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A three-point boundary-value problem for a hyperbolic equation with a non-local condition *

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Abstract

We use an energy method to solve a three-point boundary-value problem for a hyperbolic equation with a Bessel operator and an integral condition. The proof is based on an energy inequality and on the fact that the range of the operator generated is dense.

1 Introduction

In this paper, we investigate a boundary-value problem for a one-dimensional hyperbolic equation with a weighted nonlocal boundary integral condition of the form

$$\int_{l_1}^{l} \xi u(\xi,t)d\xi = E(t), \quad 0<t<T,$$

where l_1 is a real number in $(0,l)$ and $E(\cdot)$ is a given function.

Evolution problems dealing with nonlocal conditions were first studied a long time ago by Samarskii [12] and Cannon [2]. The latter author considered the problem

$$\begin{gathered} u_t - u_{xx} = 0, \quad x>0,\ t>0, \\ u(x,0) = \varphi(x), \quad x>0, \\ u(0,t) = g(t), \\ \int_0^{x(t)} u(\xi,t)dx = f(t), \end{gathered} \tag{1.1}$$

for $x(t)$ and $f(t)$ given functions. Introducing $g \equiv u(0,t)$ as the unknown, it is proved in [2] that (1.1) is equivalent to a Volterra integral equation of the second kind for the function g. The author proved the existence and uniqueness of the solution with the aid of the integral equation. Shi [11] considered weak

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solutions of the problem

$$
\begin{gathered}
u_t - u_{xx} = f + g_x, \quad (x,t) \in (0,1) \times (0,T), \\
u(x,0) = \varphi(x), \quad 0 < x < 1, \\
u_x(1,t) = 0, \quad 0 < t < T, \\
\int_0^b u(\xi,t)dx = E(t), \quad 0 < t < T
\end{gathered} \tag{1.2}
$$

and discussed the well-posedness of (1.2) in a weighted fractional Sobolev space. Along a different line, (1.2) was also considered by Ionkin [5], Makarov and Kulyev [8], and Yurchuk [13].

In this work, we are concerned with the mixed evolution problem

$$
\begin{gathered}
\mathcal{L}u = u_{tt} - \frac{1}{x}\left(xu_x\right)_x = F(x,t), \quad (x,t) \in Q, \\
\ell_1 u = u(x,0) = \varphi_1(x), \quad x \in (0,l), \\
\ell_2 u = u_t(x,0) = \varphi_2(x), \quad x \in (0,l), \\
u_x(l,t) = E_1(t), \quad t \in (0,T), \\
\int_{l_1}^{l} xu(x,t)dx = E_2(t), \quad 0 \le l_1 \le l,\ t \in (0,T),
\end{gathered} \tag{1.3}
$$

where $Q = (0,l) \times (0,T)$, with $0 < l < \infty$, $0 < T < \infty$, $F(x,t)$, $\varphi_1(x)$, $\varphi_2(x)$, $E_1(t)$, and $E_2(t)$ are known functions satisfying, for compatibility,

$$
\begin{gathered}
\varphi_1'(l) = E_1(0), \\
\int_{l_1}^{l} x\varphi_1(x)dx = E_2(0), \\
\varphi_2'(l) = E_1'(0), \\
\int_{l_1}^{l} x\varphi_2(x)dx = E_2'(0).
\end{gathered} \tag{1.4}
$$

Problem (1.3), for $l_1 = 0$, has been studied by Mesloub and Bouziani [9]. We also refer the reader to Denche and Marhoune [3] for a similar result in the parabolic case and to Yurchuk [13], Kartynik [6] and Bouziani [1] for related results in both parabolic and hyperbolic cases, where the Bessel operator was replaced by $\left(a(x,t)u_x\right)_x$. It should be noted that the used method was developed first by Ladyzhenskaya [7]. Our interest lies in proving the existence and uniqueness of a strong solution of problem (1.3). In point of view of the used method, it is preferable to transform inhomogeneous boundary conditions to homogeneous ones by introducing a new unknown function v defined as follows:

$$
v(x,t) = u(x,t) - \Phi(x,t), \tag{1.5}
$$

where

$$
\Phi(x,t) = x\big(x - \frac{4(x-l)^2}{l}\big)E_1(t) + \frac{12(x-l)^2}{l^4}E_2(t). \tag{1.6}
$$

Then problem (1.3) becomes

$$
\begin{aligned}
\mathcal{L}v &= F(x,t) - \mathcal{L}\Phi = f(x,t),\\
\ell_1 v &= \varphi_1 - \ell_1\Phi = \varphi(x),\\
\ell_2 v &= \varphi_2 - \ell_2\Phi = \psi(x)\\
v_x(l,t) &= 0,\\
\int_{l_1}^{l} xv(x,t)dx &= 0.
\end{aligned} \tag{1.7}
$$

The solution to (1.3) is then given by $u(x,t) = v(x,t) + \Phi(x,t)$.

