ON THE AXIOMATIZATION OF MATHEMATICAL UNDERSTANDING:

CONTINUOUS FUNCTIONS IN THE TRANSITION TO TOPOLOGY

by

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DEDICATION

To Briane, for the love and patience that made it possible.

To Adia, for the dreams and inspiration that made it worthwhile.

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ABSTRACT

The introduction to general topology represents a challenging transition for students of advanced mathematics. It requires the generalization of their previous understanding of ideas from fields like geometry, linear algebra, and real or complex analysis to fit within a more abstract conceptual system. Students must adopt a new lexicon of topological terms accompanied by a multitude of relationships among their underlying mathematical ideas. While some students are successful in coordinating these two related strands of understanding, many others encounter challenges as they attempt to accommodate their prior conceptual schemas within the context of the axiomatic system of topology. Although there has been increasing interest in studying students' understanding of axiomatic systems, few researchers in the field of mathematics education have explored the ways that students think about and reason with the axioms of topology. I claim that distinctions between individual cases of student reasoning in topology can offer new insights into advanced mathematical thinking and learning, both in topology and other mathematical fields of study.

To advance the research on students' reasoning in advanced mathematics, I conducted a semester-long qualitative study to illuminate how six undergraduate mathematics majors approached the transition to axiom-based reasoning in their introductory topology course. Through a series of individual clinical interviews, I observed and interpreted their mathematical activities as they completed proof tasks in the context of topologies with which they were unfamiliar. I found that they employed diverse strategies and reasoned with multiple conceptions of the open set and continuous function ideas as they embedded their formal and informal ways of understanding into schemas that would reflect the axiomatic system of topology. By exploring these participants' transformative uses of properties during the accommodation of their schemas to axiomatic contexts, this study contributes to an emerging perspective on the construction of axiomatic mathematical understanding in general.

1. INTRODUCTION

1.1 Background

> "The axiomatization of a theory represents the final stage, the culmination, of its development; it is the systematic formulation of elements that have previously been elaborated, with the aim of clarifying their logical connections."
>
> Piaget & Garcia (1983/1989)

An important goal in undergraduate mathematics education is to help students reconstruct their mathematical understanding in more precise and formal ways (Zorn, 2015). Over time, mathematics majors learn new ways to give logical structure to their previous conceptions, and increasingly align their conceptual understanding with the formal theories of the disciplines they study. In the quotation above, Garcia & Piaget (1983/1989) were describing the development of such formal theories by experts in a given field; however, the culmination of an individual student's conceptual development may also involve reformulating previously-understood mathematical ideas into an axiomatic structure.

This *axiomatization* of a student's understanding might follow a logical pathway modeled after the original thinking of the founders of a mathematical theory, especially when it is presented in such a way by her professors. However, it may also be true that a student's personal logical development for a mathematical system may differ greatly from that of the original experts, yet still achieve the same axiomatic structure in her understanding. Considering the explicitly axiomatic nature of advanced undergraduate mathematics courses, it would be useful to understand how students build their personal mathematical conceptions around formal systems of axioms.

To begin to respond to this question, I conducted a study to investigate the ways that undergraduate students related to the axiomatic system of general topology through their reasoning about continuous functions. My "background" theoretical framework was a radical constructivist (Glasersfeld, 1995) interpretation of Piaget's (1970) theory of genetic epistemology. The basic theoretical construct for this framework is the conceptual *schema*, which I considered to be the fundamental (coherent) developmental structure in a learner's mind. This structure was mediated by Tall and Vinner's (1981) distinction between the *concept image* and the *personal concept definition* within a learner's conception of a mathematical idea. I will demonstrate that during the axiomatic transition, my participants often reasoned with varying levels of coherency between their personal concept definitions, mental schemas and other elements within their concept images (e.g., exemplars, prototypes, metaphors), which then formed the basis for the development of their *re-equilibrated* understanding (Glasersfeld, 1995; Piaget, 1970; 1975).

1.1.1 Axioms in general topology

The organization of the standard axiomatic structure for point-set topology[1] has evolved over roughly two centuries of mathematical reflection, tension, and revision by leading mathematicians of those times (Koetsier & van Mill, 1999; Moore, 2008) reflecting the values of mathematical utility and efficiency that were important in the minds of its creators. The axioms of topology were designed to reflect mathematicians' intuitive understanding of certain spatial concepts, such as continuity, connectedness, and

[1] Also known as "general topology," which is distinguished from particular branches of the field such as algebraic, geometric, or differential topology. I will use the term *topology* specifically to refer to point-set topology, which is the content domain of this study.

containment (Moore, 2008). However, given the historical difficulty faced by expert mathematicians in originally establishing the axiomatic system of topology (Epple, 1998; Moore, 2008), it seems reasonable to expect novice students to face challenges when they reason about the axiomatic aspects of the subject. For example, many students may find the intellectual benefits of reasoning within axiomatic systems to be muted by the opaque and apparently arbitrary nature of the axioms themselves (de Villiers, 1986). They may encounter difficulty in reconciling the axioms with their pre-formed, perceptual and spatial understandings (Dawkins, 2016; Freudenthal, 2002). In many cases, students may also struggle with the complex mathematical logic of set theory as it is applied in topology (Narli, 2010; Zazkis & Gunn, 1997)

Topology is one of the first courses encountered by some undergraduate mathematics majors in which the content is explicitly and entirely structured around a set of axioms. Transitioning to a mathematical environment structured in this way can be challenging for undergraduate students (Dawkins, under review; de Villiers, 1986; Narli, 2010), especially when they are accustomed to reasoning within the familiar contexts of the real numbers and Euclidean space (Harel & Tall, 1991; Tall, 2013). Some of these challenges may arise from logical and didactical obstacles associated with students' pre-axiomatic experiences. But, it is also possible that students' intuition-laden perspectives may hinder their abilities to reason within the axiomatic structure of the field (Fischbein, 1987, Hazzan, 1994; Tall, 2013). This may be true for students of topology despite the axiom-centered treatment found in textbooks and classroom instruction, dis-incentivising some students to learn to reason holistically about the axiomatic system from first principles. To explore these issues, I report on an investigation into how six students in an introductory topology class reasoned with their personal conceptions of continuous functions in relation to the axioms of topology.

1.1.2 Open sets and continuous functions in general topology

Continuity is a central concept in topology. The field “deals with properties which are not destroyed by continuous transformations like bending, shrinking, stretching, and twisting” (Croom, 1989, p. 1). It has been said that “topology is essentially just the study of continuous functions” (Crossley, 2005, p. 3). However, by the time they enroll in their first topology course, students have already had significant experiences with the idea of continuous functions. Thus, students are expected to build on their previous understanding of the concept through the *assimilation* of the axiomatic structure of topology into their previous schemas; but in many cases, they must actually reconstruct their understanding through the *accommodation* of those schemas to reflect the new topological structure (Garcia & Piaget, 1989). To gather insight into how students' previous understanding influences this transition, I chose to study six participants' conceptions of continuous functions in the context of both familiar and unfamiliar topological spaces.

