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SPECTRAL ANALYSIS OF q-FRACTIONAL STURM-LIOUVILLE OPERATORS

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Abstract. In this article, we study q-fractional Sturm-Liouville operators. Using by the functional method, we pass to a new operator. Then, showing that this operator is a maximal operator and constructing a self-adjoint dilation of the maximal dissipative operator. We prove a theorem on the completeness of the system of eigenvectors and associated vectors of the dissipative q-fractional Sturm-Liouville operators.

1. Introduction

It is well known that many problems in mechanics, engineering and mathematical physics lead to the concept of completeness of root functions and basis properties of all or part of the eigenvectors and associated vectors corresponding to some operators. In many engineering applications, the Sturm-Liouville problems arise as boundary value problems. These problems and many of the associated theories were presented in 1800's (see [22, 31, 41]), and since then, the related fields such as fractional Sturm-Liouville operators and q-fractional Sturm-Liouville operators have attracted considerable interest in a variety of applied sciences and mathematics (see [21, 26, 28, 33] and the references therein).

Spectral theory is one of the major subjects of modern functional analysis and its applications in mathematics. So there has recently been a noticeable interest in spectral analysis of Sturm-Liouville boundary value problems (see [6, 10, 14, 20, 34, 38, 40] and the references therein).

There are some methods to give the completeness of non-self-adjoint (dissipative) operators, such as the method of contour integration of the resolvent, Lidskij's method, functional model method, etc. In this paper, we prove a theorem on the completeness of the system of eigenvectors and associated vectors of dissipative operators by using functional model theory that belongs to Sz.-Nagy-Foiaş. It is related with the equivalence of the Lax-Phillips scattering function technique and Sz.-Nagy-Foiaş characteristic function. By combining the results of Nagy-Foiaş [4] and Lax-Phillips [3], the characteristic function is expressed with a scattering matrix, and the dissipative operator in the spectral representation of dilation becomes

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the model. By means of different spectral representation of dilation, the given operator can be written very simply and the functional models are obtained. The eigenvalues, eigenvectors and the spectral projection of the model operator is expressed obviously by the characteristic function. In the centre of this method, the problem on the completeness of the system of eigenvectors is solved by writing the characteristic function as a factorization. That is, the factorization of the characteristic function gives us information about whether the system of all eigenvectors and associated vectors is complete or not. This approach is applied for dissipative Schrödinger operators, Sturm-Liouville operators, fractional Sturm-Liouville operators and difference Sturm-Liouville operators (see [14, 20, 32, 38, 40]).

In this article, we apply it to the q-fractional Sturm-Liouville operator. To do this, we form a new operator and show that this new operator is a maximal dissipative operator. Then, we construct a functional model of the dissipative operator by means of the incoming and outgoing spectral representations, and define its characteristic function because this makes it possible to determine the scattering matrix of dilation according to the Lax and Phillips scheme. Finally, we prove a theorem on the completeness of the system of eigenvectors and associated vectors of dissipative operators.

2. Preliminaries

In this section, we recall some basic definitions and properties of the fractional calculus theory, which are useful in the following discussion. These definitions and properties can be found in [1, 2, 5, 13, 29, 30, 32, 35, 37] and the references therein.

Let q be a positive number with $0 < q < 1$, $A \subset \mathbb{R}$,

$$A_{t,q} := \{tq^n : n \in \mathbb{N}\},\ A^*_{t,q} := A_{t,q} \cup \{0\}, \quad t > 0,$$
$$\mathcal{A}_{t,q} := \{\pm tq^n : n \in \mathbb{N}\}, \quad t > 0.$$

Let $y(\cdot)$ be a complex-valued function on A. The *q-difference operator* D_q is defined by

$$D_q y(x) = \frac{y(qx) - y(x)}{\mu(x)} \quad \text{for all } x \in A\backslash\{0\},$$

where $\mu(x) = (q-1)x$. The q-derivative at zero is defined by

$$D_q y(0) = \lim_{n\to\infty} \frac{y(q^n x) - y(0)}{q^n x} \quad (x \in A),$$

if the limit exists and does not depend on x. A *right-inverse* to D_q, the *Jackson q-integration* is given by

$$\int_0^x f(t) d_q t = x(1-q) \sum_{n=0}^{\infty} q^n f(q^n x) \quad (x \in A),$$

provided that the series converges, and

$$\int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t \quad (a, b \in A).$$

Let $L^2_q(0,a)$ be the space of all complex-valued functions defined on $[0,a]$ such that

$$\|f\| := \Big(\int_0^a |f(x)| d_q x \Big)^{1/2} < \infty.$$

The space $L_q^2(0,a)$ is a separable Hilbert space with the inner product

$$(f,g) := \int_0^a f(x)\overline{g(x)}d_qx, \quad f,g \in L_q^2(0,a), \tag{2.1}$$

and the orthonormal basis

$$\phi_n(x) = \begin{cases} \frac{1}{\sqrt{x(1-q)}}, & x = aq^n, \\ 0, & \text{otherwise}, \end{cases}$$

where $n = 0,1,2,\dots$ (see [1]).

Definition 2.1. A function f which is defined on A, $0 \in A$, is said to be q-regular at zero if

$$\lim_{n\to\infty} f(xq^n) = f(0)$$

for every $x \in A$ (see [1]).

Let $C(A)$ denote the space of all q-regular at zero functions on A. This space is a normed space with the norm function

$$\|f\| = \sup\{|f(xq^n)| : x \in A,\ n \in \mathbb{N}\}.$$

(see [1]).

Definition 2.2. A q-regular at zero function f which is defined on $A_{t,q}^*$ is said to be q-absolutely continuous if

$$\sum_{j=0}^{\infty} |f(uq^j) - f(uq^{j+1})| \le K, \quad \forall u \in A_{t,q}^*,$$

where K is a constant depending on the function f (see [1]).

The space of q-absolutely continuous functions on $A_{t,q}^*$ is denoted by $AC_q(A_{t,q}^*)$.

For $n \in \mathbb{N}$ and $\alpha, a_1, \dots, a_n \in \mathbb{C}$, the *$q$-shifted factorial*, the *multiple q-shifted factorial* and the *q-binomial coefficients* are defined by

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k),$$

$$(a_1,a_2,\dots,a_k : q) = \prod_{j=1}^{k}(a_j;q)_n,$$

$$\begin{bmatrix}\alpha\\0\end{bmatrix}_q = 1, \quad \begin{bmatrix}\alpha\\n\end{bmatrix}_q = \frac{(1-q^\alpha)(1-q^{\alpha-1})\dots(1-q^{\alpha-n+1})}{(q;q)_n},$$

respectively (see [1]). The *generalized q-shifted factorial* is defined by

$$(a;q)_\nu = \frac{(a;q)_\infty}{(aq^\nu;q)_\infty} \quad (\nu \in \mathbb{R})$$

(see [1]). The *q-Gamma function* is defined by

$$\Gamma_q(z) = \frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z}, \quad z \in \mathbb{C},\ |q| < 1$$

(see [1]).

