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BLOW-UP OF SOLUTIONS FOR AN INTEGRO-DIFFERENTIAL EQUATION WITH A NONLINEAR SOURCE

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ABSTRACT. We study the nonlinear viscoelastic wave equation

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s) ds = |u|^p u,$$

in a bounded domain, with the initial and Dirichlet boundary conditions. By modifying the method in [15], we prove that there are solutions, under some conditions on the initial data, which blow up in finite time with nonpositive initial energy as well as positive initial energy. Estimates of the lifespan of solutions are also given.

1. INTRODUCTION

In this paper we consider the initial boundary value problem for the nonlinear integro-differential equation

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s) ds = |u|^p u, \tag{1.1}$$

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in \Omega, \tag{1.2}$$

$$u(x,t) = 0,\ x \in \partial\Omega,\ t \geq 0, \tag{1.3}$$

where $\Delta = \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2}$ and Ω is a bounded domain in R^N, $N \geq 1$, with a smooth boundary $\partial\Omega$ so that the Divergence theorem can be applied. Here, g is a positive function satisfying some conditions to be specified later and $p > 0$.

When $g \equiv 0$, the equation (1.1) becomes a nonlinear wave equation. There is a large body of literature on nonexistence of global solutions and blowup for solutions with negative initial energy [1, 8, 9]. Levine [10, 11, 12] considered the interaction between linear or strong damping and the source terms and showed that solutions with negative initial energy blow up in finite time. Georgiev and Todorova [7] extended Levine's result to wave equations with nonlinear damping terms. Under some conditions, they proved that the solutions blow up in finite time provided they have sufficiently negative initial energy. This result was generalized by Levine and Serrin [13], and then by Levine and Park [14]. Vitillaro [18] combined the arguments in [7] and [13] to extend these results to the case of positive initial energy.

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On the contrary, when g is not trivial on R, (1.1) becomes a semilinear viscoelastic equation. Cavalcanti et al. [4] treated (1.1) with a localized damping mechanism acting on a part of the domain. By assuming the kernel g in the memory term decays exponentially, they obtained an exponential decay rate of the energy function. Later, Cavcalanti [6] and Berrimi and Messaoudi [2] improved this work by using different methods. Also, Cavcalcanti et al. [5] established an existence result and a decay result for problem (1.1) with nonlinear boundary damping. Regarding nonexistence, Messaoudi [17] studied problem (1.1) with an internal nonlinear damping term and showed under some restrictions on the initial energy the solutions blow up in finite time. Recently, Berrimi and Messaoudi [3] considered problem (1.1) and proved, for suitable initial data, that the solution is bounded and global and the damping caused by the integral term is enough to obtain uniform decay of solutions. However, no blow up result is discussed for this problem (1.1).

In this paper we shall deal with the blow up behavior of solutions for problem (1.1)-(1.3). We derive the blow-up properties of solutions of problem (1.1)-(1.3) with nonpositive and positive initial energy by modifying the method in [15]. The content of this paper is organized as follows. In section 2, we give some lemmas and the local existence theorem 2.4. In section 3, we define an energy function $E(t)$ and show that it is a non-increasing function of t. Then, we obtain theorem 3.5, which gives the blow-up phenomena of solutions even for positive initial energy. Estimates for the blow-up time T^* are also given.

2. Preliminary results

In this section, we shall give some lemmas which will be used throughout this work.

Lemma 2.1 (Sobolev-Poincaré inequality [16]). *If $2 \le p \le \frac{2N}{N-2}$, then*

$$\|u\|_p \le B\|\nabla u\|_2,$$

for $u \in H_0^1(\Omega)$ holds with some constant B, where $\|\cdot\|_p$ denotes the norm of $L^p(\Omega)$.