We now introduce appropriate function spaces. First let

$$
\theta(x) = \begin{cases} 1 + l_1^2)x, & \text{if } 0 < x \le l_1 \\ x + x^3, & \text{if } l_1 \le x < l \end{cases}
$$

and

$$
\Im_x v = \int_x^l v(\xi,t)d\xi, \quad \Im_x^2 v = \int_x^l \int_\xi^l v(\eta,t)d\eta d\xi.
$$

Let $L^2(Q)$ be the space of square integrable functions with the norm

$$
\|v\|^2_{L^2(Q)} = \int_Q v^2\, dx\, dt
$$

and $L^2_\theta(Q)$ be the weighted L^2-space with the norm

$$
\|v\|^2_{L^2_\theta(Q)} = \int_Q \theta(x)v^2\, dx\, dt.
$$

We then define $W^{1,0}_{\theta,2}(Q)$ to be the subspace of $L^2(Q)$ with the norm

$$
\|v\|^2_{W^{1,0}_{\theta,2}(Q)} = \|v\|^2_{L^2_\theta(Q)} + \|v_x\|^2_{L^2_\theta(Q)}
$$

and $W^{1,1}_{\theta,2}(Q)$ to be the subspace of $W^{1,0}_{\theta,2}(Q)$ whose elements satisfy $\sqrt{\theta(x)}v_t \in L^2(Q)$. In general, a function in the space $W^{q,p}_{\theta,2}(Q)$, with q,p nonnegative integers, possesses x-derivatives up to qth order in $L^2_\theta(Q)$ and t-derivatives up to pth order in $L^2_\theta(Q)$. We use also weighted subspaces on the interval $(0,l)$ such as $W^1_{\theta,2}((0,l)) = H^1_\theta((0,l))$, whose definition is analogous to the space on Q. For example, $H^1_\theta((0,l))$ is the subspace of $L^2(0,l)$ with the norm

$$
\|\varphi\|^2_{H^1_\theta((0,l))} = \|\varphi\|^2_{L^2_\theta((0,l))} + \|\varphi_x\|^2_{L^2_\theta((0,l))}\,.
$$

We associate with problem (1.7) the operator $L = (\mathcal{L}, \ell_1, \ell_2)$ whose domain of definition is $D(L)$, the set of functions $v \in L^2(Q)$ for which $v_t, v_x, v_{tt}, v_{xt}, v_{xx} \in L^2(Q)$ and satisfying conditions in (1.7). The operator L maps E into F; E is the Banach space of functions $v \in L^2(Q)$ satisfying conditions in (1.7), with the norm

$$\begin{aligned} \|v\|_E^2 &= \max_{0\le t\le T} \|v(.,\tau)\|^2_{W^{1,1}_{\theta,2}((0,l))} \\ &= \max_{0\le t\le T} \left\{ \|v(.,\tau)\|^2_{L^2_\theta((0,l))} + \|v_x(.,\tau)\|^2_{L^2_\theta((0,l))} + \|v_t(.,\tau)\|^2_{L^2_\theta((0,l))} \right\} \end{aligned}$$

and F is the Hilbert space $L^2_\theta(Q) \times H^1_\theta((0,l)) \times L^2_\theta((0,l))$, which consists of elements $\mathcal{F} = (f, \varphi, \psi)$ with the norm

$$\|\mathcal{F}\|_F^2 = \|f\|^2_{L^2_\theta(Q)} + \|\varphi\|^2_{H^1_\theta((0,l))} + \|\psi\|^2_{L^2_\theta((0,l))} . \tag{1.9}$$

Then, we establish an energy inequality:

$$\|v\|_E \le K \|Lv\|_F , \quad \forall v \in D(L), \tag{1.10}$$

and show that the operator L has a closure $\overline{L}$.

Definition 1.1 *A solution of the operator equation*

$$\overline{L}v = (f, \varphi, \psi),$$

is called a strong solution of the problem (1.7).

Since the points of the graph of the operator $\overline{L}$ are limits of sequences of points of the graph of L, we can extend the a priori estimate (1.9) to be applied to strong solutions by taking limits, that is we have the inequality

$$\|v\|_E \le K \left\|\overline{L}v\right\|_F , \quad \forall v \in D(\overline{L}). \tag{1.11}$$

From this inequality, We deduce the uniqueness of a strong solution, if it exists, and that the range of the operator $\overline{L}$ coincides with the closure of the range of L.