The field of topology is organized around a set of formal axioms that generalize students' prior understanding of topics in calculus, linear algebra, and real analysis. It is also a field involving a rich interplay between students' mathematical understanding and their physical and spatial perceptions. Thus, I chose topology as the curricular setting for my research to investigate the interaction between my participants' spatial intuitions, their prior mathematical experiences, and the axiomatic structure of topology.

Typically, in their real analysis and introductory topology classes, mathematics majors will be presented with increasingly general versions of the continuous function concept. They will consider more abstract function domains as they explore the definition for a continuous function between metric and then topological spaces. But, despite the relevance of continuous functions to the advanced mathematical curriculum, little research has been done to document the ways that undergraduate students reason with and about the underlying mathematical mechanism for continuous functions: the *open sets*. These are defined by the axioms of topology as a special family of subsets $\mathscr{T}$ of a topological space X which collectively adhere to the these axioms (Croom, 1989, p. 99):

1) The set X and the empty set $\varnothing$ belong to $\mathscr{T}$.
2) The union of any family of members of $\mathscr{T}$ is a member of $\mathscr{T}$.
3) The intersection of any finite family of members of $\mathscr{T}$ is a member of $\mathscr{T}$.

The concept of an open set is the fundamental basis for defining every other idea in general topology, including the important notion of a continuous function. Therefore, open sets will play a central role in the analyses for this study.

Open intervals on the real number line and open regions in the Euclidean plane are specific instantiations of the open set concept. By studying continuous functions in the context of these familiar domains, students gain valuable models (Alcock & Simpson, 2002; Fischbein, 1987; Tall, 2013; Tall & Vinner, 1981) as scaffolds for their understanding of the abstract mechanism that characterizes continuous functions. That mechanism is highlighted in the following characterization of continuous functions from Croom (1989, p. 115):

> A function $f:(X,\mathscr{T}) \to (Y,\mathscr{T})$ is *continuous* means that for each open set V in Y, $f^{-1}(V)$ is an open set in X.

As a special case, this includes the standard ‘epsilon-delta’ or ‘limit’ definition of a continuous function on the real numbers from real analysis, which takes the following form (Croom, 1989, p. 75):

> Let $f : A \to \mathbb{R}$ be a function from a subset A of $\mathbb{R}$ to $\mathbb{R}$ and let $a \in A$. Then f is *continuous at a* if for each positive number ε there is a positive number δ such that if $x \in A$ and $|x-a| < \lambda$, then $\left|f(x) - f(a)\right| < \varepsilon$. If f is continuous for each $a \in A$, then it is said simply that f is continuous.

By comparing these two definitions—the simple but abstract version that students will learn in topology and the technical version students learn much earlier—the challenges students face in accommodating their schemas for continuity are apparent. My research interest was to explore how introductory topology students might use such familiar examples from their prior mathematical experience, as well as their informal conceptions of physical space, to develop new understandings of the open set and continuous function concepts.

1.2 Purpose and Significance of the Study

The overarching goal of my research was to study the formation of axiomatic structures in the minds of university-level mathematics students. The long-term development of a student’s schema for an advanced mathematical notion follows a trajectory from informal, intuitive structures toward logical formalization, and culminates in axiomatization (Freudenthal, 2002; Garcia & Piaget, 1983/1989; Tall, 2013). For many students, this process may terminate the acquisition of formal understanding; however, mathematics majors are typically expected to begin the process of restructuring their knowledge around axioms during their undergraduate schooling (Zorn, 2015).

Although there has been significant research into the process of formalizing mathematical understanding (cf., Alcock & Simpson, 2002; Chin & Tall, 2000, 2002;

Morena-Armella, 2014; Dawkins, 2012; Pirie & Kieren, 1989; Tall, 1992, 2013; Zandieh & Rasmussen, 2010), less has been researched about students' attempts to achieve *axiomatization* of their understanding. While axiomatization may be a type of formalization, the cognitive changes it fosters in students are qualitatively different from earlier forms of formal understanding (de Villiers, 1996; Freudenthal, 2002; Piaget & Garcia, 1983/1989; Tall, 2013). I chose to study these cognitive transformations in the context of an introductory topology class, a relatively unexplored setting within the field of research in undergraduate mathematics education.

The research I conducted contributes theoretically to knowledge about advanced mathematical understanding in several ways. It has helped to:

- provide insight into students' transitions to axiomatic content, especially in the important context of continuous functions,
- explore learners' mental representations of mathematical *properties* in addition to mental *objects*; and,
- serve as a foundation to study more sophisticated andragogic (Knowles, 1980) techniques for the introduction of abstract mathematical content.

These contributions include both pragmatic knowledge about the learning and teaching of topology and methodological considerations for future research in undergraduate mathematics education.

1.3 Research Questions and Methodological Overview

To begin to investigate how undergraduate topology students equilibrate their continuous function schemas with an axiomatic structure, I observed an introductory topology class for one semester, and collected completed quizzes and extra credit assignments from thirty-nine student-participants. Seven of those participants were chosen for a series of three, one-on-one task-based interviews. Data from these interviews constitute the primary analytical focus for this study, and are supplemented with analysis of data from the entire class. Below are the research questions addressed by this study.

For undergraduate students in an introductory topology class:

1) What distinctions and comparisons can be made between the various ways that students manage their transition to an axiomatic understanding of continuous functions?
2) What obstacles do students face during this transition?

To build a theoretical model around answering these questions, I adopted a radical constructivist epistemological perspective (Garcia & Piaget, 1989; Piaget, 1970; Steffe & Thompson, 2000; von Glasersfeld, 1995, 2007) and employed qualitative research methods to build a new conceptual framework grounded in my data. The research design was informed by three semesters of preliminary data collection along with three cycles of data analysis in the tradition of Strauss and Corbin (1998). These initial studies led to the generation of a coding paradigm (Strauss & Corbin, 1998), which took the form of fifteen secondary research questions which I applied to the analysis of the data from the final semester of the study. Tasks were designed to elicit participants' mathematical activity in a way that could relate to and answer the coding paradigm's questions. The coding paradigm served to operationalize my conceptual framework, providing a lens to approach the initial analysis of the data.

For the main study, I conducted and reported on a series of task-based interviews with six participants from an introductory topology class. The interviews were structured to provoke revealing responses about the participants' explanatory schemes for foundational content in the class. I approached this inquiry inductively, seeking to interpret my participants' personal and collective meanings for concepts related to continuous functions. I looked for dimensions of variation in the development of their schemas both within and across participants' responses. I also looked for contrasts and comparisons in the ways that my participants negotiated new meanings in the unfamiliar context of abstract topological spaces; as well as difficulties and obstacles they encountered along the way.

I viewed the data through an analytical lens that highlighted the interplay of the axiomatic structure of topology with my participants' personal and prior understanding about continuous functions. Through this analysis, a variety of cases of distinct forms of reasoning were constituted evidence for the participants' unique ways of developing an understanding of continuity in axiomatic contexts. I report on these cases in the Results and Analysis chapters (Chapters 4 and 5).

1.4 Overview of the Analyses

The theory that emerged from this study developed in the form of two multiple-case analyses, in which the units of analysis were episodes of student reasoning rather than the individual participants themselves. These analyses resulted from the networking (Artigue, 2016; Bikner-Ahsbahs & Prediger, 2014) of two distinct, but related, theoretical traditions, chosen in response to the emergent themes from the initial coding of the data.

