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SEMI-CLASSICAL STATES FOR SCHRÖDINGER-POISSON SYSTEMS ON $\mathbb{R}^3$

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Abstract. In this article, we study the nonlinear Schrödinger-Poisson equation

$$-\epsilon^2\Delta u + V(x)u + \phi(x)u = f(u), \quad x \in \mathbb{R}^3,$$
$$-\epsilon^2\Delta\phi = u^2, \quad \lim_{|x|\to\infty}\phi(x) = 0\,.$$

Under suitable assumptions on $V(x)$ and $f(s)$, we prove the existence of ground state solution around local minima of the potential $V(x)$ as $\epsilon \to 0$. Also, we show the exponential decay of ground state solution.

1. Introduction

Consider the nonlinear Schrödinger equation:

$$i\epsilon\frac{\partial\psi}{\partial t} = -\epsilon^2\Delta\psi + \tilde{V}\psi - f(\psi) \tag{1.1}$$

coupled with the Poisson equation

$$-\epsilon^2\Delta\phi = |\psi|^2, \tag{1.2}$$

where ϵ is the planck constant, i is the imaginary unit and $\tilde{V}, \psi$ are real functions on $\mathbb{R}^3$ and represent the effective potential and electric potential respectively. $\psi(x,t) \to \mathbb{C}$ and f is supposed to satisfy $f(\alpha e^{i\theta}) = f(\alpha)e^{i\theta}$ for all $\theta, \alpha \in \mathbb{R}$. Problem (1.1), (1.2) arose from semiconductor theory; see e.g, [4, 10, 25] and the references therein for more physical background.

We are interested in standing wave solutions, namely solutions of form $\psi(x,t) = u(x)\exp(i\omega t/\epsilon)$ with $u(x) > 0$ in $\mathbb{R}^3$ and $\omega > 0$ (the frequency), then it is not difficult to see that $u(x)$ must satisfy

$$\begin{gathered} -\epsilon^2\Delta u + V(x)u + \phi(x)u = f(u), \quad x \in \mathbb{R}^3, \\ -\epsilon^2\Delta\phi = u^2, \lim_{|x|\to\infty}\phi(x) = 0. \end{gathered} \tag{1.3}$$

An interesting class of solutions of (1.3), sometimes called semi-classical states, are families solutions $u_\epsilon(x)$ which concentrate and develop a spike shape around one, or more, special points in $\mathbb{R}^3$, which vanishing elsewhere as $\epsilon \to 0$.

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Similar equations have been studied extensively by many authors concerning existence, non-existence, multiplicity when the nonlinearity $f(u) = u^p$, $1 < p < 5$ and we cite a couple of them. In [16, 17], the authors proved the existence of radially symmetric solutions concentrating on the spheres, and in [18], there is a positive bound state solution concentrating on the local minimum of the potential V. The existence of radial solution was obtained in [5] for the case $3 \le p < 5$. In [7], the authors constructed positive radially symmetric bound states of (1.3) with $1 < p < 11/7$. In [6], a related Pohozăev identity was established and the authors showed that (1.3) has nontrivial solutions in the case $0 \le p < 1$ or $p \ge 5$. In [2, 13, 23], the authors proved the existence of infinity many radially symmetric solutions. Ruiz and Vaira [24] proved the existence of multi-bump solutions whose bumps concentrating around a local minimum of the potential. Also, there are a lot of results on Schrödinger-Poisson systems with general classes of nonlinear terms. In [26], existence and nonexistence nontrivial solutions of Schrödinger-Poisson system with sign-changing potential were obtained by using variational methods. Sun, Wu and Feng [27] studied the multiplicity of positive solutions for a nonlinear Schrödinger-Poisson system when the nonlinearity $f(x, u) = Q(x)|u|^{p-2}u, 2 < p < 6$; furthermore, they showed that the number of positive solutions was dependent of the profile of $Q(x)$. In [28], the authors proved the existence and nonexistence solutions of Schrödinger-Poisson system with an asymptotically linear nonlinearity. In [29], existence and multiplicity results were established. We refer to [1, 2, 3, 8, 9, 11, 13, 14, 20, 22] for some more results on this subject.

Recently, semi-classical states for Schrödinger-Poisson systems with much more general nonlinear term have been object of interest for many authors. Bonheure, Di Cosmo and Mercuric [11] proved the existence the solutions for the weighted nonlinear Schrödinger-Poisson systems whose bumps concentrating around a circle. He and Zou [20] showed the existence and concentration of ground states for Schrödinger-Poisson equations with critical growth.

But, in most of the above papers, the potential $V(x)$ either has a limit at infinity, or is required to be radial symmetry respect to x. Motivated by some related works, the aim of this paper is to study the existence of solution of (1.3) concentrating on a given set of local minima of $V(x)$. We take the penalization arguments going back to del Pino and Felmer [21] to a wider class of the potentials $V(x)$ and nonlinearity $f(s) \in C^1(\mathbb{R}, \mathbb{R})$.

In this article, we use the following assumptions:

- (A1) $V(x) \ge V_0 > 0$ for all $x \in \mathbb{R}^3$.
- (A2) $f(s) = o(s^3)$ as $s \to 0$.
- (A3) There exists $q \in (3, 5)$ such that $\lim_{s\to+\infty} f(s)/s^q = 0$.
- (A4) There exists some $4 < \theta < q + 1$ such that

$$0 < \theta F(s) = \theta \int_0^s f(t)dt \le f(s)s \quad \text{for all } s > 0.$$

- (A5) For all $x \in \mathbb{R}^3$, $f(x, s)/s$ is nondecreasing in $s \ge 0$.

The main result of this paper reads as follows.

Theorem 1.1. *Assume that* (A1)–(A5) *hold, and that there is a bounded and compact domain* Λ *in* $\mathbb{R}^3$ *such that*

$$\inf_{x\in\Lambda} V(x) < \min_{x\in\partial\Lambda} V(x).$$

Then there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$, problem (1.3) has a positive solution u_ϵ. Moreover, u_ϵ has at most one local (hence global) maximum $x_\epsilon \in \Lambda$ such that

$$\lim_{\epsilon \to 0} V(x_\epsilon) = V_0.$$

Also, there are constants $C, c > 0$ such that

$$u_\epsilon(x) \leq C \exp(-c\frac{x - x_\epsilon}{\epsilon}). \tag{1.4}$$

Remark 1.2. (i) We point out that no restriction on the global behavior of $V(x)$ is required other than condition (A1). This is an improvement on some previous works, see, e.g.,[11] [20] and references therein.