Definition 2.3. Let $0 < \alpha \le 1$. The left-sided and right-sided Riemann-Liouville q-fractional operator are given by the formulas

$$J^{\alpha}_{q,a^+} f(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_a^x (\frac{qt}{x}; q)_{\alpha-1} f(t) d_q t, \tag{2.2}$$

$$J^{\alpha}_{q,b^-} f(x) = \frac{1}{\Gamma_q(\alpha)} \int_{qx}^b t^{\alpha-1} (\frac{qx}{t}; q)_{\alpha-1} f(t) d_q t, \tag{2.3}$$

respectively (see [32]).

Definition 2.4. Let $\alpha > 0$ and $\lceil \alpha \rceil = m$. The left-sided and right-sided Riemann-Liouville fractional q-derivatives of order α are defined, respectively, as follows:

$$D^{\alpha}_{q,a^+} f(x) = D^m_q J^{m-\alpha}_{q,a^+} f(x), \tag{2.4}$$

$$D^{\alpha}_{q,b^-} f(x) = (\frac{-1}{q})^m D^m_{q^{-1}} J^{m-\alpha}_{q,b^-} f(x). \tag{2.5}$$

Similar formulas give the left-sided and right-sided Caputo fractional q-derivatives of order α, respectively as follows:

$$^{c}D^{\alpha}_{q,a^+} f(x) = J^{m-\alpha}_{q,a^+} D^m_q f(x),$$
$$^{c}D^{\alpha}_{q,b^-} f(x) = (\frac{-1}{q})^m J^{m-\alpha}_{q,b^-} D^m_{q^{-1}} f(x)$$

(see [32]).

Theorem 2.5. *(i) The left-sided Riemann-Liouville q-fractional operator satisfies the semi-group property*

$$J^{\alpha}_{q,a^+} J^{\beta}_{q,a^+} = J^{\alpha+\beta}_{q,a^+} f(x), \quad x \in A_{q,a}, \tag{2.6}$$

for any function defined on $A_{q,a}$ and for any values of α and β.

(ii) The right-sided Riemann-Liouville q-fractional operator satisfies the semi-group property

$$J^{\alpha}_{q,b^-} J^{\beta}_{q,b^-} f(x) = J^{\alpha+\beta}_{q,b^-} f(x), \quad x \in A_{q,b},$$

for any function defined on $A_{q,b}$ and for any values of α and β (see [32]*).*

Definition 2.6. An operator T is called dissipative (accumulative) if $\operatorname{Im}(Tx, x) \ge 0$, $(\operatorname{Im}(Tx, x) \le 0)$ for all $x \in D(T)$ (see [24, 25]).

Definition 2.7. The linear operator T with domain $D(T)$ acting in the Hilbert space H is called simple if there is no invariant subspace $N \subseteq D(T)$ ($N \ne \{0\}$) of the operator T on which the restriction T to N is self-adjoint (see [16]).

Definition 2.8. A triple $(\mathcal{H}, \Lambda_1, \Lambda_2)$ is called a space of boundary values of a closed symmetric operator T on a Hilbert space H if Λ_1, Λ_2 are linear maps from $D(T^*)$ to $\mathcal{H}$ with equal deficiency numbers such that:

(i) Green's formula is valid,

$$(T^* f, g)_H - (f, T^* g)_H = (\Lambda_1 f, \Lambda_2 g)_{\mathcal{H}} - (\Lambda_2 f, \Lambda_1 g)_{\mathcal{H}}, \quad f, g \in D(A^*).$$

(ii) For any $F_1, F_2 \in \mathcal{H}$, there is a vector $f \in D(T^*)$ such that $\Lambda_1 f = F_1$, $\Lambda_2 f = F_2$ [25].

Definition 2.9. Let $T : H \to H$ be an operator acting on a Hilbert space H, and let $U : \mathcal{H} \to \mathcal{H}$ be acting on another Hilbert space $\mathcal{H} \supset H$. The operator U is called a dilation of A if

$$T^n h = P_H U^n h, \quad h \in H, n \geq 0,$$

where P_H is the orthogonal projection of $\mathcal{H}$ onto H. The space $\mathcal{H}$ is called a dilation space (see [4]).

Definition 2.10. Let T be a symmetric operator and λ a non-real number. The operator

$$V = (T - \lambda I)(T - \overline{\lambda} I)^{-1}$$

is called the Cayley transform of the operator T (see [34]).

Definition 2.11. Let H be a Hilbert space and $T : H \to H$ be a contraction operator, i.e., $\|T\| < 1$. The operator $D_T = (I - T^*T)^{1/2}$ is called the defect operator of T. The characteristic function θ_T of the contraction A is defined by

$$\theta_T(\xi) = D_{T^*}(I - \xi T^*)^{-1}(\xi - T)D_T$$

(see [4]).

Definition 2.12 ([4])**.** The analytic function $S(\lambda)$ on the upper half-plane $\mathbb{C}_+$ is called the *inner function* on $\mathbb{C}_+$ if $|S(\lambda)| \leq 1$ for all $\lambda \in \mathbb{C}_+$ and $|S(\lambda)| = 1$ for almost all $\lambda \in (-\infty, \infty)$.

Definition 2.13. A sequence of points (a_n) inside the unit disk is said to satisfy the Blaschke condition when

$$\sum_n (1 - |a_n|) < \infty.$$

Given a sequence obeying the Blaschke condition, the Blaschke product is defined as

$$B(z) = \prod_n B(a_n, z)$$

with factors

$$B(a, z) = \frac{|a|}{a} \frac{a - z}{1 - \overline{a} z}$$

provided that $a \neq 0$. Here $\overline{a}$ is the complex conjugate of a (see [4]).

Definition 2.14. For a given countable set of points a_i, $i = 1, 2, \dots.$ such that $|a_i| < 1$ and $\sum_i (1 - |a_i|) < \infty$, and an ordered family of orthogonal projections $\{P_i\}$ in E in the disk $|a_i| < 1$ we can construct the Blaschke-Potapov product

$$\Pi(\xi) = \prod_{k=1}^{\infty} \{ \frac{a_k - \xi}{1 - \overline{a_k} \xi} \frac{\overline{a_k}}{|a_k|} P_k + (I - P_k) \}$$

which is the multi-dimensional analogue of the Blaschke product (see [6]).

Definition 2.15. The logarithmic capacity of a compact set E in the complex plane is given by

$$\gamma(E) = e^{-V(E)},$$

where

$$V(E) = \inf_{\nu} \int_{E \times E} \ln \frac{1}{u - v} d\nu(u) d\nu(v)$$

and ν runs over each probability measure on E (see [12]).

Definition 2.16 ([12])**.** Let $\widetilde{E}$ be an $n-$dimensional Hilbert space ($n < \infty$). In $\widetilde{E}$ we fix an orthonormal basis $e_1, e_2, \dots, e_n$ and denote by E_k ($k = 1, 2, \dots, n$) the linear span of the vectors $e_1, e_2, \dots, e_k$. If $L \subset E_k$, then the population of $x \in E_{k-1}$ with the property

$$\operatorname{Cap}\{\lambda : \lambda \in \mathbb{C}, (x + \lambda e_k) \subset L\} > 0$$

will be shown by $\Gamma_{k-1}L$ ($\operatorname{Cap} G$ is the inner logarithmic capacity of a set $G \subset \mathbb{C}$). The Γ-capacity of a set $L \subset \widetilde{E}$ is a number

$$\Gamma - \operatorname{Cap} L := \sup \operatorname{Cap}\{\lambda : \lambda e_1 \subset \Gamma_1\Gamma_2 \dots \Gamma_{n-1}L\},$$

where the supremum is taken with respect to all orthonormal basis in $\widetilde{E}$.

