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A viability result for second-order differential inclusions *

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Abstract

We prove a viability result for the second-order differential inclusion

$$x'' \in F(x,x'), \quad (x(0),x'(0)) = (x_0,y_0) \in Q := K \times \Omega,$$

where K is a closed and Ω is an open subsets of $\mathbb{R}^m$, and is an upper semicontinuous set-valued map with compact values, such that $F(x,y) \subset \partial V(y)$, for some convex proper lower semicontinuous function V.

1 Introduction

Bressan, Cellina and Colombo [6] proved the existence of local solutions to the Cauchy problem

$$x' \in F(x), \quad x(0) = \xi \in K,$$

where F is an upper semicontinuous, cyclically monotone, and compact valued multifunction. While Rossi [15] proved a viability result for this problem. On the other hand, for the second order differential inclusion

$$x'' \in F(x,x'), \quad x(0) = x_0, \quad x'(0) = y_0,$$

existence results were obtained by many authors [1, 4, 9, 10, 13, 16]). In [12], existence results are proven for the case when $F(.,.)$ is an upper semicontinuous set-valued map with compact values, such that $F(x,y) \subset \partial V(y)$ for some convex proper lower semicontinuous function V.

The aim of this paper is to prove a viability result for the second-order differential inclusion

$$x'' \in F(x,x'), \quad (x(0),x'(0)) = (x_0,y_0) \in Q := K \times \Omega,$$

where K is a closed and Ω is an open subsets of $\mathbb{R}^m$, and $F : Q \subset \mathbb{R}^{2m} \to 2^{\mathbb{R}^m}$ is an upper semicontinuous set-valued map with compact values, such that $F(x,y) \subset \partial V(y)$, for some convex proper lower semicontinuous function V.

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2 Preliminaries and statement of main result

Let $\mathbb{R}^m$ be the m-dimensional Euclidean space with scalar product $\langle .,. \rangle$ and norm $\|.\|$. For $x \in \mathbb{R}^m$ and $\varepsilon > 0$ let

$$B_\varepsilon(x) = \{y \in \mathbb{R}^m : \|x - y\| < \varepsilon\}$$

be the open ball, centered at x with radius ε, and let $\overline{B}_\varepsilon(x)$ be its closure. Denote by B the open unit ball $B = \{x \in \mathbb{R}^m : \|x\| < 1\}$.

For $x \in \mathbb{R}^m$ and for a closed subsets $A \subset \mathbb{R}^m$ we denote by $d(x, A)$ the distance from x to A given by

$$d(x, A) = \inf\{\|x - y\| : y \in A\}.$$

Let $V : \mathbb{R}^m \to \mathbb{R}$ be a proper lower semicontinuous convex function. The multifunction $\partial V : \mathbb{R}^m \to 2^{\mathbb{R}^m}$ defined by

$$\partial V(x) = \{\xi \in \mathbb{R}^m : V(y) - V(x) \geqslant \langle \xi, y - x\rangle, \ \forall y \in \mathbb{R}^m\}$$

is called subdifferential (in the sense of convex analysis) of the function V.

We say that a multifunction $F : \mathbb{R}^m \to 2^{\mathbb{R}^m}$ is upper semicontinuous if for every $x \in \mathbb{R}^m$ and every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$F(y) \subset F(x) + B_\varepsilon(0), \quad \forall y \in B_\delta(x).$$

This definition of the upper semicontinuous multifunction is less restrictive than the usual (see Definition 1.1.1 in [3] or Definition 1.1 in [11]). Actually such a property is called (ε, δ)-upper semicontinuity (see Definition 1.2 in [11]) and it is only equivalent to the upper semicontinuity for compact-valued multifunctions (see Proposition 1.1 in [11]).

For $K \subset \mathbb{R}^m$ and $x \in K$ denote by $T_K(x)$ the Bouligand's contingent cone of K at x, defined by

$$T_K(x) = \big\{v \in \mathbb{R}^m : \liminf_{h \to 0+} \frac{d(x + hv, K)}{h} = 0\big\}.$$

For $K \subset \mathbb{R}^m$ and $(x, y) \in K \times \mathbb{R}^m$ we denote by $T_K^{(2)}(x, y)$ the second-order contingent set of K at (x, y) introduced by Ben-Tal [5] and defined by

$$T_K^{(2)}(x, y) = \big\{v \in \mathbb{R}^m : \liminf_{h \to 0+} \frac{d(x + hy + \frac{h^2}{2}v, K)}{h^2/2} = 0\big\}.$$

We remark that if $T_K^{(2)}(x, y)$ is non-empty then, necessarily, $y \in T_K(x)$.