(ii) Condition (A5) holds if $f(s)/s^3$ is an increasing function of $s > 0$. In fact, that $f(s)/s^3$ is increasing is required in [20].

This article is organized as follows: In section2, influenced by the work of del Pino and Felmer [21], we introduce a modified functional for any $\epsilon > 0$ and show it has a ground state solution $u_\epsilon(x)$. In Section3, we give the uniform boundedness of $\max_{x \in \partial\Lambda} u_\epsilon(x)$ and the critical value c_ϵ when ϵ goes to zero. In section4, we show the critical point of the modified functional which satisfies the original problem (1.3), and investigate its concentration and exponential decay behavior, which completes the proof Theorem 1.1.

Hereafter we use the following notation:

- $H^1(\mathbb{R}^3)$ is usual Sobolev space endowed with the standard scalar product and norm
$$(u, v) = \int_{\mathbb{R}^3} (\nabla u \nabla v + uv)dx; \|u\|^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2)dx.$$
- $D^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm
$$\|u\|^2_{D^{1,2}(\mathbb{R}^3)} = \int_{\mathbb{R}^3} |\nabla u|^2 dx.$$
- H^{-1} denotes the dual space of $H^1(\mathbb{R}^3)$.
- $L^q(\Omega), 1 \leq q \leq +\infty, \Omega \subseteq \mathbb{R}^3$, denotes a Lebesgue space, the norm in $L^q(\Omega)$ is denoted by $|u|_{q,\Omega}$.
- For any $R > 0$ and for any $y \in \mathbb{R}^3, B_R(y)$ denotes the ball of radius R centered at y.
- C, c are various positive constants.
- $o(1)$ denotes the quantity which tends to zero as $n \to \infty$.

It is well known that for every $u \in H^1(\mathbb{R}^3)$, the Lax-Milgram theorem implies that there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} \nabla \phi_u \nabla v dx = \int_{\mathbb{R}^3} u^2 dx, \quad \forall v \in D^{1,2}(\mathbb{R}^3),$$

where ϕ_u is a weak solution of $-\Delta\phi = u^2$ with

$$\phi_u(x) = \int_{\mathbb{R}^3} \frac{u^2(y)}{|x - y|} dy.$$

Substituting ϕ_u in (1.3), we can rewrite (1.3) as the equivalent equation

$$-\epsilon^2 u + V(x)u + \epsilon^{-2}\phi_u u = f(u). \tag{1.5}$$

Let

$$H = \{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2 dx < +\infty\}$$

be the Sobolev space endowed with the norm

$$\|u\|_\epsilon^2 = \int_{\mathbb{R}^3} (\epsilon^2 |\nabla u|^2 + V(x)u^2) dx.$$

We see that (1.5) is variational and its solutions are the critical points of the functional

$$I_\epsilon(u) = \frac{1}{2}\int_{\mathbb{R}^3} (\epsilon^2 |\nabla u|^2 + V(x)u^2) dx + \frac{1}{4\epsilon^2}\int_{\mathbb{R}^3} \phi_u(x) u^2 dx - \int_{\mathbb{R}^3} F(u) dx. \tag{1.6}$$

Clearly, under the hypotheses (A2)–(A5), we see that I_ϵ is well-defined C^1 functional. In the following proposition, we summarize some properties of ϕ_u, which are useful to study our problem.

Proposition 1.3 ([12]). *For any $u \in H^1(\mathbb{R}^3)$, we have*

- (i) *$\phi_u : H^1(\mathbb{R}^3) \to D^{1,2}(\mathbb{R}^3)$ is continuous, and maps bounded sets into bounded sets;*
- (ii) *if $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, then $\phi_{u_n} \rightharpoonup \phi_u$ in $D^{1,2}(\mathbb{R}^3)$;*
- (iii) *$\phi_u \geq 0, \|\phi_u\|_{D^{1,2}(\mathbb{R}^3)}$, and $\int_{\mathbb{R}^3} \phi_u u^2 dx \leq C\|u\|^4$;*
- (iv) *$\phi_{tu}(x) = t^2 \phi_u$ for all $t \in \mathbb{R}$.*

2. Solution of the modified equation

In this section, we find a solution u_ϵ of problem (1.3) concentrating on a given set Λ, we modify the nonlinearity $f(s)$. Here we follow an approach used by del Pino and Felmer [21].

Let $k > \frac{\theta}{\theta - 4}$, $a > 0$ be such that $\frac{f(a)}{a} = \frac{V_0}{k}$, and set

$$\tilde{f}(s) = \begin{cases} f(s), & \text{if } s \leq a, \\ \frac{V_0}{k} s, & \text{if } s > a, \end{cases} \tag{2.1}$$

and define

$$g(., s) = \chi_\Lambda f(s) + (1 - \chi_\Lambda)\tilde{f}(s), \tag{2.2}$$

where Λ is the bounded domain as in the assumptions of Theorem1.1, and χ_Λ denotes its characteristic function. It is easy to check that $g(x, s)$ satisfies the following assumptions:

- (A6) $g(x, s) = o(s^3)$ as $s \to 0$.
- (A7) There exists $q \in (3, 5)$ such that $\lim_{s \to +\infty} \frac{g(x,s)}{s^q} = 0$.
- (A8) There exists a bounded subset K of $\mathbb{R}^3$, $\mathrm{int}(K) \neq \emptyset$ such that

$$0 < \theta G(x, s) \leq g(x, s)s \quad \text{for all } x \in K, s > 0,$$

$$0 \leq 2G(x, s) \leq g(x, s)s \leq \frac{1}{k} V(x)s^2 \quad \text{for all } s > 0, x \in K^c.$$

- (A9) The function $\frac{g(x,s)}{s}$ is increasing for $s > 0$.

Now, we consider the modified equation

$$\begin{gathered} -\epsilon^2 \Delta u + V(x)u + \phi(x)u = g(x, s), \quad x \in \mathbb{R}^3, \\ -\epsilon^2 \Delta \phi = u^2, \end{gathered} \tag{2.3}$$

where V satisfies condition (A1), and g satisfies (A6)–(A9). Here we have set $\epsilon = 1$ for notational simplicity.