It is known that every set $L \subset \widetilde{E}$ of zero Γ-capacity has zero $2n$-dimensional Lebesgue measure; however, the converse is not true (see [16]).

Now denote by $[E]$ the set of all linear operators in E ($\dim E = m$). To convert $[E]$ into an m^2-dimensional Hilbert space, we give the inner product $\langle T, S\rangle = trS^*T$ for $T, S \in [E]$ (trS^*T is the trace of the operators S^*T). Hence we may give the Γ-capacity of a set in E (see [16]).

3. Dilation of q-fractional Sturm-Liouville operator

In this section, we construct a space of boundary value for minimal symmetric fractional q-Sturm-Liouville operator and describe all extensions (dissipative, accumulative, self-adjoint and other) of such operators.

By q-fractional differential expression

$$\tau_{q,\alpha}y(x) := D^{\alpha}_{q,a^-}p(x)^cD^{\alpha}_{q,0^+}y(x) + r(x)y(x), \quad \alpha \in (0, 1), \tag{3.1}$$

consider the fractional q-Sturm-Liouville equation

$$\tau_{q,\alpha}y(x) - \lambda y(x) = 0, \quad x \in A^*_{t,\alpha}, \tag{3.2}$$

where $p(x) \neq 0$ for all $x \in A^*_{t,\alpha}$ and p, r are real valued functions defined in $A^*_{t,\alpha}$. For $q \to 1$, this problem was investigated by Eryılmaz and Tuna (see [23]).

To pass from the differential expression $\tau_{q,\alpha}y$ to operators, we introduce the Hilbert space $H \subseteq L^2_q(A^*_{t,\alpha}) \cup C(A^*_{t,\alpha})$, $\alpha \in (0, 1)$ with the inner product (2.1).

Let $\mathcal{L}_0$ denote the closure of the minimal operator generated by (3.1) and $\mathcal{D}_0$ its domain. Besides, we denote by $\mathcal{D}$ the set of all functions f from H such that $f \in AC_q(A^*_{t,q})$ and $\tau_{q,\alpha}y \in H$. $\mathcal{D}$ is the domain of the maximal operator $\mathcal{L}$. Furthermore, $\mathcal{L} = \mathcal{L}^*_0$ [34].

For two arbitrary functions $y, z \in \mathcal{D}$, we have Green's identity [32]:

$$\begin{aligned} &\int_0^a [y(x)\tau_{q,\alpha}z(x) - z(x)\tau_{q,\alpha}y(x)]\, d_qx \\ &= [y(x)(J^{1-\alpha}_{q,a^-}p^cD^{\alpha}_{q,0^+}z)(\frac{x}{q}) - z(x)(J^{1-\alpha}_{q,a^-}p^cD^{\alpha}_{q,0^+}y)(\frac{x}{q})] \mid^a_{x=0} . \end{aligned} \tag{3.3}$$

Let us denote by Λ_1, Λ_2 the linear maps from $\mathcal{D}$ to $E := \mathbb{C}^2$ by the formulas

$$\Lambda_1 f = \begin{pmatrix} -y(0) \\ y(a) \end{pmatrix}, \quad \Lambda_2 f = \begin{pmatrix} J^{1-\alpha}_{q,a^-}p^cD^{\alpha}_{q,0^+}y(0) \\ J^{1-\alpha}_{q,a^-}p^cD^{\alpha}_{q,0^+}y(\frac{a}{q}) \end{pmatrix}, \quad y \in \mathcal{D}. \tag{3.4}$$

Lemma 3.1. *For arbitrary $y, z \in \mathcal{D}$, one has*

$$(\mathcal{L}y, z)_H - (y, \mathcal{L}z)_H = (\Lambda_1 y, \Lambda_2 z)_E - (\Lambda_2 y, \Lambda_1 z)_E.$$

Proof. From Green's identity and by

$$[y,z]_x = y(x)J^{1-\alpha}_{q,a^-}p^c D^\alpha_{q,0^+}\overline{z(\frac{x}{q})} - \overline{z(x)}J^{1-\alpha}_{q,a^-}p^c D^\alpha_{q,0^+}y(\frac{x}{q}),$$
$$y,z\in\mathcal{D},\ x\in[0,a],$$

we have

$$(\mathcal{L}y,z)_H - (y,\mathcal{L}z)_H = [y,z]_a - [y,z]_0.$$

Then, we obtain

$$\begin{aligned}
&(\Lambda_1 y,\Lambda_2 z)_E - (\Lambda_2 y,\Lambda_1 z)_E\\
&= -y(0)J^{1-\alpha}_{q,a^-}p^c D^\alpha_{q,0^+}\overline{z(0)} - (-\overline{z(0)})J^{1-\alpha}_{q,a^-}p^c D^\alpha_{q,0^+}y(0)\\
&\quad + y(a)J^{1-\alpha}_{q,a^-}p^c D^\alpha_{q,0^+}\overline{z(\frac{a}{q})} - z(a)J^{1-\alpha}_{q,a^-}p^c D^\alpha_{q,0^+}y(\frac{a}{q})\\
&= [y,z]_a - [y,z]_0.
\end{aligned}$$

Hence we have

$$(\mathcal{L}y,z)_H - (y,\mathcal{L}z)_H = (\Lambda_1 y,\Lambda_2 z)_E - (\Lambda_2 y,\Lambda_1 z)_E.$$

□

Theorem 3.2. *The triplet (E,Λ_1,Λ_2) defined by (3.4) is a boundary space of the operator $\mathcal{L}_0$.*

Proof. The first condition is obtained from Lemma 3.1. Now we will prove the second condition. Let $u=\begin{pmatrix}u_1\\u_2\end{pmatrix}$, $v=\begin{pmatrix}v_1\\v_2\end{pmatrix}\in E$. Then, the vector-valued function

$$y(t)=\alpha_1(t)u_1(t)+\alpha_2(t)v_1(t)+\beta_1(t)u_2(t)+\beta_2(t)v_2(t),$$

where $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, $\beta_1(\cdot)$, $\beta_2(\cdot)\in H$, satisfies the conditions

$$\begin{gathered}
\alpha_1(0)=-1,\\
J^{1-\alpha}_{q,a^-}p^c D^\alpha_{q,0^+}\alpha_1(0)=\alpha_1(a)=J^{1-\alpha}_{q,a^-}p^c D^\alpha_{q,0^+}\alpha_1(aq^{-1})=0,\\
J^{1-\alpha}_{q,a^-}p^c D^\alpha_{q,0^+}\alpha_2(0)=1,\\
\alpha_2(0)=\alpha_2(a)=J^{1-\alpha}_{q,a^-}p^c D^\alpha_{q,0^+}\alpha_2(aq^{-1})=0,\\
\beta_1(a)=1,\\
\beta_1(0)=J^{1-\alpha}_{q,a^-}p^c D^\alpha_{q,0^+}\beta_1(0)=J^{1-\alpha}_{q,a^-}p^c D^\alpha_{q,0^+}\beta_1(aq^{-1})=0,\\
J^{1-\alpha}_{q,a^-}p^c D^\alpha_{q,0^+}\beta_2(aq^{-1})=1,\\
\beta_2(0)=\beta_2(a)=J^{1-\alpha}_{q,a^-}p^c D^\alpha_{q,0^+}\beta_2(0)=0.
\end{gathered}$$