Moreover (see [4], [10], [13]), if F is upper semicontinuous with compact convex values and if $x : [0, T] \to \mathbb{R}^m$ is a solution of the Cauchy problem

$$x'' \in F(x, x'), \quad x(0) = x_0, \quad x'(0) = y_0,$$

such that $x(t) \in K$, $\forall t \in [0,T]$, then

$$(x(t), x'(t)) \in \operatorname{graph}(T_K), \quad \forall t \in [0,T),$$

hence, in particular, $(x_0, y_0) \in \operatorname{graph}(T_K)$.

For a multifunction $F : Q := K \times \Omega \subset \mathbb{R}^{2m} \to 2^{\mathbb{R}^m}$ and for any $(x_0, y_0) \in \operatorname{graph}(T_K)$ we consider the Cauchy problem

$$x'' \in F(x, x'), \quad (x(0), x'(0)) = (x_0, y_0) \in Q \tag{2.1}$$

under the following assumptions:

(H1) K is a closed and Ω and open subset of $\mathbb{R}^m$, such that

$$Q := K \times \Omega \subset \operatorname{graph}(T_K)$$

(H2) F is an upper semicontinuous compact valued multifunction such that

$$F(x, y) \cap T_K^{(2)}(x, y) \neq \varnothing, \quad \forall (x, y) \in Q;$$

(H3) There exists a proper convex and lower semicontinuous function $V : \mathbb{R}^m \to \mathbb{R}$ such that

$$F(x, y) \subset \partial V(y), \quad \forall (x, y) \in Q.$$

Remark. A convex function $V : \mathbb{R}^m \to \mathbb{R}$ is continuous in the whole space $\mathbb{R}^m$ (Corollary 10.1.1 in [14]) and almost everywhere differentiable (Theorem 25.5 in [14]). Therefore, (H3) strongly restricts the multivaluedness of F.

Definition. By viable solution of the problem (2.1) we mean any absolutely continuous function $x : [0,T] \to \mathbb{R}^m$ with absolutely continuous derivative x' such that $x(0) = x_0$, $x(0) = y_0$,

$$\begin{gathered} x''(t) \in F(x(t), x'(t)) \quad a.e. \text{ on } [0,T], \\ (x(t), x'(t)) \in Q \quad \forall t \in [0,T]. \end{gathered}$$

Our main result is the following:

Theorem 2.1 *If $F : Q \subset \mathbb{R}^{2m} \to 2^{\mathbb{R}^m}$ and $V : \mathbb{R}^m \to \mathbb{R}$ satisfy assumptions (H1)–(H3), then then for every $(x_0, y_0) \in Q$ there exist $T > 0$ and $x : [0,T] \to \mathbb{R}^m$, a viable solution of the problem* (2.1)*.*

3 Proof of the main result

We start this section with the following technical result, which will be used to prove the main result.

Lemma 3.1 *Assume $Q = K \times \Omega \subset \mathbb{R}^{2m}$ satisfies (H1), $F : Q \to 2^{\mathbb{R}^m}$ satisfies (H2), $Q_0 \subset Q$ is a compact subset and $(x_0, y_0) \in Q_0$. Then for every $k \in \mathbb{N}^*$ there exist $h_k^0 \in (0, \frac{1}{k}]$ and $u_k^0 \in \mathbb{R}^m$ such that*

$$x_0 + h_k^0 y_0 + \frac{(h_k^0)^2}{2} u_k^0 \in K, (x_0, y_0, u_k^0) \in \operatorname{graph}(F) + \frac{1}{k}(B \times B \times B).$$