The functional associated with (2.3) is

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u(x)u^2 dx - \int_{\mathbb{R}^3} G(x,u)dx, \tag{2.4}$$

which is of class C^1 in H with associated norm $\|\cdot\|_H$.

In the rest of this section, we show some lemmas related to the functional J. First, we show the functional J satisfying the mountain pass geometry.

Lemma 2.1. *The functional J satisfies the following conditions:*

(i) *There exist $\alpha, \rho > 0$ such that $J(u) \geq \alpha$ for all $\|u\|_H = \rho$.*
(ii) *There exists $e \in B^c_\rho(0)$ with $J(e) < 0$.*

Proof. (i) For any $u \in H\backslash\{0\}$ and $\varepsilon > 0$, by (A2) and (A3) there exists $C(\varepsilon) > 0$ such that

$$|f(s)| \leq \varepsilon|s| + C_\varepsilon|s|^q, \quad \forall s \in \mathbb{R}.$$

By the Sobolev embedding $H \hookrightarrow L^p(\mathbb{R}^3)$, with $p \in [2, 6]$, we have

$$\begin{aligned} J(u) &\geq \frac{1}{2}\|u\|_H^2 - \int_{\mathbb{R}^3} [\chi_\Lambda(x)F(u) + (1-\chi_\Lambda(x))\tilde{F}(u)]dx \\ &\geq \frac{1}{2}\|u\|_H^2 - \int_{\mathbb{R}^3} F(u)dx \\ &\geq \frac{1}{2}\|u\|_H^2 - \frac{\varepsilon}{2}\int_{\mathbb{R}^3} |u|^2 dx - \frac{C_\varepsilon}{q+1}|u|^{p+1}dx \\ &\geq \frac{1}{2}\|u\|_H^2 - C_1\varepsilon\|u\|_H^2 - C_2C_\varepsilon\|u\|_H^{p+1}. \end{aligned}$$

Since ε is arbitrarily small, we can choose constants α, ρ such that $J(u) \geq \rho > 0$ for all $\|u\|_H = \rho$.

(ii) By (A4), we have $F(s) \geq Cs^\theta - C$ for all $t > 0$, Choosing $u \in H\backslash\{0\}$ not negative, with its support contained in the set K, we see that

$$\begin{aligned} J(tu) &= \frac{t^2}{2}\|u\|_H^2 + \frac{t^4}{4}\int_{\mathbb{R}^3} \phi_u u^2 dx - \int_{\mathbb{R}^3} G(x,tu)dx \\ &\leq \frac{t^2}{2}\|u\|_H^2 + \frac{t^4}{4}\int_{\mathbb{R}^3} \phi_u u^2 dx - Ct^\theta \int_K u^\theta dx + C|K| < 0 \end{aligned}$$

for some $t >$ large enough. So, we can choose $e = t^*u$ for some $t^* > 0$, and (ii) follows.

By lemma 2.1 and the mountain pass theorem, there is a $(PS)_c$ sequence $\{u_n\} \subset H$ such that $J(u_n) \to c$ in H^{-1} with the minmax value

$$c = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} J(\gamma(t)), \tag{2.5}$$

where

$$\Gamma := \{\gamma \in C([0,1], H) : \gamma(0) = 0, J(\gamma(1)) < 0\}.$$

□

Lemma 2.2. *Let $\{u_n\} \subset H$ be a $(PS)_c$ sequence for $c > 0$. Then u_n has a convergent subsequence.*

Proof. First, we show that $\{u_n\}$ is bounded in H. In fact, using $(A8)$ we easily see that

$$\int_{\mathbb{R}^3}(|\nabla u_n|^2+V(x)u_n^2)dx+\int_{\mathbb{R}^3}\phi_{u_n}u_n^2dx \geq \int_K g(x,u_n)u_ndx+o(\|u_n\|_H). \tag{2.6}$$

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u_n|^2+V(x)u_n^2)dx+\frac{1}{4}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2dx\\ &=\int_{\mathbb{R}^3}G(x,u_n)dx+O(1)\\ &\leq \int_K G(x,u_n)dx+\frac{1}{2k}\int_{K^c}V(x)u_n^2dx+O(1)\end{aligned} \tag{2.7}$$

Thus, from (2.6) (2.7) and (A8) we find

$$\frac{2}{k}\int_{K^c}V(x)u_n^2dx+o(\|u_n\|_H)+O(1)\geq(1-\frac{2}{k})\int_{\mathbb{R}^3}(|\nabla u_n|^2+V(x)u_n^2)dx. \tag{2.8}$$

Then, it follows from the choice of k in $(A8)$ that $\{u_n\}$ is bounded in H.

Then there is a subsequence, still denoted by $\{u_n\}$ such that $u_n \rightharpoonup u$ weakly in H. We now prove this convergence is actually strong. In deed, it suffices to show that, given $\delta>0$, there is an $R>0$ such that

$$\limsup_{n\to\infty}\int_{B_R^c}(|\nabla u_n|^2+V(x)u_n^2)dx\leq\delta. \tag{2.9}$$

Let $\xi_R(x)\in C^\infty(\mathbb{R}^3,\mathbb{R})$ be a cut-off function such that $0\leq\xi_R\leq 1$ and

$$\xi_R(x)=\begin{cases}0 & \text{for } |x|\leq\frac{R}{2},\\ 1, & \text{for } |x|\geq R\end{cases}$$

and $|\nabla\xi_R(x)|\leq\frac{C}{R}$ for all $x\in\mathbb{R}^3$. Moreover we may assume that R is chosen so that $K\subset B_{\frac{R}{2}}$. Since $\{u_n\}$ is a bounded $(PS)_c$ sequence, we have that

$$\langle J'(u_n),\xi_Ru_n\rangle=o(1), \tag{2.10}$$

so that

$$\begin{aligned}&\int_{\mathbb{R}^3}(|\nabla u_n|^2+V(x)u_n^2)dx+\int_{\mathbb{R}^3}u_n\nabla u_n\nabla\xi_Rdx+\int_{\mathbb{R}^3}\phi_{u_n}u_n\xi_Rdx\\ &=\int_{\mathbb{R}^3}g(x,u_n)u_n\xi_Rdx+o(1)\leq\frac{1}{k}\int_{\mathbb{R}^3}V(x)u_n^2\xi_Rdx+o(1).\end{aligned} \tag{2.11}$$

We conclude that

$$\int_{B_R^c}V(x)u_n^2dx\leq\frac{C}{R}\|u_n\|_{L^2(\mathbb{R}^3)}\|\nabla u_n\|_{L^2(\mathbb{R}^3)}, \tag{2.12}$$

from where (2.9) follows. □

Lemma 2.1 implies that c defined in (2.5) is a critical value of J.

