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PERIODIC SOLUTIONS FOR NEUTRAL NONLINEAR DIFFERENTIAL EQUATIONS WITH FUNCTIONAL DELAY

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Abstract. We use Krasnoselskii's fixed point theorem to show that the nonlinear neutral differential equation with functional delay

$$x'(t) = -a(t)x(t) + c(t)x'(t - g(t)) + q\big(t, x(t), x(t - g(t)\big)$$

has a periodic solution. Also, by transforming the problem to an integral equation we are able, using the contraction mapping principle, to show that the periodic solution is unique.

1. Introduction

Motivated by the papers [1, 5, 6, 10, 11] and the references therein, we consider the nonlinear neutral differential equation

$$x'(t) = -a(t)x(t) + c(t)x'(t - g(t)) + q\big(t, x(t), x(t - g(t)\big) \tag{1.1}$$

which arises in a (food-limited) population model [2, 3, 8, 7]. In recent years many researchers have considered a particular form of equation (1.1) and studied the existence of periodic solutions and their boundedness. The authors in [2] investigated the boundedness of solution of an equation of the form (1.1) by using comparison techniques. Also, the author in [6] considered the logistic form of (1.1) with several constants delays where the functional q is linear and showed the existence of positive periodic solutions.

The purpose of this paper is to transform (1.1) to an integral equation and then use Krasnoselskii's fixed point theorem to show the existence of a periodic solution. The obtained integral equation is the sum of two mappings; one is a contraction and the other is compact.

Transforming equation (1.1) to an integral equation enables us to show the uniqueness of the periodic solution by appealing to the contraction mapping principle. For more on the existence and uniqueness of periodic solutions of equations that are similar to (1.1) but with constant delay, we refer the reader to [2, 3].

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2. Existence Of Periodic Solutions

For $T > 0$ define $P_T = \{\phi : C(R,R), \phi(t+T) = \phi(t)\}$ where $C(R,R)$ is the space of all real valued continuous functions. Then P_T is a Banach space when it is endowed with the supremum norm

$$\|x(t)\| = \max_{t\in[0,T]} |x(t|.$$

In this paper we assume that

$$a(t+T) = a(t), \quad c(t+T) = c(t), \quad g(t+T) = g(t), \quad g(t) \geq g^* > 0 \tag{2.1}$$

with $c(t)$ continuously differentiable, $g(t)$ twice continuously differentiable and g^* is constant. In [6], the author made the assumption that $a(t)$ is positive, while here we only ask that

$$\int_0^T a(s)ds > 0. \tag{2.2}$$

It is interesting to note that equation (1.1) becomes of advanced type when $g(t) < 0$. Since we are searching for periodic solutions, it is natural to ask that $q(t,x,y)$ is continuous and periodic in t and Lipschitz continuous in x and y. That is

$$q(t+T,x,y) = q(t,x,y) \tag{2.3}$$

and some positive constants L and E,

$$|q(t,x,y) - q(t,z,w)| \leq L\|x-z\| + E\|y-w\|\,. \tag{2.4}$$

Also, we assume that for all t, $0 \leq t \leq T$,

$$g'(t) \neq 1. \tag{2.5}$$

Since $g(t)$ is periodic, condition (2.5) implies that $g'(t) < 1$.

Lemma 2.1. *Suppose (2.1)-(2.2) and (2.5) hold. If $x(t) \in P_T$, then $x(t)$ is a solution of equation* (1.1) *if and only if*

$$\begin{aligned} x(t) &= \frac{c(t)}{1-g'(t)}x(t-g(t)) + \big(1 - e^{-\int_{t-T}^t a(s)ds}\big)^{-1} \\ &\quad\times \int_{t-T}^t \big[-r(u)x(u-g(u)) + q(u,x(u),x(u-g(u)))\big]e^{-\int_u^t a(s)ds}\,du\,, \end{aligned} \tag{2.6}$$

where

$$r(u) = \frac{\Big(c'(u) - c(u)a(u)\Big)\Big(1-g'(u)\Big) + g''(u)c(u)}{(1-g'(u))^2}. \tag{2.7}$$