Note that $y(\cdot)$ belongs to the set $\mathcal{D}$ and $\Lambda_1 y=u$, $\Lambda_2 y=v$. Hence the proof is complete. □

Corollary 3.3. *For any contraction K in E, the restriction of the operator $\mathcal{L}$ to the set of functions $y\in\mathcal{D}$ satisfying either the boundary conditions*

$$(K-I)\Lambda_1 y + i(K+I)\Lambda_2 y = 0 \tag{3.5}$$

or

$$(K-I)\Lambda_1 y - i(K+I)\Lambda_2 y = 0 \tag{3.6}$$

is respectively, a maximal dissipative or a maximal accumulative extension of the operator $\mathcal{L}_0$, where $\mathcal{L}_0$ is the restriction of the operator $\mathcal{L}$ to the domain $\mathcal{D}$. Conversely, every maximal dissipative (accumulative) extension of the operator $\mathcal{L}_0$ is the restriction of $\mathcal{L}$ to the set of functions $y \in \mathcal{D}$ satisfying (3.5) (3.6)*, and the extension uniquely determines the contraction K. Conditions* (3.5) (3.6)*, in which K is an isometry describe the maximal symmetric extensions of $\mathcal{L}_0$ in H. If K is unitary, these conditions define self-adjoint extensions.*

In particular, the boundary conditions

$$-y(0) + \gamma_1 J_{q,a^-}^{1-\alpha} p^c D_{q,0^+}^{\alpha} y(0) = 0, \tag{3.7}$$

$$y(a) + \gamma_2 J_{q,a^-}^{1-\alpha} p^c D_{q,0^+}^{\alpha} y(\frac{a}{q}) = 0, \tag{3.8}$$

with $\operatorname{Im}\gamma_1 \geq 0$ or $\gamma_1 = \infty$, $\operatorname{Im}\gamma_2 \geq 0$ or $\gamma_2 = \infty$ ($\operatorname{Im}\gamma_1 = 0$ or $\gamma_1 = \infty$, $\operatorname{Im}\gamma_2 = 0$ or $\gamma_2 = \infty$) describe the maximal dissipative (self-adjoint) extensions of $\mathcal{L}_0$ with separated boundary conditions.

Now we study the maximal dissipative operator $\mathcal{L}_K$, where K is the strict contraction in E generated by the expression $\tau_{q,\alpha} y$ and the boundary condition (3.5). Since K is a strict contraction, the operator $K + I$ must be invertible, and the boundary condition (3.5) is equivalent to the condition

$$\Lambda_2 y + \Omega \Lambda_1 y = 0, \tag{3.9}$$

where $\Omega = -i(K+I)^{-1}(K-I)$, $\operatorname{Im}\Omega > 0$, and $-K$ is the Cayley transform of the dissipative operator Ω. We denote $\mathcal{L}_\Omega$ $(= \mathcal{L}_K)$ the dissipative operator generated by the expression $\tau_{q,\alpha} y$ and the boundary condition (3.9).

Let

$$\Omega = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}$$

where $\operatorname{Im}\gamma_1 > 0$, $\operatorname{Im}\gamma_2 > 0$ and $\eta^2 = 2\operatorname{Im}\Omega,\ \eta > 0$. Then the boundary condition (3.9) coincides with the separated boundary conditions (3.5) and (3.6).

4. Self-adjoint dilation, incoming and outgoing spectral representations

In this section, we construct a self-adjoint dilation of the maximal dissipative q-fractional Sturm-Liouville operator and its incoming and outgoing spectral representations. Hence we determine the scattering matrix of the dilation according to the Lax and Phillips scheme ([3], [4]). Later, we construct a functional model of this operator, using incoming spectral representations. Finally, we determine the characteristic function of this operator.

Now we consider the "incoming" and "outgoing" subspaces $L^2((-\infty,0);E)$ and $L^2((0,\infty);E)$. The orthogonal sum $\mathcal{H} = L^2((-\infty,0);E) \oplus H \oplus L^2((0,\infty);E)$ is called the *main Hilbert space of the dilation.*

In the space $\mathcal{H}$, we define the operator Υ on the set $D(\Upsilon)$, where $D(\Upsilon)$ consist of vectors $w = \langle \psi_-, y, \psi_+ \rangle$, generated by the expression

$$\Upsilon\langle \psi_-, y, \psi_+ \rangle = \langle i\frac{d\psi_-}{d\xi}, \tau_{q,\alpha} y, i\frac{d\psi_+}{d\varsigma} \rangle; \tag{4.1}$$

$$\Lambda_2 y + \Omega \Lambda_1 y = \eta\psi_-(0), \Lambda_2 y + \Omega_1^* \Lambda y = \eta\psi_+(0), \eta^2 := 2\operatorname{Im}\Omega, \eta > 0, \tag{4.2}$$

where $\psi_- \in W_2^1((-\infty,0);E)$, $\psi_+ \in W_2^1((0,\infty);E)$, $y \in H$ and W_2^1 is the Sobolev space.

Theorem 4.1. *The operator Υ is self-adjoint in $\mathcal{H}$.*

Proof. We first prove that Υ is symmetric in $\mathcal{H}$. Let $f,g \in D(\Upsilon)$, $f = \langle \psi_-, y, \psi_+\rangle$ and $g = \langle \zeta_-, z, \zeta_+\rangle$. Then, we have

$$\begin{aligned}
&(\Upsilon f, g)_{\mathcal{H}} - (f, \Upsilon g)_{\mathcal{H}} \\
&= i(\psi_-(0), \zeta_-(0))_E - i(\psi_+(0), \zeta_+(0))_E + [y,z]_a - [y,z]_0.
\end{aligned} \tag{4.3}$$

By direct computation, we obtain

$$i(\psi_-(0), \zeta_-(0))_E - i(\psi_+(0), \zeta_+(0))_E + [y,z]_a - [y,z]_0 = 0.$$

Thus, $\Upsilon \subseteq \Upsilon^*$, i.e., Υ is a symmetric operator.