Lemma 2.2 ([15]). *Let $\delta > 0$ and $B(t) \in C^2(0,\infty)$ be a nonnegative function satisfying*

$$B''(t) - 4(\delta+1)B'(t) + 4(\delta+1)B(t) \ge 0. \tag{2.1}$$

If

$$B'(0) > r_2 B(0), \tag{2.2}$$

then $B'(t) > 0$ for $t > 0$, where $r_2 = 2(\delta+1) - 2\sqrt{(\delta+1)\delta}$ is the smallest root of the equation

$$r^2 - 4(\delta+1)r + 4(\delta+1) = 0.$$

Lemma 2.3 ([15]). *If $J(t)$ is a non-increasing function on $[t_0,\infty)$, $t_0 \ge 0$ and satisfies the differential inequality*

$$J'(t)^2 \ge a + bJ(t)^{2+\frac{1}{\delta}} \quad \text{for } t_0 \ge 0, \tag{2.3}$$

where $a > 0$, $\delta > 0$ and $b \in R$, then there exists a finite time T^* such that

$$\lim_{t\to T^{*-}} J(t) = 0$$

and the upper bound of T^* is estimated respectively by the following cases:

(i) If $b < 0$ *and* $J(t_0) < \min\{1, \sqrt{\frac{a}{-b}}\}$*, then*

$$T^* \le t_0 + \frac{1}{\sqrt{-b}} \ln \frac{\sqrt{\frac{a}{-b}}}{\sqrt{\frac{a}{-b}} - J(t_0)}.$$

(ii) If $b = 0$*, then*

$$T^* \le t_0 + \frac{J(t_0)}{\sqrt{a}}.$$

(iii) If $b > 0$*, then*

$$T^* \le \frac{J(t_0)}{\sqrt{a}},$$

or

$$T^* \le t_0 + 2^{\frac{3\delta+1}{2\delta}} \frac{\delta c}{\sqrt{a}} \{1 - [1 + cJ(t_0)]^{\frac{-1}{2\delta}}\},$$

where $c = (\frac{b}{a})^{\frac{\delta}{2+\delta}}$.

Now, we state the local existence theorem which is proved in [17].

Theorem 2.4 (Local existence)**.** *Let* $u_0 \in H_0^1(\Omega)$*,* $u_1 \in L^2(\Omega)$ *and* $0 < p < p^*$*, here* $p^* = \frac{2}{N-2}$*, if* $N \ge 3(\infty$*, if* $N \le 2)$*. Let* g *be a bounded* C^1 *function satisfying*

$$g(0) > 0, \quad g'(s) \le 0, \quad 1 - \int_0^\infty g(s)ds = l > 0. \tag{2.4}$$

Then problem (1.1)-(1.3) *has a unique weak solution* u *in* $C([0,T], H_0^1(\Omega))$ *with* u_t *in* $C([0,T], L^2(\Omega))$*, for some* $T > 0$*.*

3. Blow-up property

In this section, we shall discuss the blow up phenomena of problem (1.1)-(1.3). For this purpose, we make the following assumption on g:

$$\int_0^\infty g(s)ds < \frac{4\delta}{1+4\delta}, \tag{3.1}$$

here $\delta = p/4$. First, we define the energy function for the solution u of (1.1)-(1.3) by

$$\begin{aligned} E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}(1 - \int_0^t g(s)ds)\|\nabla u(t)\|_2^2 + \frac{1}{2}(g \diamond \nabla u)(t) \\ - \frac{1}{p+2}\|u\|_{p+2}^{p+2}, \end{aligned} \tag{3.2}$$

for $t \ge 0$, where $(g \diamond \nabla u)(t) = \int_0^t g(s)\|\nabla u(t) - \nabla u(s)\|_2^2 ds$

Remark. From (3.2), (2.4) and Lemma 2.1, we have

$$\begin{aligned} E(t) &\ge \frac{1}{2}\Big(1 - \int_0^t g(s)ds\Big)\|\nabla u(t)\|_2^2 + \frac{1}{2}(g \diamond \nabla u)(t) - \frac{1}{p+2}\|u\|_{p+2}^{p+2} \\ &\ge \frac{1}{2}\big(l\|\nabla u(t)\|_2^2 + (g \diamond \nabla u)(t)\big) - \frac{B_1^{p+2} l^{\frac{p+2}{2}}}{p+2}\|\nabla u\|_2^{p+2} \\ &\ge G\big[\big(l\|\nabla u(t)\|_2^2 + (g \diamond \nabla u)(t)\big)^{1/2}\big], \quad t \ge 0, \end{aligned} \tag{3.3}$$

where

$$G(\lambda) = \frac{1}{2}\lambda^2 - \frac{B_1^{p+2}}{p+2}\lambda^{p+2}, \quad B_1 = \frac{B}{\sqrt{l}}.$$

It is easy to verify that $G(\lambda)$ has a maximum at $\lambda_1 = B_1^{-\frac{p+2}{p}}$ and the maximum value is

$$E_1 = \frac{p}{2(p+2)} B_1^{-\frac{p+2}{p}}.$$

Before proving our main result, we need the following lemmas.

