Electronic Journal of Differential Equations, Vol. 2018 (2018), No. 17, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

DECAY ESTIMATES FOR THE KLEIN-GORDON EQUATION IN CURVED SPACETIME

MUHAMMET YAZICI

ABSTRACT. We consider the initial-value problem for the Klein-Gordon equation in de Sitter spacetime. We derive L^∞ decay estimates for the solution to the linear Klein-Gordon equation in de Sitter spacetime with and without source term.

1. INTRODUCTION

In this article, we consider the following initial value problem for the Klein-Gordon equation in de Sitter spacetime,

$$\begin{gathered} \partial_t^2\Phi + n\Phi_t - e^{-2t}\Delta\Phi + m^2\Phi = f(x,t), \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ \Phi(x,0) = \varphi_0(x), \quad \partial_t\Phi(x,0) = \varphi_1(x), \quad x \in \mathbb{R}^n, \end{gathered} \tag{1.1}$$

where $f \in C^\infty(\mathbb{R}^{n+1})$, φ_0, φ_1 are in Sobolev space $W^{[n/2]+1,1}(\mathbb{R}^n)$, and $m > 0$.

In Minkowski spacetime, the initial value problem for the semilinear Klein-Gordon equation

$$u_{tt} - \Delta u + m^2 u = |u|^\alpha u,$$

has been extensively investigated. The existence of global weak solutions has been obtained by Jörgens [6], Pecher [8], Brenner [3], Ginibre and Velo [4, 5]. In order for the total energy is well-defined in the energy space, one needs the assumption $\alpha < 4/(n-1)$. On the other hand, the initial value problem for so-called Higgs boson equation

$$u_{tt} - \Delta u - m^2 u = -|u|^\alpha u, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}$$

in Minkowski spacetime, and

$$\partial_t^2\Phi + nH\Phi_t - e^{-2Ht}\Delta\Phi - m^2\Phi = -|\Phi|^\alpha\Phi, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}$$

in de Sitter spacetime are studied by Yagdjian [11], and some qualitative property of the solution revealed if the global solution exists. In addition, it was shown by Baskin [1] that the initial value problem for

$$\partial_t^2 u + n\partial_t u + \frac{\partial_t\sqrt{h_t}}{\sqrt{h_t}}\partial_t u + e^{-2t}\Delta_{h_t} u + \lambda u + |u|^\alpha u = 0, \quad (y,t) \in Y \times \mathbb{R}$$

2010 *Mathematics Subject Classification.* 35L15, 35C15, 35Q75.
Key words and phrases. De sitter spacetime; Klein-Gordon eqution; fundamental solutions, L^∞ estimates.

Submitted June 16, 2017. Published January 13, 2018.

admits a small amplitude global solution in the energy space $H^1 \oplus L^2$, provided $\lambda > n^2/4$ and $\alpha = 4/(n-1)$. Here h_t is a smooth family of Riemannian metrices on compact n-dimensional manifold Y, which is characterized as an asymptotically de Sitter spacetime. In Nakamura [7], the assumption on the regularity of the initial data is weakened in the case of $m \geq n/2$. Turning back to the initial value problem (1.1), the following theorem obtained by Yagdjian [12] states the estimate in the Sobolev space $H^s(\mathbb{R}^n)$.

Theorem 1.1 ([12]). *Let $\Phi = \Phi(x,t)$ be the solution of the initial value problem*

$$\Phi_{tt} + n\Phi_t + e^{-2t}\Delta\Phi + m^2\Phi = f, \quad \Phi(x,0) = \varphi_0(x), \quad \Phi_t(x,0) = \varphi_1(x)$$

for $(x,t) \in \mathbb{R}^n \times (0,\infty)$, where $\varphi_0, \varphi_1 \in C_0^\infty(\mathbb{R}^n)$ and $f \in C^\infty(\mathbb{R}^{n+1})$. Let l be a nonnegative integer, $m < \sqrt{n^2-1}/2$ and $n \geq 2$. Then there exists a constant $C > 0$ such that

