Electronic Journal of Differential Equations, Vol. 2007(2007), No. 153, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
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CONSTRUCTION OF GREEN'S FUNCTIONS FOR THE BLACK-SCHOLES EQUATION

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Abstract. A technique is proposed for the construction of Green's functions for terminal-boundary value problems of the Black-Scholes equation. The technique permits an application to a variety of problems that vary by boundary conditions imposed. This is possible by extension of an approach that was earlier developed for partial differential equations in applied mechanics. The technique is based on the method of integral Laplace transform and the method of variation of parameters. It provides closed form analytic representations for the constructed Green's functions.

1. Introduction

The well-known function, in financial mathematics [3, 4, 6],

$$G(S,t;\widetilde{S}) = \frac{\exp(-r(T-t))}{\widetilde{S}[2\pi\sigma^2(T-t)]^{1/2}} \exp\Big(- \frac{[\ln(S/\widetilde{S}) + (r-\sigma^2/2)(T-t)]^2}{2\sigma^2(T-t)}\Big) \tag{1.1}$$

is referred to as the Green's function of the backward in time parabolic partial differential equation

$$\frac{\partial v(S,t)}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 v(S,t)}{\partial S^2} + rS\frac{\partial v(S,t)}{\partial S} - rv(S,t) = 0 \tag{1.2}$$

which is called the Black-Scholes equation [1]. To be more specific mathematically, we note that (1.1) represents the Green's function for the homogeneous terminal-boundary value problem corresponding to

$$v(S,T) = f(S) \tag{1.3}$$

$$|v(0,t)| < \infty \quad \text{and} \quad |v(\infty,t)| < \infty. \tag{1.4}$$

This problem was posed for the Black-Scholes equation in the quarter-plane $\Omega = (0 < S < \infty) \times (T > t > -\infty)$ of the S,t-plane.

In the above setting, $v = v(S,t)$ is the price of the derivative product, $f(S)$ is the pay-off function of a given derivative problem at the expiration time T, with S and t being the share price of the underlying asset and time, respectively. The parameters σ and $r > 0$ represent the volatility of the underlying asset and the risk-free interest rate, respectively. The variable $\widetilde{S} \in (0,\infty)$ in (1.1) plays the role of a *source point*.

2000 *Mathematics Subject Classification.* 35K20, 58J35.
Key words and phrases. Black-Scholes equation; Green's function.
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Submitted August 28, 2007. Published November 14, 2007.

A special comment is required as to the symbolism used in specifying the *boundary* conditions in (1.4). Both the end-points of the domain for the independent variable S represent the so-called *singular points* [5] to the Black-Scholes equation, in which case the corresponding boundary conditions cannot formally assign certain values to the solution of the governing differential equation. Instead, the conditions in (1.4) imply that the solution that we are looking for has to be bounded as the variable S approaches both zero and infinity.

The function in (1.1) represents the only Green's function for (1.2) that is available in financial mathematics for decades. This study proposes a new approach that enables one to construct Green's functions to the Black-Scholes equation not only for the boundary conditions in (1.4) but also for a variety of others. The approach flows out from a technique proposed earlier [2] for boundary value problems in applied mechanics. It is not based on the classical formalism for the diffusion equation as in [3, 4]. Instead, the emphasis is made on the parabolic single-parameter partial differential equation forward in time

$$\frac{\partial u(x,\tau)}{\partial \tau} = \frac{\partial^2 u(x,\tau)}{\partial x^2} + (c-1)\frac{\partial u(x,\tau)}{\partial x} - cu(x,\tau) \tag{1.5}$$

which is traditionally obtained [3, 6] from (1.2) by introducing new independent variables

$$x = \ln S \quad \text{and} \quad \tau = \frac{\sigma^2}{2}(T-t) \tag{1.6}$$

and setting $u(x,\tau) = v(S,t)$.

The parameter c in (1.5) is defined in terms of r and σ^2 of the Black-Scholes equation as $c = 2r/\sigma^2$.

To illustrate the effectiveness of our approach, a validation example is considered in the next section where we derive the Green's function of (1.1). After the approach is validated, it is used, in the following sections, to tackle some other terminal-boundary value problems for the Black-Scholes equation. New Green's functions are obtained none of which have earlier been presented in literature.

2. A validation example

By introducing new variables x and τ in compliance with the relations in (1.6), the terminal-boundary value problem of (1.2)-(1.4) transforms to the following initial-boundary value problem

$$u(x,0) = f(\exp x) \tag{2.1}$$

$$|u(-\infty,\tau)| < \infty, \quad |u(\infty,\tau)| < \infty \tag{2.2}$$

for (1.5) on the half-plane $(-\infty < x < \infty) \times (0 < \tau < \infty)$. Applying the Laplace transform

$$U(x;s) = L\{u(x,\tau)\} = \int_0^\infty \exp(-s\tau)u(x,\tau)d\tau$$

to the problem in (1.5), (2.1) and (2.2), one arrives at the boundary value problem

$$\frac{d^2U(x;s)}{dx^2} + (c-1)\frac{dU(x;s)}{dx} - (s+c)U(x;s) = -f(\exp x), \tag{2.3}$$

$$|U(-\infty;s)| < \infty, \quad |U(\infty;s)| < \infty \tag{2.4}$$

for the Laplace transform $U(x;s)$ of $u(x,\tau)$. Note that (2.3) is a linear nonhomogeneous ordinary differential equation with constant coefficients since s is just a parameter and $U(x;s)$ is treated as a single-variable function of x.