It is easy to check that Υ and Υ^* are generated by the same expression (4.1). Let us describe the domain of Υ^*. We shall compute the terms outside the integral sign, which are obtained via integration by parts in the bilinear form $(\Upsilon f, g)_{\mathcal{H}}$, $f \in D(\Upsilon)$, $g \in D(\Upsilon^*)$. Their sum is equal to zero, i.e.,

$$[y,z]_a - [y,z]_0 + i(\psi_-(0), \zeta_-(0))_E - i(\psi_+(0), \zeta_+(0))_E = 0. \tag{4.4}$$

Further, solving the boundary conditions (4.2) for $\Lambda_1 y$ and $\Lambda_2 y$, we find that

$$\Lambda_1 y = -i\eta^{-1}(\psi_-(0) - \psi_+(0)), \quad \Lambda_2 y = \eta\psi_-(0) + i\mathcal{L}\eta^{-1}(\psi_-(0) - \psi_+(0)).$$

Therefore, using (3.4), we find that (4.4) is equivalent to the equality

$$\begin{aligned}
&i(\psi_+(0), \zeta_+(0))_E - i(\psi_-(0), \zeta_-(0))_E \\
&= [y,z]_a - [y,z]_0 = (\Lambda_1 y, \Lambda_2 z)_E - (\Lambda_2 y, \Lambda_1 z)_E \\
&= -i(\eta^{-1}(\psi_-(0) - \psi_+(0)), \Lambda_2 z)_E - (\eta\psi_-(0), \Lambda_1 z)_E \\
&\quad - i(\mathcal{L}\eta^{-1}(\psi_-(0) - \psi_+(0)), \Lambda_1 z)_E.
\end{aligned}$$

Since the values $\psi_\mp(0)$ can be arbitrary vectors, a comparison of the coefficients of $\psi_{i\mp}(0)$ $(i = 1,2)$ on the left-hand side and the right-hand side of the last equality proves that the vector $g = \langle \zeta_-, z, \zeta_+\rangle$ satisfies the boundary conditions (4.2), namely, $\Lambda_2 z + \Omega\Lambda_1 z = \eta\zeta_-(0)$, $\Lambda_2 z + \Omega_1^*\Lambda_1 z = \eta\zeta_+(0)$. Therefore $D(\Upsilon^*) \subseteq D(\Upsilon)$, and hence $\Upsilon = \Upsilon^*$. □

Note that the self-adjoint operator Υ generates a unitary group $U_t = \exp(i\Upsilon t)$ $(t \in (-\infty,\infty))$ on $\mathcal{H}$. Let us denote by $\mathcal{P} : \mathcal{H} \to H$ and $\mathcal{P}_1 : H \to \mathcal{H}$ the mappings acting according to the formulae $\mathcal{P} : \langle \psi_-, y, \psi_+\rangle \to y$ and $\mathcal{P}_1 : y \to \langle 0, y, 0\rangle$. Let $Z_t := \mathcal{P}U_t\mathcal{P}_1$, $t \geq 0$, by using U_t. The family $\{Z_t\}$ $(t \geq 0)$ of operators is a strongly continuous semigroup of completely non-unitary contraction on H. Let us denote by B the generator of this semigroup : $By = \lim_{t\to+0}(\frac{Z_t y - y}{it})$. The domain of B consists of all vectors for which the limit exists. The operator B is dissipative. The operator Υ is called the *self-adjoint dilation* of B. Then, we have the following result.

Theorem 4.2. *The operator Υ is a self-adjoint dilation of the operator $\mathcal{L}_\Omega (= \mathcal{L}_K)$.*

Proof. It is sufficient to prove the following equality (see [6]):

$$\mathcal{P}(\Upsilon - \lambda I)^{-1}\mathcal{P}_1 y = (\mathcal{L}_\Omega - \lambda I)^{-1} y, \quad y \in H, \ \operatorname{Im} h < 0. \tag{4.5}$$

We set $(\Upsilon - \lambda I)^{-1}\mathcal{P}_1 y = g = \langle \zeta_-, z, \zeta_+\rangle$. Then $(\Upsilon - \lambda I)g = \mathcal{P}_1 y$, and hence $\tau_{q,\alpha} z - \lambda z = y$, $\zeta_-(\xi) = \zeta_-(0)e^{-i\lambda\xi}$ and $\zeta_+(\xi) = \zeta_+(0)e^{-i\lambda\xi}$. Since $g \in D(\Upsilon)$, we have $\zeta_- \in W_2^1((-\infty,0);E)$. Thus it follows that $\zeta_-(0) = 0$ and consequently, z satisfies the boundary condition $\Lambda_2 z + \Omega\Lambda_1 z = 0$. Therefore $z \in D(\mathcal{L}_\Omega)$, and since the point λ with $\operatorname{Im}\lambda < 0$ cannot be an eigenvalue of the dissipative operator, it follows that $z = (\mathcal{L}_\Omega - \lambda I)^{-1}y$. Thus

$$(\Upsilon - \lambda I)^{-1}\mathcal{P}_1 y = \langle 0, (\mathcal{L}_\Omega - \lambda I)^{-1}y, \eta^{-1}(\Lambda_2 y + \Omega^*\Lambda_1 y)e^{-i\lambda\xi}\rangle$$

for $y \in H$ and $\operatorname{Im}\lambda < 0$. By applying the mapping $\mathcal{P}$, we obtain

$$\begin{aligned}(\mathcal{L}_\Omega - \lambda I)^{-1} &= \mathcal{P}(\Upsilon - \lambda I)^{-1}\mathcal{P}_1 = -i\mathcal{P}\int_0^\infty U_t e^{-i\lambda t}dt\mathcal{P}_1\\ &= -i\int_0^\infty Z_t e^{-i\lambda t}dt = (B - \lambda I)^{-1}, \quad \operatorname{Im}\lambda < 0,\end{aligned}$$

i.e., $\mathcal{L}_\Omega = B$. □

On the other hand, the unitary group $\{U_t\}$ has an important property which makes it possible to apply it to the Lax-Phillips theory (see [3]). It has orthogonal incoming and outgoing subspaces $D_- = \langle L^2(-\infty,0), 0, 0\rangle$ and $D_+ = \langle 0, 0, L^2(0,\infty)\rangle$, and they have the following properties.

Lemma 4.3. *$U_t D_- \subset D_-$, $t \le 0$ and $U_t D_+ \subset D_+$, $t \ge 0$.*

Proof. We will just prove for D_+ since the proof for D_- is similar. Set $\mathcal{R}_\lambda = (\Upsilon - \lambda I)^{-1}$. Then, for all λ, with $\operatorname{Im}\lambda < 0$, we have

$$\mathcal{R}_\lambda f = \langle 0, 0, -ie^{-i\lambda\xi}\int_0^\xi e^{i\lambda s}\psi_+(s)ds\rangle, \quad f = \langle 0, 0, \psi_+\rangle \in D_+.$$

Hence we have $\mathcal{R}_\lambda f \in D_+$. If $g \perp D_+$, then we obtain

$$0 = (\mathcal{R}_\lambda f, g)_{\mathcal{H}} = -i\int_0^\infty e^{-i\lambda t}(U_t f, g)_{\mathcal{H}}dt, \qquad \operatorname{Im}\lambda < 0.$$

Thus we have $(U_t f, g)_{\mathcal{H}} = 0$ for all $t \ge 0$, i.e., $U_t D_+ \subset D_+$ for $t \ge 0$. □

Lemma 4.4. $\cap_{t\le 0} U_t D_- = \cap_{t\ge 0} U_t D_+ = \{0\}$.