$$\begin{aligned}
&\|(-\Delta)^{-s}\Phi(\cdot,t)\|_{W^{l,q}(\mathbb{R}^n)} \\
&\leq Ce^{(M-\frac{n}{2})t}(1-e^{-t})^{\left(2s-n\left(\frac{1}{p}-\frac{1}{q}\right)\right)} \left\{\|\varphi_0\|_{W^{l,p}(\mathbb{R}^n)} + (1-e^{-t})\|\varphi_1\|_{W^{l,p}(\mathbb{R}^n)}\right\} \\
&\quad + Ce^{-(\frac{n}{2}-M)t}\int_0^t e^{(\frac{n}{2}-M)b}e^{-b\left(2s-n\left(\frac{1}{p}-\frac{1}{q}\right)\right)}\|f(\cdot,b)\|_{W^{l,p}(\mathbb{R}^n)}db
\end{aligned} \tag{1.2}$$

for all $t > 0$, provided that $1 < p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$,

$$\frac{1}{2}(n+1)(\frac{1}{p} - \frac{1}{q}) \leq 2s \leq n(\frac{1}{p} - \frac{1}{q}) < 2s + 1.$$

Here we have set $M = \sqrt{\frac{n^2}{4} - m^2}$.

Moreover, Galstian and Yagdjian [13] showed similar estimates to the initial value problem for

$$\Phi_{tt} + n\Phi_t - e^{-2t}A(x,\partial_x)\Phi + m^2\Phi = f, \quad t > 0, \ x \in \mathbb{R}^n, \tag{1.3}$$

in the Besov space $B_p^{s,q}$, where $A(x,\partial_x) = \sum_{|\alpha|\leq 2} a_\alpha(x)\partial_x^\alpha$ is a second-order negative elliptic differential operator with real coefficients $a_\alpha \in \mathcal{B}^\infty$ and m in the set $(0, \sqrt{n^2-1}/2) \cup [n/2, \infty)$. Here, $\mathcal{B}^\infty$ denotes the space of all C^∞ functions with uniformly bounded derivatives of all orders. The case $m \in (\sqrt{n^2-1}/2, n/2)$ is also considered by Yagdjian [14] in the Besov space.

In this article, we are interested in the case of $0 < m < \sqrt{n^2-1}/2$. Decay estimate is an important tool to prove the global existence for nonlinear partial differential equations. The limiting case $q = \infty$ (i.e. $p = 1$) for the decay estimate is excluded in Theorem 1.1. We remark that the decay rate for the L^∞ decay estimate is faster than the decay rate for the L^2 decay estimate. Therefore, by using the L^∞ decay estimate, we prove the following theorem.

Theorem 1.2. *Let $\Phi = \Phi(x,t)$ be the solution of the initial value problem*

$$\Phi_{tt} + n\Phi_t + e^{-2t}\Delta\Phi + m^2\Phi = f, \quad \Phi(x,0) = \varphi_0(x), \quad \Phi_t(x,0) = \varphi_1(x)$$

for $(x,t) \in \mathbb{R}^n \times (0,\infty)$, where $\varphi_0, \varphi_1 \in C_0^\infty(\mathbb{R}^n)$ and $f \in C^\infty(\mathbb{R}^{n+1})$. Let l be a nonnegative integer, $m < \sqrt{n^2-1}/2$ and $n \geq 2$. Then there exists a constant

$C>0$ such that

$$\begin{aligned}\|\Phi(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} &\le Ce^{(M-\frac{n}{2})t}\left\{\|\varphi_0\|_{W^{[n/2]+1,1}(\mathbb{R}^n)}+\|\varphi_1\|_{W^{[n/2]+1,1}(\mathbb{R}^n)}\right\}\\ &\quad + Ce^{-(\frac{n}{2}-M)t}\int_0^t e^{(\frac{n}{2}-M)b}\|f(\cdot,b)\|_{W^{[n/2]+1,1}(\mathbb{R}^n)}db,\end{aligned} \tag{1.4}$$

for all $t>0$. Here we have set $M=\sqrt{\frac{n^2}{4}-m^2}$.