To find a fundamental set of solutions to the homogeneous equation corresponding to (2.3), consider its characteristic equation

$$k^2+(c-1)k-(s+c)=0$$

whose roots are

$$k_1=\alpha+\omega, \quad k_2=\alpha-\omega$$

where $\omega=(s+\beta)^{1/2}$, while the parameters α and β are defined in terms of c as

$$\alpha=\frac{1-c}{2}, \quad \beta=(\frac{1+c}{2})^2 \tag{2.5}$$

This yields two linearly independent particular solutions to the homogeneous equation corresponding to (2.3) as

$$U_1(x;s)=\exp(\alpha+\omega)x, \quad U_2(x;s)=\exp(\alpha-\omega)x$$

with their linear combination

$$U(x;s)=A(x;s)\exp(\alpha+\omega)x+B(x;s)\exp(\alpha-\omega)x \tag{2.6}$$

representing, according to the method of variation of parameters, the general solution to (2.3). Following the procedure of this method, one arrives at the well-posed system

$$\begin{pmatrix} \exp(\alpha+\omega)x & \exp(\alpha-\omega)x \\ (\alpha+\omega)\exp(\alpha+\omega)x & (\alpha-\omega)\exp(\alpha-\omega)x \end{pmatrix} \begin{pmatrix} A'(x;s) \\ B'(x;s) \end{pmatrix} = \begin{pmatrix} 0 \\ -f(\exp x) \end{pmatrix}$$

of linear algebraic equations in the derivatives with respect to x of the coefficients $A(x;s)$ and $B(x;s)$ of the linear combination in (2.6). The solution of the above system is obtained as

$$A'(x;s)=-\frac{\exp(-(\alpha+\omega)x)}{2\omega}f(\exp x), \quad B'(x;s)=\frac{\exp(-(\alpha-\omega)x)}{2\omega}f(\exp x)$$

Upon integration, the coefficients $A(x;s)$ and $B(x;s)$ are found in the form

$$\begin{aligned} A(x;s)&=-\frac{1}{2\omega}\int_{-\infty}^{x}\exp(-(\alpha+\omega)\xi)f(\exp\xi)d\xi+M(s), \\ B(x;s)&=\frac{1}{2\omega}\int_{-\infty}^{x}\exp(-(\alpha-\omega)\xi)f(\exp\xi)d\xi+N(s) \end{aligned}$$

Substitution of these in (2.6) yields the general solution to (2.3) in the form

$$\begin{aligned} U(x;s)=&\frac{1}{2\omega}\int_{-\infty}^{x}\exp\alpha(x-\xi)[\exp\omega(\xi-x)-\exp\omega(x-\xi)]f(\exp\xi)d\xi \\ &+M(s)\exp(\alpha+\omega)x+N(s)\exp(\alpha-\omega)x \end{aligned} \tag{2.7}$$

The 'constants of integration' $M(s)$ and $N(s)$ can be obtained upon satisfying the boundary conditions of (2.4). Omitting details, we have

$$N(s)=0, \quad M(s)=\frac{1}{2\omega}\int_{-\infty}^{\infty}\exp(-(\alpha+\omega)\xi)f(\exp\xi)d\xi$$

Upon substituting these in (2.7), one obtains the solution to the boundary value problem in (2.3) and (2.4) in the form

$$\begin{aligned} U(x;s) &= \int_{-\infty}^{x} \frac{\exp\alpha(x-\xi)}{2\omega}[\exp\omega(\xi-x)-\exp\omega(x-\xi)]f(\exp\xi)d\xi \\ &\quad+\int_{-\infty}^{\infty}\frac{\exp\alpha(x-\xi)}{2\omega}\exp\omega(x-\xi)f(\exp\xi)d\xi \end{aligned}$$

which can be rewritten in a compact *single-integral* form as

$$U(x;s) = \int_{-\infty}^{\infty}\frac{\exp\alpha(x-\xi)}{2\omega}\exp(-\omega|x-\xi|)f(\exp\xi)d\xi \tag{2.8}$$

The solution $u(x,\tau)$ to the initial-boundary value problem stated by (1.5), (2.1) and (2.2) can be obtained from $U(x;s)$ with the aid of the inverse Laplace transform. In doing so, we keep in mind that the parameter ω has earlier been introduced in terms of the parameter s of the Laplace transform as $\omega = (s+\beta)^{1/2}$. This yields

$$\begin{aligned} u(x,\tau) &= L^{-1}\{U(x;s)\} \\ &= \int_{-\infty}^{\infty}\exp\alpha(x-\xi)L^{-1}\{\frac{\exp(-(s+\beta)^{1/2}|x-\xi|)}{2(s+\beta)^{1/2}}\}f(\exp\xi)d\xi \\ &= \int_{-\infty}^{\infty}\frac{\exp\alpha(x-\xi)\exp(-\beta\tau)}{2(\pi\tau)^{1/2}}\exp(-\frac{(x-\xi)^2}{4\tau})f(\exp\xi)d\xi \end{aligned} \tag{2.9}$$

To obtain the solution $v(S,t)$ to the setting in (1.2)-(1.4), we make the backward substitutions in compliance with the relations of (1.6). This implies that the variables x, τ and ξ ought to be replaced with S, t and $\widetilde{S}$, respectively as

$$x=\ln S, \quad \tau=\frac{\sigma^2}{2}(T-t), \quad \xi=\ln\widetilde{S}$$

The differential of the variable of integration ξ in (2.9) converts to the form

$$d\xi = \frac{1}{\widetilde{S}}d\widetilde{S}$$

while the interval of integration $(-\infty,\infty)$ in (2.9) transforms, according to the relation $\xi=\ln\widetilde{S}$, to the interval $[0,\infty)$ with respect to $\widetilde{S}$. With all this in mind, one arrives at the solution to the terminal-boundary value problem in (1.2)-(1.4) as

$$v(S,t) = \int_0^{\infty}\frac{1}{\sigma\widetilde{S}[2\pi(T-t)]^{1/2}}\exp\Big(\alpha\ln(S/\widetilde{S})-\beta\frac{\sigma^2}{2}(T-t)-\frac{[\ln(S/\widetilde{S})]^2}{2\sigma^2(T-t)}\Big)f(\widetilde{S})d\widetilde{S} \tag{2.10}$$

revealing the Green's function to the problem in (1.2)-(1.4) in the form

$$G(S,t;\widetilde{S}) = \frac{1}{\widetilde{S}[2\pi\sigma^2(T-t)]^{1/2}}\exp\Big(\alpha\ln(S/\widetilde{S})-\beta\frac{\sigma^2}{2}(T-t)-\frac{[\ln(S/\widetilde{S})]^2}{2\sigma^2(T-t)}\Big) \tag{2.11}$$