Proof. Let us define the mapping $\mathcal{P}^+ : \mathcal{H} \to L^2((0,\infty);E)$ and the mapping $\mathcal{P}_1^+ : L^2((0,\infty);E) \to D_+$ as $\mathcal{P}^+ : \langle \psi_-, y, \psi_+\rangle \to \psi_+$ and $\mathcal{P}_1^+ : \psi \to \langle 0, 0, \psi\rangle$, respectively. We consider that the semigroup of isometries $U_t^+ := \mathcal{P}^+ U_t \mathcal{P}_1^+$ $(t \ge 0)$ is a one-sided shift in $L^2((0,\infty);E)$. Indeed, the generator of the semigroup of the one-sided shift V_t in $L^2((0,\infty);E)$ is the differential operator $i\frac{d}{d\xi}$ with the boundary condition $\psi(0) = 0$. On the other hand, the generator S of the semigroup of isometries U_t^+ $(t \ge 0)$ is the operator

$$S\psi = \mathcal{P}^+\Upsilon\mathcal{P}_1^+\psi = \mathcal{P}^+\Upsilon\langle 0, 0, \psi\rangle = \mathcal{P}^+\langle 0, 0, i\frac{d\psi}{d\xi}\rangle = i\frac{d\psi}{d\xi},$$

where $\psi \in W_2^1((0,\infty);E)$ and $\psi(0) = 0$. Since a semigroup is uniquely determined by its generator, it follows that $U_t^+ = V_t$, and hence we obtain

$$\cap_{t\ge 0} U_t D_+ = \langle 0, 0, \cap_{t\le 0} V_t L^2((0,\infty);E)\rangle = \{0\}.$$

□

Lemma 4.5. *The operator $\mathcal{L}_\Omega$ is simple.*

Proof. Let $H' \subset H$ be a nontrivial subspace in which $\mathcal{L}_\Omega$ induces a self-adjoint operator $\mathcal{L}'_\Omega$ with domain $D(\mathcal{L}'_\Omega) = H' \cap D(\mathcal{L}_\Omega)$. If $f \in D(\mathcal{L}'_\Omega)$, then $f \in D(\mathcal{L}^*_\Omega)$ and

$$\begin{aligned} 0 &= \frac{d}{dt}\|e^{i\mathcal{L}_\Omega t} f\|_H^2 = \frac{d}{dt}(e^{i\mathcal{L}_\Omega t} f, e^{i\mathcal{L}_\Omega t} f)_H \\ &= -2(\operatorname{Im}\Omega\Lambda_1 e^{i\mathcal{L}_\Omega t} f, \Lambda_1 e^{i\mathcal{L}_\Omega t} f)_E. \end{aligned}$$

Consequently, $\Lambda_1 e^{i\mathcal{L}_\Omega t} f = 0$. For eigenvectors $y \in H'$ of the operator $\mathcal{L}_\Omega$ we have $\Lambda_1 y(\lambda) = 0$. Using this result with the boundary condition $\Lambda_2 y + \Omega\Lambda_1 y = 0$, we have $\Lambda_2 y = 0$, i.e., $y = 0$. Since all the solutions of $\tau_{q,\alpha} y = \lambda y$ belong to H, from this it can be concluded that the resolvent operator $R_\lambda(\mathcal{L}_\Omega)$ is compact, and the spectrum of $\mathcal{L}_\Omega$ is purely discrete. Consequently, by the theorem on expansion in the eigenvectors of the self-adjoint operator $\mathcal{L}'_\Omega$, we obtain $H' = \{0\}$. Hence the operator $\mathcal{L}_\Omega$ is simple. □

Now we set

$$H_- = \overline{\cup_{t\geq 0} U_t D_-}, \quad H_+ = \overline{\cup_{t\leq 0} U_t D_+}.$$

Lemma 4.6. *The equality $H_- + H_+ = \mathcal{H}$ holds.*

Proof. From Lemma 4.5, it is easy to show that the subspace $\mathcal{H}' = \mathcal{H} \ominus (H_- + H_+)$ is invariant relative to the group $\{U_t\}$, and has the form $\mathcal{H}' = \langle 0, H', 0\rangle$ where H' is a subspace of H. Therefore, if the subspace $\mathcal{H}'$ (and hence also H') were nontrivial, then the unitary group $\{U'_t\}$ restricted to this subspace would be a unitary part of the group $\{U_t\}$, and hence the restriction $\mathcal{L}'_\Omega$ of $\mathcal{L}_\Omega$ to H' would be a self-adjoint operator in H'. Since the operator $\mathcal{L}_\Omega$ is simple, it follows that $H' = \{0\}$. □

Suppose that $\chi(\lambda)$ and $\omega(\lambda)$ are the solutions of $\tau_{q,\alpha} y = \lambda y$, satisfying the conditions

$$\chi(0,\lambda) = 0, \quad J^{1-\alpha}_{q,a^-} p^c D^\alpha_{q,0^+}\chi(0,\lambda) = -1, \quad \omega(0,\lambda) = 1, \quad J^{1-\alpha}_{q,a^-} p^c D^\alpha_{q,0^+}\omega(0,\lambda) = 0.$$

We denote by $m(\lambda)$ the matrix-valued function satisfying the conditions

$$m(\lambda)\Lambda_1\chi = \Lambda_2\chi, \quad m(\lambda)\Lambda_1\omega = \Lambda_2\omega.$$

$m(\lambda)$ is a meromorphic function on the complex plane $\mathbb{C}$ with a countable number of poles on the real axis. Furthermore, it is possible to show that the function $m(\lambda)$ possesses the following properties: $\operatorname{Im} m(\lambda) \leq 0$ for all $\operatorname{Im}\lambda \neq 0$, and $m^*(\lambda) = m(\overline{\lambda})$ for all $\lambda \in \mathbb{C}$, except the real poles $m(\lambda)$.

We denote by $\mu_j(x,\lambda)$ and $\nu_j(x,\lambda)$ $(j = 1,2)$ the solutions of the system $\tau_{q,\alpha} y = \lambda y$, which satisfy the conditions

$$\Lambda_1\mu_j = (m(\lambda) + \Omega)^{-1}\eta e_j, \quad \Lambda_1\nu_j = (m(\lambda) + \Omega^*)^{-1}\eta e_j \quad (j = 1,2),$$

where $\{e_1, e_2\}$ is an orthonormal basis for E.

We set

$$U^-_{\lambda j}(x,\xi,\rho) = \langle e^{-i\lambda\xi} e_j, \mu_j(x,\lambda), \eta^{-1}(m + \Omega^*)(m + \Omega)^{-1}\eta e^{-i\lambda\rho} e_j\rangle \quad (j = 1,2).$$

We note that the vectors $U^-_{\lambda j}(x,\xi,\rho)$ $(j = 1,2)$ for real λ do not belong to the space $\mathcal{H}$. However, $U^-_{\lambda j}(x,\xi,\rho)$ $(j = 1,2)$ satisfies the equation $\Upsilon U = \lambda U$ and the corresponding boundary conditions for the operator Υ.

By means of the vector $U_{\lambda j}^{-}(x,\xi,\rho)$ $(j=1,2)$, we define the transformation $\mathcal{F}_{-}: f \to \widetilde{f_{-}}(\lambda)$ by

$$(\mathcal{F}_{-}f)(\lambda) := \widetilde{f_{-}}(\lambda) := \frac{1}{\sqrt{2\pi}} \sum_{j=1}^{2} (f, U_{\lambda j}^{-})_{\mathcal{H}} e_j$$

on the vectors $f = \langle \psi_{-}, y, \psi_{+} \rangle$ in which ψ_{-}, ψ_{+}, y are smooth, compactly supported functions.