Proposition 1.2 *The operator L admits a closure.*

The proof of this proposition is similar to that in [9]; therefore we omit it.

2 A priori bound

This section is devoted to the proof of the uniqueness and continuous dependence of the solution on the given data.

Theorem 2.1 *For any function $v \in D(L)$, we have the inequality*

$$\|v\|_E \le c \|Lv\|_F , \tag{2.1}$$

where the positive constant c is independent of the function v.

Proof We define

$$Mv = \begin{cases} x(1+l_1^2)v_t & \text{if } 0<x<l_1 \\ (x+x^3)v_t - x\Im_x^2(\xi v_t) + x\Im_x(\xi^2 v_t) & \text{if } l_1<x<l\,. \end{cases}$$

Then we perform the scalar product in $L^2(Q^\tau)$ of equation (1.7) and Mv to get

$$\begin{aligned} &\int_{Q^\tau}\theta(x)v_tv_{tt}\,dx\,dt - \int_0^\tau\int_0^{l_1}(l_1^2+1)(xv_x)_xv_t\,dx\,dt \\ &-\int_0^\tau\int_{l_1}^l(x^2+1)(xv_x)_xv_t\,dx\,dt - \int_0^\tau\int_{l_1}^l xv_{tt}\Im_x^2(\xi v_t)\,dx\,dt \\ &+\int_0^\tau\int_{l_1}^l(xv_x)_x\Im_x^2(\xi v_t)\,dx\,dt + \int_0^\tau\int_{l_1}^l xv_{tt}\Im_x(\xi^2v_t)\,dx\,dt \\ &-\int_0^\tau\int_{l_1}^l(xv_x)_x\Im_x(\xi^2v_t)\,dx\,dt \\ &= \int_{Q^\tau}\theta(x)v_t\mathcal{L}v dxdt - \int_0^\tau\int_{l_1}^l x\mathcal{L}v\Im_x^2(\xi v_t)\,dx\,dt + \int_0^\tau\int_{l_1}^l x\mathcal{L}v\Im_x(\xi^2v_t)\,dx\,dt. \end{aligned} \tag{2.2}$$

Integrating by parts each term of (2.2) and using conditions (1.7), we obtain the following equations:

$$\int_{Q^\tau}\theta(x)v_tv_{tt}\,dx\,dt = \frac12\int_0^l\theta(x)v_t^2(x,\tau)dx - \frac12\int_0^l\theta(x)\psi^2(x,\tau)dx\,, \tag{2.3}$$

$$\begin{aligned} &-\int_0^\tau\int_0^{l_1}(l_1^2+1)(xv_x)_xv_t\,dx\,dt \\ &= \frac12\int_0^{l_1}(l_1^2+1)xv_x^2(x,\tau)dx - \frac12\int_0^{l_1}(l_1^2+1)x\varphi_x^2dx \\ &\quad -\int_0^\tau(l_1^2+1)l_1v_t(l_1,t)v_x(l_1,t)dt\,, \end{aligned} \tag{2.4}$$

$$\begin{aligned} &-\int_0^\tau\int_{l_1}^l(x^2+1)(xv_x)_xv_t\,dx\,dt \\ &= \frac12\int_{l_1}^l(x^3+x)v_x^2(x,\tau)dx - \frac12\int_{l_1}^l(x^3+x)\varphi_x^2dx \\ &\quad +2\int_0^\tau\int_{l_1}^l x^2v_xv_t\,dx\,dt + \int_0^\tau(l_1^2+1)l_1v_t(l_1,t)v_x(l_1,t)dt\,, \end{aligned} \tag{2.5}$$

$$-\int_0^\tau\int_{l_1}^l xv_{tt}\Im_x^2(\xi v_t)\,dx\,dt = \frac12\int_{l_1}^l(\Im_x(\xi v_t(\xi,\tau)))^2dx - \frac12\int_{l_1}^l(\Im_x(\xi\psi))^2dx\,, \tag{2.6}$$

$$\begin{aligned}
&\int_0^\tau\int_{l_1}^l (xv_x)_x \Im_x^2(\xi v_t)\,dx\,dt\\
&= -l_1\int_0^\tau\int_{l_1}^l x^2 v_x(l_1,t)v_t\,dx\,dt + \int_0^\tau\int_{l_1}^l xv_x\Im_x(\xi v_t)\,dx\,dt\,, \qquad (2.7)
\end{aligned}$$