Proof. Let $(x,y)\in Q$ be fixed. Since by (H2), $F(x,y)\cap T_K^{(2)}(x,y)\neq\emptyset$, there exists $v=v_{(x,y)}\in F(x,y)$ such that
$$\liminf_{h\to 0_+}\frac{d(x+hy+\frac{h^2}{2}v,K)}{h^2/2}=0.$$
Hence, for every $k\in\mathbb{N}^*$ there exists $h_k=h_k(x,y)\in(0,\frac{1}{k}]$ such that
$$d(x+h_ky+\frac{h_k^2}{2}v,K)<\frac{h_k^2}{4k}. \tag{3.1}$$
By the continuity of the map $(a,b)\to d(a+h_kb+\frac{h_k^2}{2}v,K)$ it follows that
$$N(x,y)=\big\{(a,b):d(a+h_kb+\frac{h_k^2}{2}v,K)<\frac{h_k^2}{4k}\big\}$$
is an open set and, by (3.1), it contains (x,y). Then there exists $r:=r(x,y)\in(0,\frac{1}{k})$ such that $B_r(x,y)\subset N(x,y)$. Since Q_0 is compact there exists a finite subset $\{(x_j,y_j)\in Q:1\leqslant j\leqslant m\}$ such that
$$Q_0\subset\bigcup_{j=1}^m B_{r_j}(x_j,y_j).$$
We set
$$h_0(k):=\min\{h_k(x_j,y_j):j\in\{1,\dots,m\}\}.$$
Since $(x_0,y_0)\in Q_0$, there exists $j_0\in\{1,2,\dots m\}$ such that
$$(x_0,y_0)\in B_{r_{j_0}}(x_{j_0},y_{j_0})\subset N(x_{j_0},y_{j_0}). \tag{3.2}$$
Denote by $h_k^0:=h_k(x_{j_0},y_{j_0})$ and remark that, by (3.1) and (3.2), one has $h_k^0\in[h_0(k),\frac{1}{k}]$ and there exists $z_0\in K$ such that we have that
$$\frac{d(x_0+h_k^0y_0+\frac{(h_k^0)^2}{2}v_0,z_0)}{(h_k^0)^2/2}\leqslant\frac{d(x_0+h_ky_0+\frac{(h_k^0)^2}{2}v_0,K)}{(h_k^0)^2/2}+\frac{1}{2k}<\frac{1}{k},$$
hence
$$\|\frac{z_0-x_0-h_k^0y_0}{(h_k^0)^2/2}-v_0\|<\frac{1}{k}. \tag{3.3}$$
Let
$$u_k^0:=\frac{z_0-x_0-h_k^0y_0}{(h_k^0)^2/2}.$$
Then
$$x_0+h_k^0y_0+\frac{(h_k^0)^2}{2}u_k^0\in K.$$
By (3.3) and (3.2) we get successively:
$$\|u_k-v_0\|<\frac{1}{k},$$
$$d((x_0,y_0),(x_{j_0},y_{j_0}))\leqslant r_{j_0}<\frac{1}{k},$$
hence $(x_0,y_0,u_k^0)\in\operatorname{graph}(F)+\frac{1}{k}(B\times B\times B)$. $\square$

Proof of Theorem 2.1 Let $(x_0, y_0) \in Q \subset \operatorname{graph}(T_K)$. Since $\Omega \subset \mathbb{R}^m$ is an open subset, there exist $r > 0$ such that $\overline{B}_r(y_0) \subset \Omega$.

We set $Q_0 := \overline{B}_r(x_0, y_0) \cap (K \times \overline{B}_r(y_0))$. Since Q_0 is a compact set, by the upper semicontinuity of F and Proposition 1.1.3 in [3], we have that

$$F(Q_0) := \bigcup_{(x,y)\in Q_0} F(x,y)$$

is a compact set, hence there exists $M > 0$ such that:

$$\sup\{\|v\| : v \in F(x,y), \quad (x,y) \in Q_0\} \leqslant M.$$

Let

$$T = \min\Big\{\frac{r}{2(M+1)}, \sqrt{\frac{r}{M+1}}, \frac{r}{2(\|y_0\|+1)}\Big\}. \tag{3.4}$$