Lemma 3.1 ([17]). *Assume that the conditions of theorem 2.4 hold and let u be a solution of* (1.1)-(1.3). *Then $E(t)$ is a non-increasing function on $[0,T]$ and*

$$E'(t) = \frac{1}{2}(g' \diamond \nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|_2^2, \tag{3.4}$$

for almost every $t \in [0,T]$.

Proof. Multiplying (1.1) by u_t and integrating it over Ω, and integrating by parts, we obtain (3.4) for any regular solution. Then by density arguments, we have the result. □

Lemma 3.2. *Suppose that the conditions of theorem 2.4 hold. Let u be a solution of* (1.1)-(1.3) *with initial data satisfying $E(0) < E_1$ and $l^{\frac{1}{2}}\|\nabla u_0\|_2 > \lambda_1$, then there exists $\lambda_2 > \lambda_1$ such that*

$$l\|\nabla u(t)\|_2^2 + (g \diamond \nabla u)(t) \geq \lambda_2^2, \quad \textit{for } t > 0. \tag{3.5}$$

Proof. From the definition of $G(\lambda)$, we see that $G(\lambda)$ is increasing in $(0, \lambda_1)$ and decreasing in (λ_1, ∞), and $G(\lambda) \to -\infty$, as $\lambda \to \infty$. Since $E(0) < E_1$, there exist λ_2' and λ_2 such that $\lambda_2' < \lambda_1 < \lambda_2$ and $G(\lambda_2') = G(\lambda_2) = E(0)$. When $l^{1/2}\|\nabla u_0\|_2 > \lambda_1$, by (3.3), we have

$$G(l^{1/2}\|\nabla u_0\|_2) \leq E(0) = G(\lambda_2).$$

This implies $l^{1/2}\|\nabla u_0\|_2 > \lambda_2$. To establish (3.5), we suppose by contradiction that

$$l\|\nabla u(t)\|_2^2 + (g \diamond \nabla u)(t) < \lambda_2^2,$$

for some $t_0 > 0$.
Case 1: If $\lambda_2' < (l\|\nabla u(t_0)\|_2^2 + (g \diamond \nabla u)(t_0))^{\frac{1}{2}} < \lambda_2$, then

$$G((l\|\nabla u(t_0)\|_2^2 + (g \diamond \nabla u)(t_0))^{1/2}) > E(0) \geq E(t_0),$$

which contradicts (3.3).
Case 2: If $(l\|\nabla u(t_0)\|_2^2 + (g \diamond \nabla u)(t_0))^{1/2} < \lambda_2'$, then by the continuity of $(l\|\nabla u(t)\|_2^2 + (g \diamond \nabla u)(t))^{1/2}$, there exists $0 < t_1 < t_0$ such that

$$\lambda_2' < (l\|\nabla u(t_1)\|_2^2 + (g \diamond \nabla u)(t_1))^{1/2} < \lambda_2.$$

Then

$$G((l\|\nabla u(t_1)\|_2^2 + (g \diamond \nabla u)(t_1))^{1/2}) > E(0) \geq E(t_1).$$

This is a contradiction. □

Definition. A solution u of (1.1)-(1.3) is said to blowup if there exists a finite time T^* such that

$$\lim_{t\to T^{*-}}\Big(\int_\Omega u^2dx\Big)^{-1}=0. \tag{3.6}$$

For short notation, we define

$$a(t)=\int_\Omega u^2dx,\quad t\geq 0. \tag{3.7}$$

Lemma 3.3. *Assume that the conditions of theorems 2.4 and* (3.1) *hold and let u be a solution of* (1.1)-(1.3)*, then we have*

$$a''(t)-4(\delta+1)\int_\Omega u_t^2dx\geq Q_1(t), \tag{3.8}$$

where $Q_1(t)=(-4-8\delta)E(0)+m(l\|\nabla u\|_2^2+(g\diamond\nabla u)(t))$, $m=(1+4\delta)-1/l>0$.