Here, $W^{k,p}(\mathbb{R}^n)=\{u\in L^p(\mathbb{R}^n): D^\alpha u\in L^p(\mathbb{R}^n),\ |\alpha|\le k\}$, denotes a Sobolev space with the norm

$$\|u\|_{W^{k,p}(\mathbb{R}^n)}=\Big(\sum_{|\alpha|\le k}\int_{\mathbb{R}^n}|D^\alpha u|^p\Big)^{1/p},\quad (1\le p<\infty),$$
$$\|u\|_{W^{k,\infty}(\mathbb{R}^n)}=\sum_{|\alpha|\le k}\operatorname{ess\,sup}_{\mathbb{R}^n}|D^\alpha u|.$$

2. Preliminaries

Throughout this article, the positive constants which may change, are denoted by the same letters C. We prepare some inequalities for proving Theorem 1.2. First of all, we introduce the hypergeometric function $F(a,b;c;\zeta)$ and study its property. It is defined by the power series

$$F(a,b;c;\zeta)=\sum_{n=0}^\infty\frac{(a)_n(b)_n}{(c)_n}\frac{\zeta^n}{n!},\quad |\zeta|<1,$$

where $a,b,c\in\mathbb{C}$ with $c\ne 0,-1,-2,\dots$, and we denote

$$(a)_0=1,$$
$$(a)_n=\Gamma(a+n)/\Gamma(a)=a(a+1)\dots(a+n-1),\quad n=1,2,3,\dots.$$

Here Γ is the gamma function (see, e.g. [2]).

We remark that there exists a constant $C>0$ such that

$$|F(a,b;c;\zeta)|\le C \tag{2.1}$$

for all $\zeta\in[0,1]$ if $\operatorname{Re}(c-b-a)>0$ for $a,b,c\in\mathbb{C}$ with $c\ne 0,-1,-2,\dots$ (see e.g. [9] and references therein).

3. Fundamental solutions of the linear Klein-Gordon equation

We separate the initial value problem (1.1) into two parts. First, we consider the Klein-Gordon equation without source term:

$$\begin{gathered}\partial_t^2\Phi+n\Phi_t-e^{-2t}\Delta\Phi+m^2\Phi=0,\quad (x,t)\in\mathbb{R}^n\times\mathbb{R},\\ \Phi(x,0)=\varphi_0(x),\quad \partial_t\Phi(x,0)=\varphi_1(x),\quad x\in\mathbb{R}^n,\end{gathered} \tag{3.1}$$

where $\Phi(x,0)=\varphi_0,\Phi(x,0)=\varphi_1\in C_0(\mathbb{R}^n)$. Next, we consider the Klein-Gordon equation with source term,

$$\begin{gathered}\partial_t^2\Phi+n\Phi_t-e^{-2t}\Delta\Phi+m^2\Phi=f(x,t),\quad (x,t)\in\mathbb{R}^n\times\mathbb{R},\\ \Phi(x,0)=0,\quad \partial_t\Phi(x,0)=0,\end{gathered} \tag{3.2}$$

where $f \in C^\infty(\mathbb{R}^{n+1})$. For $(x_0,t_0) \in \mathbb{R}^{n+1}$, the forward and backward light cones are defined as

$$D_+(x_0,t_0) := \left\{(x,t) \in \mathbb{R}^{n+1} : t \ge t_0,\ |x-x_0| \le e^{-t_0} - e^{-t}\right\},$$
$$D_-(x_0,t_0) := \left\{(x,t) \in \mathbb{R}^{n+1} : t \le t_0,\ |x-x_0| \le e^{-t} - e^{-t_0}\right\}.$$

