\sigma_0 > 0$ such that

$$0 < a \leq Q(x) \leq \max_{\mathbb{R}^N} Q(x) = 1, \quad \text{for } x \in \mathbb{R}^N, \tag{1.4}$$

$$Q(|x|) - Q(1) < 0 \quad \text{for } 0 < \||x| - 1| \leq 2\sigma_0. \tag{1.5}$$

For convenience, and without loss of generality, we have set the maximum value of Q to be 1. Roughly speaking, Q is bounded from above and below by positive constants and has an isolated global maximum point at $|x| = 1$. Now we state the main theorem to be proved in this article.

Theorem 1.1. *Assume $N \geq 2$, $p \in (2, 2+\frac{4}{N})$. Assume* (A1) *is satisfied. Let $k \geq 1$ be an integer. Then there exists $\varepsilon_k > 0$ such that for all $0 < \varepsilon < \varepsilon_k$,* (1.1) *has a non-radial positive solution $u_{\varepsilon,k}$ satisfying $\lim_{\varepsilon\to 0} J_\varepsilon(u_{\varepsilon,k}) = c(k^{\frac{2-p}{2}}, 1)$. In particularly, as $\varepsilon \to 0$,* (1.1) *has more and more non-radial positive solutions.*

We remark that we can distinguish these solutions by separating their energies using the asymptotic limits $c(k^{\frac{2-p}{2}}, 1)$ which is proved to be different for different k in Lemma 2.1.

Finally, we summarize our work in terms of results and methods. We give a new local minimization scheme which tracks down non-radial bound state solutions of multi-bump type. This is motivated by the fact that in general the known global minimization method cannot give these k-bump type concentrated solutions for $k \geq 3$ (this is particularly true for the dimension $N = 3$ as will be discussed in Section 2). Our method allows to establish k-bump type solutions for any integer k by taking ε small.

The article is organized as follows. In Section 2 we develop a basic formula for the minimization problem $c(d, \mu)$ which will be used frequently in our proof later.Then we introduce a local minimization scheme which will be used to construct $k-$bump solutions. In Section 3 we first prove several lemmas in preparation of the proof for the main theorem and we close with the proof of the main theorem. In the Appendix we give the proof of a result in Section 2 showing that the global minimization approach cannot give multi-bump type solutions in general.

2. Variational formulation and some technical results

Using scalings we first give a formula for the ground state energy in the constant potential case. Here again for $d > 0$ and $\mu > 0$, we define

$$c(d, \mu) = \inf_{u\in H^1(\mathbb{R}^N), \|u\|_2^2=\mu} \Big(\frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - \frac{d}{p}\int_{\mathbb{R}^N} |u|^p dx\Big). \tag{2.1}$$

We formulate the dependency of $c(d, \mu)$ in terms of d and μ, using and extending the results of [13, 17].

Let $\omega > 0$ be the unique positive radially symmetric solution of

$$-\Delta\omega + \omega = \omega^{p-1}, \quad \omega > 0, \quad \text{in } \mathbb{R}^N. \tag{2.2}$$

The ground state solution ω above is useful for the proof of the following lemma.

Lemma 2.1. *Let $d > 0$, $\mu > 0$, $2 < p < 2 + \frac{4}{N}$. Then*

$$c(d, \mu) = -c_0 d^{\frac{4}{4-N(p-2)}} \mu^{\frac{4-(N-2)(p-2)}{4-N(p-2)}}, \tag{2.3}$$

where $c_0 = \frac{[4-N(p-2)]\|\omega\|_2^{\frac{-4}{4-N(p-2)}}}{4[4-(N-2)(p-2)]}$.

Proof. First by [17, Proposition 2.3] we have $c(d, 1) = c(1, 1)d^{\frac{4}{4-N(p-2)}}$. Similar proof of this gives us $c(d, \mu) = c(1, \mu)d^{\frac{4}{4-N(p-2)}}$. For $\lambda > 0$, we use a scaling of w where $u^\lambda(x) = \lambda^{\frac{2}{p-2}}\omega(\lambda x)$. From (2.2) we have

$$-\Delta u^\lambda + \lambda^2 u^\lambda = \big(u^\lambda\big)^{p-1}, \tag{2.4}$$

$$\|u^\lambda\|_2^2 = \lambda^{\frac{4-N(p-2)}{p-2}} \|\omega\|_2^2. \tag{2.5}$$

Set $\mu = \lambda^{\frac{4-N(p-2)}{p-2}}\|\omega\|_2^2$. Then $\lambda^2 = (\frac{\mu}{\|\omega\|_2^2})^{\frac{2(p-2)}{4-N(p-2)}}$, i.e., u^λ satisfies $\|u^\lambda\|_2^2 = \mu$ and

$$-\Delta u^\lambda + (\frac{\mu}{\|\omega\|_2^2})^{\frac{2(p-2)}{4-N(p-2)}} u^\lambda = \big(u^\lambda\big)^{p-1}. \tag{2.6}$$

By [13], we have almost everywhere in $\mu > 0$,

$$\frac{\partial}{\partial\mu} c(1,\mu) = -\frac{1}{4}\Big(\frac{\mu}{\|\omega\|_2^2}\Big)^{\frac{2(p-2)}{4-N(p-2)}}. \tag{2.7}$$

Then integrating (2.7) with respect to μ, we can obtain

$$c(1,\mu_2) - c(1,\mu_1) = -\frac{1}{4}\int_{\mu_1}^{\mu_2}\Big(\frac{\mu}{\|\omega\|_2^2}\Big)^{\frac{2(p-2)}{4-N(p-2)}} d\mu.$$