The first section of the results is an APOS (Action Process Object Schema) analysis of the open set construct, examining how the participants defined and used the central concept that underlies continuous functions. APOS is a neo-Piagetian theory of conceptual development that has been applied primarily to undergraduate mathematics education (Arnon, et al., 2014; Dubinsky, 1991). This theory's main constructs and mechanisms were useful in differentiating the approaches used by the participants to define the open set idea, and how they reasoned with this idea in the context of tasks related to continuous functions.

The second section of the results is a radical constructivist conceptual analysis (Glasersfeld, 1995) that applies a conceptual framework involving a range of theories in cognitive linguistics and psychology to explore the origins of the operations within my participants' schemas for continuous functions. These theories were borrowed from three major schools of thought: 1) the prototype models (Dry & Storms, et al., 2010; Rosch & Mervis, 1975), exemplar models (Hampton, 2003; Heit & Barsalou, 1996, Verbeemen, et al., 2007) and rules-based models (Ross, 1996) of memory categorization theory; 2) theories of the metaphorical and metonymical structure of understanding (Dogan-Dunlap, 2007; Lakoff, 1980; Presmeg, 1997; Sfard, 1991; Zandieh & Rasmussen, 2010); and 3) theories of embodied cognition (Arzarello & Robutti, 2001; Lakoff & Nuñez, 2000; Nemirovsky, 2003; Piaget & Garcia, 1989; Glasersfeld, 1995). These frameworks were useful for describing the informal modes of reasoning I observed in my participants' responses, clarifying the roles of these linguistic mechanisms and experiential biases in advanced mathematical thinking.

Together, these two analyses contribute to a better understanding of the ways that some students reason within axiomatic systems, as well as the processes involved in reconstructing their conceptions of the specific ideas related to continuous functions in general topology.

1.5 Summary and Chapter Overview

Advanced mathematics courses often place new demands on undergraduate students to reorganize their cognitive schemas in a way that reflects the axiomatic structures of their respective disciplines. The research presented here helps to understand how some students went about re-building their personal mathematical conceptions to account for the formal system of axioms that defines a topology. To do so, I investigated the way my participants accommodated their schemas for open sets and continuous functions as they were introduced to increasingly abstract versions of these concepts throughout the semester.

I reported my research in seven chapters, which includes a detailed explanation of the preliminary research done for this study and two chapters of analysis and discussion. In Chapter 2, I will review the relevant research behind the analytical frameworks outlined above, as well as content-specific literature from the field of undergraduate mathematics education. For a description of the methods I used to develop the interview tasks and coding paradigm, see Appendix E. The methods used to collect and analyze the data for the main study are explained in Chapter 3. I reported the analysis and results in two chapters. Chapter 4 is presented as an APOS analysis of the open set concept, while Chapter 5 details a radical constructivist conceptual analysis of continuity. I conclude with recommendations for future research in Chapter 6.

2. LITERATURE REVIEW

2.1 Introduction

Since the turn of the last century, modern mathematics has rested on the notion of axiomatic systems to organize the foundational ideas of the discipline (Zach, 2007). These deductive systems consist of collections of declarative statements, or axioms, whose logical interactions describe the properties and relationships of the primitive[2] elements within them. Additional properties can be logically deduced from the axioms, without the direct need for subjective criteria, representational forms, or human intuition. Historically, the axiomatization of the concept of continuous functions led to the development of the important field of topology (Moore, 1995) and is central to the exploration of invariant properties between topological spaces; especially through its role in the definition of homeomorphism (Croom, 1989).

Continuity itself is a mathematical property that may be possessed by some examples of any student's concept of a function. However, continuity is also defined as a complex set of relationships between various sub-properties, such as the property of being open and/or closed for sets. These sub-properties play a role in the determination of limit points, interior points, and boundary points of the images and pre-images of sets under a given function. Each of these constructs is part of a larger axiomatic system, composed of collections of mathematical properties and interconnected through the interactions of those properties. Describing and analyzing the interactions between these collections of properties led 19th and 20th century mathematicians to build an abstract definition of continuity that was detached from spatial perceptions and the real number

[2] Axiomatic systems are built from basic, undefined elements, known as *primitives* (Tarski & Tarski, 1994).

context (Moore, 2008). In this way, their intuitive understanding of space became formalized and later axiomatized. Similarly, I will argue that the study of such properties and their relationships will be useful to effectively model students' transitions toward axiomatic formalism in undergraduate mathematics classrooms.

The research presented here began as an attempt to illuminate the transition that learners might face when embedding their informal and more formal ways of understanding within axiomatic structures. This contributes to an emerging theoretical perspective on the construction of axiomatic mathematical understanding, by exploring transformations in how undergraduate mathematics students use and represent the properties of cognitively-represented objects during the reconstruction of their concept images (Tall & Vinner, 1981) for continuous functions in topological contexts. A student's abstraction and later instantiation of those properties plays a role in the development of her axiomatic knowledge structures (Freudenthal, 2002; Tall, 2013). Therefore, I chose to explore my participants' specific mental representations of mathematical properties and perceptual attributes (of mental or mathematical objects). Instead of focusing directly on the reflective abstractions (Piaget, 1970; Simon, Tzur, Heinz, & Kinzel, 2004) that lead to students' reification of mental objects (Arnon, et al. 2014; Sfard, 1994), I conducted a detailed survey of the underlying properties those abstractions may entail.

2.2 Research on Mathematical Understanding

The notion of mathematical understanding is complex and difficult to study for several reasons. First, as a cognitive representation of subjective, internal experience (Glasersfeld, 1995), it is necessarily rooted in both psychology and epistemology, rather than within the confines of mathematics itself (Lakoff & Nuñez, 2000; Piaget & Garcia, 1983/1989). This requires those within the tradition of mathematics education research to examine advances in these separate fields for help in establishing meaningful and useful theories of their own. Mathematical understanding is also difficult to study because of the nature of mathematics itself. Mathematic knowledge is both common to the human experience and a highly abstract and specialized field of inquiry. As Freudenthal (2002) warns, "…to the view and mind of most people, mathematics, though deeply rooted in common sense, is more remote from it than anything else" (p. 9). Mathematical understanding is a broad construct, incorporating humanity's grasp of the most basic mathematical constructions, which most of us are capable of; up to the most advanced, which few of us may reach.

The focus of the present study is on *advanced* mathematical thinking and understanding, which has been distinguished from earlier forms of mathematical thinking in the literature. Tall (2002) describes this distinction in the following way:

> The move from elementary to advanced mathematical thinking involves a significant transition: that from *describing* to *defining*, from *convincing* to *proving* in a logical manner based on those definitions. This transition requires a cognitive reconstruction which is seen during the university students' initial struggle with formal abstractions as they tackle the first year of university. It is the transition from the *coherence* of elementary mathematics to the *consequence* of advanced mathematics, based on abstract entities which the individual must construct through deductions from formal definitions. (Tall, 2002, p. 20, italics in original)

Harel & Sowder (2005) linked the development of advanced mathematical thinking to the presence of "epistemological obstacles" (Brousseau, 1997; Sierpinska, 1994). Dreyfus (2002) argued that "reflection about one's mathematical experience…is a characteristic of advanced mathematical thinking" (p. 25); but pointed out that "there is no sharp distinction many of the processes of elementary and advanced mathematical thinking" (p. 26). The distinctions made by the scholars above will serve to delineate the expectations faced by my participants within the social environment of the classroom. However, it was not assumed that the participants' mathematical behaviors would remain consistently at an advanced level.