Remark 2.3. Similar to the proof of lemma 2.2, it is not difficult to see that c can be characterized as

$$c=\inf_{u\in H\setminus\{0\}}\sup_{t\geq 0}J(tu).$$

Since the modified function g satisfies assumptions (A6)–(A9), the results of the above yield the following lemma.

Lemma 2.4. *For any $\epsilon > 0$, there exists a critical point for J_ϵ at level*

$$J_\epsilon(u_\epsilon) = c_\epsilon = \inf_{\gamma_\epsilon\in\Gamma}\max_{0\le t\le 1} J_\epsilon(\gamma(t)), \tag{2.13}$$

where

$$\begin{gathered}\Gamma_\epsilon := \{\gamma \in C([0,1], H) : \gamma(0) = 0, J_\epsilon(\gamma(1)) < 0\},\\ J_\epsilon(u) = \frac{1}{2}\int_{\mathbb{R}^3}(\epsilon^2|\nabla u|^2 + V(x)u^2)dx + \frac{1}{4\epsilon^2}\int_{\mathbb{R}^3}\phi_u u^2 dx - \int_{\mathbb{R}^3} G(x,u)dx.\end{gathered} \tag{2.14}$$

3. Some estimates

To show that the solution u_ϵ found in lemma 2.4 satisfies the original problem and concentrates at some point in Λ, we need to study the behavior of u_ϵ as $\epsilon \to 0$. We begin with an energy estimate.

Proposition 3.1 (Upper estimate of the critical value)**.** *For ϵ small enough, the critical value c_ϵ defined* (2.13) *satisfies*

$$c_\epsilon = J_\epsilon(u_\epsilon) \le \epsilon^3(c_0 + o(1)) \quad \text{as } \epsilon \to 0. \tag{3.1}$$

Moreover, there exists $C > 0$ such that

$$\int_{\mathbb{R}^3}(\epsilon^2|\nabla u_\epsilon|^2 + V(x)u_\epsilon^2)dx \le C\epsilon^3. \tag{3.2}$$

Proof. Set $V_0 = \min_\Lambda V = V(x_0)$, and let

$$c_0 := \inf_{\gamma\in\Gamma}\max_{0\le t\le 1} I_0(\gamma(t)), \tag{3.3}$$

where

$$\begin{gathered}I_0(u) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + V_0u^2)dx + \frac{1}{4}\int_{\mathbb{R}^3}\phi_u u^2 dx - \int_{\mathbb{R}^3} F(u)dx,\\ \Gamma := \{\gamma \in C([0,1], H^1(\mathbb{R}^3)) : \gamma(0) = 0, I_0(\gamma(1)) < 0\}.\end{gathered}$$

From (3.3), for any $\delta > 0$, there exists a continuous path $\gamma_\delta : [0,1] \to H^1(\mathbb{R}^3)$ such that $\gamma_\delta(0) = 0, I_0(\gamma_\delta(1)) < 0$ and

$$c_0 \le \max_{0\le t\le 1} I_0(\gamma_\delta(t)) \le c_0 + \delta.$$

Let η be a smooth cut-off function with support in Λ such that $0 \le \eta \le 1, \eta = 1$ in a neighborhood of x_0 and $|\nabla\eta| \le C$. We consider the path

$$\bar{\gamma_\delta}(t)(x) = \eta(x)\gamma_\delta(t)(\frac{x - x_0}{\epsilon}).$$

Setting

$$\bar{\gamma_\delta}(t)(x) := v_t(\frac{x - x_0}{\epsilon}),$$

we compute, by a charge of variable

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}^3}[\epsilon^2|\nabla\bar{\gamma_\delta}(t)|^2 + V(x)\bar{\gamma_\delta}(t)^2]dx - \int_{\mathbb{R}^3} G(x, \bar{\gamma_\delta}(t))dx\\ &= \epsilon^3\frac{1}{2}[|\nabla v_t(x)|^2 + V(x_0 + \epsilon x)v_t^2(x)]dx - \epsilon^3\int_{\mathbb{R}^3} G(x_0 + \epsilon x, v_t(x))dx.\end{aligned} \tag{3.4}$$

The Hardy-Littlewood Sobolev inequality leads to

$$\int_{\mathbb{R}^3}\phi_{\bar{\gamma_\delta}(t)(x)}\bar{\gamma_\delta}(t)(x)^2dx = \int_{\mathbb{R}^3}\Big[\int_{\mathbb{R}^3}\frac{\bar{\gamma_\delta}(t)(y)^2}{|x-y|}\bar{\gamma_\delta}(t)dy\Big]dx$$

$$
\begin{aligned}
&= \epsilon^5 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{v_t^2(x) v_t^2(y)}{|x-y|}\, dx\, dy \\
&= \epsilon^5 \int_{\mathbb{R}^3} \phi_{v_t} v_t^2 dx.
\end{aligned}
$$

For ϵ small enough, we obtain

$$
\epsilon^{-3} J_\epsilon(\bar{\gamma_\delta}(t)) \to I_0(\gamma_\delta(t)) + o(1).
$$

It follows that ϵ small enough, $\bar{\gamma_\delta}$ belongs to the class of paths Γ_ϵ defined by (2.14). We deduce that

$$
\epsilon^{-3} c_\epsilon \le \epsilon^{-3} \max_{0 \le t \le 1} J_\epsilon(\bar{\gamma_\delta}(t)) \to \max_{0 \le t \le 1} I_0(\bar{\gamma_\delta}(t)) + o(1) \le (c_0 + \delta) + o(1).
$$

Since $\delta > 0$ is arbitrary, (3.1) is proved.