Using $c(1,\mu_1) \to 0$ as $\mu_1 \to 0$, we obtain

$$c(1,\mu) = -\frac{1}{4}\frac{\|\omega\|_2^{\frac{-4}{4-N(p-2)}}\mu^{\frac{4-(N-2)(p-2)}{4-N(p-2)}}}{\frac{4-(N-2)(p-2)}{4-N(p-2)}} = -c_0\mu^{\frac{4-(N-2)(p-2)}{4-N(p-2)}}.$$

□

Lemma 2.2. *For k fixed there exists $\delta_k > 0$, for every $0 < b \le \delta_k$, it holds*

$$c(1,b) + c(k^{\frac{2-p}{2}}, 1-b) > c(k^{\frac{2-p}{2}}, 1). \tag{2.8}$$

Proof. Consider a continuous function of b, $f(b) = c(1,b) + c(k^{\frac{2-p}{2}}, 1-b)$, where $0 \le b \le 1$. Then the result follows by a direct computation from (2.3). □

Since in some cases the global minimizers do not produce multi-bump solutions, we introduce the method of local minimization to construct k-bump solutions. Before presenting the results, we introduce some notation. Let $k \ge 2$ be an integer, we define a subgroup of $O(2)$

$$\widetilde{G_k} = \Big\{g, g^2, \cdots, g^k = Id : g = \begin{pmatrix} \cos\frac{2\pi}{k} & -\sin\frac{2\pi}{k} \\ \sin\frac{2\pi}{k} & \cos\frac{2\pi}{k} \end{pmatrix}\Big\}. \tag{2.9}$$

Note that $\widetilde{G_k}$ acts on $\mathbb{R}^2$ as rotation. Then to guarantee the invariance of the group in the three-dimensional or higher dimensional cases, a group G is defined as follows,

$$G = \widetilde{G_k} \times \mathbb{Z}_2. \tag{2.10}$$

Here $\mathbb{Z}_2$ is the reflection about the plane of x_1, x_2. In view of the above statement, we require to establish a function space as follows

$$H^1_G(\mathbb{R}^N) = \{u \in H^1(\mathbb{R}^N) : \text{u is } G\text{-invariant}\}. \tag{2.11}$$

In other words, if $u \in H^1_G$, then u is $\widetilde{G_k}$-invariant with respect to (x_1, x_2) and u is even in $(x_3, \dots, x_N)$. The solutions of (1.1) are constructed in G-invariant subspaces, so that they are G-invariant.

To prove our main result, the existence of solutions in Theorem 1.1, we set up a local minimization scheme. Choose $\delta > 0$ such that

$$0 < \delta < \min\{\frac{\delta_k}{2}, \frac{\sigma_0}{2}\}. \tag{2.12}$$

Here δ_k is from Lemma 2.2 and σ_0 is from condition (A1). We define

$$O_{\delta,\varepsilon} = \{u \in H^1_G(\mathbb{R}^N) \colon \int_{\mathbb{R}^N} u^2 = 1, \gamma_\varepsilon(u) \ge 1-\delta\} \tag{2.13}$$

where

$$\gamma_\varepsilon(u) = \int_{T_{\frac{1+\sigma_0}{\varepsilon}} \setminus T_{\frac{1-\sigma_0}{\varepsilon}}} |u|^2 dx. \tag{2.14}$$

Here for $R > 0$, $T_R = \{x \in \mathbb{R}^N : |Px| < R\}$ and $P : \mathbb{R}^N \to \mathbb{R}^2$ is the linear projection. Finally, we define a local minimization problem which will give us desired k-bump solutions

$$c(O_{\delta,\varepsilon}) = \inf_{u \in O_{\delta,\varepsilon}, \|u\|_2^2 = 1} J_\varepsilon(u). \tag{2.15}$$

We will locate k-bump solutions as minimizers in $O_{\delta,\varepsilon}$ of the energy functional J_ε, therefore as local minimizers in the full space for J_ε on the L^2 constraint. More precisely, we will show that for $\varepsilon > 0$ small, $c(O_{\delta,\varepsilon})$ is attained at an interior point u_ε of $O_{\delta,\varepsilon}$, and therefore u_ε is a critical point of J_ε on $\|u\|_2^2 = 1$ and a solution of (1.1). Then we establish an asymptotic energy estimate of $J_\varepsilon(u_\varepsilon)$ so we may distinguish these solutions in k for $\varepsilon > 0$ small.

We finish this section by pointing out why a local minimization argument is necessary for constructing these k-bumped solutions. We show in general the global minimization cannot give k-bumped solutions.

Proposition 2.3. *Assume $N = 3$, $Q(x) = Q(|x|)$ is continuous with $0 < a \leq Q(x) \leq 1$, Q attains its maximum at $|x| = 1$, and*

$$\limsup_{|x| \to \infty} Q(x) := q_\infty < 1.$$

Then for $\varepsilon > 0$ small, the minimum of J_ε with the constraint $\|u\|_2^2 = 1$ is achieved. If $Q(0) > k^{\frac{2-p}{2}}$, then the minimizers u_ε concentrate at the origin as $\varepsilon \to 0$.

The proof uses some results from [17] and is presented in the Appendix.