$$\begin{aligned}
&\int_0^\tau\int_{l_1}^l xv_{tt}\Im_x(\xi^2 v_t)\,dx\,dt\\
&= -\frac{1}{2}\int_{l_1}^l (\Im_x(\xi v_t(\xi,\tau)))^2 dx + \frac{1}{2}\int_{l_1}^l (\Im_x(\xi\psi))^2 dx + \int_0^\tau\int_{l_1}^l x^3 v_x v_t\,dx\,dt\\
&\quad -\int_0^\tau\int_{l_1}^l xv_x\Im_x(\xi v_t)\,dx\,dt + \int_0^\tau\int_{l_1}^l x^2\Im_x(\xi v_t)\mathcal{L}v\,dx\,dt\,, \qquad (2.8)
\end{aligned}$$

$$\begin{aligned}
&-\int_0^\tau\int_{l_1}^l (xv_x)_x\Im_x(\xi^2 v_t)\,dx\,dt\\
&= l_1\int_0^\tau\int_{l_1}^l x^2 v_x(l_1,t)v_t\,dx\,dt - \int_0^\tau\int_{l_1}^l x^3 v_x v_t\,dx\,dt\,. \qquad (2.9)
\end{aligned}$$

Substituting (2.3)-(2.9) in (2.2) yields

$$\begin{aligned}
&\frac{1}{2}\int_0^l \theta(x)v_t^2(x,\tau)dx + \frac{1}{2}\int_0^l \theta(x)v_x^2(x,\tau)dx\\
&= \frac{1}{2}\int_0^l \theta(x)\psi^2 dx + \frac{1}{2}\int_0^l \theta(x)\varphi_x^2 dx - 2\int_0^\tau\int_{l_1}^l x^2 v_x v_t\,dx\,dt\\
&\quad + \int_0^\tau\int_{l_1}^l x\mathcal{L}v\Im_x(\xi^2 v_t)\,dx\,dt - \int_0^\tau\int_{l_1}^l x^2\mathcal{L}v\Im_x(\xi v_t)\,dx\,dt\\
&\quad + \int_{Q^\tau}\theta(x)v_t\mathcal{L}v\,dx\,dt - \int_0^\tau\int_{l_1}^l x\mathcal{L}v\Im_x^2(\xi v_t)\,dx\,dt. \qquad (2.10)
\end{aligned}$$

Using Young's inequality and

$$\int_{l_1}^l (\Im_x^2 v)^2 dx \le \frac{(l-l_1)^2}{2}\int_{l_1}^l (\Im_x v)^2 dx,$$

to estimate the last five terms on the right-hand side of (2.10), we obtain the following inequalities:

$$-2\int_0^\tau\int_{l_1}^l x^2 v_x v_t\,dx\,dt \le \int_0^\tau\int_{l_1}^l xv_x^2\,dx\,dt + \int_0^\tau\int_{l_1}^l x^3 v_t^2\,dx\,dt, \qquad (2.11)$$

$$\int_0^\tau\int_{l_1}^l x\mathcal{L}v\Im_x(\xi^2 v_t)\,dx\,dt$$

$$\leq \frac{(l-l_1)}{2}\int_0^\tau\int_{l_1}^l x(\mathcal{L}v)^2\,dx\,dt + \frac{(l-l_1)^5}{4}\int_0^\tau\int_{l_1}^l xv_t^2\,dx\,dt \tag{2.12}$$

$$\begin{aligned}
&-\int_0^\tau\int_{l_1}^l x^2\mathcal{L}v\Im_x(\xi v_t)\,dx\,dt\\
&\leq \frac{(l-l_1)^3}{2}\int_0^\tau\int_{l_1}^l x(\mathcal{L}v)^2\,dx\,dt + \frac{(l-l_1)^3}{4}\int_0^\tau\int_{l_1}^l xv_t^2\,dx\,dt,
\end{aligned} \tag{2.13}$$

$$\int_{Q^\tau}\theta(x)v_t\mathcal{L}v\,dx\,dt \leq \frac{1}{2}\int_{Q^\tau}\theta(x)v_t^2\,dx\,dt + \frac{1}{2}\int_{Q^\tau}\theta(x)(\mathcal{L}v)^2\,dx\,dt, \tag{2.14}$$

$$\begin{aligned}
&-\int_0^\tau\int_{l_1}^l x\mathcal{L}v\Im_x^2(\xi v_t)\,dx\,dt\\
&\leq \frac{(l-l_1)^3}{4}\int_0^\tau\int_{l_1}^l (\Im_x(\xi v_t))^2\,dx\,dt + \frac{l-l_1}{2}\int_0^\tau\int_{l_1}^l x(\mathcal{L}v)^2\,dx\,dt\\
&\leq \frac{(l-l_1)^5}{8}\int_0^\tau\int_{l_1}^l xv_t^2\,dx\,dt + \frac{l-l_1}{2}\int_{Q^\tau}\theta(x)(\mathcal{L}v)^2\,dx\,dt.
\end{aligned} \tag{2.15}$$