Despite the long history of research into elementary and intermediate mathematical understanding (Bruner, 1973; Dewey, 1910; Fischbein, 1999; Freudenthal, 1973; Piaget, 1970, 1975; Skemp, 1979; Vygotsky, 1978), studies involved in probing advanced mathematical understanding have only begun to emerge in the past few decades (see Arnon, et al., 2014; Dawkins, 2015; Freudenthal, 2002; Harel & Sowder, 2005; Hershkowitz, Schwarz, Dreyfus, 2001; Lakoff & Nuñez, 2000; Luneta, & Makonye, 2013; Simon & Tzur, 2004; Tall, 2002, 2013). These important first steps have led the way for a more comprehensive account of students' formalization of advanced mathematical ideas. A significant research effort is now underway to develop a theory for the basis of humans' formal understanding of advanced mathematical concepts (Arnon, et al. 2014; Lakoff & Nuñez, 2000; Garcia & Piaget, 1983/1989; Pirie & Kieren, 1989; Sierpinska, 1987, 1994; Sfard, 1994; Tall, 2013). While the vocabulary, methods and assumptions of these researchers have varied, common themes have evolved from the diverse body of work that has grown out of this endeavor.

The long-term development of a student's understanding of a mathematical notion typically follows a trajectory from informal, sensori-motor intuitions toward mathematical structures, which may then be logically formalized, potentially culminating in *axiomatization* (Freudenthal, 2002; Piaget & Garcia, 1989; Tall, 2013). Piaget and Garcia (1983/1989) described this transition as being essential to a "constructivist epistemology":

> …it is a mechanism that leads from intra-object (object analysis) to inter-object (analyzing relations or transformations) to trans-object (building of structures) levels of analysis. That this dialectic triad can be found in all domains and at all levels of development seems to us to constitute the principal result of our comparative efforts...The unavoidable intra, inter, trans sequence clearly demonstrates the constructivist, dialectical nature of cognitive activities. (p. 28)

This *intra-, inter-, trans* framework has been used in various ways within the mathematics education research community (Arnon, et al., 2014; Baker, Cooley, Trigueros, 2000; Clark, et al., 1997; Paraguez & Oktaç, 2010; Trigueros & Martinez-Planell, 2009); it will serve as a theoretical model for the participants' long-term shifts in understanding about the concepts involved in learning topology, in particular, the concepts of open set and continuous function. My research was primarily concerned with the ways that students make the final transition to the trans-object level of analysis, which I will refer to here as *axiomatization*. I will refer to the transition to the inter-object level of analysis as *formalization;* and I will follow Freudenthal (2002) in using the term *mathematization* to describe the general activity of organizing perceptual reality in mathematical ways.

Pirie & Kieren (1989) developed a recursive model for mathematical understanding which reflects the above transitions in a student's understanding: mathematization, formalization, and axiomatization. Figure 1 presents the Pirie-Kieren model's eight layers (or modes) of understanding: "Primitive Knowing, Image Making, Image Having, Property Noticing, Formalising, Observing, Structuring, and Inventising" (Pirie & Martin, 2000, p. 129). Elaborating on this model, Pirie & Martin (2000) explained how the Pirie-Kieren model embeds earlier forms of mathematical understanding into more abstract ways of understanding:

> …Each layer contains all previous layers and is included in all subsequent layers. This set of unfolding layers suggests that any more formal or abstract layer of understanding action enfolds, unfolds from, and is connected to inner, less formal, less sophisticated, less abstract, and more local ways of acting. (p. 129)

Therefore, a student's growth in mathematical understanding entails a continual return to previous modes of understanding, a behavior they called "folding back":

> When faced with a problem or question at any level, which is not immediately solvable, one needs to *fold back* to an inner level in order to extend one's current, inadequate understanding. This returned-to inner level activity, however, is not identical to the original inner level actions; it is now informed and shaped by outer level interests and understandings. (Pirie & Kieren, 1994a)

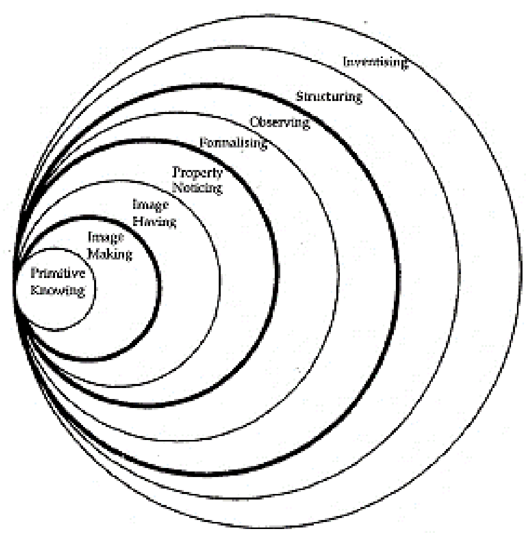

Figure 1. Pirie and Kieren's (1989) recursive model for growth in understanding mathematics. Each successive layer contains all previous layers, connecting less formal understanding with more formal understanding through the process of "folding back."

By locating informal and perceptual ways of knowing as central aspects of the process of mathematical formalization, Pirie and Kieren's theory provides a useful model for development of abstract mathematical understanding. The mechanism of folding back then ensures the continue significance of these forms of meaning-making, even during advanced mathematical activity. Folding back allows students to "collect" (Matin & Pirie, 2000) previous ways of understanding and "reviewing or reading it anew in light of the needs of current mathematical actions" (p. 131).

For my study, it is important to note the importance of the shift from "property noticing" to "formalization." Pirie and Kieren (1994b) called this a "metamorphic transformation," which is indicated in
Figure 1, along with two other such transformations, by the three bold circles. The outermost circle represents a metamorphic transformation from "observing" to "structuring" activities, which Pirie and Kieren (1994b) compared to Sfard's (1991) notion of the duality between "operational" and "structural" conceptions.

Sfard (1991) and Dubinsky (1991) were both early proponents of similar theories that collectively came to be known as "reification theory" (Confrey & Costa, 1996; Simon & Tzur, 1994). As stated above, Sfard's (1991) operational-structural duality offered a way to characterize distinct conceptions according to the manner in which subject reasons with them. Meanwhile, Dubinsky's (1991) Action Process Object Schema (APOS) theory presented a hierarchy of mental structures that were built up through a series of "mechanisms" that came about through the Piagetian (1970) notion of "reflective abstraction" (for a more complete description of APOS theory, see Section 2.5.2). While reification theories involve learning about mathematics at all levels of mathematical development, they have been especially useful in describing the formalization of mathematical activity, during which the “advanced mathematical constructs are totally inaccessible to our senses” (Sfard, 1991, p. 3).