3. Asymptotic estimates and the proof of Theorem 1.1

Recall that for proving Theorem 1.1 we need to prove that the local minimization problem (2.15) is solvable. We first give an auxiliary result on another minimization problem. First define

$$T_R = \{x \in \mathbb{R}^N : |Px| < R\}, \tag{3.1}$$

where $P : \mathbb{R}^N \to \mathbb{R}^2$ is the linear projection, and

$$T_R^c = \mathbb{R}^N \setminus T_R. \tag{3.2}$$

Base on the property of solutions to be constructed and to be compared, we establish a space for $R > 0$,

$$X = X_R = H^1_{0,G}(T_R^c) = \{u \in H^1_0(T_R^c) : gu = u, \forall g \in G\}. \tag{3.3}$$

For $R > 0$ and $b > 0$, set

$$S(R, b) = \inf_{u \in X, \|u\|_2^2 = b} \Big(\frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p dx\Big). \tag{3.4}$$

Next we state some asymptotic estimates for $S(R, b)$.

Lemma 3.1. *In the setting of* (3.3) *and* (3.4)*, the following hold:*

(i) *Let* $R_n \to \infty$*, as* $n \to \infty$ *and* $(u_n) \subset X_{R_n}$ *be such that* $\lim_{n\to\infty} \|u_n\|_2^2 = b$ *and*

$$\lim_{n\to\infty} \Big(\frac{1}{2}\int_{T^c_{R_n}} |\nabla u_n|^2 - \frac{1}{p}\int_{T^c_{R_n}} |u_n|^p dx\Big) = A \leq \lim_{R\to\infty} S(R,b). \tag{3.5}$$

Then $A = c(k^{\frac{2-p}{2}}, b)$ *and up to a subsequence, for every* $\alpha > 0$*, there exists* $r > 0$ *and* $(y_n) \subset T^c_{R_n}$ *such that*

$$\liminf_{n\to\infty} \int_{B_r(y_n)} |u_n|^2 dx \geq \frac{b}{k} - \alpha. \tag{3.6}$$

(ii) $\lim_{R\to\infty} S(R,b) = c(k^{\frac{2-p}{2}}, b)$.

Proof. By using special testing functions, we can easily obtain that for any $R > 0$,

$$S(R,b) \leq c(k^{\frac{2-p}{2}}, b).$$

Since $S(R,b)$ is non-decreasing as $R \to \infty$, $\lim_{R\to\infty} S(R,b)$ exists. Now choose a sequence $R_n \to \infty$ and $(u_n) \subset X_{R_n}$, such that $\lim_{n\to\infty} \|u_n\|_2^2 = b$ and $A \leq \lim_{R\to\infty} S(R,b)$. By the concentration-compactness principle [8, 9], we have three possibilities.

If vanishing occurs, then for any $r > 0$, it holds

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_r(y)} |u_n|^2 dx = 0.$$

It follows from [8, 9] that $u_n \to 0$ in $L^p(\mathbb{R}^N)$ for $2 < p < 2^* = \frac{2N}{N-2}$. Then we obtain $\lim_{n\to\infty}(\frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_n|^2 - \frac{1}{p}\int_{\mathbb{R}^N} |u_n|^p dx) \geq 0$. This is a contradiction with $c(k^{\frac{2-p}{2}}, 1) < 0$.

Next because of the symmetry we can only expect compactness of the sequence module the symmetry. If there exists $b_1 > 0$ and $(y_n) \subset T^c_{R_n}$ such that for any $\alpha > 0$, there exists $r > 0$,

$$\liminf_{n\to\infty} \int_{B_r(y_n)} |u_n|^2 dx \geq b_1 - \alpha. \tag{3.7}$$

Since u_n is radial, one can get

$$\int_{B_r(gy_n)} |u_n|^2 dx \geq b_1 - \alpha.$$

Now we claim that $kb_1 = b$. If $kb_1 < b$, let $\eta = \eta(t)$ be a smooth non-increasing function on $[0, +\infty)$ such that $\eta(t) = 1$ for $t \in [0,1]$, $\eta(t) = 0$ for $t \geq 2$ and $|\eta'(x)| \leq 2$. Write $\eta^c(t) = 1 - \eta(t)$. In (3.7) we choose $\alpha_m \to 0$ and $r_m \to \infty$. Then we we can find u_{n_m} and y_{n_m} such that $\int_{B_r(y_n)} |u_{n_m}|^2 dx \geq b_1 - \alpha_m$. We still name this subsequence u_{n_m} as u_n for simplicity of notations. Define

$$v_n(x) = \sum_{i=1}^{k} \eta\big(\frac{|x - g^i y_n|}{r_n}\big) u_n(x), \tag{3.8}$$

$$\omega_n = u_n(x) - v_n. \tag{3.9}$$

Hence, $\|v_n(x)\|_2^2 \to kb_1$ and $\|\omega_n(x)\|_2^2 \to b - kb_1$, as $n \to \infty$. Then