$$
\begin{aligned}
J_\epsilon(u_\epsilon) &= \frac{1}{2} \int_{\mathbb{R}^3} (\epsilon^2 |\nabla u_\epsilon|^2 + V(x) u_\epsilon^2) dx + \frac{1}{4\epsilon^2} \int_{\mathbb{R}^3} \phi_{u_\epsilon} u_\epsilon^2 dx - \int_{\mathbb{R}^3} G(x, u_\epsilon) dx \\
&= \frac{1}{2} \int_{\mathbb{R}^3} (\epsilon^2 |\nabla u_\epsilon|^2 + V(x) u_\epsilon^2) dx + \frac{1}{4\epsilon^2} \int_{\mathbb{R}^3} \phi_{u_\epsilon} u_\epsilon^2 dx - \int_K G(x, u_\epsilon) dx \\
&\quad - \int_{\mathbb{R}^3 \setminus \{K\}} G(x, u_\epsilon) dx \qquad (3.5)\\
&\ge \frac{1}{4} \int_{\mathbb{R}^3} (\epsilon^2 |\nabla u_\epsilon|^2 + V(x) u_\epsilon^2) dx - \frac{1}{2k} \int_{\mathbb{R}^3} V(x) u_\epsilon^2 dx \\
&\ge (\frac{1}{4} - \frac{1}{2k}) \int_{\mathbb{R}^3} (\epsilon^2 |\nabla u_\epsilon|^2 + V(x) u_\epsilon^2) dx,
\end{aligned}
$$

where $C = \frac{1}{4} - \frac{1}{2k} > 0$ thanks to the choice of k. Combining (3.1) and (3.5), it is easy to obtain (3.2). $\square$

Next, we give a proposition that is the crucial step in the proof of Theorem 1.1.

Proposition 3.2.

$$
\lim_{\epsilon \to 0} \max_{\partial \Lambda} u_\epsilon(x) = 0. \tag{3.6}
$$

Moreover, for all ϵ sufficiently small enough, u_ϵ possesses one local maximum $x_\epsilon \in \Lambda$ and we must have

$$
\lim_{\epsilon \to 0} V(x_\epsilon) = V_0 = \min_{x \in \Lambda} V(x). \tag{3.7}
$$

Proof. To prove this proposition we establish that If $\epsilon_n \to 0$ and $x_n \in \Lambda$ are such that $u_{\epsilon_n} \ge b > 0$, then

$$
\lim_{n \to \infty} V(x_n) = V_0. \tag{3.8}
$$

We take three steps to prove this claim.

Step1: We argue by contradiction. Thus we assume, passing to a subsequence, that $x_n \to x^* \in \bar{\Lambda}$ and

$$
V(x^*) > V_0. \tag{3.9}
$$

We consider the sequence $v_n(x) = u_{\epsilon_n}(x_n + \epsilon_n x)$. A simple computation shows

$$
\epsilon^2 \phi_{v_n}(x) = \phi_{u_{\epsilon_n}}(x_n + \epsilon_n x).
$$

The function v_n satisfies the equation

$$
-\Delta v_n + V(x_n + \epsilon_n x) v_n + \phi_{v_n} v_n = g(x_n + \epsilon_n x, v_n), x \in \Omega_n, \tag{3.10}
$$

where $\Omega_n = \epsilon_n^{-1}\{H - x_n\}$. As a consequence of (3.2), we see that v_n is bounded in $H^1(\mathbb{R}^3)$, and from elliptic estimates, we deduce that there exists $v \in H^1(\mathbb{R}^3)$ such that

$$v_n \to v \quad \text{in } C^2_{\rm loc}(\mathbb{R}^3).$$

Let $\chi_n(x) = \chi_\Lambda(x_n + \epsilon_n x)$, then $\chi_n(x) \rightharpoonup \chi$ in any $L^p(\mathbb{R}^3)$ over compacts with $0 \le \chi \le 1$. Now, we claim that

$$\int_{\mathbb{R}^3} \phi_{v_n} v_n \varphi dx \to \int_{\mathbb{R}^3} \phi_v v \varphi dx \quad \text{for any } \varphi \in C_0^\infty(\mathbb{R}^3).$$

In fact, we can assume support $\varphi \subset \Omega$, where Ω is a bounded domain. Then

$$\begin{aligned}
&|\int_{\mathbb{R}^3} \phi_{v_n} v_n \varphi dx - \int_{\mathbb{R}^3} \phi_v v \varphi dx| \\
&= |\int_{\mathbb{R}^3} \phi_{v_n}(v_n - v)\varphi dx + \int_{\mathbb{R}^3} (\phi_{v_n} - \phi_v) v\varphi dx| \\
&\le |\int_{\mathbb{R}^3} \phi_{v_n}(v_n - v)\varphi dx| + |\int_{\mathbb{R}^3} (\phi_{v_n} - \phi_v) v\varphi dx| \\
&\le |\phi_{v_n}|_{L_6(\Omega)} |v_n - v|_{L_2(\Omega)} |\varphi|_{L_3(\Omega)} + o(1) \to o(1).
\end{aligned}$$

Therefore, v satisfies the limiting equation

$$-\Delta v + V(x^*)v + \phi_v v = \bar{g}(x, v), \tag{3.11}$$

where

$$\bar{g}(x, s) = \chi(x) f(s) + (1 - \chi(x))\tilde{f}(s).$$

Associated with (3.11) we have functional $\bar{J} : H^1(\mathbb{R}^3) \to \mathbb{R}$ defined as

$$\begin{aligned}
\bar{J}(u) = \frac{1}{2}\int_{\mathbb{R}^3} [|\nabla u|^2 + V(x^*)u^2] dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 dx \\
- \int_{\mathbb{R}^3} \bar{G}(x, u) dx, u \in H^1(\mathbb{R}^3),
\end{aligned} \tag{3.12}$$

where $\bar{G}(x, s) = \int_0^s \bar{g}(x, t) dt$. Then v is a critical point of $\bar{J}$. Set

$$\begin{aligned}
J_n(u) = \frac{1}{2}\int_{\Omega_n} [|\nabla u|^2 + V(x_n + \epsilon_n x)u^2] dx + \frac{1}{4}\int_{\Omega_n} \phi_u u^2 dx \\
- \int_{\Omega_n} G(x_n + \epsilon_n x, u) dx, u \in H_0^1(\Omega_n).
\end{aligned}$$

Then $J_n(v_n) = \epsilon_n^{-3} J_{\epsilon_n}(u_{\epsilon_n})$. So the key step in the proof of proposition is the following step.