We shall prove the existence of a viable solution of the problem (2.1) defined on the interval $[0, T]$. Since $(x_0, y_0) \in Q_0$ then, by Lemma 3.1, there exist $h_k^0 \in [h_0(k), \frac{1}{k}]$ and $u_k^0 \in \mathbb{R}^m$ such that

$$x_0 + h_k^0 y_0 + \frac{1}{2}(h_k^0)^2 u_k^0 \in K$$

and $(x_0, y_0, u_k^0) \in \operatorname{graph}(F) + \frac{1}{k}(B \times B \times B)$. Define

$$\begin{aligned} x_k^1 &:= x_0 + h_k^0 y_0 + \frac{1}{2}(h_k^0)^2 u_k^0;\\ y_k^1 &:= y_0 + h_k^0 u_k^0. \end{aligned} \tag{3.5}$$

We remark that if $h_k^0 < T$ then

$$\|x_k^1 - x_0\| \leqslant h_k^0\|y_0\| + \frac{1}{2}(h_k^0)^2\|u_k^0\| < h_k^0\|y_0\| + \frac{1}{2}(h_k^0)^2(M+1),$$
$$\|y_k^1 - y_0\| = h_k^0\|u_k^0\| < h_k^0(M+1),$$

and by the choice of T we get

$$\|x_k^1 - x_0\| < r, \quad \|y_k^1 - y_0\| < r.$$

Therefore $(x_k^1, y_k^1) \in Q_0$ and by Lemma 3.1, there exist $h_k^1 \in [h_0(k), \frac{1}{k}]$ and $u_k^1 \in \mathbb{R}^m$ such that

$$x_k^1 + h_k^1 y_k^1 + \frac{1}{2}(h_k^1)^2 u_k^1 \in K,$$
$$(x_k^1, y_k^1, u_k^1) \in \operatorname{graph}(F) + \frac{1}{k}(B \times B \times B).$$

We claim that, for each $k \in \mathbb{N}^*$, there exist $m(k) \in \mathbb{N}^*$ and h_k^p, x_k^p, y_k^p, u_k^p, such that for every $p \in \{2, \dots, m(k) - 1\}$, we have that:

(i) $\sum_{j=0}^{m(k)-1} h_k^j \leqslant T < \sum_{j=0}^{m(k)} h_k^j$

(ii)

$$\begin{aligned} x_k^p =& x_k^0 + (\sum_{i=0}^{p-1} h_k^i) y_0 + \frac{1}{2}\sum_{i=0}^{p-1}(h_k^i)^2 u_k^i + \sum_{i=0}^{p-2}\sum_{j=i+1}^{p-1} h_k^i h_k^j u_k^i, \\ y_k^p =& y_k^0 + \sum_{i=0}^{p-1} h_k^i u_k^i; \end{aligned}$$

(iii) $(x_k^p, y_k^p) \in Q_0$

(iv) $(x_k^p, y_k^p, u_k^p) \in \operatorname{graph}(F) + \frac{1}{k}(B \times B \times B)$.

If $h_k^0 + h_k^1 \geq T$ then we set $m(k) = 1$. Assume that $h_k^0 + h_k^1 < T$ and define

$$\begin{aligned} x_k^2 :=& x_k^1 + h_k^1 y_k^1 + \frac{1}{2}(h_k^1)^2 u_k^1, \\ y_k^2 :=& y_k^1 + h_k^1 u_k^1. \end{aligned} \tag{3.6}$$

Then by (3.5) and (3.6) we have that

$$\begin{aligned} x_k^2 :=& x_k^0 + (h_k^0 + h_k^1) y_k^0 + \frac{1}{2}(h_k^0)^2 u_k^1 + (h_k^1)^2 u_k^1 + h_k^0 h_k^1 u_k^0, \\ y_k^2 :=& y_k^0 + h_k^0 u_k^0 + h_k^1 u_k^1 \end{aligned}$$

and since $h_k^0 + h_k^1 \leq T$ and

$$\begin{aligned} \|x_k^2 - x_0\| \leqslant& (h_k^0 + h_k^1)\|y_k^0\| + \frac{1}{2}(h_k^0)^2\|u_k^0\| + \frac{1}{2}(h_k^1)^2\|u_k^1\| + h_k^0 h_k^1 \|u_k^0\| \\ <& (h_k^0 + h_k^1)\|y_k^0\| + \frac{1}{2}(h_k^0 + h_k^1)^2 (M+1), \end{aligned}$$

it follows

$$\|x_k^2 - x_0\| < r, \|y_k^2 - y_0\| < r,$$

hence $(x_k^2, y_k^2) \in Q_0$.