$A + o(1)$

$$
\begin{aligned}
&= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_n|^2 - \frac{1}{p}\int_{\mathbb{R}^N} |u_n|^p dx \\
&= \Big(\frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_n|^2 - \frac{1}{p}\int_{\mathbb{R}^N} |v_n|^p dx\Big) + \Big(\frac{1}{2}\int_{\mathbb{R}^N} |\nabla \omega_n|^2 - \frac{1}{p}\int_{\mathbb{R}^N} |\omega_n|^p dx\Big) - \frac{C_0}{r_n} \\
&= \Big(\frac{\|v_n\|_2^2}{2b}\int_{\mathbb{R}^N} \Big|\frac{\sqrt{b}\nabla v_n}{\|}v_n\|_2\Big|^2 - \frac{\|v_n\|_2^p}{pb^{p/2}}\int_{\mathbb{R}^N} \Big|\frac{\sqrt{b}v_n}{\|v_n\|}_2\Big|^p dx\Big) \\
&\quad + \Big(\frac{\|\omega_n\|_2^2}{2b}\int_{\mathbb{R}^N} \Big|\frac{\sqrt{b}\nabla \omega_n}{\|\omega_n\|_2}\Big|^2 - \frac{\|\omega_n\|_2^p}{pb^{p/2}}\int_{\mathbb{R}^N} \Big|\frac{\sqrt{b}\omega_n}{\|\omega_n\|}_2\Big|^p dx\Big) - \frac{C_0}{r_n} \\
&\ge \frac{\|v_n\|_2^2}{b} S(R_n, b) + \Big(\frac{\|v_n\|_2^2}{pb} - \frac{\|v_n\|_2^p}{pb^{p/2}}\Big)\int_{\mathbb{R}^N} \Big|\frac{\sqrt{b}v_n}{\|v_n\|_2}\Big|^p dx + \frac{\|\omega_n\|_2^2}{b} S(R_n, b) - \frac{C_0}{r_n}.
\end{aligned}
$$

Here C_0 is independent of n. Sending $n \to \infty$, this implies

$$
A \ge \frac{kb_1}{b}A + \frac{b-kb_1}{b}A + \frac{1}{p}\Big[\frac{kb_1}{b} - \Big(\frac{kb_1}{b}\Big)^{p/2}\Big] \lim_{n\to\infty}\int_{\mathbb{R}^N} \Big|\frac{\sqrt{b}v_n}{\|v_n\|_2}\Big|^p dx.
$$

Then we deduce that $kb_1 = b$, since otherwise using $\lim_{n\to\infty}\int_{\mathbb{R}^N} |u_n|^p dx \neq 0$ we obtain a contradiction with the last term in the formula above positive.

Finally,choose $\alpha_n \to 0, r_n \to \infty$,by doing a cut-off function which is similar to (3.8),we can get $\|v_n\|_2^2 \to b$, as $n \to \infty$. We can repeat the method above, using $\lim_{n\to\infty}\|v_n\|_2^2 = b$ and Lemma 2.1 we obtain

$$
\begin{aligned}
A + o(1) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_n|^2 - \frac{1}{p}\int_{\mathbb{R}^N} |u_n|^p dx + o(1) \\
&\ge \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_n|^2 - \frac{1}{p}\int_{\mathbb{R}^N} |v_n|^p dx + o(1) \\
&\ge kc(1, \frac{b}{k}) + o(1) \\
&= c(k^{\frac{2-p}{2}}, b) + o(1).
\end{aligned}
$$

Letting $n \to \infty$, $o(1) \to 0$, we obtain the result (i). The assertion (ii) follows from (i) readily. $\square$

For the minimization problem $c(O_{\delta,\varepsilon})$, we will use the following asymptotic estimates.

Lemma 3.2. (i) *Let $\varepsilon_n \to 0$ and $(u_n) \subset \overline{O_{\delta,\varepsilon}}$ such that $\limsup_{n\to\infty} J_{\varepsilon_n}(u_n) \le c(k^{\frac{2-p}{2}}, 1)$. Then there exists $(y_n) \subset \mathbb{R}^2 \times \{\overrightarrow{0}\}$ satisfying for any $0 < \sigma \le \sigma_0$ and*

$$
\limsup_{n\to\infty} \operatorname{dist}(y_n, T_{\frac{1+\sigma_0}{\varepsilon_n}} \setminus T_{\frac{1-\sigma_0}{\varepsilon_n}}) < \infty \tag{3.10}
$$

and for any $\alpha > 0$, there exists $R > 0$ such that

$$
\lim_{n\to\infty}\int_{B_R(y_n)} |u_n|^2 dx \ge \frac{1}{k} - \alpha. \tag{3.11}
$$

(ii) $\lim_{\varepsilon\to 0} c(O_{\delta,\varepsilon}) = c(k^{\frac{2-p}{2}}, 1)$.

Proof. Let $(u_n) \subset O_{\delta,\varepsilon_n}$ be such that $A = \lim_{n\to\infty} J_{\varepsilon_n}(u_n) \le c(k^{\frac{2-p}{2}}, 1)$. Then for any $0 < t \le 1 - \sigma_0$, we have

$$
\limsup_{n\to\infty}\int_{T_{t/\varepsilon_n}} |u_n|^2 dx = \limsup_{n\to\infty}\Big(1 - \int_{T^c_{t/\varepsilon_n}} |u_n|^2 dx\Big)
$$

$$
\begin{aligned}
&= 1 - \liminf_{n\to\infty} \int_{T_{t/\varepsilon_n}} |u_n|^2 dx \\
&\le 1 - (1-\delta) = \delta.
\end{aligned}
$$

Next we select a sequence $R_{m+1} = 2R_m \to \infty$, as $m \to \infty$. Up to a subsequence, we suppose that

$$
b_m = \limsup_{n\to\infty} \int_{T_{R_m}} |u_n|^2 dx, \quad b_0 = \lim_{m\to\infty} b_m.
$$

Then using the fact that $u_n \subset O_{\delta,\varepsilon_n}$ we obtain $b_0 \le \delta$. Choose $n_m \to \infty$, as $m \to \infty$ such that

$$
\big|b_m - \int_{T_{R_m}} |u_{n_m}|^2 dx\big| \le \frac{1}{m}, \quad \Big|b_{m+1} - \int_{T_{R_{m+1}}} |u_{n_m}|^2 dx\Big| \le \frac{1}{m}.
$$