Proof. Let $x(t) \in P_T$ be a solution of (1.1). Multiply both sides of (1.1) by $\exp\int_0^t a(s)ds$ and then integrate from $t-T$ to t to obtain

$$\begin{aligned} &\int_{t-T}^t \big[x(u)e^{\int_0^u a(s)ds}\big]'du \\ &= \int_{t-T}^t \big[c(u)x'(u-g(u)) + q(u,x(u),x(u-g(u)))\big]e^{\int_0^u a(s)ds}du. \end{aligned}$$

As a consequence, we arrive at

$$\begin{aligned} & x(t)e^{\int_0^t a(s)ds} - x(t-T)e^{\int_0^{t-T} a(s)ds} \\ &= \int_{t-T}^{t} \big[c(u)x'(u-g(u)) + q(u,x(u),x(u-g(u)))\big]e^{\int_0^u a(s)ds}du. \end{aligned}$$

By dividing both sides of the above equation by $\exp(\int_0^t a(s)ds)$ and the fact that $x(t) = x(t-T)$, we obtain

$$\begin{aligned} x(t) &= \big(1 - e^{-\int_{t-T}^t a(s)ds}\big)^{-1} \\ &\quad \times \int_{t-T}^{t} \big[c(u))x'(u-g(u)) + q(u,x(u),x(u-g(u)))\big]e^{-\int_u^t a(s)ds}du\,. \end{aligned} \tag{2.8}$$

Rewrite

$$\begin{aligned} & \int_{t-T}^{t} c(u)x'(u-g(u))e^{-\int_u^t a(s)ds}du \\ &= \int_{t-T}^{t} \frac{c(u)x'(u-g(u))(1-g'(u))}{(1-g'(u))}e^{-\int_u^t a(s)ds}du. \end{aligned}$$

Integration by parts on the above integral with

$$U = \frac{c(u)}{1-g'(u)}e^{-\int_u^t a(s)ds}, \quad \text{and} \quad dV = x'(u-g(u))(1-g'(u))du$$

we obtain

$$\begin{aligned} & \int_{t-T}^{t} c(u)x'(u-g(u))e^{-\int_u^t a(s)ds}du \\ &= \frac{c(t)}{1-g'(t)}x(t-g(t))\big(1 - e^{-\int_{t-T}^t a(s)ds}\big) - \int_{t-T}^{t} r(u)e^{-\int_u^t a(s)ds}x(u-g(u))du\,, \end{aligned} \tag{2.9}$$

where $r(u)$ is given by (2.7). Then substituting (2.9) into (2.8) completes the proof. $\square$

Define the mapping $H : P_T \to P_T$ by

$$\begin{aligned} (H\varphi)(t) =& \frac{c(t)}{1-g'(t)}\varphi(t-g(t)) + \big(1 - e^{-\int_{t-T}^t a(s)ds}\big)^{-1} \\ & \times \int_{t-T}^{t} \big[-r(u)\varphi(u-g(u)) + q(u,\varphi(u),\varphi(u-g(u)))\big]e^{-\int_u^t a(s)ds}du\,. \end{aligned} \tag{2.10}$$

Next we state Krasnoselskii's fixed point theorem which enables us to prove the existence of a periodic solution. For its proof we refer the reader to [9].

Theorem 2.2 (Krasnoselskii). *Let $\mathbb{M}$ be a closed convex nonempty subset of a Banach space $(\mathbb{B}, \|\cdot\|)$. Suppose that A and B maps $\mathbb{M}$ into $\mathbb{B}$ such that*

(i) *$x, y \in \mathbb{M}$, implies $Ax + By \in \mathbb{M}$,*
(ii) *A is compact and continuous,*
(iii) *B is a contraction mapping.*

Then there exists $z \in \mathbb{M}$ with $z = Az + Bz$.

Note that to apply the above theorem we need to construct two mappings; one is a contraction and the other is compact. Therefore, we express equation (2.10) as

$$(H\varphi)(t) = (B\varphi)(t) + (A\varphi)(t)$$

where $A, B : P_T \to P_T$ are given by

$$(B\varphi)(t) = \frac{c(t)}{1-g'(t)}\varphi(t-g(t)) \tag{2.11}$$

and

$$\begin{aligned}(A\varphi)(t) &= (1-e^{-\int_{t-T}^{t} a(s)ds})^{-1}\\ &\quad\times \int_{t-T}^{t}\big[-r(u)\varphi(u-g(u)) + q(u,\varphi(u),\varphi(u-g(u)))\big]e^{-\int_u^t a(s)ds}du.\end{aligned} \tag{2.12}$$

Lemma 2.3. *Suppose (2.1)-(2.5) hold. Then $A : P_T \to P_T$, as defined by (2.12), is compact provided that*

$$\max_{t\in[0,T]}\Big|\frac{c'(t)}{1-g'(t)} + \frac{g''(t)}{(1-g'(t))^2}\Big| \le Q \tag{2.13}$$

for some positive constant Q.