Proof. Form (3.7), we have

$$a'(t)=2\int_\Omega uu_tdx \tag{3.9}$$

and by (1.1) and the Divergence theorem, we get

$$a''(t)=2\|u_t\|_2^2-2\|\nabla u\|_2^2+2\|u\|_{p+2}^{p+2}+2\int_0^t\int_\Omega g(t-s)\nabla u(s)\cdot\nabla u(t)dxds. \tag{3.10}$$

Then, using (3.4), we obtain

$$\begin{aligned}
&a''(t)-4(\delta+1)\|u_t\|_2^2\\
&\geq(-4-8\delta)E(0)+2\big[1-\frac{2+4\delta}{p+2}\big]\|u\|_{p+2}^{p+2}+4\delta\|\nabla u(t)\|_2^2\\
&\quad-(2+4\delta)\int_0^t g(s)ds\|\nabla u(t)\|_2^2+2\int_0^t\int_\Omega g(t-s)\nabla u(s)\cdot\nabla u(t)dxds\\
&\quad-(2+4\delta)\int_0^t(g'\diamond\nabla u)(t)dt+(2+4\delta)(g\diamond\nabla u)(t).
\end{aligned} \tag{3.11}$$

It follows from Hölder's inequality and Young's inequality that

$$\begin{aligned}
&\int_\Omega\int_0^t g(t-s)\nabla u(s)\cdot\nabla u(t)dsdx\\
&=\int_\Omega\int_0^t g(t-s)\nabla u(t)\cdot(\nabla u(s)-\nabla u(t))dsdx+\int_0^t g(t-s)ds\|\nabla u(t)\|_2^2\\
&\geq-[\frac12(g\diamond\nabla u)(t)+\frac12\int_0^t g(s)ds\|\nabla u(t)\|_2^2]+\int_0^t g(s)ds\|\nabla u(t)\|_2^2.
\end{aligned}$$

Hence, (3.11) becomes

$$\begin{aligned}
&a''(t)-4(\delta+1)\|u_t\|_2^2\\
&\geq(-4-8\delta)E(0)+\Big(4\delta-(1+4\delta)\int_0^t g(s)ds\Big)\|\nabla u(t)\|_2^2\\
&\quad+(1+4\delta)(g\diamond\nabla u)(t)-(2+4\delta)\int_0^t(g'\diamond\nabla u)(t)dt.
\end{aligned}$$

Therefore, by (2.4) and (3.1), we obtain

$$\begin{aligned}
&a''(t) - 4(\delta+1)\|u_t\|_2^2 \\
&\geq (-4-8\delta)E(0) + (4\delta - (1+4\delta)(1-l))\|\nabla u(t)\|_2^2 + (1+4\delta)(g \diamond \nabla u)(t) \\
&\geq (-4-8\delta)E(0) + m(l\|\nabla u(t)\|_2^2 + (g \diamond \nabla u)(t)),
\end{aligned}$$

where $m = (1+4\delta) - \frac{1}{l} > 0$.

Now, we consider different cases on the sign of the initial energy $E(0)$.

Case 1: If $E(0) < 0$, then from (3.8), we have

$$a'(t) \geq a'(0) - 4(1+2\delta)E(0)t, \quad t \geq 0.$$

Thus we get $a'(t) > 0$ for $t > t^*$, where

$$t^* = \max\big\{\frac{a'(0)}{4(1+2\delta)E(0)}, 0\big\}. \tag{3.12}$$

Case 2: If $E(0) = 0$, then $a''(t) \geq 0$ for $t \geq 0$. Furthermore, if $a'(0) > 0$, then $a'(t) > 0$, $t \geq 0$.

Case 3: If $0 < E(0) < \frac{m}{p}E_1$ and $l^{1/2}\|\nabla u_0\|_2 > \lambda_1$, then using lemma 3.2, we see that

$$\begin{aligned}
Q_1(t) &= (-4-8\delta)E(0) + m(l\|\nabla u\|_2^2 + (g \diamond \nabla u)(t)) \\
&\geq (-4-8\delta)E(0) + m\lambda_2^2 \\
&> (-4-8\delta)E(0) + m\frac{2(p+2)}{p}E_1 \\
&= (4+8\delta)(-E(0) + \frac{m}{4+8\delta}\frac{2(p+2)}{p}E_1) \\
&= (4+8\delta)(-E(0) + \frac{m}{p}E_1).
\end{aligned}$$