2.2.1 Formalization of mathematical understanding

Formal mathematical reasoning has been characterized in a variety of ways. It has been said to involve: the use of quantified statements and formal deduction (Arnon, et al., 2014; Pinto & Tall, 2002), communication in the formal-symbolic, or mathematical register (Halliday, M., 1978; O’ Halloran, K. L., 2000; Schleppegrell, M. 2007), and a greater reliance on the use of defining properties for the categorization of concepts (Alcock & Simpson, 2002). On the other hand, the notion of informal reasoning often denotes a stronger dependence on components of the concept image that are not based on the definition (Dawkins, 2012; Tall & Vinner, 1981; Vinner, 2002), such as: exemplars of a category (Alcock, & Simpson, 2002; Hazzan, 1994; Zazkis & Leikin, 2008), semantic representations (Weber & Alcock, 2004), or prototypical examples (Pinto & Tall, 2002; Tall & Bakar, 1991).

The formalization of understanding has been described as a transition whereby a student recognizes the structural relations among the properties of a mathematical object. A new construct, or “mental object” is synthesized, which reflects this structure and enables action on the concept as a structural whole (Pirie & Kieren, 1989; Sfard, 1994; Tall, 2013). Harel & Tall (1991) named this process “formal abstraction” and described it as the abstraction of “the form of the new concept through the selection of generative properties of one or more specific situations” (p. 39). So, formal abstraction describes a transition related to the thinking subject's interpretation (and selection) of the properties of mathematical situations.

Regarding a student's formalization of mathematics, Pirie & Kieren (1994b) claimed that the "formalising" level, or mode, of understanding would "embed less formal understanding levels called primitive knowing, image making, image having and property noticing" (p. 39). These four modes of understanding were claimed to represent the fundamental, "local" understanding of students in a particular context. "Formalising" was said to occur through the abstraction of "informal patterns" and common features, leading the way for the student to observe theorems and structure their mathematical understanding in new ways.

2.2.2 Axiomatization of mathematical understanding

While there has been significant research into the process of formalizing mathematical understanding (Alcock & Simpson, 2002; Morena-Armella, 2014; Dawkins, 2012; Pirie & Kieren, 1989; Tall, 1992, 2013; Zandieh & Rasmussen, 2010), less has been researched about students' attempts to achieve *axiomatic* formalization of their understanding. That is, for a student to restructure her formal mathematical understanding in such a way that it reflects an underlying axiomatic structure. To distinguish this from non-axiomatic formal understanding, I will refer to such a student's conception as *axiomatic understanding* (as I will show, axiomatic understanding may also be informal, or formal to varying degrees). I will refer to a student's reasoning with an axiomatic structure as *axiomatic reasoning*, whether it is of a formal nature or not. In a cognitive sense, to *axiomatize* will be to restructure one's understanding around an underlying system of axioms.

Freudenthal (2002) used the term *axiomatizing* to describe the process of reorganizing fields of mathematical research through the abstraction of key properties, which become form-reflecting structures, rather than content-embedded features, within the new structure of the field. He described axiomatization as the "reshuffling of an area of knowledge so that ends are chosen as starting points; and conversely, using proven properties as definitions to prove what was a definition originally…structuring 'upside down…'" (p. 30). Thus, axiomatizing requires a dramatic shift in how students think about mathematical properties.

Axiomatization has often been characterized as a later, or "final", stage in the formalization of mathematical processes, both from a cognitive perspective (Tall, 2012; Yannotta, 2013) and from an historical perspective (de Villiers, 1986; Freudenthal, 2002). This is especially true in the field of topology (Epple, 1998; Moore, G., 2008). However, it may be useful for educators to recognize the informal work that learners must accomplish before such formal structures can be usefully constructed. Relating axiomatic mathematical structures to their intuitive development, Fischbein (1987) highlighted the informal processes that precede the evolution of an axiomatic theory:

> …the axiomatic structure is the final state attained by a body of mathematical knowledge *after* the body of knowledge has already been obtained by other means than mere deduction. These procedures refer to heuristic and inductive processes similar to those which intervene in empirical sciences. (p. 23, italics in original)

Those "heuristic and inductive processes" that were used by the originators of axiomatic mathematical fields may offer similar affordances to students of topology, or any other axiomatically structured content, as they attempt to accommodate their conceptual schemas in relation to the given axioms. Fischbein (1987) pointed out the usefulness of having such experientially real interpretations in the context of reasoning about consistency (lack of contradiction) in axiomatic systems:

> The practical procedure is to resort to a *model*, a particular interpretation which would appear *psychologically* as a structured reality and with regard to which, contradictions would become salient. When we say *structured* we mean *intrinsically coherent*, as real objects are…if one does not resort to any such internally structured interpretation and one deals only with symbols, the meanings of which are defined by axioms, it is possible to overlook the contradiction. (p. 22)

However, it may also be true that such informal processes constrain students' understanding, or lead to the same "epistemological obstacles" (Brousseau, 1997; Sierpinska, 1994) as were encountered in the original formulation of the field. Indeed, I found evidence that informal ways of thinking both benefitted and constrained my participants in different situations as they worked on tasks concerning topological content.

The transition to axiomatic reasoning and understanding can be difficult for students who are accustomed to concrete foundations for the mathematics they have studied, but who have difficulty finding those foundations during professors' formal treatment of the subject matter. Although axiomatization has been hailed as a powerful way to organize mathematical theory (Freudenthal, 2002; Garcia & Piaget, 1983/1989), few researchers have examined the learning processes by which students make this transition in their own minds. Recently, some researchers have documented difficulties students may face when attempting to reconstruct mathematical properties axiomatically (Dawkins, 2016; Fischbein, 1987; Freudenthal, 2002; Harel & Tall, 1991; Tall, 2013; Yannotta, 2013). They emphasized that students' prior conceptions may obstruct the process of reconstructing their schemas in a new way. For example, Harel & Tall (1991, p. 23) argued that the "construction [of an abstract concept] is guided by the properties which hold in the original mathematical concepts from which it is abstracted, but judgement of the truth of these properties must be suspended until they are deduced from the definition." Similarly, Tall (2013, p. 203) expressed concern that "the novice reader will not be able to read the axioms without making implicit mental associations with previous mathematical ideas."

Students' previous mathematical conceptions seem to be obstructive in such cases. In mathematical domains that model highly perceptual attributes of human experience, obstacles may also lie within students' spatially and perceptually based intuitions. For example, Alcock and Simpson (2002) claimed that in a real analysis class, students' access to relevant visual representations provided them with an informal way to gain understanding of the concepts, but in turn diminished their intellectual need for engaging with the complexity of the formal mathematical content being represented. They concluded: "So paradoxically, analysis may be difficult not only because the material is complex *per se*, but because it is initially *less* 'abstract' than other beginning subjects in advanced mathematics" (p. 34). Similarly, I will show how my participants' sensori-motor experiences influenced the reconstruction of their schemas in axiomatic ways.

2.3 Semantic and Semiotic Contributions to the Research

Piaget (1970) described the function of language in the representation of meaning as significant but incomplete without other, non-linguistic forms of representation.