Assume that h_k^q x_k^q, y_k^q u_k^q, have been constructed for $q \leqslant p$ satisfying (ii)–(iv) and that we construct h_k^{p+1}, x_k^{p+1}, y_k^{p+1}, u_k^{p+1} satisfying such properties. Since $(x_k^p, y_k^p) \in Q_0$, by lemma 2, there exist $h_k^p \in [h_0(k), \frac{1}{k}]$ and $u_k^p \in \mathbb{R}^m$ such that

$$\begin{gathered} x_k^p + h_k^p y_k^p + \frac{1}{2}(h_k^p)^2 u_k^p \in K, \\ (x_k^p, y_k^p, u_k^p) \in \operatorname{graph}(F) + \frac{1}{k}(B \times B \times B). \end{gathered}$$

If $h_k^0 + h_k^1 + \cdots + h_k^p \geqslant T$ then we set $m(k) = p$. Assume that $h_k^0 + h_k^1 + \cdots + h_k^p < T$ and define

$$\begin{aligned} x_k^{p+1} :=& x_k^p + h_k^p y_k^p + \frac{1}{2}(h_k^p)^2 u_k^p, \\ y_k^{p+1} :=& y_k^p + h_k^p u_k^p. \end{aligned} \tag{3.7}$$

Then, by the above equations and (ii), we obtain that

$$\begin{aligned} x_k^{p+1} &= x_k^p + h_k^p y_k^p + \frac{1}{2}(h_k^p)^2 u_k^p = x_k^0 + \big(\sum_{i=0}^{p-1} h_k^i\big) y_0 + \frac{1}{2}\sum_{i=0}^{p-1}(h_k^i)^2 u_k^i \\ &\quad + \frac{1}{2}\sum_{i=0}^{p-1}(h_k^i)^2 u_k^i + \sum_{i=0}^{p-2}\sum_{j=i+1}^{p-1} h_k^i h_k^j u_k^i + h_k^p \sum_{i=0}^{p-1} h_k^i u_k^i + \frac{1}{2}(h_k^p)^2 u_k^p \\ &= x_k^0 + \big(\sum_{i=0}^{p} h_k^i\big) y_0 + \frac{1}{2}\sum_{i=0}^{p}(h_k^i)^2 u_k^i + \sum_{i=0}^{p-1}\sum_{j=i+1}^{p} h_k^i h_k^j u_k^i \end{aligned}$$

and

$$y_k^{p+1} := y_k^p + h_k^p u_k^p = y_k^0 + \sum_{i=0}^{p-1} h_k^i u_k^i + h_k^p u_k^p = y_k^0 + \sum_{i=0}^{p} h_k^i u_k^i.$$

Therefore,

$$\begin{aligned} \|x_k^{p+1} - x_0\| &\leqslant \big(\sum_{i=0}^{p} h_k^i\big)\|y_0\| + \frac{1}{2}\sum_{i=0}^{p}(h_k^i)^2\|u_k^i\| + \sum_{i=0}^{p-1}\sum_{j=i+1}^{p} h_k^i h_k^j \|u_k^i\| \\ &\leqslant \big(\sum_{i=0}^{p} h_k^i\big)\|y_0\| + \frac{M+1}{2}\big(\sum_{i=0}^{p} h_k^i\big)^2 \end{aligned}$$

and

$$\|y_k^{p+1} - x_0\| \leqslant \sum_{i=0}^{p} h_k^i \|u_0^i\| \leqslant (M+1)\big(\sum_{i=0}^{p} h_k^i\big).$$

Since $\sum_{i=0}^{p} h_k^i < T$ one obtains that

$$\|x_k^{p+1} - x_0\| < r, \quad \|y_k^{p+1} - x_0\| < r,$$

hence $(x_k^{p+1}, y_k^{p+1}) \in Q_0$.