The function introduced by Yagdjian [9], [12] is

$$\begin{aligned} E(x,t;x_0,t_0;M) &:= (4e^{-t_0-t})^{-M}\left((e^{-t}+e^{-t_0})^2 - |x-x_0|^2\right)^{-\frac{1}{2}+M} \\ &\quad \times F\left(\frac{1}{2}-M, \frac{1}{2}-M; 1; \frac{(e^{-t_0}-e^{-t})^2 - |x-x_0|^2}{(e^{-t_0}+e^{-t})^2 - |x-x_0|^2}\right), \end{aligned}$$

for $(x,t) \in D_+(x_0,t_0) \cup D_-(x_0,t_0)$, where $M = \sqrt{\frac{n^2}{4} - m^2}$ and $(x-x_0)^2 = (x-x_0).(x-x_0)$ for $x, x_0 \in \mathbb{R}^n$. The kernels $K_0(z,t;M)$ and $K_1(z,t;M)$ are given by Yagdjian [9], [12] as follows

$$\begin{aligned} &K_0(z,t;M) \\ &:= -\Big[\frac{\partial}{\partial b} E(z,t;0,b;M)\Big]_{b=0} \\ &= (4e^{-t})^{-M}\left((1+e^{-t})^2 - z^2\right)^{M-\frac{1}{2}}\left((1-e^{-t})^2 - z^2\right)^{-1}\Big[\big(e^{-t}-1 \\ &\quad + M(e^{-2t}-1-z^2)\big)F\Big(\frac{1}{2}-M, \frac{1}{2}-M; 1; \frac{(1-e^{-t})^2 - z^2}{(1+e^{-t})^2 - z^2}\Big) \\ &\quad + (1-e^{-2t}+z^2)\Big(\frac{1}{2}+M\Big)F\Big(-\frac{1}{2}-M, \frac{1}{2}-M; 1; \frac{(1-e^{-t})^2 - z^2}{(1+e^{-t})^2 - z^2}\Big)\Big] \end{aligned}$$

and

$$\begin{aligned} &K_1(z,t;M) \\ &:= E(z,t;0,0;M) \\ &= (4e^{-t})^{-M}\left((1+e^{-t})^2 - z^2\right)^{-\frac{1}{2}+M} F\Big(\frac{1}{2}-M, \frac{1}{2}-M; 1; \frac{(1-e^{-t})^2 - z^2}{(1+e^{-t})^2 - z^2}\Big), \end{aligned}$$

where $0 \le z \le 1 - e^{-t}$. The solution $\Phi = \Phi(x,t)$ of the initial value problem

$$\Phi_{tt} + n\Phi_t - e^{-2t}\Delta\Phi + m^2\Phi = 0, \quad \Phi(x,0) = \varphi_0(x), \quad \Phi_t(x,0) = \varphi_1(x), \tag{3.3}$$

with $\varphi_0,\ \varphi_1 \in C_0^\infty(\mathbb{R}^n)$ is given by Yagdjian-Galstian [9, 10] as follows

$$\begin{aligned} &\Phi(x,t) \\ &= e^{-\frac{n-1}{2}t} v_{\varphi_0}(x,\phi(t)) \\ &\quad + e^{-nt/2}\int_0^1 v_{\varphi_0}(x,\phi(t)s)\left(2K_0(\phi(t)s,t;M) + nK_1(\phi(t)s,t;M)\right)\phi(t)ds \\ &\quad + e^{-nt/2}\int_0^1 v_{\varphi_1}(x,\phi(t)s)(2K_1(\phi(t)s,t;M))\phi(t)ds, \end{aligned} \tag{3.4}$$

where $\phi(t) := 1 - e^{-t}$ with $t > 0$. Here, for $\varphi \in C_0^\infty(\mathbb{R}^n)$, $v_\varphi(x,t)$ denotes the solution of

$$v_{tt} - \Delta v = 0, \quad v(x,0) = \varphi(x), \quad v_t(x,0) = 0, \quad (x,t) \in \mathbb{R}^n \times (0,\infty). \tag{3.5}$$

Moreover, the solution $\Phi = \Phi(x,t)$ of the initial value problem

$$\Phi_{tt} + n\Phi_t - e^{-2t}\Delta\Phi + m^2\Phi = f, \quad \Phi(x,0) = 0, \quad \Phi_t(x,0) = 0, \tag{3.6}$$

with $f \in C^\infty(\mathbb{R}^{n+1})$ is given by Yagdjian-Galstian [9, 10] as follows

$$\Phi(x,t) = 2e^{-nt/2} \int_0^t db \int_0^{e^{-b}-e^{-t}} dr e^{\frac{n}{2}b} v(x,r;b) E(r,t;0,b;M), \tag{3.7}$$

where $v(x,t;b)$ is the solution to the following initial value problem for the wave equation

$$v_{tt} - \Delta v = 0, \quad v(x,0;b) = f(x,b), \quad v_t(x,0;b) = 0, \quad (x,t) \in \mathbb{R}^n \times (0,\infty), \tag{3.8}$$

where $b > 0$.