Thus, from (3.8), we have

$$a''(t) \geq Q_1(t) > C_1 > 0, \quad t > 0, \tag{3.13}$$

where $C_1 = (4+8\delta)(-E(0) + \frac{m}{p}E_1)$. Hence, we get $a'(t) > 0$ for $t > t_1^*$, where

$$t_1^* = \max\{\frac{-a'(0)}{C_1}, 0\}. \tag{3.14}$$

Case 4: If $E(0) \geq \frac{m}{p}E_1$, using Hölder's inequality and Young's inequality, we get

$$a'(t) \leq \|u\|_2^2 + \|u_t\|_2^2.$$

Hence, from (3.8), we have

$$a''(t) - 4(\delta+1)a'(t) + 4(\delta+1)a(t) + (4+8\delta)E(0) \geq 0.$$

Let

$$b(t) = a(t) + \frac{(1+2\delta)E(0)}{1+\delta}, \quad t > 0.$$

Then $b(t)$ satisfies (2.1). By (2.2), we see that if

$$a'(0) > r_2\big[a(0) + \frac{(1+2\delta)E(0)}{1+\delta}\big], \tag{3.15}$$

then $a'(t) > 0$, $t > 0$. $\square$

Consequently, we have the following result.

Lemma 3.4. *Assume that the conditions of theorem 2.4 and* (3.1) *hold, and that either one of the following four conditions is satisfied:*

(i) $E(0) < 0$,
(ii) $E(0) = 0$ *and* $a'(0) > 0$,
(iii) $0 < E(0) < \frac{m}{p}E_1$ *and* $l^{1/2}\|\nabla u_0\|_2 > \lambda_1$
(iv) $\frac{m}{p}E_1 \le E(0)$ *and* (3.15) *holds.*

Then $a'(t) > 0$ *for* $t > t_0$*, where* $t_0 = t^*$ *is given by* (3.12) *in case 1,* $t_0 = 0$ *in cases 2 and 4, and* $t_0 = t_1^*$ *is given by* (3.14) *in case 3.*

Hereafter, we will find an estimate for the life span of $a(t)$. Let

$$J(t) = a(t)^{-\delta}, \quad \text{for } t \ge 0. \tag{3.16}$$

Then we have

$$\begin{gathered} J'(t) = -\delta J(t)^{1+\frac{1}{\delta}} a'(t), \\ J''(t) = -\delta J(t)^{1+\frac{2}{\delta}} V(t), \end{gathered} \tag{3.17}$$

where

$$V(t) = a''(t)a(t) - (1+\delta)(a'(t))^2. \tag{3.18}$$

Using (3.8) and exploiting Hölder's inequality on $a(t)$, we deduce that

$$\begin{aligned} V(t) &\ge [Q_1(t) + 4(1+\delta)\|u_t\|_2^2]a(t) - 4(1+\delta)\|u\|_2^2\|u_t\|_2^2 \\ &= Q_1(t)J(t)^{-\frac{1}{\delta}}. \end{aligned}$$

Therefore, (3.17) yields

$$J''(t) \le -\delta Q_1(t) J(t)^{1+\frac{1}{\delta}}, \quad t \ge t_0. \tag{3.19}$$

Theorem 3.5. *Assume the conditions of theorem 2.4 and* (3.1) *hold, and that either one of the following four conditions is satisfied:*

(i) $E(0) < 0$,
(ii) $E(0) = 0$ *and* $a'(0) > 0$,
(iii) $0 < E(0) < \frac{m}{p}E_1$ *and* $l^{\frac{1}{2}}\|\nabla u_0\|_2 > \lambda_1$,
(iv) $\frac{m}{p}E_1 \le E(0) < \frac{a'(t_0)^2}{8a(t_0)}$ *and* (3.15) *holds.*

Then the solution u *blows up at finite time* T^* *in the sense of* (3.6). *Moreover, the upper bounds for* T^* *can be estimated according to the sign of* $E(0)$*: In case (1),*

$$T^* \le t_0 - \frac{J(t_0)}{J'(t_0)}.$$

Furthermore, if $J(t_0) < \min\{1, \sqrt{\frac{\alpha}{-\beta}}\}$*, then*

$$T^* \le t_0 + \frac{1}{\sqrt{-\beta}} \ln \frac{\sqrt{\frac{\alpha}{-\beta}}}{\sqrt{\frac{\alpha}{-\beta}} - J(t_0)}.$$