We remark that this iterative process is finite because $h_k^p \in [h_0(k), \frac{1}{k}]$, implies the existence of an integer $m(k)$ such that

$$h_k^0 + h_k^1 + \dots + h_k^{m(k)-1} \leqslant T < h_k^0 + h_k^1 + \dots + h_k^{m(k)-1} + h_k^{m(k)}.$$

By (iv), for every $k \in \mathbb{N}^*$ and every $p \in \{0, 1, \dots, m(k)\}$ there exists $(a_k^p, b_k^p, v_k^p) \in \operatorname{graph}(F)$ such that

$$\|x_k^p - a_k^p\| < \frac{1}{k}, \quad \|y_k^p - b_k^p\| < \frac{1}{k}, \quad \|u_k^p - v_k^p\| < \frac{1}{k}; \tag{3.8}$$

hence,

$$\begin{gathered} \|x_k^p\| \leqslant \|x_k^p - x_0\| + \|x_0\| \leqslant \frac{1}{k} + \|x_0\| \leqslant 1 + \|x_0\|, \\ \|y_k^p\| \leqslant \|y_k^p - y_0\| + \|y_0\| \leqslant \frac{1}{k} + \|y_0\| \leqslant 1 + \|y_0\|, \\ \|u_k^p\| \leqslant \|u_k^p - v_k^p\| + \|v_k^p\| \leqslant \frac{1}{k} + M \leqslant 1 + M. \end{gathered} \tag{3.9}$$

Let us set

$$t_k^p = h_k^0 + h_k^1 + \cdots + h_k^{p-1}, \quad t_k^0 = 0.$$

We remark that for all $k \in \mathbb{N}^*$ and all $p \in \{1, \dots, m(k)\}$, we have

$$t_k^p - t_k^{p-1} < \frac{1}{k} \quad \text{and} \quad t_k^{m(k)-1} \leqslant T < t_k^{m(k)}. \tag{3.10}$$

For each $k \geqslant 1$ and for $p \in \{1, \dots, m(k)\}$ we set $I_k^p = [t_k^{p-1}, t_k^p]$ and for $t \in I_k^p$ we define

$$x_k(t) = x_k^{p-1} + (t - t_k^{p-1})y_k^{p-1} + \frac{1}{2}(t - t_k^{p-1})^2 u_k^{p-1}. \tag{3.11}$$

Then

$$\begin{gathered} x_k'(t) = y_k^{p-1} + (t - t_k^{p-1})u_k^{p-1}, \quad \forall t \in I_k^p, \\ x_k''(t) = u_k^{p-1}, \quad \forall t \in I_k^p, \end{gathered} \tag{3.12}$$

hence, by (3.9), for all $t \in [0, T]$, we obtain

$$\begin{aligned} \|x_k''(t)\| &\leqslant \|u_k^{p-1}\| < M + 1 \\ \|x_k'(t)\| &\leqslant \|y_k^{p-1}\| + (t - t_k^p)\|u_k^p\| < \|y_0\| + M + 2 \\ \|x_k(t)\| &\leqslant \|x_k^{p-1}\| + (t - t_k^{p-1})\|y_k^{p-1}\| + \frac{1}{2}(t - t_k^{p-1})^2 \|u_k^{p-1}\| \\ &\leqslant \|x_0\| + \|y_0\| + M + 3. \end{aligned} \tag{3.13}$$

Moreover, for all $t \in [0, T]$ we have that

$$(x_k(t), x_k'(t), x_k''(t)) \in (x_k^p, y_k^p, u_k^p) + \frac{\|y_0\| + M + 2}{k} B \times \frac{M + 1}{k} B \times \{0\};$$

hence, by (iv), we have

$$(x_k(t), x_k'(t), x_k''(t)) \in \operatorname{graph}(F) + \varepsilon(k)(B \times B \times \{0\}), \tag{3.14}$$

where $\varepsilon(k) \to 0$ when $k \to \infty$. Then, by (3.11), (3.12) and (3.13), we obtain that $(x_k'')_k$ is bounded in $L^2([0, T], \mathbb{R}^m)$, $(x_k')_k$ and $(x_k)_k$ are bounded in $C([0, T], \mathbb{R}^m)$ and equi-Lipschitzian, hence, by Theorem 0.3.4 in [3] there exist a subsequence (again denoted by $(x_k)_k$) and an absolutely continuous function $x : [0, T] \to \mathbb{R}^m$ such that

(a) $(x_k)_k$ converge uniformly to x

(b) $(x_k')_k$ converge uniformly to x'

(c) $(x_k'')_k$ converge weakly in $L^2([0, T], \mathbb{R}^m)$ to x''.