> Language is certainly not the exclusive means of representation. It is only one aspect of the general function…the semiotic function. This function is the ability to represent

> something by a sign or a symbol or another object. In addition to language the semiotic function includes gestures,…deferred imitation,…drawing, painting, modeling. It includes mental imagery… Language is but one among these many aspects of the semiotic function, even though it is in most instances the most important. (Piaget, 1970, p.45-46)

In this section, the linguistic and non-linguistic forms of building meaningful re-presentations will be reviewed to derive the constructs used in the conceptual framework (see Section 2.4).

2.3.1 The multi-semiotic mathematics register

Coining the term *register* as it pertains to linguistic analysis, Halliday (1978, p. 195) described “a set of meanings that is appropriate to a particular function of language, together with the words and structures which express these meanings.” The concept of a register provides a useful lens for investigating the influence of specialized modes of communication on students’ thinking and learning. From a historical perspective, such technical uses of language can be said to have “evolved as clusters of features that differ from other registers (e.g. literary texts, news reports, casual conversation)” (O’Halloran, 2015). These specialized modes of communication help to store dense conceptual meanings in an abbreviated format, which bears little structural resemblance to other common language registers. This was Halliday’s basis for distinguishing the “mathematics register” from that of everyday, or natural, language:

> …in the sense of the meanings that belong to the language of mathematics (the mathematical use of natural language, that is: not mathematics itself), and that a language must express if it is being used for mathematical purposes. (p. 195)

While common uses of mathematics rely on everyday language, mathematics in academic settings require students to learn to use language in new ways, and build new meanings, as they interact within a social context. Halliday (1978) suggested that there are many ways, other than the creation of new words, that a language can contribute new meanings to our understanding. These included new "styles of meaning and modes of argument…and of combining existing elements into new combinations" (p.195). Thus, learning the mathematics register requires more than an in-depth study of new vocabulary, but in fact demands that students begin to reason and communicate in novel ways.

More recently, researchers following Halliday's tradition (see Schleppegrell, 2007) have developed and extended his initial theory by examining and incorporating other meaning-enriched signifiers into a "multimodal…approach to the mathematics register, where language is considered as one resource, often a secondary one, which operates in conjunction with mathematical symbolism and images to create meaning in mathematics" (O'Halloran, 2015, p. 64). In fact, there are several alternative forms of language that constitute the mathematics register, of which, verbal and written mathematical language is only one aspect. "The discourse of mathematics integrates scientific English with two key resources which were instrumental in ordering the physical world; namely, mathematical symbolism and mathematical images…" (O'Halloran, 2015, p. 66). Similarly, Schleppegrell (2007, p.141) pointed out that: "mathematics draws on multiple semiotic (meaning-creating) systems to construct knowledge: symbols, oral language, written language, and visual representations such as graphs and diagrams."

Therefore, adapting to the "multi-semiotic" mathematics register requires students to internalize new vocabulary terms, altered forms of argumentation, qualitatively different forms of meaning-making, specialized symbolic notation, and content-laden images. Each of these modes of thought play a role in mathematical cognition and communication. To communicate and reason mathematically in academic settings, each of these modes of thought and expression must be learned and coordinated effectively.

O'Halloran (2015) described the complexities involved in learning to use the mathematics register, which include "interlocking definitions (where concepts are related in larger theoretical frameworks), technical taxonomies (the classification systems which underlie technical terms), and special expressions (i.e. specialized vocabulary), all of which are common features of scientific discourse" (p. 66). However, some researchers (Halliday, 1978; Schleppegrell, 2007) have argued that the primary obstacles to learning the mathematics register stem from its grammatical structure, rather than the actual vocabulary itself. These grammatical challenges include "dense noun phrases that participate in relational processes, and the precise meanings of conjunctions and implicit logical relationships that link elements in mathematics discourse" (Schleppegrell, 2007, p. 139). O'Halloran also pointed to related difficulties concerning "the larger theoretical framework within which technical terms are defined and the metaphorical nature of the discourse of argumentation in which these terms are used, rather than individual terms themselves" (O'Halloran, 2015, p. 65).

2.3.2 Concept definitions and images

Tall & Vinner (1981) illustrated an important distinction between a student's *concept image* and the *concept definition*. They defined the concept image to consist "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (p. 152). They contrasted this with the *concept definition*, which was defined to be the "form of words used to specify that concept" (p. 152), and pointed out that these constructs may differ for an individual student at any given time. Tall and Vinner (1981) further delineated the student's *personal concept definition* from the *formal concept definition*, "the latter being a concept definition which is accepted by the mathematical community at large" (p. 152). These distinctions provided a terminology to discuss my participants defining activities, as well as the other structures (e.g., examples, hierarchies, and metaphors) that influenced how my participants reasoned about the tasks' content.

Zandieh & Rasmussen (2010) used Tall & Vinner's (1981) distinction to develop the Defining as a Mathematical Activity (DMA) framework within Freudenthal's (1973) instructional design theory of Realistic Mathematics Education (RME). The DMA framework was created as a way to chart stages of defining activity determined by the interplay of the concept definition and concept images of students. Dawkins (2012) expanded on this to incorporate the interplay of metaphor in building formal reasoning as well. I used the DMA framework as one of my selection guidelines for choosing participants, and as a theoretical perspective when analyzing my participants' defining activities. However, I did not adopt RME or attempt to study design instruction as a research goal.

An important element within any individual's concept image for a mathematical idea is the set of examples she associates with the idea. Watson and Mason (2002) introduced the term "example-space" (p. 242) to describe the set of examples a student has access to for reasoning about a mathematical concept. Watson & Mason (2005) expanded this notion to describe an *example space* as a set of examples for a specific concept, delineating several types of example space categorized by function. These included (Watson & Mason, 2005, as cited in Zazkis & Leikin, 2008, p. 132):

- *Situated* (local), *personal* (individual) *example spaces*, triggered by a task, cues and environment as well as by recent experience;
- *Personal potential example space*, from which a local space is drawn, consisting of a person's past experience (even though not explicitly remembered or recalled), and which may not be structured in ways which afford easy access;
- *Conventional example space*, as generally understood by mathematicians and as displayed in textbooks, into which the teacher hopes to induct his or her students;
- *A collective and situated example space*, local to a classroom or other group at a particular time, that acts as a *local conventional space.*

The concept of example space has been used widely in the field of research in mathematics education. Goldenberg and Mason (2008) added further structure to the example space construct by discussing the "dimensions of possible variation" and "range of permissible change" for examples used by students. Stylianides & Stylianides (2009) discussed the use of examples in proving statements; and Weber & Alcock (2004) examined differences between students who reason syntactically and those who use semantic instantiations (such as example spaces) during proof tasks.

A student's example space may be used in several different ways. The student may use examples to 1) instantiate and study a definition (Parameswaran, 2010), 2) examine *non*-examples of a mathematical phenomenon (Bogomolny, 2007; Tsamir, Tirosh, & Levenson, 2008), or 3) to extract properties from a prototypical example within it (Hazzan, 1994; Pimm & Mason, 1984). Fischbein (1987) named the latter use a "paradigmatic model" of the phenomenon it describes; which Zandieh & Knapp (2006) adapted to explain the notion of a "paradigmatic metonymy." That is, "our tendency to see a whole class of objects or an entire concept through the knowledge of a particular example or a submodel that exemplifies the concept or class" (p. 3).