4. Proof of Theorem 1.2

We derive L^∞ estimates for the linear Klein-Gordon equation in de Sitter spacetime. We apply the following two lemmas to prove the theorem.

Lemma 4.1. *Let $M > 1/2$ and $\phi(t) = 1 - e^{-t}$. Then*

$$\int_0^1 (1+\phi(t)s)^{-\frac{n-1}{2}} |K_1(\phi(t)s,t;M)| \phi(t) ds \le C_M e^{Mt} \tag{4.1}$$

for all $t > 0$.

Proof. Changing the variable by $1+\phi(t)s = r$ and using the definition of the kernel K_1, we obtain

$$\begin{aligned}
&\int_0^1 (1+\phi(t)s)^{-\frac{n-1}{2}} |K_1(\phi(t)s,t;M)| \phi(t) ds \\
&= 4^{-M} e^{Mt} \int_1^{2-e^{-t}} r^{-\frac{n-1}{2}} ((1+e^{-t})^2 - (r-1)^2)^{-\frac{1}{2}+M} \\
&\quad \times |F\left(\frac{1}{2}-M, \frac{1}{2}-M; 1; \frac{(1-e^{-t})^2 - (r-1)^2}{(1+e^{-t})^2 - (r-1)^2}\right)| dr \\
&\le Ce^{-Mt} \int_0^{e^t-1} ((e^t+1)^2 - y^2)^{-\frac{1}{2}+M} \Big| F\Big(\frac{1}{2}-M, \frac{1}{2}-M; 1; \frac{(e^t-1)^2-y^2}{(e^t+1)^2-y^2}\Big)\Big| dy,
\end{aligned}$$

where we have changed the variable by $e^t(r-1) = y$ in the last inequality. Since $M > 1/2$, by (2.1) the hypergeometric function in the last inequality is bounded, and hence

$$\begin{aligned}
\int_0^1 (1+\phi(t)s)^{-\frac{n-1}{2}} |K_1(\phi(t)s,t;M)| \phi(t) ds &\le Ce^{-Mt} \int_0^{e^t-1} ((e^t+1)^2 - y^2)^{-\frac{1}{2}+M} dy \\
&\le C_M e^{-Mt} (e^t+1)^{2M-1} (e^t-1),
\end{aligned}$$

which leads to (4.1). $\square$

Lemma 4.2. *Let $M > 1/2$ and $\phi(t) = 1 - e^{-t}$. Then*

$$\int_0^1 (1+\phi(t)s)^{-\frac{n-1}{2}} |K_0(\phi(t)s,t;M)| \phi(t) ds \le C_M e^{Mt} \tag{4.2}$$

for all $t > 0$.

Proof. Similarly to the proof of Lemma 4.1, we obtain

$$\begin{aligned}
&\int_0^1 (1+\phi(t)s)^{-\frac{n-1}{2}} |K_0(\phi(t)s,t;M)|\phi(t)ds \\
&\le C_M e^{-Mt} \int_0^{e^t-1} ((e^t+1)^2-y^2)^{M-\frac12} \left((e^t-1)^2-y^2\right)^{-1} \\
&\quad \times \Big|\Big[(e^t-e^{2t}+M(1-e^{2t}-y^2))F\Big(\frac12-M,\frac12-M;1;\frac{(e^t-1)^2-y^2}{(e^t+1)^2-y^2}\Big) \\
&\quad + (e^{2t}-1+y^2)(\frac12+M)F\Big(-\frac12-M,\frac12-M;1;\frac{(e^t-1)^2-y^2}{(e^t+1)^2-y^2}\Big)\Big]\Big|dy.
\end{aligned}$$