By (H3) and Theorem 1.4.1 in [3] we get that

$$x''(t) \in \operatorname{co} F(x(t), x'(t)) \subset \partial V(x'(t)), a.e. \text{ on } [0, T],$$

where co stands for the closed convex hull; hence, by Lemma 3.3 in [7], we obtain that

$$\frac{d}{dt}V(x'(t)) = \|x''(t)\|^2, a.e. \text{ on } [0,T];$$

hence

$$V(x'(T)) - V(x'(0)) = \int_0^T \|x''(t)\|^2 dt. \tag{3.15}$$

On the other hand, since $x_k''(t) = u_k^{p-1}$, $\forall t \in I_k^p$, by (iv), there exist a_k^{p-1}, b_k^{p-1}, $z_k^{p-1} \in \frac{1}{k}B$, such that

$$u_k^{p-1} - z_k^{p-1} \in F(x_k^{p-1} - a_k^{p-1}, y_k^{p-1} - b_k^{p-1}) \subset \partial V(y_k^{p-1} - b_k^{p-1}), \forall k \in \mathbb{N}^* \tag{3.16}$$

and so the properties of the subdifferential of a convex function imply that, for every $p < m(k)$, and for every $k \in \mathbb{N}^*$ we have

$$\begin{aligned}
&V(x_k'(t_k^p) - b_k^p) - V(x_k'(t_k^{p-1}) - b_k^{p-1}) \geqslant \\
&\quad \geqslant \langle u_k^{p-1} - z_k^{p-1}, x_k'(t_k^p) - x_k'(t_k^{p-1}) + b_k^{p-1} - b_k^p \rangle = \\
&\quad = \langle u_k^{p-1} - z_k^{p-1}, \int_{t_k^{p-1}}^{t_k^p} x_k''(t)dt \rangle + \langle u_k^{p-1} - z_k^{p-1}, b_k^{p-1} - b_k^p \rangle = \\
&\quad = \int_{t_k^{p-1}}^{t_k^p} \|x_k''(t)\|^2 dt - \langle z_k^{p-1}, \int_{t_k^{p-1}}^{t_k^p} x_k''(t)dt \rangle + \langle u_k^{p-1} - z_k^{p-1}, b_k^{p-1} - b_k^p \rangle;
\end{aligned}$$

hence

$$\begin{aligned}
&V(x_k'(t_k^p) - b_k^p) - V(x_k'(t_k^{p-1}) - b_k^{p-1}) \\
&\geqslant \int_{t_k^{p-1}}^{t_k^p} \|x_k''(t)\|^2 dt - \langle z_k^{p-1}, \int_{t_k^{p-1}}^{t_k^p} x_k''(t)dt \rangle + \langle u_k^{p-1} - z_k^{p-1}, b_k^{p-1} - b_k^p \rangle.
\end{aligned} \tag{3.17}$$

Analogously if $T \in I_k^{m(k)}$, then by (3.10) we have

$$\begin{aligned}
&V(x_k'(T)) - V(x_k'(t_k^{m(k)-1}) - b_k^{m(k)-1}) \\
&\quad \geqslant \langle u_k^{m(k)-1} - z_k^{m(k)-1}, \int_{t_k^{m(k)-1}}^T x_k''(t)dt + b_k^{m(k)-1} \rangle \\
&\quad = \int_{t_k^{m(k)-1}}^T \|x_k''(t)\|^2 dt - \langle z_k^{m(k)-1}, \int_{t_k^{m(k)-1}}^T x_k''(t)dt \rangle \\
&\quad\quad + \langle u_k^{m(k)-1} - z_k^{m(k)-1}, b_k^{m(k)-1} \rangle.
\end{aligned} \tag{3.18}$$

By adding the $m(k) - 1$ inequalities from (3.17) and the inequality from (3.18), we get

$$V(x_k'(T)) - V(y_0 - b_k^0) \geqslant \int_0^T \|x_k''(t)\|^2 dt + \alpha(k), \tag{3.19}$$

where

$$\alpha(k) = -\sum_{p=1}^{m(k)-1}\langle z_k^{p-1}, \int_{t_k^{p-1}}^{t_k^p} x_k''(t)dt\rangle + \sum_{p=1}^{m(k)-1}\langle u_k^{p-1}-z_k^{p-1}, b_k^{p-1}-b_k^p\rangle$$
$$- \langle z_k^{m(k)-1}, \int_{t_k^{m(k)-1}}^{T} x_k''(t)dt\rangle + \langle u_k^{m(k)-1}-z_k^{m(k)-1}, b_k^{m(k)-1}\rangle.$$