We also have

$$\frac{1}{2}\int_0^l\theta(x)v^2(x,\tau)dx \leq \frac{1}{2}\int_0^l\theta(x)\varphi^2dx + \frac{1}{2}\int_{Q^\tau}\theta(x)v^2\,dx\,dt + \frac{1}{2}\int_{Q^\tau}\theta(x)v_t^2\,dx\,dt. \tag{2.16}$$

Indeed, we have

$$\frac{\partial u^2}{\partial t} = 2uu_t,$$

multiplying both sides by $\theta(x)$ then integrating with respect to t from 0 to τ, and using Young's inequality, we obtain

$$\theta(x)v^2(x,\tau) - \theta(x)\varphi^2(x) = 2\int_0^\tau\theta(x)vv_tdt \leq \int_0^\tau\theta(x)v^2dt + \int_0^\tau\theta(x)v_t^2dt.$$

Multiplying by (1/2) and integration of both sides of this last inequality with respect to x from 0 to l yields (2.16). Substituting (2.11)-(2.15) in (2.10) and adding the resulting inequality with (2.16), each side, gives

$$\begin{aligned}
&\frac{1}{2}\int_0^l\theta(x)v_t^2(x,\tau)dx + \frac{1}{2}\int_0^l\theta(x)v_x^2(x,\tau)dx + \frac{1}{2}\int_0^l\theta(x)v^2(x,\tau)dx\\
&\leq \frac{1}{2}\int_0^l\theta(x)\psi^2dx + \frac{1}{2}\int_0^l\theta(x)\varphi_x^2dx + \frac{1}{2}\int_0^l\theta(x)\varphi^2dx\\
&\quad \int_0^\tau\int_{l_1}^l xv_x^2\,dx\,dt + \big(\frac{3(l-l_1)^5}{8} + \frac{(l-l_1)^3}{4}\big)\int_0^\tau\int_{l_1}^l xv_t^2\,dx\,dt
\end{aligned}$$

$$
\begin{aligned}
&\int_{Q^\tau} \theta(x) v_t^2 \,dx\,dt + \int_0^\tau \int_{l_1}^{l} x^3 v_t^2 \,dx\,dt + \frac{1}{2}\int_{Q^\tau} \theta(x) v^2 \,dx\,dt \\
&+\Big(\frac{1}{2} + \frac{(l-l_1)}{2}\Big) \int_{Q^\tau} \theta(x)(\mathcal{L}v)^2 \,dx\,dt \\
&+\Big(\frac{(l-l_1)^3}{2} + \frac{(l-l_1)}{2}\Big) \int_{Q^\tau} x(\mathcal{L}v)^2 \,dx\,dt.
\end{aligned} \tag{2.17}
$$

When we add the term $\int_0^\tau \int_{l_1}^l x^3 v_x^2 \,dx\,dt + \int_0^\tau \int_0^{l_1} (1+l_1^2) x v_x^2 \,dx\,dt$ to the right-hand side of (2.17) and use the definition of $\theta(x)$, (2.17) takes the form

$$
\begin{aligned}
&\int_0^l \theta(x) v_t^2(x,\tau) dx + \int_0^l \theta(x) v_x^2(x,\tau) dx + \int_0^l \theta(x) v^2(x,\tau) dx \\
&\leq \; K\Big(\int_0^l \theta(x)\psi^2 dx + \int_0^l \theta(x)\varphi_x^2 dx + \int_0^l \theta(x)\varphi^2 dx \\
&\quad + \int_{Q^\tau} \theta(x)(\mathcal{L}v)^2 \,dx\,dt + \int_{Q^\tau} \theta(x) v^2 \,dx\,dt \\
&\quad \int_{Q^\tau} \theta(x) v_x^2 \,dx\,dt + \int_{Q^\tau} \theta(x) v_t^2 \,dx\,dt \Big),
\end{aligned} \tag{2.18}
$$

where $K = \max\{c_1, c_2\}$, $c_1 = \max\big\{3 + \frac{3(l-l_1)^5}{4} + \frac{3(l-l_1)^3}{2}, 5\big\}$, and $c_2 = 1 + 2(l - l_1) + (l-l_1)^3$. By [4, Lemma 7.1], we obtain, from inequality (2.18),

$$
\begin{aligned}
\|v(x,\tau)\|^2_{W^{1,1}_{\theta,2}((0,l))} &\leq Ke^{K\tau}\Big\{ \|\varphi\|^2_{H^1_\theta((0,l))} + \|\psi\|^2_{L^2_\theta((0,l))} + \|\mathcal{L}v\|^2_{L^2_\theta(Q^\tau)} \Big\} \\
&\leq Ke^{KT}\Big\{ \|\varphi\|^2_{H^1_\theta((0,l))} + \|\psi\|^2_{L^2_\theta((0,l))} + \|\mathcal{L}v\|^2_{L^2_\theta(Q)} \Big\}.
\end{aligned}
$$

By taking the supremum with respect to τ, over $[0,T]$, the energy inequality (2.1) follows with $c = \sqrt{K} e^{KT/2}$.