As with a student's example space, it has also been shown that categorization hierarchies (Zazkis & Gunn, 1997), metaphors (Lakoff & Johnson, 1980; Lakoff, 1987; Johnson, 1987), prototypical representations (Pinto & Tall, 1999; Tall & Vinner, 1981; Vinner, 2002), and other elements of the concept image can play a significant role in student reasoning. Considering the many different cognitive structures that may form a student's concept image, and the tendency to use those structures as bases for reasoning by students and experts alike (Inglis & Mejia-Ramos, 2008; Parameswaran, 2010; Pinto & Tall, 1999; Sandefur, Mason, Stylianides, & Watson, 2013; Weber, 2008; Weber, Inglis & Mejia-Ramos, 2014), it is necessary to consider how semantic structures in the concept image play a role in mathematical reasoning. Alcock & Simpson (2011) introduced the notion of concept consistency. This involves students' ability to understand "that there should be a single mechanism for judging whether a mathematical object is a member of a set (whether or not that mechanism is a formal definition)."

2.3.3 Categorization of concepts

Learning the taxonomic structures and theoretical frameworks that undergird the different semiotic components of the mathematics register is often challenging for students of advanced mathematics. Therefore, prevailing theories surrounding the psychology of human categorization should be considered basic frameworks for research in mathematics education. For the past several decades, cognitive psychologists have sought to understand how learned categories are stored in memory and how they are used in decisions related to categorization. Independent lines of research toward this goal have yielded a robust exchange between alternative schools of thought.

2.3.3.1 Classical categorization models

Early, 'classical' theories situated the mental processes associated with categorization in a manner similar to scientific, taxonomic, or mathematical categorization, where category boundaries are fixed and well-defined, and where there are no members of a category that are more 'typical' than any other. Categories were thought to be represented in memory as a collection of properties that were necessary and sufficient for inclusion in the category; and categorizing simply required one to verify the applicability of each property expressed in the definition (Verbeemen, et al., 2007).

Rosch & Mervis (1975) explained the classical perspective in this way:

> …when describing categories analytically, most traditions of thought have treated category membership as a digital, all-or-none phenomenon. That is, much work in philosophy, psychology, linguistics, and anthropology assumes that categories are logical bounded entities, membership in which is defined by an item's possession of a simple set of criterial features, in which all instances possessing the criterial attributes have a full and equal degree of membership. (p. 573-574)

This model of categorization is analogous to the categorical structure of a formal mathematical system. Category boundaries are fixed and precisely delineated by criteria based on the defining mathematical properties. When educators and textbooks present pre-established theorems and concise proofs that are based solely on deductive reasoning from given definitions, they are implicitly modelling this classical formation of categories. Such an approach may send unintended messages to students about how best to reason in mathematical settings (Raman, 2002, 2004), since a classical taxonomy for mathematical concepts may not be the ideal, or even a necessarily useful, form of concept representation for students.

Research in mathematics education has shown that both novice and expert mathematicians often successfully reason in informal ways that do not strictly depend on the classical model's definitions and fixed categories (see Easdown, 2009; Knapp, 2005; Weber & Alcock, 2004). Studies have found that students often use prototypes (Alcock & Simpson, 2002; Pinto & Tall, 2002), metaphor (Dawkins, 2012; Dogan-Dunlap, 2007), metonymy (Presemeg, 1992, 1998; Zandieh & Knapp, 2006) and conceptual blending (Zandieh, Roh, & Knapp, 2014) as a basis for reasoning, making categorical decisions, and discovering key ideas in proof. Research has also found that expert mathematicians often rely on semantic instantiations as well (Parameswaran, 2010; Sfard, 1994; Weber, 2008; Weber, Inglis, & Mejia-Ramos, 2014). On a more fundamental level, Lakoff and Nuñez (2000) claimed that all concept representation ultimately relies on embodied metaphors, reflecting our sensory and perceptual experiences of the world.

2.3.3.2 Prototype categorization models

The classical view of category formation was upended in the 1970's and 1980's, with the work of Eleanor Rosch, George Lakoff and others (see Johnson, 1987; Lakoff, 1987; Lakoff & Johnson, 1980; Mervis & Rosch, 1981; Rosch, 1973; Rosch & Mervis, 1975; Rosch, et al., 1976), who studied semantic categorization in natural language. Their body of work indicated that, in cognition, category boundaries are less deterministic than had been supposed in classical theories of categorization. Theoretical contributions included the following:

- Learners define categories through "natural prototypes," generated by the mean distribution of attributes across all members of the category (Rosch, 1973);

- Categories exhibit a “graded structure,” with their members falling along gradients of rated typicality (Rosch & Mervis, 1975);
- Naturally-learned categories are constructed on “family resemblances” rather than criterial features (Rosch & Mervis, 1975);
- “Basic” levels of categorization exist, which contain as much information as needed to discern real world differences, but no more (Rosch, et al., 1976); and,
- Metaphors extend the meaning and range of applicability of prototypically defined concepts (Lakoff & Johnson, 1980; Lakoff, 1987).

2.3.3.2.1 Mental prototypes

Rosch (1973, p. 329) posited that human learners cognitively generate “natural categories” based on prototypes abstracted from a category’s “distribution of attributes” among its members. A prototype for a natural category acting as a “summary representation is assumed to be built up through abstraction over previously encountered concept members” (Verbeemen, et al., 2007). These prototypes then “operate in classification and recognition of instances” with the important effect of modifying the perceived “correlational structure of the environment” through “selective ignorance and exaggeration of the attributes and structure of that environment” (Rosch, et al., 1976, p. 435). In other words, humans most readily build categories by forming abstract representations around collections of observed properties, which are weighted based on the individual’s experiences with, and recognition of, those properties. These “natural prototypes” are then used as comparative models when classifying new phenomena.

2.3.3.2.2 Prototypes in mathematics education

Several researchers in mathematics education have referenced prototype effects they observed in their students' reasoning (see Ferrari, 2003; Hazzan, 1994; Mason & Pimm, 1984; Pinto & Tall, 2002; Presmeg, 1992; Tall & Bakar, 1991). Pinto & Tall (2002) examined a case of student reasoning with a prototypical pictorial representation of a convergent sequence:

> This visual representation is not, however, a simple picture of an increasing sequence' bounded above nor a 'decreasing sequence' bounded below nor even an alternating sequence oscillating above and below the limit. It moves up and down in a more general manner. He uses it not as a specific picture, but as a *generic* picture, one that represents the sequence in a manner that is as general as he possibly can make it. (p. 3)

Presmeg (1992) pointed out difficulties that students may encounter with such prototypical representations, including:

1) Non-recognition of a figure when it does not conform to a prototype; or

2) Introduction of extraneous properties which are present in the figure but not necessarily in the general case. (p. 600)

These difficulties may arise because "the essential and random features of the diagram (or image) are not separated." She illustrated this with an example of a student drawing a prototypical image of a parabola, which in her participant's mind stood for the class of all downward-opening parabolas. However, the image drawn was of a parabola with the *y*-axis as the axis of symmetry, which caused this student "great difficulty" as he attempted to reason about parabolas without that property.