In case (2),

$$T^* \le t_0 - \frac{J(t_0)}{J'(t_0)} \quad \text{or} \quad T^* \le t_0 + \frac{J(t_0)}{\sqrt{\alpha}}.$$

In case (3),

$$T^* \le t_0 - \frac{J(t_0)}{J'(t_0)}.$$

Furthermore, if $J(t_0) < \min\{1, \sqrt{\frac{\alpha_1}{-\beta_1}}\}$, *then*

$$T^* \le t_0 + \frac{1}{\sqrt{-\beta_1}} \ln \frac{\sqrt{\frac{\alpha_1}{-\beta_1}}}{\sqrt{\frac{\alpha_1}{-\beta_1}} - J(t_0)}.$$

In case (4),

$$T^* \le \frac{J(t_0)}{\sqrt{\alpha}} \quad or \quad T^* \le t_0 + 2^{\frac{3\delta+1}{2\delta}} \frac{\delta c}{\sqrt{\alpha}} \{1 - [1 + cJ(t_0)]^{\frac{-1}{2\delta}}\},$$

where $c = (\frac{\beta}{\alpha})^{\frac{\delta}{2+\delta}}$. *Here* α, β, α_1 *and* β_1 *are given in* (3.21)-(3.24), *respectively. Note that in case 1,* $t_0 = t^*$ *is given in* (3.12) *and* $t_0 = 0$ *in cases 2 and 4, and in case 3,* $t_0 = t_1^*$ *is given in* (3.14).

Proof. (1) For $E(0) \le 0$, from (3.19),

$$J''(t) \le \delta(4 + 8\delta)E(0)J(t)^{1+\frac{1}{\delta}}. \tag{3.20}$$

Note that by lemma 3.4, $J'(t) < 0$ for $t > t_0$. Multiplying (3.20) by $J'(t)$ and integrating it from t_0 to t, we have

$$J'(t)^2 \ge \alpha + \beta J(t)^{2+\frac{1}{\delta}} \quad \text{for } t \ge t_0,$$

where

$$\alpha = \delta^2 J(t_0)^{2+\frac{2}{\delta}} \left[a'(t_0)^2 - 8E(0)J(t_0)^{\frac{-1}{\delta}}\right] > 0. \tag{3.21}$$

and

$$\beta = 8\delta^2 E(0). \tag{3.22}$$

Then by lemma 2.3, there exists a finite time T^* such that $\lim_{t\to T^{*-}} J(t) = 0$ and this will imply that $\lim_{t\to T^{*-}} (\int_\Omega u^2 dx)^{-1} = 0$.
(2) For the case $0 < E(0) < \frac{m}{p}E_1$, from (3.19) and (3.13), we have

$$J''(t) \le -\delta C_1 J(t)^{1+\frac{1}{\delta}} \quad \text{for } t \ge t_0.$$

Then using the same arguments as in (1), we have

$$J'(t)^2 \ge \alpha_1 + \beta_1 J(t)^{2+\frac{1}{\delta}} \text{ for } t \ge t_0,$$

where

$$\alpha_1 = \delta^2 J(t_0)^{2+\frac{2}{\delta}} [a'(t_0)^2 + \frac{2C_1}{1+2\delta} J(t_0)^{\frac{-1}{\delta}}] > 0. \tag{3.23}$$

and

$$\beta_1 = -\frac{2C_1\delta^2}{1+2\delta}. \tag{3.24}$$

Thus, by lemma 2.3, there exists a finite time T^* such that $\lim_{t\to T^{*-}} (\int_\Omega u^2 dx)^{-1} = 0$.
(3) For the case $\frac{m}{p}E_1 \le E(0)$. Applying the same discussion as in part (1), we also have the equalities (3.21) and (3.22). In this way, we observe that

$$\alpha > 0 \quad \text{if and only if} \quad E(0) < \frac{a'(t_0)^2}{8a(t_0)}.$$

Hence, by lemma 2.3, there exists a finite time T^* such that $\lim_{t\to T^{*-}} (\int_\Omega u^2 dx)^{-1} = 0$. $\square$

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