Lemma 4.7. *The transformation $\mathcal{F}_{-}$ maps isometrically H_{-} onto $L^2((-\infty,\infty);E)$. For all vectors $f,g \in H_{-}$ the Parseval equality and the inversion formulae hold:*

$$(f,g)_{\mathcal{H}} = (\widetilde{f_{-}}, \widetilde{g_{-}})_{L^2} = \int_{-\infty}^{\infty} \sum_{j=1}^{2} \widetilde{f_{j-}}(\lambda) \overline{\widetilde{g_{j-}}(\lambda)} d\lambda,$$

$$f = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{j=1}^{2} \widetilde{f_{j-}}(\lambda) U_{\lambda j}^{-} d\lambda,$$

where $\widetilde{f_{-}}(\lambda) = (\mathcal{F}_{-}f)(\lambda)$ and $\widetilde{g_{-}}(\lambda) = (\mathcal{F}_{-}g)(\lambda)$.

Proof. By the Paley-Wiener theorem, we obtain

$$\widetilde{f_{j-}}(\lambda) = \frac{1}{\sqrt{2\pi}} (f, U_{\lambda j}^{-})_{\mathcal{H}} = \frac{1}{2\pi} \int_{-\infty}^{0} (\psi_{-}(\xi), e^{-i\lambda\xi} e_j)_E d\xi \in H_{-}^2(E),$$

where $f = \langle \psi_{-}, 0, 0 \rangle$, $g = \langle \zeta_{+}, 0, 0 \rangle \in D_{-}$. If we use the Parseval equality for Fourier integrals, then we obtain

$$(f,g)_{\mathcal{H}} = \int_t -\infty^{\infty} (\psi_{-}(\xi), \zeta_{-}(\xi))_E d\xi = \int_{-\infty}^{\infty} (\widetilde{f_{-}}(\lambda), \widetilde{g_{-}}(\lambda))_E d\lambda = (\mathcal{F}_{-}f, \mathcal{F}_{-}g)_{L^2},$$

where $H_{\pm}^2(E)$ denote the Hardy classes in $L^2((-\infty,\infty);E)$ consisting of the functions analytically extendible to the upper and lower half-planes, respectively. We now extend the Parseval equality to the whole of H_{-}. We consider in H_{-} the dense set H'_{-} consisting of smooth, compactly supported functions in $D_{-}: f \in H'_{-}$ if $f = U_T\ f_0$, $f_0 = \langle \psi_{-}, 0, 0 \rangle$, $\psi_{-} \in C_0^{\infty}((-\infty,0);E)$, where $T = T_f$ is a nonnegative number depending on f. If $f,g \in H'_{-}$, then for $T > T_f$ and $T > T_g$ we have $U_{-T}f$, $U_{-T}g \in D_{-}$. Moreover, the first components of these vectors belong to $C_0^{\infty}((-\infty,0);E)$. Since the operators U_t $(t \in (-\infty,\infty))$ are unitary, by the equality

$$\mathcal{F}_{-}U_t f = \frac{1}{\sqrt{2\pi}} \sum_{j=1}^{2} (U_t f, U_{\lambda j}^{-})_{\mathcal{H}} e_j = e^{i\lambda t} \mathcal{F}_{-} f,$$

we have

$$\begin{aligned} (f,g)_{\mathcal{H}} &= (U_{-T}f, U_{-T}g)_{\mathcal{H}} = (\mathcal{F}_{-}U_{-T}f, \mathcal{F}_{-}U_{-T}g)_{L^2} \\ &= (e^{-i\lambda T}\mathcal{F}_{-}f, e^{-i\lambda T}\mathcal{F}_{-}g)_{L^2} = (\widetilde{f}, \widetilde{g})_{L^2}. \end{aligned}$$

By taking the closure, we obtain the Parseval equality for the space H_{-}. The inversion formula is obtained from the Parseval equality if all integrals in it are considered as limits in the mean of integrals over finite intervals. Finally, we obtain the desired result

$$\mathcal{F}_{-}H_{-} = \overline{\cup_{t \geq 0} \mathcal{F}_{-}U_t D_{-}} = \overline{\cup_{t \geq 0} e^{i\lambda t} H_{-}^2} = L^2((-\infty,\infty);E).$$

□

Now we set

$$U^+_{\lambda j}(x,\xi,\rho) = \langle S_\Omega(\lambda)e^{-i\lambda\xi}e_j, \nu_j(x,\lambda), e^{-i\lambda\rho}e_j\rangle \quad (j=1,2),$$

where

$$S_\Omega(\lambda) = \eta^{-1}(m(\lambda)+\Omega)(m(\lambda)+\Omega^*)^{-1}\eta. \tag{4.6}$$

We note that the vectors $U^+_{\lambda j}(x,\xi,\rho)$ for real λ do not belong to the space $\mathcal{H}$. However, $U^+_{\lambda j}(x,\xi,\rho)$ satisfies the equation $\Upsilon U = \lambda U$ and the corresponding boundary conditions for the operator Υ. With the help of the vector $U^+_{\lambda j}(x,\xi,\rho)$, we define the transformation $\mathcal{F}_+ : f \to \widetilde{f_+}(\lambda)$ by

$$(\mathcal{F}_+ f)(\lambda) := \widetilde{f_+}(\lambda) := \sum_{j=1}^{2} \widetilde{f_{j+}}(\lambda)e_j := \frac{1}{\sqrt{2\pi}}\sum_{j=1}^{2}(f, U^+_{\lambda j})_{\mathcal{H}} e_j$$

on the vectors $f = \langle \psi_-, y, \psi_+\rangle$ in which ψ_-, ψ_+ and y are smooth, compactly supported functions.

Lemma 4.8. *The transformation $\mathcal{F}_+$ isometrically maps H_+ onto $L^2((-\infty,\infty))$. For all vectors $f, g \in H_+$ the Parseval equality and the inversion formula hold:*

$$(f,g)_{\mathcal{H}} = (\widetilde{f_+}, \widetilde{g_+})_{L^2} = \int_{-\infty}^{\infty}\sum_{j=1}^{2}\widetilde{f_{j+}}(\lambda)\overline{\widetilde{g_{j+}}(\lambda)}d\lambda,$$

$$f = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sum_{j=1}^{2}\widetilde{f_{j+}}(\lambda)U^+_{\lambda j}d\lambda,$$

where $\widetilde{f_+}(\lambda) = (\mathcal{F}_+ f)(\lambda)$ and $\widetilde{g_+}(\lambda) = (\mathcal{F}_+ g)(\lambda)$.

The proof of the above lemma is similar to that of Lemma 4.7, and it is omitted. It is clear that the matrix-valued function $S_\Omega(\lambda)$ is meromorphic in $\mathbb{C}$ and all the poles are in the lower half-plane. From (4.6), we obtain $\|S_\Omega(\lambda)\| \le 1$ for $\operatorname{Im}\lambda > 0$ and $S_\Omega(\lambda)$ is the unitary matrix for all $\lambda \in \mathbb{R}$. Therefore, we have

$$U^+_{\lambda j} = \sum_{k=1}^{2} S_{jk}(\lambda)U^-_{\lambda k} \quad (j=1,2,\dots,n), \tag{4.7}$$

where S_{jk} $(j,k=1,2,\dots,n)$ are elements of the matrix $S_\Omega(\lambda)$. From Lemmas 4.7 and 4.8, we obtain $H_- = H_+$. With Lemma 4.6, this shows that $H_- = H_+ = \mathcal{H}$. Therefore we have proved the following lemma for the incoming and outgoing subspaces (for D_- and D_+).