Proof. A change of variable in (2.12) shows that $(A\varphi)(t+T) = (A\varphi)(t)$. To see that A is continuous, we let $\varphi, \psi \in P_T$ with $\|\varphi\| \le C$ and $\|\psi\| \le C$. Let

$$\eta = \max_{t\in[0,T]}|(1-e^{-\int_{t-T}^{t} a(s)ds})^{-1}|, \quad \beta = \max_{t\in[0,T]}|r(t)|, \quad \gamma = \max_{u\in[t-T,t]} e^{-\int_u^t a(s)ds}. \tag{2.14}$$

Given $\epsilon > 0$, take $\delta = \epsilon/M$ such that $\|\varphi - \psi\| < \delta$. By making use of (2.4) into (2.12) we get

$$\begin{aligned}\big\|\big(A\varphi(t)\big) - \big(A\psi(t)\big)\big\| &\le \gamma\eta\int_{t-T}^{t}\big[L\|\varphi-\psi\| + E\|\varphi-\psi\| + \beta\|\varphi-\psi\|\big]du\\ &\le M\|\varphi-\psi\| < \epsilon\end{aligned}$$

where L, E are given by (2.4) and $M = T\gamma\eta[\beta+L+E]$. This proves A is continuous. To show A is compact, we let $\varphi_n \in P_T$ with $\|\varphi_n\| \le R$, where n is a positive integer and $R > 0$. Observe that in view of (2.4) we arrive at

$$\begin{aligned}|q(t,x,y)| &= |q(t,x,y) - q(t,0,0) + q(t,0,0)|\\ &\le |q(t,x,y) - q(t,0,0)| + |q(t,0,0)|\\ &\le L\|x\| + E\|y\| + \alpha\end{aligned}$$

where $\alpha = |q(t,0,0)|$. Hence, if A is given by (2.12) we obtain that

$$\|A(\varphi_n(t))\| \le D$$

for some positive constant D. Now, it can be easily checked that

$$\begin{aligned}(A\varphi_n)'(t) = -a(t)\varphi_n(t) - \big[\frac{c\prime(t)}{1-g'(t)} + \frac{g''(t)}{(1-g'(t))^2}\big]\varphi_n(t-g(t))\\ + q(t,\varphi_n(t),\varphi_n(t-g(t))).\end{aligned}$$

Hence by invoking (2.13) we obtain $\|(A\varphi_n)'(t)\| \le F$, for some positive constant F. Thus the sequence $(A\varphi_n)$ is uniformly bounded and equi-continuous. The Arzela-Ascoli theorem implies that $(A\varphi_{n_k})$ uniformly converges to a continuous T-periodic function φ^*. Thus A is compact. □

Lemma 2.4. *Let B be defined by (2.11) and*

$$\Big\|\frac{c(t)}{1-g'(t)}\Big\| \le \zeta < 1. \tag{2.15}$$

Then B is a contraction.

Proof. For $\varphi, \psi \in P_T$, we have

$$\begin{aligned}\|B(\varphi) - B(\psi)\| &= \max_{t\in[0,T]} |B(\varphi) - B(\psi)| \\ &= \max_{t\in[0,T]} \Big|\frac{c(t)}{1-g'(t)}\Big|\big|\varphi(t-g(t)) - \psi(t-g(t)\big| \\ &\le \zeta\|\varphi-\psi\|.\end{aligned}$$