Step2: $\liminf_{n\to\infty} J_n(v_n) \ge \bar{J}(v)$. In particular, $\bar{J}(v) \le c_0$, where c_0 is given by (3.3).

Proof: Write

$$h_n = \frac{1}{2}[|\nabla v_n|^2 + V(x_n + \epsilon_n x)v_n^2] + \frac{1}{4}\phi_{v_n} v_n^2 - \bar{G}(x_n + \epsilon_n x, u).$$

Then, choose $R > 0$, since v_n converges in the C^1 sense over compacts to v, we have

$$\lim_{n\to\infty} \int_{B_R} h_n dx = \frac{1}{2}\int_{B_R} [|\nabla v|^2 + V(x^*)v^2] dx + \frac{1}{4}\int_{B_R} \phi_v v^2 dx - \int_{B_R} \bar{G}(x, v) dx.$$

Since $v \in H^1(\mathbb{R}^3)$, for each $\delta > 0$, we have

$$\lim_{n\to\infty} \int_{B_R} h_n dx \geq \bar{J}(v) - \delta, \tag{3.13}$$

provided that R was chosen sufficiently large. Then it only suffices to check that for large enough R

$$\lim_{n\to\infty} \int_{\Omega_n \setminus B_R} h_n dx \geq -\delta. \tag{3.14}$$

For any fixed $R > 0$, let $\xi_R(x) \in C_0^\infty(\mathbb{R}^3, \mathbb{R})$ be a cut-off function such that

$$\xi_R(x) = \begin{cases} 0 & \text{for } |x| \leq R-1, \\ 1, & \text{for } |x| \geq R, \end{cases}$$

and $|\nabla \xi_R(x)| \leq \frac{C}{R}$ for all $x \in \mathbb{R}^3$ and $C > 0$ is a constant.

We use $w_n = \xi_R v_n \in H^1(\Omega_n)$ as a test function for $J_n'(v_n) = 0$ to obtain

$$\begin{aligned} 0 = J_n'(v_n)w_n &= E_n + \int_{\Omega_n \setminus B_R} (2h_n + g_n)dx + \int_{\Omega_n \setminus B_R} \phi_{v_n} v_n^2 dx \\ &\leq E_n + \int_{\Omega_n \setminus B_R} 2h_n dx + \int_{\Omega_n \setminus B_R} \phi_{v_n} v_n^2 dx, \end{aligned} \tag{3.15}$$

where $g_n = 2G(x_n + \epsilon_n x, v_n) - g(x_n + \epsilon_n x, v_n)v_n$, and E_n is given by

$$\begin{aligned} E_n &= \int_{B_R \setminus B_{R-1}} [\nabla v_n \nabla(\xi_R v_n) + V(x_n + \epsilon_n x)\xi_R v_n^2 + \phi_{v_n} v_n^2 \xi_R] dx \\ &= \int_{B_R \setminus B_{R-1}} g(x_n + \epsilon_n x, v_n)\xi_R v_n dx. \end{aligned} \tag{3.16}$$

Since v_n is bounded in $H^1(\mathbb{R}^3)$, it follows that $\int_{\mathbb{R}^3} \phi_{v_n} v_n^2 dx \leq C\|u_n\|^4$. The fact that $v \in H^1(\mathbb{R}^3)$ implies that for given $\delta > 0$, there exists $R > 0$ sufficiently large such that

$$\lim_{n\to\infty} |E_n| \leq \delta, \quad \int_{\Omega_n \setminus B_R} \phi_{v_n} v_n^2 dx \leq \delta.$$

On the other hand, the definition of g together with the properties of f give that $g_n \leq 0$. Using this in (3.15), (3.14) follows, and the proof of step2 is complete.

Step3: Now, we are ready to obtain a contradiction with (3.8). Since v is a critical point of $\bar{J}$, and $\bar{g}$ satisfies $(A9)$, we have that

$$\bar{J}(v) = \max_{t>0} \bar{J}(tv). \tag{3.17}$$

Then since $f(s) \geq \tilde{f}(s)$ for all $s > 0$ we have

$$\bar{J}(v) \geq \inf_{u \in H^1(\mathbb{R}^3)\setminus\{0\}} \sup_{\tau > 0} I^*(\tau u) \Delta q c^*, \tag{3.18}$$

where

$$I^*(u) = \frac{1}{2} \int_{\mathbb{R}^3} [|\nabla u|^2 + V(x^*)u^2] dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u u^2 dx - \int_{\mathbb{R}^3} F(u) dx. \tag{3.19}$$

But, since $V(x^*) > V_0$, we have $c^* > c_0$; hence $\bar{J}(v) > c^*$, which contradicts step 2, and the proof of the claim, i.e. (3.8) is follows.

To conclude the proof of proposition 3.2, we show that u_ϵ has at most one maximum point in Λ. The proofs rely on the the arguments carried out in step2 and so we sketch it. By contradiction, assume that, the existence of sequence $\epsilon_n \to 0$

such that u_{ϵ_n} has two distinct maxima x_n^1 and x_n^2 in Λ. Set $v_n(x) = u_{\epsilon_n}(x_n^1 + \epsilon_n x)$, and it is easy to check that $\epsilon_n^{-1}(x_n^2 - x_n^1)$ is a maximum point of $v_n(x)$, two cases occur.

Case 1: $\epsilon_n^{-1}(x_n^2 - x_n^1)$ is bounded. From (3.2) and elliptic estimates, up to a subsequence, $v_n \to v$ uniformly over compacts, where $v \in H^1(\mathbb{R}^3)$ maximizes at zero and solves $-\Delta v + V(x^1)v + \phi_v v = f(v)$, here $x^1 = \lim_{n\to\infty} x_n^1$. Since $\epsilon_n^{-1}(x_n^2 - x_n^1)$ is bounded and hence, up to a subsequence, it converges to some $p \in \mathbb{R}^3$. So we conclude that $p = 0$; therefore $\epsilon_n^{-1}(x_n^2 - x_n^1) \in B_r$ for n large enough, which is impossible since 0 is the only critical point of v in B_r.