From [12], we have

$$\begin{aligned}
&\int_0^{z-1} ((z+1)^2-y^2)^{M-\frac12} \left((z-1)^2-y^2\right)^{-1} \\
&\quad \times \Big|\Big[(z-z^2+M(1-z^2-y^2))F(\frac12-M,\frac12-M;1;\frac{(z-1)^2-y^2}{(z+1)^2-y^2}) \\
&\quad + (z^2-1+y^2)(\frac12+M)F(-\frac12-M,\frac12-M;1;\frac{(z-1)^2-y^2}{(z+1)^2-y^2})\Big]\Big|dy \\
&\le C_M(z+1)^{2M}
\end{aligned} \tag{4.3}$$

for all $z\in[1,\infty)$. Hence (4.3) leads to (4.2). This completes the proof. □

Proof of Theorem 1.2. First we consider the solution of the initial value problem (3.1). In the case of $\varphi_1=0$, from (3.4), we have

$$\begin{aligned}
\Phi(x,t) &= e^{-\frac{n-1}{2}t} v_{\varphi_0}(x,\phi(t)) + e^{-nt/2}\int_0^1 v_{\varphi_0}(x,\phi(t)s)(2K_0(\phi(t)s,t;M) \\
&\quad + nK_1(\phi(t)s,t;M))\phi(t)ds.
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
&\|\Phi(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \\
&\le e^{-\frac{n-1}{2}t}\|v_{\varphi_0}(\cdot,\phi(t))\|_{L^\infty(\mathbb{R}^n)} + e^{-nt/2}\int_0^1 \|v_{\varphi_0}(\cdot,\phi(t)s)\|_{L^\infty(\mathbb{R}^n)} \\
&\quad \times |(2K_0(\phi(t)s,t;M)+nK_1(\phi(t)s,t;M))|\phi(t)ds.
\end{aligned} \tag{4.4}$$

As is well known, the solution $v(x,t)$ of the initial value problem (3.5) satisfies

$$\|v(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \le C(1+t)^{-\frac{n-1}{2}}\|\varphi\|_{W^{[n/2]+1,1}(\mathbb{R}^n)} \tag{4.5}$$

for $t\ge 0$, if $n\ge 2$ (see e.g. [15]). For all $t\ge 0$, we have

$$\begin{aligned}
e^{-\frac{n-1}{2}t}\|v_{\varphi_0}(\cdot,\phi(t)\|_{L^\infty(\mathbb{R}^n)} &\le Ce^{-\frac{n-1}{2}t}(1+\phi(t))^{-\frac{n-1}{2}}\|\varphi_0\|_{W^{[n/2]+1,1}(\mathbb{R}^n)} \\
&\le Ce^{-\frac{n-1}{2}t}\|\varphi_0\|_{W^{[n/2]+1,1}(\mathbb{R}^n)}.
\end{aligned}$$

Hence, we obtain

$$e^{-\frac{n-1}{2}t}\|v_{\varphi_0}(\cdot,\phi(t)\|_{L^\infty(\mathbb{R}^n)} \le Ce^{-\frac{n-1}{2}t}\|\varphi_0\|_{W^{[n/2]+1,1}(\mathbb{R}^n)}. \tag{4.6}$$

On the other hand, we obtain

$$
\begin{aligned}
& e^{-nt/2}\int_0^1 \|v_{\varphi_0}(\cdot,\phi(t)s)\|_{L^\infty(\mathbb{R}^n)}|(2K_0(\phi(t)s,t;M)+nK_1(\phi(t)s,t;M))|\phi(t)ds \\
& \le C\|\varphi_0\|_{W^{[n/2]+1,1}(\mathbb{R}^n)}e^{-nt/2}\int_0^1 (1+\phi(t)s)^{-\frac{n-1}{2}}|(2K_0(\phi(t)s,t;M) \\
& \quad + nK_1(\phi(t)s,t;M))|\phi(t)ds.
\end{aligned}
\tag{4.7}
$$