It is not evident that the above representation for $G(S,t;\widetilde{S})$ and the one in (1.1) are identical. To verify their identity, we express α and β in (2.11) in terms of the original parameters σ^2 and r of the Black-Scholes equation as

$$\alpha=\frac{\sigma^2/2-r}{\sigma^2}, \quad \beta=(\frac{r+\sigma^2/2}{\sigma^2})^2$$

and then rewrite (2.11) as

$$\begin{aligned} G(S,t;\widetilde{S}) = \frac{1}{\widetilde{S}[2\pi\sigma^2(T-t)]^{1/2}} \exp\Big(& \frac{\sigma^2/2 - r}{\sigma^2} \ln(S/\widetilde{S}) \\ & - \frac{(r+\sigma^2/2)^2}{2\sigma^2}(T-t) - \frac{[\ln(S/\widetilde{S})]^2}{2\sigma^2(T-t)}\Big) \end{aligned} \tag{2.12}$$

Multiplying the above expression by the product of the two factors

$$\exp(-r(T-t)) \exp r(T-t)$$

which is identically equal to one, we leave the first of these factors (the negative exponent) in its current form, combine the second factor with the existing exponential term in (2.12) and rewrite subsequently the latter as

$$\begin{aligned} G(S,t;\widetilde{S}) = \frac{\exp(-r(T-t))}{\widetilde{S}[2\pi\sigma^2(T-t)]^{1/2}} \exp\Big(& -\frac{r-\sigma^2/2}{\sigma^2} \ln(S/\widetilde{S}) \\ & + r(T-t) - \frac{(r+\sigma^2/2)^2}{2\sigma^2}(T-t) - \frac{[\ln(S/\widetilde{S})]^2}{2\sigma^2(T-t)}\Big) \end{aligned}$$

Combining the second and the third additive terms in the argument of the extended exponential function, we reduce the latter to the form

$$\exp\Big(-\frac{r-\sigma^2/2}{\sigma^2}\ln(S/\widetilde{S}) - \frac{(r-\sigma^2/2)^2}{2\sigma^2}(T-t) - \frac{[\ln(S/\widetilde{S})]^2}{2\sigma^2(T-t)}\Big)$$

which can immediately be transformed into

$$\exp\Big(-\frac{[\ln(S/\widetilde{S})]^2 + 2(r-\sigma^2/2)(T-t)\ln(S/\widetilde{S}) + (r-\sigma^2/2)^2(T-t)^2}{2\sigma^2(T-t)}\Big)$$

It is evident that the numerator in the argument of this exponential function represents a complete square, reducing the above to

$$\exp\Big(-\frac{[\ln(S/\widetilde{S}\ ) + (r-\sigma^2/2)(T-t)]^2}{2\sigma^2(T-t)}\Big)$$

Thus, the representation in (2.11) is, indeed, identical to that of (1.1). This implies that the function that we came up with in (2.11) does really represent the Green's function to the terminal-boundary value problem in (1.2)-(1.4). In other words, our approach is proven productive, and in the next section we bring a convincing justification of its successful applicability to other terminal-boundary value problems for the Black-Scholes equation.

3. Other Green's functions

Two particular terminal-boundary value problems with different boundary conditions imposed are considered as an illustration to the assertion made in the previous section.

3.1. **Dirichlet boundary conditions.** As the first example, consider a terminal-boundary value problem stated for (1.2) in the semi-infinite strip $\Omega = (S_1 < S < S_2) \times (T > t > -\infty)$ of the S, t -plane. Let the terminal condition be given by (1.3), while the Dirichlet boundary conditions

$$v(S_1, t) = 0, \quad v(S_2, t) = 0 \tag{3.1}$$

are imposed on the edges $S = S_1$ and $S = S_2$ of Ω.

Note that the above setting for the Black-Scholes equation sounds quite practical for the financial engineering, whereas its Green's function is not yet available in literature.

By the transformations of (1.6), the setting in (1.2), (1.3) and (3.1) converts to the following initial-boundary value problem

$$u(x, 0) = f(\exp x), \tag{3.2}$$

$$u(a, \tau) = 0, \quad u(b, \tau) = 0 \tag{3.3}$$

for (1.5) on the semi-infinite strip $(a < x < b) \times (0 < \tau < \infty)$ in the x, τ-plane, on which the region Ω maps by the change of variables introduced in (1.6). The parameters a and b are determined in terms of S_1 and S_2 as

$$a = \ln S_1, \quad b = \ln S_2$$

The Laplace transform applied to the setting in (1.5), (3.2) and (3.3) converts the latter into the boundary value problem

$$\frac{d^2U(x;s)}{dx^2} + (c-1)\frac{dU(x;s)}{dx} - (s+c)U(x;s) = -f(\exp x), \tag{3.4}$$

$$U(a;s) = 0, \quad U(b;s) = 0 \tag{3.5}$$

for the Laplace transform $U(x;s)$ of $u(x,\tau)$.