The a priori bound (1.10) leads to the following results.

Corollary 2.2 *If a strong solution of the problem (1.7) exists, it is unique and depends continuously on the data $\mathcal{F} = (f, \varphi, \psi) \in F$.*

Corollary 2.3 *The range $R(\overline{L})$ of the operator $\overline{L}$ is closed and coincides with the set $\overline{R(L)}$ and $\overline{L}^{-1}\mathcal{F} = \overline{L^{-1}}\mathcal{F}$ where $\overline{L^{-1}}$ is the continuous extension of L^{-1} from $R(L)$ to $\overline{R(L)}$.*

3 Existence of a solution

The main result in this paper reads as follows.

Theorem 3.1 *For each $f \in L^2_\theta(Q)$, $\varphi \in H^1_\theta((0,l))$, $\psi \in L^2_\theta((0,l))$, there exists a unique strong solution $v = \overline{L}^{-1}\mathcal{F} = \overline{L^{-1}}\mathcal{F}$ of problem (1.7) satisfying the estimate*

$$
\max_{0 \leq t \leq T} \|v(.,\tau)\|^2_{W^{1,1}_{\theta,2}((0,l))} \leq c^2 \Big(\|f\|^2_{L^2_\theta(Q)} + \|\varphi\|^2_{H^1_\theta((0,l))} + \|\psi\|^2_{L^2_\theta((0,l))} \Big) \tag{3.1}
$$

where c is a positive constant independent of v.

Remark 3.2 *According to corollary 2.3, to prove the existence of the solution in the sense of Definition 1.1, for any $(f, \varphi, \psi) \in F$, it is sufficient to prove that $R(L)^{\perp} = \{0\}$. For this purpose we need the following statement.*

Proposition 3.3 *Let $D_0(L) = \{v \in D(L) : \ell_1 v = \ell_2 v = 0\}$. If for all ω in $L^2(Q)$ and all v in $D_0(L)$,*

$$\int_Q \omega \mathcal{L} v \, dx \, dt = 0, \tag{3.2}$$

then ω vanishes almost everywhere in Q.

Proof Assume that relation (3.2) holds for any function $v \in D_0(L)$. Using this fact, (3.2) can be expressed in a special form. First define the function β by the formula

$$\beta(x,t) = \int_t^T \omega(x,\tau)\, d\tau \,. \tag{3.3}$$

Let v_{tt} be a solution of

$$\beta = \begin{cases} x l_1 v_{tt}, & \text{if } 0 \le x < l_1 \\ \frac{1}{2}(x^2 + x l_1) v_{tt} + x \Im_x(\xi v_t), & \text{if } l_1 < x < l \end{cases} \tag{3.4}$$

and let

$$v = \begin{cases} 0, & \text{if } 0 \le t \le s \\ \int_s^t (t-\tau) v_{\tau\tau} d\tau, & \text{if } s \le t \le T. \end{cases} \tag{3.5}$$

It follows that

$$\omega = \begin{cases} -x l_1 v_{ttt}, & \text{if } 0 \le x < l_1 \\ -\frac{1}{2}(x^2 + x l_1) v_{ttt} - x \Im_x(\xi v_{tt}), & \text{if } l_1 < x < l. \end{cases} \tag{3.6}$$

By [10, Lemma 4.2], the function v defined by the relations (3.4) and (3.5) has derivatives with respect to t up to the third order belonging to the space $L^2(Q_s)$, where $Q_s = (0,l) \times (s,T)$. By replacing the function ω, given by its representation (3.6), in (3.2) we get

$$\begin{aligned} &- \int_s^T \int_0^{l_1} l_1 x v_{ttt} \big(v_{tt} - \frac{1}{x}(x v_x)_x\big)\, dx\, dt \\ -\frac{1}{2} &\int_s^T \int_{l_1}^{l} (l_1 x + x^2) v_{ttt} \big(v_{tt} - \frac{1}{x}(x v_x)_x\big)\, dx\, dt \\ &- \int_s^T \int_{l_1}^{l} x \Im_x(\xi v_{tt}) \big(v_{tt} - \frac{1}{x}(x v_x)_x\big)\, dx\, dt \quad = \quad 0. \end{aligned} \tag{3.7}$$