Since

$$\begin{aligned}|\alpha(k)| &\leqslant \sum_{p=1}^{m(k)-1}|\langle z_k^{p-1}, \int_{t_k^{p-1}}^{t_k^p} x_k''(t)dt\rangle| + \sum_{p=1}^{m(k)-1}|\langle u_k^{p-1}-z_k^{p-1}, b_k^{p-1}-b_k^p\rangle| + \\ &\quad + |\langle z_k^{m(k)}, \int_{t_k^{m(k)-1}}^{T} x_k''(t)dt\rangle| + |\langle u_k^{m(k)-1}-z_k^{m(k)-1}, b_k^{m(k)-1}\rangle| \\ &\leqslant \sum_{p=1}^{m(k)-1}\|z_k^{p-1}\|\|\int_{t_k^{p-1}}^{t_k^p} x_k''(t)dt\| + \sum_{p=1}^{m(k)-1}\|u_k^{p-1}-z_k^{p-1}\|\|b_k^{p-1}-b_k^p\| \\ &\quad + \|z_k^{m(k)}\|\|\int_{t_k^{m(k)-1}}^{T} x_k''(t)dt\| + \|u_k^{m(k)-1}-z_k^{m(k)-1}\|\|b_k^{m(k)-1}\| \\ &\leqslant \frac{(M+2)(3m(k)-1)}{k}\end{aligned}$$

it following that $\alpha(k)\to 0$ when $k\to\infty$; hence, by (3.19), we passing to the limit for $k\to\infty$, we obtain

$$V(x'(T)) - V(y_0) \geqslant \limsup_{k\to\infty}\int_0^T \|x_k''(t)\|^2dt. \tag{3.20}$$

Therefore, by (3.15) and (3.20),

$$\int_0^T \|x''(t)\|^2dt \geqslant \limsup_{k\to\infty}\int_0^T \|x_k''(t)\|^2dt$$

and, since $(x'')_k$ converges weakly in $L^2([0,T],\mathbb{R}^m)$ to x'', by applying Proposition III.30 in [8], we obtain that $(x'')_k$ converge strongly in $L^2([0,T],\mathbb{R}^m)$ to x'', hence a subsequence again denoted by $(x'')_k$ converge poinwise a.e. to x''. Since by (3.14)

$$\lim_{k\to\infty} d((x_k(t),x_k'(t),x_k''(t)),\mathrm{graph}(F)) = 0,$$

and since by (H2) the graph of F is closed ([3], Proposition 1.1.2), we have that

$$x''(t)\in F(x(t),x'(t)) \quad a.e. \text{ on } [0,T].$$

It remains to prove that $(x(t),x'(t))\in Q$, $\forall t\in[0,T]$. Indeed, by (3.11), (3.12), and (3.13), we have that

$$\|x_k(t)-x_k^p\| < \frac{\|y_0\|+M+2}{k}, \quad \|x_k'(t)-y_k^p\| < \frac{M+1}{k},$$

hence

$$\lim_{k\to\infty} d((x_k(t), x'_k(t)), (x_k^p, y_k^p)) = 0.$$

Since, $(x_k^p, y_k^p) \in Q_0, \forall k \in \mathbb{N}^*$, by (a) and (b) we have that

$$\lim_{k\to\infty} d((x(t), x'(t)), (x_k(t), x'_k(t))) = 0.$$

On the other hand

$$\begin{aligned} & d((x(t), x'(t)), Q_0) \\ & \leqslant d((x(t), x'(t)), (x_k(t), x'_k(t))) + d((x_k(t), x'_k(t)), (x_k^p, y_k^p)) + d((x_k^p, y_k^p), Q_0); \end{aligned} \tag{3.21}$$

hence, by passing to the limit we obtain that

$$d((x(t), x'(t)), Q_0) = 0, \quad \forall t \in [0, T].$$

Since Q_0 is closed, we obtain that $(x(t), x'(t)) \in Q_0$, for all $t \in [0, T]$, which completes the proof. □

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