In compliance with the method of variation of parameters, the general solution to (3.4) is found, in this case, as

$$\begin{aligned} U(x,s) = &\int_a^x \frac{\exp\alpha(x-\xi)}{2\omega}[\exp\omega(\xi - x) - \exp\omega(x-\xi)]f(\exp\xi)d\xi \\ &+ M(s)\exp(\alpha+\omega)x + N(s)\exp(\alpha-\omega)x \end{aligned} \tag{3.6}$$

where the parameter ω is defined as $\omega = (s+\beta)^{1/2}$.

Satisfying the boundary conditions of (3.5) yields the system of linear algebraic equations

$$\begin{pmatrix} \exp(\alpha+\omega)a & \exp(\alpha-\omega)a \\ \exp(\alpha+\omega)b & \exp(\alpha-\omega)b \end{pmatrix} \begin{pmatrix} M(s) \\ N(s) \end{pmatrix} = \begin{pmatrix} 0 \\ \Psi(s) \end{pmatrix}$$

in $M(s)$ and $N(s)$. Here

$$\Psi(s) = -\int_a^b \frac{1}{2\omega}[\exp(\alpha-\omega)(b-\xi) - \exp(\alpha+\omega)(b-\xi)]f(\exp\xi)d\xi$$

Solving the above system, we obtain

$$\begin{aligned} M(s) = \int_a^b &\frac{\exp(\alpha-\omega)a\exp\alpha(b-\xi)}{2\omega[\exp\omega(a-b) - \exp\omega(b-a)]} \\ &\times [\exp\omega(\xi-b) - \exp\omega(b-\xi)]f(\exp\xi)d\xi \end{aligned}$$

and

$$N(s) = -\int_a^b \frac{\exp(\alpha+\omega)a \exp\alpha(b-\xi)}{2\omega[\exp\omega(a-b) - \exp\omega(b-a)]} \times [\exp\omega(\xi-b) - \exp\omega(b-\xi)] f(\exp\xi) d\xi$$

Upon substituting these in (3.6), the latter reads

$$\begin{aligned} U(x,s) = &\int_a^x \frac{\exp\alpha(x-\xi)}{2\omega}[\exp\omega(\xi-x) - \exp\omega(x-\xi)] f(\exp\xi) d\xi \\ &+ \int_a^b \frac{\exp\alpha(x-\xi)[\exp\omega(x-a) - \exp\omega(a-x)]}{2\omega[\exp\omega(a-b) - \exp\omega(b-a)]} \\ &\times [\exp\omega(\xi-b) - \exp\omega(b-\xi)] f(\exp\xi) d\xi \end{aligned}$$

which can be expressed in a *single-integral* form as

$$\begin{aligned} U(x;s) = &\int_a^b \frac{\exp\alpha(x-\xi)}{2\omega[\exp\omega(a-b) - \exp\omega(b-a)]} \\ &\times \Big\{ \exp\omega[(x+\xi)-(a+b)] + \exp\omega[(a+b)-(x+\xi)] \\ &- \exp\omega[|x-\xi| + (a-b)] - \exp\omega[(b-a) - |x-\xi|] \Big\} f(\exp\xi) d\xi \end{aligned}$$

Transforming the bracket factor in the denominator as

$$\exp\omega(a-b) - \exp\omega(b-a) = -\exp\omega(b-a)[1-\exp 2\omega(a-b)]$$

we rewrite the above representation for $U(x;s)$ as

$$\begin{aligned} U(x;s) = &-\int_a^b \frac{\exp\alpha(x-\xi)}{2\omega\exp\omega(b-a)[1-\exp 2\omega(a-b)]} \\ &\times \{\exp\omega[(x+\xi)-(a+b)] + \exp\omega[(a+b)-(x+\xi)] \\ &- \exp\omega[|x-\xi| + (a-b)] - \exp\omega[(b-a) - |x-\xi|]\} f(\exp\xi) d\xi \end{aligned} \tag{3.7}$$

The inverse Laplace transform of $U(x;s)$ is problematic if the latter is kept in its current form. Therefore, we adjust it first by representing the factor

$$[1-\exp 2\omega(a-b)]^{-1}$$

in the integrand of (3.7) as a geometric series,

$$\frac{1}{1-\exp 2\omega(a-b)} = \sum_{n=0}^{\infty} \exp 2n\omega(a-b)$$

whose common ratio $\exp 2\omega(a-b)$ represents a negative exponential function ($a < b$) and is, therefore, less than one. This transforms (3.7) to

$$\begin{aligned} U(x;s) = &\int_a^b \frac{\exp\alpha(x-\xi)}{2\omega} \sum_{n=0}^{\infty} \{\exp\omega[|x-\xi| - 2(n+1)(b-a)] \\ &+ \exp\omega[2(n+1)(a-b) - |x-\xi|] - \exp\omega[2n(a-b) - 2b + (x+\xi)] \\ &- \exp\omega[2n(a-b) + 2a - (x+\xi)]\} f(\exp\xi) d\xi \end{aligned}$$

and the inverse Laplace transform of the above can be accomplished in the term-by-term manner. This yields the solution $u(x,\tau)$ to the initial-boundary value problem in (1.5), (3.2) and (3.3) in the form

$$
\begin{aligned}
u(x,\tau) &= L^{-1}\{U(x,s)\}\\
&= \int_a^b \frac{\exp\alpha(x-\xi)\exp(-\beta\tau)}{2(\pi\tau)^{1/2}} \sum_{n=0}^{\infty}\Big\{\exp\Big(-\frac{[|x-\xi|+2(n+1)(a-b)]^2}{4\tau}\Big)\\
&\quad + \exp\Big(-\frac{[|x-\xi|-2n(a-b)]^2}{4\tau}\Big) - \exp\Big(-\frac{[2b-(x+\xi)-2n(a-b)]^2}{4\tau}\Big)\\
&\quad - \exp\Big(-\frac{[(x+\xi)-2a-2n(a-b)]^2}{4\tau}\Big)\Big\} f(\exp\xi)d\xi
\end{aligned}
$$

which converts to a more compact form by rearranging the summation in the above series. This yields

$$
\begin{aligned}
u(x,\tau) = \int_a^b &\frac{\exp\alpha(x-\xi)\exp(-\beta\tau)}{2(\pi\tau)^{1/2}} \sum_{m=-\infty}^{\infty}\Big\{\exp\Big(-\frac{[|x-\xi|+2m(a-b)]^2}{4\tau}\Big)\\
&- \exp\Big(-\frac{[2b-(x+\xi)-2m(a-b)]^2}{4\tau}\Big)\Big\} f(\exp\xi)d\xi
\end{aligned}
$$