By doing a cut-off function which is similar to the proof of Lemma 3.1, we obtain two sequences v_n and ω_n satisfying

$$
\lim_{n\to\infty} \int_{\mathbb{R}^N} |v_n|^2 dx = b_0, \quad \lim_{n\to\infty} \int_{\mathbb{R}^N} |\omega_n|^2 dx = 1 - b_0.
$$

Note that $(\omega_n) \subset H^1_{0,G}(T^c_{R_n})$. Then by Lemma 3.1, we obtain

$$
\lim_{n\to\infty} J_{\varepsilon_n}(\omega_n) \ge \lim_{n\to\infty} S(R_n, 1-b_0) = c(k^{\frac{2-p}{2}}, 1-b_0).
$$

Claim 1: $b_0 = 0$. Direct computation shows that there exists some $C > 0$ be such that

$$
J_{\varepsilon_n}(u_n) - J_{\varepsilon_n}(v_n) - J_{\varepsilon_n}(\omega_n) \ge -\frac{C}{R_n}.
$$

It follows that

$$
\begin{aligned}
A &= \lim_{n\to\infty} J_{\varepsilon_n}(u_n) \\
&\ge \lim_{n\to\infty} (J_{\varepsilon_n}(v_n) + J_{\varepsilon_n}(\omega_n) - \frac{C}{R_n}) \\
&\ge c(1, b_0) + c(k^{\frac{2-p}{2}}, 1-b_0) + o(1).
\end{aligned}
$$

This implies $c(k^{\frac{2-p}{2}}, 1) \ge c(1, b_0) + c(k^{\frac{2-p}{2}}, 1-b_0)$. By Lemma 2.2, since $0 < b_0 \le \delta_k$, we have

$$
c(1, b_0) + c(k^{\frac{2-p}{2}}, 1-b_0) > c(k^{\frac{2-p}{2}}, 1),
$$

which is a contradiction. Thus, $b_0 = 0$ and Claim 1 is proved.

Using Lemma 3.1, we conclude that for any $\alpha > 0$, there exists $R > 0$ and $(y_n) \subset \mathbb{R}^N$ be such that

$$
\lim_{n\to\infty} \inf_{y_n \in \mathbb{R}^N} \int_{B_R(y_n)} |u_n|^2 dx \ge \frac{1}{k} - \alpha. \tag{3.12}
$$

Claim 2: $\lim_{n\to\infty} |P^\perp y_n| \le C$ where $P^\perp = Id - P$. If this claim is not the case, then up to a subsequence, we can assume that $\lim_{n\to\infty} |P^\perp y_n| = \infty$. Since u_n is G-invariant, then there exists $g \in G$ such that $|gP^\perp y_n - P^\perp y_n| \to \infty$ as $n \to \infty$. This gives a contradiction with the above estimate and $\int_{\mathbb{R}^N} |u_n|^2 dx = 1$.

Claim 3: There exists constant $C > 0$ such that

$$\limsup_{n\to\infty} \operatorname{dist}(y_n, {\varepsilon_n}^{-1}(A_{\sigma_0})) \le C, \tag{3.13}$$

where $A_{\sigma_0} = \{x : 1-\sigma_0 \le |Px| \le 1+\sigma_0\}$. Assume the claim is not true, then up to a subsequence we have $\operatorname{dist}(y_n, {\varepsilon_n}^{-1}(A_{\sigma_0})) = \infty$.

We consider two cases. The first is $|y_n| < \frac{1-\sigma_0}{\varepsilon_n}$, then $\frac{1-\sigma_0}{\varepsilon_n} - |y_n| \to \infty$; it follows that

$$\int_{T_{\frac{1+\sigma_0}{\varepsilon_n}} \setminus T_{\frac{1-\sigma_0}{\varepsilon_n}}} |u_n|^2 dx \le 1 - \sum_{i=1}^k \int_{B_R(g^i y_n)} |u_n|^2 dx \le k\alpha.$$

Let α be sufficiently small, then it is a contradiction to $\gamma(u_n) \ge 1-\delta$. For the other case we have $|y_n| > \frac{1+\sigma_0}{\varepsilon_n}$. The proof is analogous to the above.

Claim 4: For every $\sigma_0 > \sigma > 0$, there exists $C_\sigma > 0$ be such that

$$\limsup_{n\to\infty} \operatorname{dist}(y_n, {\varepsilon_n}^{-1}(A_\sigma)) \le C_\sigma. \tag{3.14}$$

If the claim is not true, then there exists $\sigma \in (0, \sigma_0)$ and up to a subsequence, $(y_n) \subset {\varepsilon_n}^{-1}(A_\sigma)$ such that $\limsup_{n\to\infty} \operatorname{dist}(y_n, {\varepsilon_n}^{-1}(A_\sigma)) = \infty$. Then we have $|y_n| - \frac{1+\sigma}{\varepsilon_n} \to \infty$ or $\frac{1-\sigma}{\varepsilon_n} - |y_n| \to \infty$.

Now we only consider the case $|y_n| - \frac{1+\sigma}{\varepsilon_n} \to \infty$, the proof of the other case is similar. Using condition (A1), and the continuity of $Q(x)$, there exists $0 < a < 1$ be such that

$$Q(|x|) \le a, \quad \text{for } \sigma \le \big||x| - 1\big| \le 2\sigma_0.$$

i.e.,

$$Q(\varepsilon|y|) \le a, \quad \text{for } \frac{\sigma}{\varepsilon} \le \big||y| - \frac{1}{\varepsilon}\big| \le \frac{2\sigma_0}{\varepsilon}.$$

Then $\frac{1+\sigma}{\varepsilon} \le |y| \le \frac{1+2\sigma_0}{\varepsilon}$.