Case2: $\epsilon_n^{-1}(x_n^2 - x_n^1)$ is unbounded. Let $\tilde{v_n}(x) = u_{\epsilon_n}(\epsilon_n x + x_n^2)$, then there exists $\tilde{v}$ such that $\tilde{v}$ is the solution of $-\Delta v + V(x^2)v + \phi_v v = f(v)$, here $x^2 = \lim_{n\to\infty} x_n^2$. Note that $|\epsilon_n^{-1}(x_n^2 - x_n^1)| \to +\infty$, then for any $R > 0$ the balls $\tilde{B}_R \cap \bar{B}^\epsilon = \emptyset$, where $\bar{B}^\epsilon = \tilde{B}_R(\epsilon_n^{-1}(x_n^2 - x_n^1))$, repeat the arguments of step2, we find that for any $\nu > 0$ it is possible to choose that $R > 0$ large enough such that

$$\lim_{n\to\infty} \int_{\bar{B}^\epsilon} h_n dx \geq \tilde{J}(\tilde{v}) - \nu, \tag{3.20}$$

where

$$\tilde{J}(u) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x^2)u^2)dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 dx - \int_{\mathbb{R}^3} F(u)dx,$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^3\setminus(B_R\cup B^\epsilon)} h_n dx \geq -\nu. \tag{3.21}$$

Similar to the argument in (3.13), we obtain

$$\lim_{n\to\infty} \int_{B_R} h_n dx \geq J_1(v) - \delta, \tag{3.22}$$

where

$$J_1(u) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x^1)u^2)dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 dx - \int_{\mathbb{R}^3} F(u)dx.$$

From (3.22),(3.20) and (3.21) we conclude that

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} h_n dx \geq J_1(v) + \tilde{J}(\tilde{v}) - 3\nu. \tag{3.23}$$

Since ν is arbitrary we find that

$$\epsilon_n^{-3} J_{\epsilon_n}(u_{\epsilon_n}) = \lim_{n\to\infty} J_n(v_n) \geq J_1(v) + \tilde{J}(\tilde{v}) \geq 2c_0,$$

which contradicts (3.1). The proof of proposition 3.2 is now complete. □

4. Proof of Theorem 1.1

In this section, we shall prove the existence, concentration, and exponential decay of ground state solution of (1.3) for small ϵ.

Proof of Theorem 1.1. By proposition 3.2, there exists ϵ_0 such that for $0 < \epsilon < \epsilon_0$,

$$u_\epsilon(x) < a \quad \text{for all } x \in \partial\Lambda. \tag{4.1}$$

The function $u_\epsilon \in H$ solves the equation

$$-\epsilon^2 \Delta u + V(x)u + \epsilon^{-2}\phi_u u = g(x, u). \tag{4.2}$$

Choose $(u_\epsilon - a)_+$ as a test function in (4.2), after integration by parts one gets

$$\begin{aligned}\int_{\mathbb{R}^3\setminus\{\Lambda\}} \Big[&\epsilon^2|\nabla(u_\epsilon - a)_+|^2 + c(x)(u_\epsilon - a)_+^2 + \epsilon^{-2}\phi_{u_\epsilon}u_\epsilon(u_\epsilon - a)_+ \\ &+ c(x)a(u_\epsilon - a)_+\Big]dx = 0,\end{aligned} \tag{4.3}$$

where

$$c(x) = V(x) - \frac{g(x, u_\epsilon(x))}{u_\epsilon(x)}.$$

The definition of g yields that $c(x) > 0$ in $\mathbb{R}^3\setminus\{\Lambda\}$, hence all terms in (4.3) are zero. We conclude in particular

$$u_\epsilon(x) \le a \quad \text{for all } \mathbb{R}^3\setminus\{\Lambda\}.$$

Consequently, u_ϵ is a solution to equation (1.3), and by proposition 3.2, we know that the maximum value of u_ϵ is achieved at a point $x_\epsilon \in \Lambda$ and it is away from zero. To obtain (1.4), we need the following proposition, which is a very particular version of [15, Theorem 8.17]. □

Proposition 4.1 ([15])**.** *Suppose that $t > 3$, $h \in L^{t/2}(\Omega)$ and $u \in H^1(\Omega)$ satisfies in the weak sense*

$$-\Delta u \le h(x) \quad \text{in } \Omega,$$

where Ω is an open subset of $\mathbb{R}^3$. Then, for any ball $B_{2R}(y) \subset \Omega$,

$$\sup_{x\in B_R(y)} u(x) \le C(\|u^+\|_{L^2(B_{2R}(y))} + \|h\|_{L^{t/2}(B_{2R}(y))}),$$

where C depends on t and R.

Lemma 4.2. *Let $v_\epsilon(x) = u_\epsilon(x_\epsilon + \epsilon x)$, where x_ϵ is the unique maximum of u_ϵ, then there exists $\epsilon^* > 0$ such that $\lim_{|x|\to\infty} v_\epsilon(x) = 0$ uniformly on $\epsilon \in (0, \epsilon^*)$.*

Proof. Since $u_\epsilon(x)$ is the solution of (1.3), by (3.2) then

$$\|v_\epsilon\|_H \le C, \tag{4.4}$$

and also $v_\epsilon(x)$ satisfies

$$-\Delta v_\epsilon + V(x_\epsilon + \epsilon x)v_\epsilon(x) + \phi_{v_\epsilon}v_\epsilon = f(v_\epsilon).$$

Now, for any sequence $\epsilon_n \to 0$, there is a subsequence such that

$$x_{\epsilon_n} \to \bar{x}; V(\bar{x}) = V_0.$$

From (4.4) and elliptic estimates, we know that this subsequence can be chosen in such a way that $v_{\epsilon_n} \to v$ uniformly over compacts, where $v \in H^1(\mathbb{R}^3)$ solves

$$-\Delta v + V_0 v + \phi_v = f(v). \tag{4.5}$$

Next, we prove that $v_{\epsilon_n} \to v \in H^1(\mathbb{R}^3)$. Since $\tilde{f}(s) \le f(s)$ for all $s \ge 0$, by (3.1) we have

$$I_n(v_{\epsilon_n}) \le \epsilon_n^{-3} J_{\epsilon_n}(u_{\epsilon_n}) \le c_0,$$

where

$$\begin{aligned}I_n(u) &= \frac{1}{2}\int_{\Omega_n}[|\nabla u|^2 + V(x_{\epsilon_n} + \epsilon_n x)u^2]dx + \frac{1}{4}\int_{\Omega_n}\phi_u u^2 dx \\ &\quad - \int_{\Omega_n} F(x_{\epsilon_n} + \epsilon_n x, u)dx, \Omega_n \\ &= \epsilon_n^{-1}\{\mathbb{R}^3 - x_{\epsilon_n}\}.\end{aligned} \tag{4.6}$$