In compliance with the relations in (1.6), the solution $v(S,t)$ to the setting in (1.2), (1.3) and (3.1) can be attained by the backward replacement of the variables x, τ and ξ with S, t and $\widetilde{S}$, respectively. Similarly to the analogous replacement that has been performed in Section 2 (with α and β replaced with the original parameters r and σ^2 of the Black-Scholes equation), we obtain $v(S,t)$ in the form

$$
\begin{aligned}
v(S,t) = \int_{S_1}^{S_2} &\frac{\exp\big(-\frac{r-\sigma^2/2}{\sigma^2}\ln(S/\widetilde{S}) - \frac{(r+\sigma^2/2)^2}{2\sigma^2}(T-t)\big)}{\widetilde{S}[2\pi\sigma^2(T-t)]^{1/2}}\\
&\times \sum_{m=-\infty}^{\infty}\Big\{\exp\Big(-\frac{[\ln(S/\widetilde{S})+2m\ln(S_1/S_2)]^2}{2\sigma^2(T-t)}\Big)\\
&- \exp\Big(-\frac{[\ln(S_2^2/S\widetilde{S})-2m\ln(S_1/S_2)]^2}{2\sigma^2(T-t)}\Big)\Big\} f(\widetilde{S})d\widetilde{S}
\end{aligned}
$$

which can be transformed, by combining the logarithmic components in the series factor. This yields

$$
\begin{aligned}
v(S,t) = \int_{S_1}^{S_2} &\frac{\exp\big(-\frac{r-\sigma^2/2}{\sigma^2}\ln(S/\widetilde{S}) - \frac{(r+\sigma^2/2)^2}{2\sigma^2}(T-t)\big)}{\widetilde{S}[2\pi\sigma^2(T-t)]^{1/2}}\\
&\times \sum_{m=-\infty}^{\infty}\Big\{\exp\Big(-\frac{[\ln(SS_1^{2m}/\widetilde{S}S_2^{2m})]^2}{2\sigma^2(T-t)}\Big)\\
&- \exp\Big(-\frac{[\ln(S_2^{2(m+1)}/S\widetilde{S}S_1^{2m})]^2}{2\sigma^2(T-t)}\Big)\Big\} f(\widetilde{S})d\widetilde{S}
\end{aligned}
\tag{3.8}
$$

Thus, the kernel in the above integral,

$$
\begin{aligned}
G(S,t;\widetilde{S}) = {} & \frac{\exp\left(-\frac{r-\sigma^2/2}{\sigma^2}\ln(S/\widetilde{S}) - \frac{(r+\sigma^2/2)^2}{2\sigma^2}(T-t)\right)}{\widetilde{S}[2\pi\sigma^2(T-t)]^{1/2}} \\
& \times \sum_{m=-\infty}^{\infty} \Big\{ \exp\Big(-\frac{[\ln(SS_1^{2m}/\widetilde{S}S_2^{2m})]^2}{2\sigma^2(T-t)} \Big) \\
& - \exp\Big(-\frac{[\ln(S_2^{2(m+1)}/S\widetilde{S}S_1^{2m})]^2}{2\sigma^2(T-t)} \Big) \Big\},
\end{aligned} \tag{3.9}
$$

represents the Green's function to the setting in (1.2), (1.3) and (3.1). The series in this representation converges at a *high* rate unless the term $(T-t)$ is very small. This implies that, in computing values of $G(S,t;\widetilde{S})$, an accuracy level required for applications can, in most cases, be attained by appropriately truncating the series in (3.9) to a partial sum.

3.2. **Mixed boundary conditions.** Note that the qualitative theory of partial differential equations [5] claims that if $G(S,t;\widetilde{S})$ is the Green's function to the setting in, say, (1.2), (1.3) and (3.1), then the solution to this problem can be written in the integral form

$$
v(S,t) = \int_{S_1}^{S_2} G(S,t;\widetilde{S})\ f(\widetilde{S})\, d\widetilde{S} \tag{3.10}
$$

This observation determined our strategy in the development of Sections 2 and 3.1 The strategy can also be applied while obtaining a Green's function to the setting in (1.2) and (1.3), with the following boundary conditions

$$
|v(0,t)| < \infty, \quad \frac{\partial v(D,t)}{\partial S} + \varrho v(D,t) = 0, \quad \varrho \ge 0 \tag{3.11}
$$

imposed on the boundary fragments $S=0$ and $S=D$ of the semi-infinite strip $\Omega = (0 < S < D) \times (T > t > -\infty)$. Indeed, if we manage to find the solution to the problem in (1.2), (1.3) and (3.11) in an *integral form* like that in (3.10), then the kernel of the integral represents the Green's function that we are looking for.

The second condition in (3.11) is referred to, in mathematical physics, as either *mixed* or *Robin* type. To our best knowledge, mixed boundary conditions have never been considered yet in association with the Black-Scholes equation. It is even unclear if such problem settings are timely for financial engineering. But from mathematics stand-point, they do not look unfeasible and could possibly find realistic applications in the field of finance in years to come.