From Lemma 4.1 and Lemma 4.2, we have

$$
e^{-nt/2}\int_0^1 (1+\phi(t)s)^{-\frac{n-1}{2}}|2K_0(\phi(t)s,t;M)|\phi(t)ds \le Ce^{(M-\frac{n}{2})t}, \tag{4.8}
$$

$$
e^{-nt/2}\int_0^1 (1+\phi(t)s)^{-\frac{n-1}{2}}|nK_1(\phi(t)s,t;M)|\phi(t)ds \le Ce^{(M-\frac{n}{2})t}. \tag{4.9}
$$

Hence, from (4.6), (4.8) and (4.9) we obtain

$$
\|\Phi(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \le Ce^{(M-\frac{n}{2})t}\|\varphi_0\|_{W^{[n/2]+1,1}(\mathbb{R}^n)} \tag{4.10}
$$

when $\varphi_1=0$. For the case $\varphi_0=0$, we have

$$
\|\Phi(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \le Ce^{(M-\frac{n}{2})t}\|\varphi_1\|_{W^{[n/2]+1,1}(\mathbb{R}^n)} \tag{4.11}
$$

in a similar way.

Next, we consider the solution of the initial value problem (3.2). From (3.7) and the definition of $E(x,t;x_0,t_0;M)$ we have

$$
\begin{aligned}
& \Phi(x,t) \\
& = 2e^{-nt/2}\int_0^t db\int_0^{e^{-b}-e^{-t}} dre^{\frac{n}{2}b}v(x,r;b)4^{-M}e^{M(b+t)}((e^{-t}+e^{-b})^2-r^2)^{-\frac{1}{2}+M} \\
& \quad \times F(\frac{1}{2}-M,\frac{1}{2}-M;1;\frac{(e^{-b}-e^{-t})^2-r^2}{(e^{-b}+e^{-t})^2-r^2})dr,
\end{aligned}
$$

where v is the solution of (3.8). From (4.5), we obtain

$$
\|v(\cdot,r;b)\|_{L^\infty(\mathbb{R}^n)} \le C(1+r)^{-\frac{n-1}{2}}\|f(\cdot,b)\|_{W^{[n/2]+1,1}(\mathbb{R}^n)}
$$

for all $r>0$. Hence,

$$
\begin{aligned}
\|\Phi(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} & \le C_Me^{-nt/2}e^{Mt}\int_0^t e^{\frac{n}{2}b}e^{Mb}\|f(\cdot,b)\|_{W^{[n/2]+1,1}(\mathbb{R}^n)}db \\
& \quad \times \int_0^{e^{-b}-e^{-t}} (1+r)^{-\frac{n-1}{2}}((e^{-t}+e^{-b})^2-r^2)^{-\frac{1}{2}+M} \\
& \quad \times |F(\frac{1}{2}-M,\frac{1}{2}-M;1;\frac{(e^{-b}-e^{-t})^2-r^2}{(e^{-b}+e^{-t})^2-r^2})|dr \\
& \le C_Me^{-nt/2}e^{Mt}\int_0^t e^{\frac{n}{2}b}e^{Mb}\|f(\cdot,b)\|_{W^{[n/2]+1,1}(\mathbb{R}^n)}db \\
& \quad \times \int_0^{e^{-b}-e^{-t}} ((e^{-t}+e^{-b})^2-r^2)^{-\frac{1}{2}+M} \\
& \quad \times |F(\frac{1}{2}-M,\frac{1}{2}-M;1;\frac{(e^{-b}-e^{-t})^2-r^2}{(e^{-b}+e^{-t})^2-r^2})|dr.
\end{aligned}
$$

If we change the variable by $r = e^{-t}y$, then we obtain

$$\begin{aligned}\|\Phi(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \le C_M e^{-nt/2} e^{-Mt} \int_0^t e^{\frac{n}{2}b} e^{Mb} \|f(\cdot,b)\|_{W^{[n/2]+1,1}(\mathbb{R}^n)} db \\ \times \int_0^{e^{t-b}-1} \big((e^{t-b}+1)^2 - y^2\big)^{-\frac12+M} \\ \times |F(\frac12 - M, \frac12 - M; 1; \frac{(e^{t-b}-1)^2-y^2}{(e^{t-b}+1)^2-y^2})| dy.\end{aligned} \tag{4.12}$$