In light of conditions (1.7) and the special form of v given by relations (3.4), (3.5), we integrate by parts each term of (3.7) to obtain the following equations:

$$\begin{aligned}
&-\int_s^T\int_0^{l_1} l_1 x v_{ttt}\big(v_{tt}-\frac{1}{x}(xv_x)_x\big)\,dx\,dt \\
&= \frac{1}{2}\int_0^{l_1} l_1 x v_{tt}^2(x,s)dx+\frac{1}{2}\int_0^{l_1} l_1 x v_{tx}^2(x,T)dx-\int_s^T l_1^2 v_{tx}(l_1,t)v_{tt}(l_1,t)dt,
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
&-\frac{1}{2}\int_s^T\int_{l_1}^{l}(l_1x+x^2)v_{ttt}\big(v_{tt}-\frac{1}{x}(xv_x)_x\big)\,dx\,dt \\
&= \frac{1}{4}\int_{l_1}^{l}(l_1x+x^2)v_{tt}^2(x,s)dx+\frac{1}{4}\int_{l_1}^{l}(l_1x+x^2)v_{tx}^2(x,T)dx \\
&\quad +\int_s^T l_1^2 v_{tx}(l_1,t)v_{tt}(l_1,t)dt+\frac{1}{2}\int_s^T\int_{l_1}^{l} x v_{tx}v_{tt}\,dx\,dt,
\end{aligned} \tag{3.9}$$

$$-\int_s^T\int_{l_1}^{l} x\Im_x(\xi v_{tt})\big(v_{tt}-\frac{1}{x}(xv_x)_x\big)\,dx\,dt=\int_s^T\int_{l_1}^{l} x^2 v_x v_{tt}\,dx\,dt. \tag{3.10}$$

Substituting (3.8)-(3.10) in (3.7) yields

$$\begin{aligned}
&\frac{l_1}{2}\int_0^{l_1} x v_{tt}^2(x,s)dx+\frac{l_1}{2}\int_0^{l_1} x v_{tx}^2(x,T)dx \\
&+\frac{1}{4}\int_{l_1}^{l}(l_1x+x^2)v_{tt}^2(x,s)dx+\frac{1}{4}\int_{l_1}^{l}(l_1x+x^2)v_{tx}^2(x,T)dx \\
&= -\frac{1}{2}\int_s^T\int_{l_1}^{l} x v_{tt}v_{tx}\,dx\,dt-\int_s^T\int_{l_1}^{l} x^2 v_{tt}v_x\,dx\,dt\,.
\end{aligned} \tag{3.11}$$

Using Young's and Poincare's inequalities, we estimate the right-hand side of (3.11) as follows

$$-\frac{1}{2}\int_s^T\int_{l_1}^{l} x v_{tt}v_{tx}\,dx\,dt\le\frac{1}{4}\int_s^T\int_{l_1}^{l} x v_{tx}^2\,dx\,dt+\frac{1}{4}\int_s^T\int_{l_1}^{l} x v_{tt}^2\,dx\,dt, \tag{3.12}$$

$$\begin{aligned}
-\int_s^T\int_{l_1}^{l} x^2 v_{tt}v_x\,dx\,dt &\le \frac{1}{2}\int_s^T\int_{l_1}^{l} x^2 v_x^2\,dx\,dt+\frac{1}{2}\int_s^T\int_{l_1}^{l} x^2 v_{tt}^2\,dx\,dt \\
&\le \frac{d}{2}\int_s^T\int_{l_1}^{l} x^2 v_{xt}^2\,dx\,dt+\frac{1}{2}\int_s^T\int_{l_1}^{l} x^2 v_{tt}^2\,dx\,dt
\end{aligned} \tag{3.13}$$

Combining (3.11)-(3.13), we arrive at

$$\int_0^{l_1} x v_{tt}^2(x,s)dx+\int_0^{l_1} x v_{tx}^2(x,T)dx$$

$$
\begin{aligned}
&+\int_{l_1}^{l}(x+x^2)v_{tt}^2(x,s)dx+\int_{l_1}^{l}(x+x^2)v_{tx}^2(x,T)dx \\
&\leq \delta\Big(\int_s^T\int_{l_1}^{l}(x+x^2)v_{tt}^2\,dx\,dt+\int_s^T\int_{l_1}^{l}(x+x^2)v_{tx}^2\,dx\,dt\Big),
\end{aligned}
\tag{3.14}
$$

where $\delta = 2\max\{d,1\}/\min\{l_1,1\}$. When we add to the right-hand side of (3.14) the quantity