Hence B defines a contraction mapping with contraction constant ζ. □

Theorem 2.5. *Let $\alpha = \|q(t,0,0)\|$. Let η, β and γ be given by* (2.14). *Suppose* (2.1)–(2.5), (2.13) *and* (2.15) *hold. Suppose there is a positive constant G such that all solutions $x(t)$ of* (1.1), *$x(t) \in P_T$ satisfy $|x(t)| \le G$, the inequality*

$$\big\{\zeta + \eta\gamma T(\beta + L + E)\big\}G + \eta\gamma T\alpha \le G \tag{2.16}$$

holds. Then equation(1.1) *has a $T-$periodic solution.*

Proof. Define $\mathbb{M} = \{\varphi \in P_T : \|\varphi\| \le G\}$. Then Lemma 2.3 implies $A : P_T \to P_T$ and A is compact and continuous. Also, from Lemma 2.4, the mapping B is a contraction and it is clear that $B : P_T \to P_T$. Next, we show that if $\varphi, \psi \in \mathbb{M}$, we have $\|A\phi + B\psi\| \le G$. Let $\varphi, \psi \in \mathbb{M}$ with $\|\varphi\|, \|\psi\| \le G$. Then from (2.11)-(2.12) and the fact that $|q(t,x,y)| \le L\|x\| + E\|y\| + \alpha$, we have

$$\begin{aligned}\Big\|\Big(A\varphi(t)\Big) + \Big(B\psi(t)\Big)\Big\| &\le \gamma\eta\int_{t-T}^{t}\big[L\|\varphi\| + E\|\varphi\| + \beta\|\varphi\| + \alpha\big]du + \zeta\|\psi\| \\ &\le \big\{\zeta + \eta\gamma T(\beta + L + E)\big\}G + \eta\gamma T\alpha \le G.\end{aligned}$$

We see that all the conditions of Krasnoselskii's theorem are satisfied on the set $\mathbb{M}$. Thus there exists a fixed point z in $\mathbb{M}$ such that $z = Az + Bz$. By Lemma 2.1, this fixed point is a solution of (1.1). Hence (1.1) has a $T-$periodic solution. □

Theorem 2.6. *Suppose* (2.1)*–eqrefe2.5,* (2.13) *and* (2.15) *hold. Let η, β and γ be given by* (2.14). *If*

$$\zeta + T\gamma\eta\big(\beta + L + E\big) < 1,$$

then equation (1.1) *has a unique T-periodic solution.*

Proof. Let the mapping H be given by (2.10). For $\varphi, \psi \in P_T$, in view of (2.10), we have

$$\begin{aligned}&\big\|\big(H\varphi(t)\big) - \big(H\psi(t)\big)\big\| \\ &\le \zeta\|\varphi-\psi\| + \gamma\eta\int_{t-T}^{t}\big[L\|\varphi-\psi\| + E\|\varphi-\psi\| + \beta\|\varphi-\psi\|\big]du \\ &\le \big[\zeta + T\gamma\eta\big(\beta + L + E\big)\big]\|\varphi-\psi\|.\end{aligned}$$

This completes the proof. $\square$

We finish this paper with an example in which we give criteria that enable us to determine the a priori bound G in Theorem 2.5.

Example 2.7. Consider (1.1) along with conditions (2.1)–(2.5) and (2.13)–(2.15). Suppose that $a(t) \neq 0$ for all $t \in [0,T]$. Set

$$\rho = \min_{t\in[0,T]} |a(t)|, \quad \delta = \max_{t\in[0,T]} k(t),$$

where

$$k(t) = \frac{c'(t)\big(1-g'(t)\big) + g''(t)c(t)}{(1-g'(t))^2}.$$

Suppose $1 - \|c\| > 0$. If

$$\rho(1-\|c\|) > (1-\|c\|)(\delta + L + E) + T\rho(\|a\| + L + E)$$

holds and G which is defined by

$$G = \frac{\alpha(1-\|c\| + T\rho)}{\rho(1-\|c\|) - (1-\|c\|)(\delta + L + E) - T\rho(\|a\| + L + E)}$$

satisfies inequality (2.16), then (1.1) has a T-periodic solution.

Proof. Let the mappings A and B be defined by (2.12) and (2.11), respectively. Let $x(t) \in P_T$. An integration of equation (1.1) from 0 to T yields,

$$x(T) - x(0) = \int_0^T [-a(t)x(t) + c(t)x'(t-g(t)) + q(t, x(t), x(t-g(t)))]dt. \quad (2.17)$$

Rewrite and then integrate by parts to obtain,

$$\begin{aligned}\int_0^T c(t)x'(t-g(t))dt &= \int_0^T \frac{c(t)x'(t-g(t))(1-g'(t))}{(1-g'(t))}dt \\ &= -\int_0^T k(t)x(t-g(t))dt.\end{aligned}$$