Upon introducing new variables x and τ as suggested in (1.6), one converts the setting in (1.2), (1.3) and (3.11) to the initial-boundary value problem

$$
u(x,0) = f(\exp x) \tag{3.12}
$$

$$
|u(-\infty,\tau)| < \infty, \quad \frac{\partial u(b,\tau)}{\partial x} + \overline{\varrho} u(b,\tau) = 0 \tag{3.13}
$$

for (1.5) on the quarter-plane $(-\infty < x < b) \times (0 < \tau < \infty)$. The parameters b and $\overline{\varrho}$ in (3.12) are defined in terms of the initial data in the original problem as

$$
b = \ln D, \quad \overline{\varrho} = D\varrho
$$

Applying the Laplace transform to the problem in (1.5), (3.12) and (3.13), one arrives at the boundary value problem

$$|U(-\infty;s)| < \infty, \quad \frac{dU(b;s)}{dx} + \overline{\varrho}U(b;s) = 0 \tag{3.14}$$

for the equation in (3.4).

Aiming at the solution to the problem in (1.2), (1.3) and (3.11) in an integral form, we apply the method of variation of parameters to the problem in (3.4) and (3.14). This gives the general solution of (3.4) in the form

$$\begin{aligned} U(x;s) = {} & \frac{1}{2\omega}\int_{-\infty}^{x} \exp\alpha(x-\xi)[\exp\omega(\xi-x) - \exp\omega(x-\xi)]f(\exp\xi)d\xi \\ & + M(s)\exp(\alpha+\omega)x + N(s)\exp(\alpha-\omega)x \end{aligned} \tag{3.15}$$

To determine the functions $M(s)$ and $N(s)$, we take advantage of the boundary conditions of (3.14). When x approaches negative infinity, the integral component in (3.15) vanishes, while the $M(s)$-containing component approaches zero. Hence, for the first condition in (3.14) to hold, $N(s)$ ought to be zero

$$N(s) = 0 \tag{3.16}$$

because the exponential factor in the $N(s)$-containing component in (3.15) is unbounded as x approaches negative infinity.

In light of (3.16), the derivative of $U(x;s)$ reads as

$$\begin{aligned} \frac{dU(x;s)}{dx} = {} & \frac{1}{2\omega}\int_{-\infty}^{x} [(\alpha-\omega)\exp\omega(\xi-x) - (\alpha+\omega)\exp\omega(x-\xi)] \\ & \times \exp\alpha(x-\xi)f(\exp\xi)d\xi + M(s)(\alpha+\omega)\exp(\alpha+\omega)x \end{aligned}$$

So, the second condition in (3.14) yields the following equation in $M(s)$,

$$\begin{aligned} & \frac{1}{2\omega}\int_{-\infty}^{b} [(\alpha-\omega)\exp\omega(\xi-b) - (\alpha+\omega)\exp\omega(b-\xi)] \\ & \times \exp\alpha(b-\xi)f(\exp\xi)d\xi + M(s)(\alpha+\omega)\exp(\alpha+\omega)b \\ & + \frac{\overline{\varrho}}{2\omega}\int_{-\infty}^{b} \exp\alpha(b-\xi)[\exp\omega(\xi-b) - \exp\omega(b-\xi)]f(\exp\xi)d\xi \\ & + \overline{\varrho}M(s)\exp(\alpha+\omega)b = 0 \end{aligned}$$

from which $M(s)$ is found as

$$M(s) = -\frac{1}{2\omega}\int_{-\infty}^{b} [\frac{(\overline{\varrho}+\alpha)-\omega}{(\overline{\varrho}+\alpha)+\omega}\exp\omega(\xi-b) - \exp\omega(b-\xi)]\exp(-\alpha\xi-\omega b)f(\exp\xi)d\xi$$

Substituting now the above expression for $M(s)$ in (3.15) and taking into account (3.16), one obtains the solution to the boundary value problem in (3.4) and (3.14) as

$$\begin{aligned} U(x;s) = {} & \frac{1}{2\omega}\int_{-\infty}^{x} \exp\alpha(x-\xi)[\exp\omega(\xi-x) - \exp\omega(x-\xi)]f(\exp\xi)d\xi \\ & - \frac{1}{2\omega}\int_{-\infty}^{b} [\frac{(\overline{\varrho}+\alpha)-\omega}{(\overline{\varrho}+\alpha)+\omega}\exp\omega(\xi-b) - \exp\omega(b-\xi)] \\ & \times \exp\alpha(x-\xi)\exp\omega(x-b)f(\exp\xi)d\xi \end{aligned}$$

To obtain the inverse Laplace transform of $U(x;s)$, $u(x,\tau) = L^{-1}\{U(x;s)\}$ which represents the solution to the initial-boundary value problem in (1.5), (3.12) and (3.13), we simplify the above expression for $U(x;s)$. Proceeding through a tedious but quite straightforward algebra, one obtains a more compact form for $U(x;s)$ as

$$\begin{aligned} U(x;s) = \int_{-\infty}^{b} &[\exp(-\omega|x-\xi|) - \frac{(\overline{\varrho}+\alpha)-\omega}{(\overline{\varrho}+\alpha)+\omega}\exp\omega(x+\xi-2b)] \\ &\times \frac{\exp\alpha(x-\xi)}{2\omega} f(\exp\xi)d\xi \end{aligned} \tag{3.17}$$

which is not, unfortunately, convenient yet for the immediate inverse Laplace transform. To facilitate the latter, we rewrite $U(x;s)$ in the equivalent form

$$\begin{aligned} U(x;s) = \int_{-\infty}^{b} &\Big\{\exp(-\omega|x-\xi|) - [\frac{2(\overline{\varrho}+\alpha)}{(\overline{\varrho}+\alpha)+\omega} - 1]\exp\omega(x+\xi-2b)\Big\} \\ &\times \frac{\exp\alpha(x-\xi)}{2\omega} f(\exp\xi)d\xi \end{aligned}$$

or, recalling the expression for ω in terms of the parameter s of the Laplace transform, the above reads as

$$\begin{aligned} U(x;s) = \int_{-\infty}^{b} &\frac{\exp\alpha(x-\xi)}{2}\Big\{\frac{\exp(-(s+\beta)^{1/2}|x-\xi|)}{(s+\beta)^{1/2}} \\ &- [\frac{2\Phi}{(\Phi+(s+\beta)^{1/2})} - 1]\frac{\exp((s+\beta)^{1/2}(x+\xi-2b))}{(s+\beta)^{1/2}}\Big\} f(\exp\xi)d\xi \end{aligned} \tag{3.18}$$

where we introduced, for compactness, $\Phi = \overline{\varrho}+\alpha$.