Since $M > 1/2$, by (2.1), we have the following estimate for the second integral of (4.12),

$$\begin{aligned}&\int_0^{e^{t-b}-1} \big((e^{t-b}+1)^2 - y^2\big)^{-\frac12+M} \Big|F(\frac12 - M, \frac12 - M; 1; \frac{(e^{t-b}-1)^2-y^2}{(e^{t-b}+1)^2-y^2})\Big| dy \\ &\le C_M \int_0^{e^{t-b}-1} \big((e^{t-b}+1)^2 - y^2\big)^{-\frac12+M} dy \\ &\le C_M \big(e^{t-b}+1\big)^{2M-1} \big(e^{t-b}-1\big) \\ &\le C_M \big(e^{t-b}+1\big)^{2M} \\ &\le C_M e^{2M(t-b)},\end{aligned}$$

for $b < t$. Thus, we have

$$\|\Phi(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \le C_M e^{-(\frac{n}{2}-M)t} \int_0^t e^{(\frac{n}{2}-M)b} \|f(\cdot,b)\|_{W^{[n/2]+1,1}(\mathbb{R}^n)} db. \tag{4.13}$$

Hence (4.10), (4.11) and (4.13) lead to (1.4). This completes the proof. $\square$

Acknowledgments. The author would like to express sincerely thanks to the ananymous referee for several comments to revise the paper.

References

[1] D. Baskin; Strichartz Estimates on Asymptotically de Sitter Spaces, *Annales Henri Poincarè*, **14** (2013), Issue 2, 221–252.

[2] H. Bateman, A. Erdelyi; *Higher Transcendental Functions*, **1,2**, McGraw-Hill, New York, 1953.

[3] P. Brenner; On the existence of global smooth solutions of certain semilinear hyperbolic equations, *Math. Z.*, **167 (2)** (1979), 99–135.

[4] J. Ginibre, G. Velo; The global Cauchy problem for the nonlinear Klein-Gordon equation, *Math Z.*, **189(4)** (1985), 487–505.

[5] J. Ginibre, G. Velo,; The global Cauchy problem for the nonlinear Klein-Gordon equation II, *Ann. Inst. H. Poincarè Anal. Linèaire*, **6(1)** (1989), 15–35.

[6] K. Jörgens; Das Anfangswertproblem im Grossen fur eine Klasse nichtlinearer Wellengleichungen, *Math. Z.*, **77** (1961), 295–308.

[7] M. Nakamura; The Cauchy problem for semi-linear Klein-Gordon equations in de Sitter spacetime, *J. Math. Anal. Appl.*, **410** (2014), 445–454.

[8] H. Pecher; L^p-Abschützungen und klassische Lösungen für nichtlineare Wellengleichungen. I, *Math. Z.*, **150** (1976), 159–183.

[9] K. Yagdjian, A. Galstian; Fundamental solutions for the KleinGordon equation in de Sitter spacetime, *Comm. Math. Phys.*, **285** (2009), 293–344.

[10] K. Yagdjian; The semilinear Klein-Gordon equation in de Sitter spacetime, *Discrete Contin. Dyn. Syst. Ser. S* **2** (3) (2009), 679–696.

[11] K. Yagdjian; On the global solutions of the Higgs boson equation, *Comm. Partial Differential Equations*, **37** (2012), 447–478.

[12] K. Yagdjian; Global existence of the scalar field in de Sitter spacetime, *J. Math. Anal. Appl.*, **396** (2012), 323–344.
[13] A. Galstian, K. Yagdjian; Global in time existence of self-interacting scalar field in de Sitter spacetimes, *Nonlinear Anal. Real World Appl.*, **34** (2017), 110–139.
[14] K. Yagdjian; Global existence of the self-interacting scalar field in the de Sitter universe, 2017. arXiv: 1706.07703v2.
[15] W. von Wahl; L^p-decay rates for homogeneous wave-equations, *Math. Z.*, **120** (1971), 93–106.

Muhammet Yazici
Department of Mathematics, Faculty of Sciences, Karadeniz Technical University, Trabzon, 61080, Turkey
E-mail address: `m.yazici@ktu.edu.tr`