As a consequence, (2.17) becomes

$$\int_0^T a(t)x(t)dt = \int_0^T [k(t)x(t-g(t)) - q(t, x(t), x(t-g(t)))]dt,$$

from which, it implies that there exists a $t^* \in (0,T)$ such that

$$Ta(t^*)x(t^*) = \int_0^T [k(t)x(t-g(t)) - q(t, x(t), x(t-g(t)))]dt.$$

Then

$$T|a(t^*)||x(t^*)| \leq \int_0^T [|k(t)||x(t-g(t))| + |q(t, x(t), x(t-g(t)))|]dt.$$

By taking the maximum over $t \in [0,T]$, from the above inequality, we obtain

$$\|x(t^*)\| \leq \frac{1}{\rho}\Big(\delta + L + E\Big)\|x\| + \frac{\alpha}{\rho}. \quad (2.18)$$

Since for all $t \in [0,T]$,

$$x(t) = x(t^*) + \int_{t^*}^t x'(s)ds,$$

we have

$$|x(t)| \leq |x(t^*)| + \int_0^t |x'(s)|ds.$$

Taking the maximum over $t \in [0, T]$ and using (2.18), yield

$$\begin{aligned} \|x(t)\| &\leq \|x(t^*)\| + \int_0^t \|x'(s)\|ds \\ &\leq \frac{1}{\rho}\big(\delta + L + E\big)\|x\| + \frac{\alpha}{\rho} + T\|x'\|. \end{aligned} \tag{2.19}$$

Taking the norm in (1.1) yields

$$\|x'\| \leq \frac{(\|a\| + L + E)\|x\| + \alpha}{1 - \|c\|}. \tag{2.20}$$

Substitution of (2.20) into (2.19), yields that for all $x(t) \in P_T$, $\|x(t)\| \leq G$. Define $\mathbb{M} = \{\varphi \in P_T : \|\varphi\| \leq G\}$. Then by Theorem 2.5, (1.1) has a T-periodic solution. □

Remark. The author of this paper has studied the asymptotic stability of the zero solution of (1.1) using fixed point theory; see [11]. However, the question of uniform asymptotic stability of the zero solution of (1.1) remains open.

References

[1] T. A. Burton, *Basic neutral equations of advanced type*, Nonlinear Analysis, Methods and applications, Vol. 31, No. 3/4, pp. 295-310, 1998.

[2] J. Cao and G. He, *Discussion of periodic solutions for p-th order delayed neutral differential equations*, Applied Mathematics and Computation, 129 2-3(2002), 391-405.

[3] J. Cao and G. He, *Further investigate on periodic solutions of p-th order delayed neutral differential equations*, Applied Mathematics and Computation, 132 2-3(2002), 231-248.

[4] T. Ding, R. Iannaci, and F. Zanolin, *On Periodic solutions of sublinear Duffing equations*, J. Math. Anaysis, 158,(1991) 316-332.

[5] K. Gopalsamy, X. He, and L. Wen, *On a periodic neutral logistic equation*, Glasgow Math. J. 33 (1991), 281-286.

[6] K. Gopalsamy, and B. G. Zhang, *On a neutral delay-logistic equation*, Dynamics stability systems 2 (1988), 183-195.

[7] Gyori and G. Ladas, *Positive solutions of integro-differential equations with unbounded delay*, J. Integral Equations and Appl 4 (3), (1992), 377-390.

[8] Y. Kuang and A. Feldstein, *Boundedness of solutions of a nonlinear nonautonomous neutral delay equations*, J. Math. Anal. Appl. 156 (1991), 293-304.

[9] L. Y. Kun, *Periodic solution of a periodic neutral delay equation*, J. Math. Anal. Appl. 214 (1997), 11-21.

[10] E. C. Pielou, *Mathematical Ecology*, Wisely-Interscience, New York,1977.

[11] Y. R. Raffoul, *Stability in neutral nonlinear differential equations with functional delays using fixed point Theory*, to appear in Mathematical and Computer Modelling.

[12] D. R. Smart, *Fixed points theorems*, Cambridge univ. Press, Cambridge, 1980.

[13] Gen-Qian Wang, and Sui Sun Cheng, *A priori bounds for periodic solutions of a delay Rayleigh equation*, Applied Mathematics Letters, 12 (1999)41-44.

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