The inverse Laplace transform of $U(x;s)$ from (3.18) represents the solution $u(x,\tau)$ to the initial-boundary value problem in (1.5), (3.12) and (3.13). It is found in the form

$$\begin{aligned} u(x,\tau) = \int_{-\infty}^{b} &\Big\{\frac{1}{2(\pi\tau)^{1/2}}\Big[\exp\big(-\frac{(x-\xi)^2}{4\tau}\big) + \exp\big(-\frac{(x+\xi-2b)^2}{4\tau}\big)\Big] \\ &- \Phi\exp(\Phi^2\tau - \Phi(x+\xi-2b))\operatorname{erfc}\Big(\Phi\tau^{1/2} - \frac{x+\xi-2b}{2\tau^{1/2}}\Big)\Big\} \\ &\times \exp\alpha(x-\xi)\exp(-\beta\tau)f(\exp\xi)d\xi \end{aligned} \tag{3.19}$$

where the $\operatorname{erfc}(\cdot)$ represents the *complementary error function*

$$\operatorname{erfc}(\varphi) = \frac{2}{\pi^{1/2}}\int_{\varphi}^{\infty} e^{-x^2}dx\,.$$

The solution $v(S,t)$ to the terminal-boundary value problem in (1.2), (1.3) and (3.11) can be obtained from (3.19) by making the backward substitution of the variables in compliance with the relations of (1.6). This implies

$$\begin{aligned} v(S,t) = \int_0^D & \frac{1}{\widetilde{S}} \exp\Big(\alpha \ln(\frac{S}{\widetilde{S}}) - \beta\frac{\sigma^2}{2}(T-t)\Big) \\ & \times \Big\{ \frac{1}{[2\pi\sigma^2(T-t)]^{1/2}} \Big[\exp\Big(- \frac{[\ln(S/\widetilde{S})]^2}{2\sigma^2(T-t)}\Big) + \exp\Big(- \frac{[\ln(S\widetilde{S}/D^2)]^2}{2\sigma^2(T-t)}\Big)\Big] \\ & - \Phi\exp(\Phi^2\sigma^2(T-t)/2 - \Phi\ln(S\widetilde{S}/D^2)) \\ & \times \operatorname{erfc}(\frac{\Phi}{2}[2\sigma^2(T-t)]^{1/2} - \frac{\ln(S\widetilde{S}/D^2)}{[2\sigma^2(T-t)]^{1/2}})\Big\} f(\widetilde{S})d\widetilde{S} \end{aligned}$$

From this, one arrives at a conclusion that the kernel $G(S,t;\widetilde{S})$ of the above integral represents the Green's function to the problem in (1.2), (1.3) and (3.11). After a trivial algebra, $G(S,t;\widetilde{S})$ can be presented in the form

$$\begin{aligned} G(S,t;\widetilde{S}) = & \frac{1}{\widetilde{S}}(\frac{S}{\widetilde{S}})^\alpha \exp\big(- \beta\frac{\sigma^2}{2}(T-t)\big) \\ & \times \Big\{ \frac{1}{[2\pi\sigma^2(T-t)]^{1/2}} \Big[\exp\Big(- \frac{[\ln(S/\widetilde{S})]^2}{2\sigma^2(T-t)}\Big) + \exp\Big(- \frac{[\ln(S\widetilde{S}/D^2)]^2}{2\sigma^2(T-t)}\Big)\Big] \\ & - \Phi(\frac{S\widetilde{S}}{D^2})^{-\Phi} \exp(\Phi^2\sigma^2(T-t)/2) \\ & \times \operatorname{erfc}\Big(\frac{\Phi}{2}[2\sigma^2(T-t)]^{1/2} - \frac{\ln(S\widetilde{S}/D^2)}{[2\sigma^2(T-t)]^{1/2}}\Big)\Big\} \end{aligned} \tag{3.20}$$

Summarizing all the notations introduced at various stages of the present development, we can express the parameters α, β and Φ in (3.20) in terms of the original parameters σ^2 and r of the Black-Scholes equation and the parameters D and ϱ as

$$\alpha = \frac{\sigma^2/2 - r}{\sigma^2}, \quad \beta = \Big(\frac{r+\sigma^2/2}{\sigma^2}\Big)^2, \quad \Phi = D\varrho + \alpha \tag{3.21}$$