$$
\delta\int_s^T\int_0^{l_1} xv_{tt}^2\,dx\,dt+\delta\int_s^T\int_0^{l_1} xv_{tx}^2\,dx\,dt,
$$

and define the function

$$
\rho(x)=\begin{cases} x & \text{if } 0<x<l_1 \\ x+x^2 & \text{if } l_1<x<l \end{cases}
$$

we deduce, from (3.14), that

$$
\begin{aligned}
&\int_0^l \rho(x)v_{tt}^2(x,s)dx+\int_0^l \rho(x)v_{tx}^2(x,T)dx \\
&\leq \delta\{\int_{Q_s}\rho(x)v_{tt}^2\,dx\,dt+\int_{Q_s}\rho(x)v_{tx}^2\,dx\,dt\}.
\end{aligned}
\tag{3.15}
$$

This inequality is basic in our proof. To use it, we introduce the new function

$$
\eta(x,t)=\int_t^T v_{\tau\tau}d\tau.
$$

Then

$$
v_t(x,t)=\eta(x,s)-\eta(x,t),\quad v_t(x,T)=\eta(x,s).
$$

Thus inequality (3.15) becomes

$$
\begin{aligned}
&\int_0^l \rho(x)v_{tt}^2(x,s)dx+(1-2\delta(T-s))\int_0^l \rho(x)\eta_x^2(x,s)dx \\
&\leq 2\delta\Big\{\int_s^T\int_0^l \rho(x)v_{tt}^2\,dx\,dt+\int_s^T\int_0^l \rho(x)\eta_x^2(x,t)\,dx\,dt\Big\}.
\end{aligned}
\tag{3.16}
$$

Hence, when $s_0>0$ satisfies $T-s_0=1/4\delta$, (3.16) implies

$$
\begin{aligned}
&\int_0^l \rho(x)v_{tt}^2(x,s)dx+\int_0^l \rho(x)\eta_x^2(x,s)dx \\
&\leq 4\delta\Big\{\int_s^T\int_0^l \rho(x)v_{tt}^2\,dx\,dt+\int_s^T\int_0^l \rho(x)\eta_x^2(x,t)\,dx\,dt\Big\}
\end{aligned}
\tag{3.17}
$$

for all $s\in[T-s_0,T]$. If, in (3.17) we put

$$
g(s)=\int_s^T\int_0^l \rho(x)v_{tt}^2\,dx\,dt+\int_s^T\int_0^l \rho(x)\eta_x^2(x,t)\,dx\,dt,
$$

then we have $\frac{-dg}{ds} \le 4\delta g(s)$, from which it follows that

$$\frac{-d}{ds}\left(g(s)\exp(4\delta s)\right) \le 0.$$

Integrating this equation over (s,T) and taking in account that $g(T)=0$, we obtain

$$g(s)\exp(4\delta s) \le 0.$$

This inequality guarantees that $g(s)=0$ for all $s \in [T-s_0, T]$, which implies that $v_{tt}=0$ on Q_s where $s \in [T-s_0, T]$. Hence it follows, from (3.6), that $\omega \equiv 0$ almost everywhere on Q_{T-s_0}. Proceeding this way step by step along the rectangle with side s_0, we prove that $\omega \equiv 0$ almost everywhere on Q. This completes the proof of the Proposition 3.3.

Proof of Theorem 3.1 Suppose that for some $W=(\omega,\omega_1,\omega_2) \in R(L)^{\perp}$,

$$(\mathcal{L}v,\omega)_{L^2_\theta(Q)} + (\ell_1 v,\omega_1)_{H^1_\theta((0,l))} + (\ell_2 v,\omega_2)_{L^2_\theta((0,l))} = 0. \tag{3.18}$$

Then we must prove that $W \equiv 0$. Putting $v \in D_0(L)$ into (3.18), we have

$$(\mathcal{L}v,\omega)_{L^2_\theta(Q)} = 0.$$

Hence Proposition 3.3 implies that $\omega \equiv 0$. Thus (3.18) takes the form

$$(\ell_1 v,\omega_1)_{H^1_\theta((0,l))} + (\ell_2 v,\omega_2)_{L^2_\theta((0,l))} = 0, \quad \forall v \in D(L). \tag{3.19}$$

Since the quantities $\ell_1 v$ and $\ell_2 v$ can vanish independently and the ranges of the trace operators ℓ_1 and ℓ_2 are dense in the spaces $H^1_\theta((0,l))$ and $L^2_\theta((0,l))$ respectively, the equation (3.19) implies that $\omega_1 \equiv 0$, $\omega_2 \equiv 0$. Hence $W \equiv 0$. The proof of Theorem 3.1 is established.

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