Because of $B_R(y_n) \subset (T_{\frac{1+2\sigma_0}{\varepsilon_n}} \setminus T_{\frac{1+\sigma}{\varepsilon_n}})$, we have $Q(\varepsilon_n x) \le a$, for $x \in B_R(y_n)$. Next, set $v_n(x) = \sum_{i=1}^k \eta(\frac{|x-g^i y_n|}{R_n}) u_n(x)$ satisfying $\lim_{n\to\infty} \|v_n\|_2^2 = 1$ and $(v_n) \subset H^1_{0,G}(T_{\frac{1+2\sigma_0}{\varepsilon_n}})$. Then

$$\begin{aligned}
& c(k^{\frac{2-p}{2}}, 1) \\
& \ge \lim_{n\to\infty} J_{\varepsilon_n}(u_n) \\
& = \lim_{n\to\infty} J_{\varepsilon_n}(v_n) + o(1) \\
& = \lim_{n\to\infty} (\frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_n|^2 - \frac{1}{p}\int_{\mathbb{R}^N} |v_n|^p dx) + \lim_{n\to\infty} \frac{1}{p}\int_{\mathbb{R}^N} (1 - Q(\varepsilon x))|v_n|^p dx + o(1) \\
& \ge c(k^{\frac{2-p}{2}}, 1) + \frac{1-a}{p} \lim_{n\to\infty} \int_{\mathbb{R}^N} |v_n|^p dx \\
& > c(k^{\frac{2-p}{2}}, 1).
\end{aligned}$$

This is a contradiction. Thus, the proof of claim 4 is complete and the part (i) is proved.

For part (ii), we note that it is easy to see using testing function, we have

$$\limsup_{\varepsilon\to 0} c(O_{\delta,\varepsilon}) \le c(k^{\frac{2-p}{2}}, 1).$$

Then the assertion (ii) follows from the assertion (i). □

We remark that for fixed *varepsilon* > 0, the function $\gamma_\varepsilon(u)$ is continuous in u, and $\lim_{n\to\infty}\gamma_{\varepsilon_n}(u_n) = 1$ for the sequence in Lemma 3.2.

Lemma 3.3. *For every $\alpha > 0$, there exists $R = R(\alpha) > 0$ and $\varepsilon = \varepsilon(\alpha) > 0$ for any $0 < \varepsilon < \varepsilon(\alpha)$, if $(u_n) \subset O_{\delta,\varepsilon}$ is a minimizing sequence for $c(O_{\delta,\varepsilon})$, then*

$$\liminf_{n\to\infty}\int_{\{x:\varepsilon^{-1}(1-\frac{\delta}{4})\le|Px|\le\varepsilon^{-1}(1+\frac{\delta}{4})\}\cap\{x:|P^\perp x|\le R\}}|u_n|^2dx \ge 1-\alpha. \tag{3.15}$$

Proof. If the assertion is not true, then there exists $\alpha_0 > 0$ for any $R_m \to \infty$ and $\varepsilon_m \to 0$, a minimizing sequence $(u_{m,n})_{n=1}^\infty$ (of fixed m) for $c(O_{\delta,\varepsilon_m})$ such that

$$\liminf_{n\to\infty}\int_{\{x\ |\ \varepsilon_m{}^{-1}(1-\frac{\delta}{4})\le|Px|\le\varepsilon_m{}^{-1}(1+\frac{\delta}{4})\}\cap\{x\ |P^\perp x|\le R_m\}}|u_{m,n}|^2dx < 1-\alpha_0.$$

Next, we choose a sequence $n_m \to \infty$ such that

$$\lim_{m\to\infty}\int_{\{x|\varepsilon_m{}^{-1}(1-\frac{\delta}{4})\le|Px|\le\varepsilon_m{}^{-1}(1+\frac{\delta}{4})\}\cap\{x||P^\perp x|\le R_m\}}|u_{m,n_m}|^2dx < 1-\alpha_0, \tag{3.16}$$

$$\lim_{m\to\infty}J_{\varepsilon_m}(u_{m,n_m}) = c(k^{\frac{2-p}{2}},1). \tag{3.17}$$

For convenience, here we denote u_{m,n_m} by u_m. Applying Lemma 3.2 to (u_m), then there exists $(y_m) \subset \mathbb{R}^N$ and $\varepsilon_m|y_m| \to 1$ for every $\alpha > 0$, there exists $R > 0$ such that

$$\lim_{m\to\infty}\int_{B_R(y_m)}|u_m|^2dx \ge \frac{1}{k}-\frac{\alpha}{2k}.$$

Take $\alpha = \alpha_0$, the above statement also holds for some $R_0 > 0$. Using the fact that for any $\sigma > 0$, there exists $C_\sigma > 0$ such that

$$\limsup_{m\to\infty}\operatorname{dist}(y_m,\varepsilon_m^{-1}(A_\sigma)) \le C_\sigma < \infty.$$

Then we can assume that

$$(y_m) \subset \big\{x : \varepsilon_m{}^{-1}(1-\frac{\delta}{8}) \le |Px| \le \varepsilon_m{}^{-1}(1+\frac{\delta}{8})\big\}.$$