3.3. **Particular cases.** Note that the problem statement in (1.2), (1.3) and (3.11) allows two particular cases that might be of interest in option pricing valuations. One of such cases occurs when the parameter ϱ is set to equal zero transforming the boundary conditions in (3.11) into

$$|v(0,t)| < \infty, \quad \frac{\partial v(D,t)}{\partial S} = 0 \tag{3.22}$$

The Green's function for the problem in (1.2), (1.3) and (3.22),

$$\begin{aligned} G(S,t;\widetilde{S}) = & \frac{1}{\widetilde{S}}(\frac{S}{\widetilde{S}})^\alpha \exp\big(- \beta\frac{\sigma^2}{2}(T-t)\big) \\ & \times \Big\{ \frac{1}{[2\pi\sigma^2(T-t)]^{1/2}} \Big[\exp\Big(- \frac{[\ln(S/\widetilde{S})]^2}{2\sigma^2(T-t)}\Big) + \exp\Big(- \frac{[\ln(S\widetilde{S}/D^2)]^2}{2\sigma^2(T-t)}\Big)\Big] \\ & - \alpha(\frac{S\widetilde{S}}{D^2})^{-\alpha} \exp(\alpha^2\sigma^2(T-t)/2) \\ & \times \operatorname{erfc}\Big(\frac{\alpha}{2}[2\sigma^2(T-t)]^{1/2} - \frac{\ln(S\widetilde{S}/D^2)}{[2\sigma^2(T-t)]^{1/2}}\Big)\Big\} \end{aligned} \tag{3.23}$$

immediately arises from that of (3.20) when the parameter Φ is replaced with α. Indeed, as it follows from (3.21), if $\varrho = 0$, then $\Phi = \alpha$.

The second particular case of the problem statement in (1.2), (1.3) and (3.11) occurs when the parameter ϱ approaches infinity, which transforms the boundary conditions in (3.11) into

$$|v(0,t)| < \infty, \quad v(D,t) = 0\,. \tag{3.24}$$

It is difficult to directly obtain Green's function to the terminal-boundary value problem in (1.2), (1.3) and (3.24) from that of (3.20). The point is that taking a limit in the latter as ϱ approaches infinity is not a trivial exercise. That is why an alternative route is suggested. We revisit (3.17) for $U(x;s)$ in the development of the previous section and, observing that

$$\lim_{\varrho\to\infty}\frac{(\overline{\varrho}+\alpha)-\omega}{(\overline{\varrho}+\alpha)+\omega} = \lim_{\varrho\to\infty}\frac{(D\varrho+\alpha)-\omega}{(D\varrho+\alpha)+\omega} = 1$$

we rewrite (3.17) for the setting in (1.2), (1.3) and (3.24) as

$$\begin{aligned} U(x;s) &= \int_{-\infty}^{b}\frac{\exp\alpha(x-\xi)}{2\omega}[\exp(-\omega|x-\xi| - \exp\omega(x+\xi-2b)]f(\exp\xi)d\xi \\ &= \int_{-\infty}^{b}\frac{\exp\alpha(x-\xi)}{2}\Big[\frac{\exp(-(s+\beta)^{1/2}|x-\xi|)}{(s+\beta)^{1/2}} \\ &\quad - \frac{\exp((s+\beta)^{1/2}(x+\xi-2b))}{(s+\beta)^{1/2}}\Big]f(\exp\xi)d\xi \end{aligned}$$

Taking the inverse Laplace transform of the above expression, we obtain

$$\begin{aligned} u(x,\tau) = \int_{-\infty}^{b}&\Big[\exp\Big(-\frac{(x-\xi)^2}{4\tau}\Big) - \exp\Big(-\frac{(x+\xi-2b)^2}{4\tau}\Big)\Big] \\ &\times\frac{\exp\alpha(x-\xi)\exp(-\beta\tau)}{2(\pi\tau)^{1/2}}f(\exp\xi)d\xi \end{aligned}$$

allowing the solution to the problem in (1.2), (1.3) and (3.24) as

$$\begin{aligned} v(S,t) = \int_0^D &\frac{1}{\widetilde{S}\big[2\pi\sigma^2(T-t)\big]^{1/2}}\exp\Big(\alpha\ln(\frac{S}{\widetilde{S}}) - \beta\frac{\sigma^2}{2}(T-t)\Big) \\ &\times\Big[\exp\Big(-\frac{[\ln(S/\widetilde{S})]^2}{2\sigma^2(T-t)}\Big) - \exp\Big(-\frac{[\ln(S\widetilde{S}/D^2)]^2}{2\sigma^2(T-t)}\Big)\Big]f(\widetilde{S})d\widetilde{S} \end{aligned}$$

that can easily be simplified to

$$\begin{aligned} v(S,t) = \int_0^D &\frac{1}{\widetilde{S}}(\frac{S}{\widetilde{S}})^\alpha\frac{\exp(-\beta\sigma^2(T-t)/2)}{[2\pi\sigma^2(T-t)]^{1/2}} \\ &\times\Big[\exp\Big(-\frac{[\ln(S/\widetilde{S})]^2}{2\sigma^2(T-t)}\Big) - \exp\Big(-\frac{[\ln(S\widetilde{S}/D^2)]^2}{2\sigma^2(T-t)}\Big)\Big]f(\widetilde{S})d\widetilde{S} \end{aligned}$$

from which it follows that

$$\begin{aligned} G(S,t;\widetilde{S}\ ) = &\frac{1}{\widetilde{S}}(\frac{S}{\widetilde{S}})^\alpha\frac{\exp\big(-\beta\sigma^2(T-t)/2\big)}{[2\pi\sigma^2(T-t)]^{1/2}} \\ &\times\Big[\exp\Big(-\frac{[\ln(S/\widetilde{S})]^2}{2\sigma^2(T-t)}\Big) - \exp\Big(-\frac{[\ln(S\widetilde{S}/D^2)]^2}{2\sigma^2(T-t)}\Big)\Big] \end{aligned}$$

represents the closed form of the Green's function to the Black-Scholes equation satisfying the boundary conditions in (3.24). Note that the relations in (3.21) bring the expressions of the parameters α and β in terms of $\sigma^2$and r.

Conclusion. Compact analytic representations are derived for Green's functions for the Black-Scholes equation. These and other Green's functions, whose compact forms can be obtained by the approach suggested in the present study, are easily accessible for both theoretical analysis and numerical work in the field. They can readily be used in solving a variety of practical problem settings in financial engineering.

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