Lemma 4.9. $\overline{\cup_{t\ge 0}U_t D_-} = \overline{\cup_{t\le 0}U_t D_+} = \mathcal{H}$.

Lemma 4.10. $D_- \perp D_+$.

The proof of the above lemma is straightforward, hence omitted. Thus the transformation $\mathcal{F}_-$ isometrically maps H_- onto $L^2((-\infty,\infty);E)$ with the subspace D_- mapped onto $H^2_-(E)$,and the operators U_t are transformed into the operators of multiplication by $e^{i\lambda t}$. This means that $\mathcal{F}_-$ is the incoming spectral representation for the group $\{U_t\}$. Similarly, $\mathcal{F}_+$ is the outgoing spectral representation for the group $\{U_t\}$. It follows that the passage from the $\mathcal{F}_-$ representation of an element $f \in \mathcal{H}$ to its $\mathcal{F}_+$ representation is accomplished as $\widetilde{f_+}(\lambda) = S_\Omega^{-1}(\lambda)\widetilde{f_-}(\lambda)$. Consequently, according to [3], we have proved the following result.

Theorem 4.11. *The function $S_\Omega^{-1}(\lambda)$ is the scattering matrix of the group $\{U_t\}$ (of the self-adjoint operator Υ).*

Let $S(\lambda)$ be an arbitrary non-constant inner function on the upper half-plane. Let us define K by the formula $K = H_+^2 \ominus SH_+^2$. It is clear that $K \neq \{0\}$ is a subspace of the Hilbert space H_+^2. We consider the semigroup of operators Z_t $(t \geq 0)$ acting in K according to the formula

$$Z_t\varphi = \mathcal{P}[e^{i\lambda t}\varphi], \quad \varphi = \varphi(\lambda) \in K,$$

where $\mathcal{P}$ is the orthogonal projection from H_+^2 onto K. The generator of the semigroup $\{Z_t\}$ is denoted by

$$T\varphi = \lim_{t\to+0}\Big(\frac{Z_t\varphi - \varphi}{it}\Big),$$

where T is a maximal dissipative operator acting in K, and with domain $D(T)$ consisting of all the functions $\varphi \in K$ such that the limit exists. The operator T is called a *model dissipative operator*. This model dissipative operator is a special case of a more general model dissipative operator constructed by Nagy and Foiaş [4], which is associated with the names of Lax-Phillips [3]. Here the basic assertion is that $S(\lambda)$ is the *characteristic function* of the operator T.

Let $K = \langle 0, H, 0\rangle$, so that $\mathcal{H} = D_- \oplus K \oplus D_+$. From the explicit form of the unitary transformation $\mathcal{F}_-$ under the mapping $\mathcal{F}_-$, we obtain

$$\begin{gathered}
\mathcal{H} \to L^2((-\infty,\infty);E), \quad f \to \widetilde{f_-}(\lambda) = (\mathcal{F}_- f)(\lambda),\\
D_- \to H_-^2(E), \quad D_+ \to S_\Omega H_+^2(E),\\
K \to H_+^2(E) \ominus S_\Omega H_+^2(E),\\
U_t \to (\mathcal{F}_- U_t \mathcal{F}_-^{-1}\widetilde{f_-})(\lambda) = e^{i\lambda t}\widetilde{f_-}(\lambda).
\end{gathered} \tag{4.8}$$

The formulas in (4.8) show that the operator $\mathcal{L}_\Omega(\mathcal{L}_K)$ is unitarily equivalent to the model dissipative operator with the characteristic function $S_\Omega(\lambda)$. Since the characteristic functions of unitary equivalent dissipative operators coincide (see [4]), we have thus proved following theorem.

Theorem 4.12. *The function $S_\Omega(\lambda)$ defined by (4.6) coincides with the characteristic function of the maximal dissipative operator $\mathcal{L}_\Omega(\mathcal{L}_K)$.*

5. Completeness of root vectors

In this section, we prove that all the root vectors of the maximal dissipative q-fractional Sturm-Liouville operator are complete. We know that the absence of the singular factor in the factorization of the characteristic function guarantees the completeness of the system of root vectors of maximal dissipative operators ([4]). We will prove that the characteristic function of the maximal dissipative q-fractional Sturm-Liouville operator is a Blaschke-Potapov product.

Lemma 5.1. *The characteristic function $\widetilde{S}_K(\lambda)$ of the operator $\mathcal{L}_K$ has the form*

$$\widetilde{S}_K(\lambda) := S_\Omega(\lambda)$$

$$=X_1(I-K_1K_1^*)^{1/2}(\Theta(\xi)-K_1)(I-K_1^*\Theta(\xi))^{-1}(I-K_1K_1^*)^{1/2}X_2,$$

where $K_1=-K$ is the Cayley transformation of the dissipative operator Ω, and $\Theta(\xi)$ is the Cayley transformation of the matrix-valued function $m(\lambda)$, where

$$\begin{gathered}\xi=(\lambda-i)(\lambda+i)^{-1},\\ X_1:=(\operatorname{Im}\Omega)^{-1/2}(I-K_1)^{-1}(I-K_1K_1^*)^{1/2},\\ X_2:=(I-K_1^*K_1)^{-1/2}(I-K_1^*)^{-1}(\operatorname{Im}\Omega)^{1/2},\\ |\det X_1||\det X_2|=1.\end{gathered}$$

Recall that the inner matrix-valued function $\widetilde{S}_K(\lambda)$ is a Blaschke-Potapov product if and only if $\det \widetilde{S}_K(\lambda)$ is a Blaschke product (see [11, 4]). By Lemma 5.1, the characteristic function $\widetilde{S}_K(\lambda)$is a Blaschke-Potapov product if and only if the matrix-valued function

$$X_K(\xi)=(I-K_1K_1^*)^{1/2}(\Theta(\xi)-K_1)(I-K_1^*\Theta(\xi))^{-1}(I-K_1K_1^*)^{1/2}$$

is a Blaschke-Potapov product in the unit disk.

We use the following result of [11].

Lemma 5.2. *Let $X(\xi)$ ($|\xi|<1$) be a analytic function with the values to be contractive operators in $[E]$ ($\|X(\xi)\|\le 1$). Then for Γ-quasi-every strictly contractive operators (i.e., for all strictly contractive $K\in[E]$ possible with the exception of a set of Γ of zero capacity) the inner part of the contractive function*

$$X_K(\xi)=(I-K_1K_1^*)^{1/2}(X(\xi)-K_1)(I-K_1^*X(\xi))^{-1}(I-K_1K_1^*)^{1/2}$$

is a Blaschke-Potapov product.

By summing all the obtained results for the dissipative operator $\mathcal{L}_K(\mathcal{L}_\Omega)$, we have proved the following result.

Theorem 5.3. *ForΓ-quasi-every strictly contractive $K\in[E]$ the characteristic function $\widetilde{S}_K(\lambda)$ of the dissipative operator $\mathcal{L}_K$ is a Blaschke-Potapov product, and the spectrum of $\mathcal{L}_K$ is purely discrete and belongs to the open upper half-plane. For Γ-quasi-every strictly contractive $K\in[E]$ the operator $\mathcal{L}_K$ has a countable number of isolated eigenvalues with finite multiplicity and limit points at infinity, and the system of eigenvectors and associated vectors (or root vectors) of this operator is complete in the space H.*

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