On the other hand, using Fatou's lemma and the weak limit of v_{ϵ_n},

$$\begin{aligned}
I_n(v_{\epsilon_n}) &= I_n(v_{\epsilon_n}) - \frac{1}{4}\langle I_n'(v_{\epsilon_n}), v_{\epsilon_n}\rangle \\
&= \frac{1}{4}\int_{\Omega_n}[|\nabla v_{\epsilon_n}|^2 + V(x_{\epsilon_n}+\epsilon_n x)v_{\epsilon_n}^2]dx + \frac{1}{4}\int_{\Omega_n}[f(v_{\epsilon_n})v_{\epsilon_n} - 4F(v_{\epsilon_n})]dx \\
&\geq \frac{1}{4}\int_{\Omega_n}[|\nabla v_{\epsilon_n}|^2 + V_0 v_{\epsilon_n}^2]dx + \frac{1}{4}\int_{\Omega_n}[f(v_{\epsilon_n})v_{\epsilon_n} - 4F(v_{\epsilon_n})]dx \\
&\geq \frac{1}{4}\int_{\mathbb{R}^3}[|\nabla v|^2 + V_0 v^2]dx + \frac{1}{4}\int_{\mathbb{R}^3}[f(v)v - 4F(v)]dx \\
&= I_0(v) - \frac{1}{4}\langle I_0'(v), v\rangle \geq c_0.
\end{aligned}$$

So, $I_n(v_{\epsilon_n}) \to c_0$ as $n\to\infty$, and it is easy to verify from the above inequalities,

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}(|\nabla v_{\epsilon_n}|^2 + V_0 v_{\epsilon_n}^2)dx = \int_{\mathbb{R}^3}(|\nabla v|^2 + V_0 v^2)dx.$$

Therefore, using that $v_{\epsilon_n} \rightharpoonup v$ weakly in $H^1(\mathbb{R}^3)$, we conclude $v_{\epsilon_n} \to v$ in $H^1(\mathbb{R}^3)$. As a consequence of the above limit, we have

$$\lim_{R\to\infty}\int_{|x|\geq R} v_{\epsilon_n}^2\,dx = 0. \tag{4.7}$$

Applying proposition 4.1 in the inequality

$$-\Delta v_{\epsilon_n} \leq -\Delta v_{\epsilon_n} + V(\epsilon_n x + x_{\epsilon_n})v_{\epsilon_n} + \phi_{v_{\epsilon_n}} v_{\epsilon_n} = h_n(x)\Delta q f(v_{\epsilon_n}) \quad \text{in } \mathbb{R}^3,$$

we have that for some $t>3$, $\|h_n\|_{\frac{t}{2}} \leq C$ for all n. Moreover,

$$\sup_{x\in B_R(y)} v_{\epsilon_n}(x) \leq C(\|v_{\epsilon_n}\|_{L^2(B_{2R}(y))} + \|h_n\|_{L^{t/2}(B_{2R}(y))}) \quad \text{for all } y\in\mathbb{R}^3,$$

which implies that $\|v_{\epsilon_n}\|_{L^\infty(\mathbb{R}^3)}$ is uniformly bounded. Then by (4.7), we have

$$\lim_{|x|\to\infty} v_{\epsilon_n}(x) = 0 \quad \text{uniformly on } n\in\mathbb{N}.$$

Consequently, there exists $\epsilon^*>0$ such that

$$\lim_{|x|\to\infty} v_\epsilon(x) = 0 \quad \text{uniformly on } \epsilon\in(0,\epsilon^*)\,.$$

□

To show the exponential decay of u_ϵ, we only need the following result involving of v_ϵ.

Lemma 4.3. *There exist constants $C>0$ and $c>0$ such that*

$$v_\epsilon(x) \leq Ce^{-c|x|} \quad \textit{for all } x\in\mathbb{R}^3.$$

Proof. By lemma 4.2 and (A2), there exists $R_1>0$ such that

$$\frac{f(v_\epsilon(x))}{v_\epsilon(x)} \leq \frac{V_0}{2} \quad \text{for all } |x|\geq R_1,\ \epsilon\in(0,\epsilon^*). \tag{4.8}$$

Fix $\omega(x) = Ce^{-c|x|}$ with $c^2 < V_0/2$ and $Ce^{-cR_1} \geq v_\epsilon$ for all $|x| = R_1$. It is easy to check that

$$\Delta\omega \leq c^2\omega \quad \text{for all } |x|\neq 0. \tag{4.9}$$

So

$$-\Delta v_\epsilon + V_0 v_\epsilon \le -\Delta v_\epsilon + V_0 v_\epsilon + \phi_{v_\epsilon} v_\epsilon = f(v_\epsilon) \le \frac{V_0}{2} v_\epsilon \quad \text{for all } |x| > R_1. \tag{4.10}$$

Define $\omega_\epsilon = \omega - v_\epsilon$. Using (4.10) and (4.9), we obtain

$$-\Delta\omega_\epsilon + \frac{V_0}{2}\omega_\epsilon \ge 0, \quad \text{in } |x| \le R_1, \omega_\epsilon \ge 0, \quad \text{on } |x| = R_1, \lim_{|x|\to\infty} \omega_\epsilon(x) = 0.$$

The classical maximum principle implies that $\omega_\epsilon \ge 0$ in $|x| \ge R_1$ and by the work in [19], we conclude that

$$v_\epsilon(x) \le Ce^{-c|x|} \quad \text{for all } |x| \ge R_1, \epsilon \in (0, \epsilon^*).$$

By the definition of v_ϵ and lemma 4.3, we have

$$u_\epsilon(x) = v_\epsilon(\frac{x - x_\epsilon}{\epsilon}) \le Cexp(-c\frac{|x - x_\epsilon|}{\epsilon})$$

for all $x \in \mathbb{R}^3, \epsilon \in (0, \epsilon^*)$. The proof of Theorem 1.1 is complete. □

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