It follows that

$$B_{R_0}(y_m) \subset \big\{x : \varepsilon_m{}^{-1}(1-\frac{\delta}{4}) \le |Px| \le \varepsilon_m{}^{-1}(1+\frac{\delta}{4})\big\}$$

and

$$\begin{aligned}
&\liminf_{n\to\infty}\int_{\big\{x:\varepsilon_m^{-1}(1-\frac{\delta}{4})\le|Px|\le\varepsilon_m^{-1}(1+\frac{\delta}{4})\big\}\cap\{x:|P^\perp x|\le R_m\}}|u_m|^2dx\\
&\ge\sum_{i=1}^k\int_{B_{R_0}(g^iy_m)}|u_m|^2dx\\
&\ge k(\frac{1}{k}-\frac{\alpha_0}{2k}) \ge 1-\frac{\alpha_0}{2},
\end{aligned}$$

which contradicts (3.16). Thus, the proof is complete. ☐

Lemma 3.4. *There exists $\varepsilon_k > 0$ such that for any $0 < \varepsilon < \varepsilon_k$, $c(O_{\delta,\varepsilon})$ is attained in the interior of $O_{\delta,\varepsilon}$.*

Proof. First we choose $\varepsilon_1 > 0$ such that for any $0 < \varepsilon < \varepsilon_1$, it follows from Lemma 3.2 that $c(O_{\delta,\varepsilon}) \le \frac{1}{2}c(k^{\frac{2-p}{2}}, 1)$. Next, it follows from (2.3) that

$$c(k^{\frac{2-p}{2}}, \mu) = c(k^{\frac{2-p}{2}}, 1)\mu^{1+\theta}, \quad \text{where } \theta = \frac{2(p-2)}{4-N(p-2)} > 0.$$

Then there exists $\mu_0 > 0$ such that ${\mu_0}^{\theta} \le 1/2$. Furthermore, applying Lemma 3.3 with $\alpha_0 = \frac{1}{2}\min\{\mu_0, \delta\}$, there exists $R_0 > 0$ and $\varepsilon_2 > 0$, for all $0 < \varepsilon < \varepsilon_2$, if $(u_n) \subset O_{\delta,\varepsilon}$ is a minimizing sequence for $c(O_{\delta,\varepsilon})$ such that

$$\liminf_{n\to\infty} \int_{\{x:\varepsilon^{-1}(1-\frac{\delta}{4})\le|Px|\le\varepsilon^{-1}(1+\frac{\delta}{4})\}\cap\{x:|P^{\perp}x|\le R_0\}} |u_n|^2 dx \ge 1-\alpha_0.$$

Now, we set $\varepsilon_k = \min\{\varepsilon_1, \varepsilon_2\} > 0$, then fix $0 < \varepsilon < \varepsilon_k$ and let $(u_n) \subset O_{\delta,\varepsilon}$ be a minimizing sequence of $c(O_{\delta,\varepsilon})$. We will show this for a subsequence $u_n \to u$ in $L^2(\mathbb{R}^N)$.

Choose a sequence $R_{m+1} = 2R_m \to \infty$, as $m \to \infty$. Then up to a subsequence, we obtain

$$\lim_{n\to\infty} \int_{B_{R_m}(0)} |u_n|^2 dx = b_m.$$

It is noteworthy that $b_m \ge 1-\delta$. Then it suffices to show $\lim_{m\to\infty} b_m = 1$. To the contrary, we assume $\lim_{m\to\infty} b_m = b < 1$, it may produce a contradiction as follows. Choose a sequence $n_m \to \infty$, as $m \to \infty$ such that

$$\Big|b_m - \int_{B_{R_m}(0)} |u_{n_m}|^2 dx\Big| \le \frac{1}{m}, \quad \Big|b_{m+1} - \int_{B_{R_{m+1}}(0)} |u_{n_m}|^2 dx\Big| \le \frac{1}{m}.$$

For simplicity of notation, we denote the sequence $\{u_{n_m}\}_{m=1}^{\infty}$ as $\{u_m\}$. Then we define

$$v_m(x) = \eta(\frac{|x|}{R_m})u_m(x), \quad \omega_m(x) = \eta^c(\frac{|x|}{R_m})u_m(x).$$

It implies that

$$\lim_{m\to\infty} \int_{\mathbb{R}^N} |v_m|^2 dx = b,$$
$$\lim_{m\to\infty} \int_{\mathbb{R}^N} |\omega_m|^2 dx = 1-b.$$

Next we claim that $v_m/\|v_m\|_2 \in O_{\delta,\varepsilon}$. In fact, it follows from (3.15) that

$$\lim_{n\to\infty} \int_{\{x:\varepsilon^{-1}(1-\frac{\delta}{4})\le|Px|\le\varepsilon^{-1}(1+\frac{\delta}{4})\}\cap\{x:|P^{\perp}x|\le R_0\}} |u_m|^2 dx \ge 1-\alpha_0.$$

It follows from $(u_n) \subset O_{\delta,\varepsilon}$ and $1-b < \delta$ that

$$\int_{T_{\frac{1+\sigma_0}{\varepsilon}} \setminus T_{\frac{1-\sigma_0}{\varepsilon}}} \frac{v_m^2}{\|v_m\|_2^2} dx \ge \frac{1}{\|v_m\|_2^2} \int_{T_{\frac{1+\sigma_0}{\varepsilon}} \setminus T_{\frac{1-\sigma_0}{\varepsilon}}} |u_n|^2 dx \ge 1-\alpha_0 > 1-\delta.$$

Hence, the proof of the claim is complete.

Consequently,

$$\begin{aligned} &c(O_{\delta,\varepsilon}) + o_m(1) \\ &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_m|^2 - \frac{1}{p}\int_{\mathbb{R}^N} Q(\varepsilon x)|u_m|^p dx \end{aligned}$$

$$
\begin{aligned}
&\geq (\frac{1}{2}\int_{\mathbb{R}^N}|\nabla v_m|^2 - \frac{1}{p}\int_{\mathbb{R}^N}Q(\varepsilon x)|v_m|^p dx)\\
&\quad + \Big(\frac{1}{2}\int_{\mathbb{R}^N}|\nabla \omega_m|^2 - \frac{1}{p}\int_{\mathbb{R}^N}Q(\varepsilon x)|\omega_m|^p dx\Big) - \frac{C}{R_m}\\
&\geq \frac{\|v_m\|_2^2}{2}\int_{\mathbb{R}^N}\frac{|\nabla v_m|^2}{\|v_m\|_2^2} - \frac{\|v_m\|_2^p}{p}\int_{\mathbb{R}^N}Q(\varepsilon x)\frac{v_m^p}{\|v_m\|_2^p}dx\\
&\quad + \Big(\frac{1}{2}\int_{\mathbb{R}^N}|\nabla \omega_m|^2 - \frac{1}{p}\int_{\mathbb{R}^N}Q(\varepsilon x)|\omega_m|^p dx\Big) - \frac{C}{R_m}\\
&\geq \|v_m\|_2^2 c(O_{\delta,\varepsilon}) + \frac{\|v_m\|_2^2 - \|v_m\|_2^p}{p}\int_{\mathbb{R}^N}Q(\varepsilon x)\frac{v_m^p}{\|v_m\|_2^p}dx + c(k^{\frac{2-p}{2}}, 1-b) - \frac{C}{R_m}.
\end{aligned}
$$

Sending $m \to \infty$, where C is a constant independent of R, we obtain

$$
(1-b)c(O_{\delta,\varepsilon}) \geq c(k^{\frac{2-p}{2}}, 1-b).
$$

From the choice at the beginning, we derive that

$$
\frac{1-b}{2}c(k^{\frac{2-p}{2}}, 1) \geq c(k^{\frac{2-p}{2}}, 1)(1-b)^{1+\theta}.
$$

This gives $1/2 \leq (1-b)^\theta$ which yields a contradiction with $1-b \leq \alpha_0 \leq \frac{1}{2}\mu_0$. Thus,$1-b=0$ or $b=1$. Therefore, we proved $u_n \to u$ in $L^2(\mathbb{R}^N)$. Then $u_n \to u$ in $L^p(\mathbb{R}^N)$ by interpolation. Using the weakly lower semi-continuity, we deduce

$$
c(O_{\delta,\varepsilon}) \leq J_\varepsilon(u) = \lim_{n\to\infty} J_\varepsilon(u_n) = c(O_{\delta,\varepsilon}).
$$

By the choice $\alpha_0 \leq \frac{\delta}{2}$, we have u is in the interior of $O_{\delta,\varepsilon}$. This completes the proof. $\square$

Proof of Theorem 1.1. The existence part of non-radial positive solution for each k follows from Lemma 3.4 and the energy asymptotic as $\varepsilon \to 0$, $\lim_{\varepsilon\to 0} J_\varepsilon(u_{\varepsilon,k}) = c(k^{\frac{2-p}{2}}, 1)$, follows from Lemma 3.2. $\square$

4. Appendix

Here we sketch the proof Proposition 2.3. We define a global minimization problem with constraint

$$
c(\varepsilon) = \inf_{u\in H_G^1(\mathbb{R}^3), \|u\|_2^2=1} J_\varepsilon(u). \tag{4.1}
$$

First using suitable test functions we easily have $\limsup_{\varepsilon\to 0} c(\varepsilon) \leq c(k^{\frac{2-p}{2}}, 1)$. Under the conditions of Proposition 2.3, using [17, Lemma 4.2] for $\varepsilon > 0$ small enough, $c(\varepsilon)$ is achieved at some u_ε. Now we analyze the asymptotic behavior of u_ε as $\varepsilon \to 0$. Take a sequence $\varepsilon_n \to 0$ and let $u_n := u_{\varepsilon_n}$. Applying the concentration compactness principle to (u_n), we can easily rule out the vanishing since $c(k^{\frac{2-p}{2}}, 1) < 0$. Now using the assumption $Q(0) > k^{\frac{2-p}{2}}$ and using testing functions concentrating at the origin we have $\limsup_{\varepsilon\to 0} c(\varepsilon) \leq c(Q(0), 1) < c(k^{\frac{2-p}{2}}, 1)$. Now we claim u_n weakly converges to $u \neq 0$. Otherwise, by doing a cut-off we obtain a sequence $v_n \in H_{0,G}^1(T_{R_n}^c)$ with $R_n \to \infty$. Using Lemma 3.1 we have $\limsup_{\varepsilon\to 0} c(\varepsilon) \geq c(k^{\frac{2-p}{2}}, 1)$, a contradiction. Now if $\|u\|_2^2 < 1$ we may use Brezis-Lieb Lemma to get a contradiction again. Thus compactness holds for the sequence (u_n). Then it follows that $\limsup_{\varepsilon\to 0} c(\varepsilon) = c(Q(0), 1)$ and for any $\alpha > 0$ there is $R > 0$ such that $\liminf_{n\to\infty} \int_{\mathbb{R}^N} u_n^2 dx \geq 1 - \alpha$. That is, u_n concentrate at the origin as $n \to \infty$.

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