Prototypes are not the only way that students of mathematics may reason in non-classical ways. In the next section, I will address alternative models for categorization and illustrate their uses in the learning of mathematics.

2.3.3.3 Alternative categorization models

In response to the prototype theory of Rosch and her colleagues, other cognitive psychologists have hypothesized alternative mechanisms for human categorization. These researchers have proposed a variety of exemplar and rule-based categorization models, and tested them against both the classical and prototype theories.

2.3.3.3.1 Exemplar models

One major line of research, based on experiments in category learning, has modelled the formation and delineation of categories as though subjects continuously store and review non-abstracted memories of interactions with category members, matching new phenomena to one or more of these *exemplars* for future categorization (Hampton, 2003). Exemplar models vary on their level of detail abstraction, depending on theoretical differences between the various researchers who subscribe to this school of thought.

> At one extreme, exemplar representations may involve no abstraction at all. In this version, representations consist only of specific memory traces of previously encountered instances (e.g., Reed, 1972). At the other extreme, an exemplar representation might be a family resemblance representation that abstracts across different specific instances (Komatsu, 1992). An intermediate position was taken by Rosch (1975) who assumed that only the most typical exemplars are activated as category representation in memory. (Verbeemen, et al., 2007, p. 538)

As the authors above noted, prototype theorists did not rule out the use of exemplars in categorization. In their broad definition of the prototype concept, Rosch &

Mervis (1975) included individual, non-abstracted category members as well as abstract and idiosyncratic representations of a category. It was hypothesized that categories might be instantiated as individual examples to be used as prototypes during reasoning and problem-solving; or, they may be represented by abstract models exhibiting a collection of properties that are considered relevant by the thinking subject.

2.3.3.3.2 Rule-based models

Rules-based categorization models assume that learners operate with pre-determined rules, or criteria, when evaluating category membership for new objects. This is like classical categorization theories in that the rules may serve to strictly delineate categories. However, rules-based categorization may differ in that the rules are assumed to be varied and learner-generated, rather than strictly defined by nominal systems that are external to the learner.

Rouder and Ratcliff (2006) compared human categorization activities that occurred according to rules-based criteria and exemplar comparisons. They reported that participants tended to base their categories on rules when the stimuli were similar and "confusable," and by comparison to known exemplars when the stimuli were "distinct" (p. 9). They concluded that categorization may rely on rules initially, until "suitable features for discriminating stimuli" have been learned, allowing for stimuli to "be stored as exemplars and used to categorize novel stimuli without recourse to rules" (p. 9).

2.3.3.3.3 The exemplar-prototype spectrum

Exemplar and prototype models may not be as oppositional to one another as they seem. Barsalou (1990) argued that exemplar and prototype representation models only differ in so far as information about individual exemplars is retained or discarded. In a prototype model, information is presumed to be lost; whereas in an exemplar model, many detailed features of past experiences with members of a category remain accessible in memory. Hampton (2003, p. 1254) elaborated: "the key difference between prototype and exemplar models of classification learning is that prototype models involve abstraction over exemplars to represent the central tendency and the variability within the category." In a prototype, "individual exemplars are therefore merged into a single generic representation," so that "information about higher-order covariance of features across exemplars is lost."

Verbeemen, et al. (2007, p. 549) argued that neither exemplar nor prototype models capture the whole story. Instead, they applied a "Varying Abstraction Framework," which is "a family of models that ranges from complete abstraction (i.e., a single summary representation) to no abstraction at all (i.e., every exemplar is represented in memory and activated in categorization decisions)." This framework allows for "intermediate levels of abstraction, in which some exemplars are merged together in a single representation and other exemplars are kept apart in separate representations." These intermediate models outperformed both the prototype and exemplar models "in explaining categorization decisions for natural categories."

Humans categorize phenomena in different ways depending on various factors. Such factors may include: past interactions with the category members (Ross, 1996), contextual effects (Hampton, 2003), and the overall differentiation of the categories (Smith & Minda, 1998). For example, Smith and Minda (1998) found that "prototype models do better at explaining learning when the stimuli are more complex, when there are more of them, and when the category structure is well differentiated" (as cited in Hampton, 2003, p. 1254).

> Broadly speaking, where a category has many exemplars, and the differentiation between categories is relatively easy (large distances between categories relative to variance within), then there tends to be more evidence for the formation of a category based around a prototype (a single point in the stimulus space) rather than exemplars. (Hampton, 2003, p. 1254)

The findings of Smith and Minda (1998), Rouder and Ratcliff (2006), and Verbeemen, et al., (2007) are based on comparisons of three models for the formation of categories—exemplar, prototype and rules-based models. Together the results illustrate an increasingly broad view of category formation.

2.3.3.4 <u>Models of student categorization in mathematics education</u>

Examining the distinction between exemplar, prototype, and rules-based categorization models in the context of mathematics education, Alcock and Simpson (2002) contrasted three approaches to mathematical reasoning used by undergraduate students in a real analysis course. These included: 1) generalizing from a "generic example or prototype," 2) abstracting a salient property from their "prototype," and 3) working deductively from definitions.

Although the authors used the term "prototype" in their description of the first approach, this way of thinking may be said to align more closely with an exemplar model of category formation as defined in the works cited above. In the authors' example illustrating this approach, their participant instantiated an individual example of a specific convergent sequence $\sum\frac{(1/2)^n}{n}$ as a representative of a larger class of a series $\sum\frac{(x)^n}{n}$ to reason about a related series (p.29). In this sense, the exemplar model may be a better descriptor for this approach than the prototype model, as a single example was generated and explored in full detail. Such an approach allows students to examine the criteria for a known category member, and to make comparisons that enable further classification of category instances.

On the other hand, the same participant was later observed to draw a pair of generic, monotonic, converging sequences, one increasing and one decreasing (p. 30), which the authors interpreted as the second form of reasoning listed above—abstracting a property from a prototype. Monotonicity was apparently a property that this participant considered central to the notion of a convergent sequence; and, by varying this attribute along the dimension of *increasing versus decreasing*, she seemed to persuade herself that convergent sequences must be monotonic. In another example chosen by Alcock and Simpson to illustrate this "prototype" approach, a participant began by drawing an abstract representation of a convergent sequence—this one chosen intentionally as a non-monotonic sequence (p. 30). By doing so, the participant could illustrate the properties he believed were central to the definition of a convergent sequence and demonstrate which properties were irrelevant.

The examples of visual representations described above closely align with a prototype model for category formation, especially considering their lack of detail specificity yet clear representation of key, salient properties. Such an approach seems to give students the flexibility to focus on individual properties while screening out unnecessary details.

The third form of student reasoning observed by Alcock & Simpson (2002), which involved the use of deductive inference and definitions, is more closely related to a rules-based or classical taxonomy. The participant they described to exemplify this approach used the defining property for sequence convergence, without reference to detailed exemplars or abstract prototypes. The authors argued that the first two approaches operate by extracting and examining properties from a pre-existing category; whereas, the third approach inverts the relationship between properties and categories, so that "the defining property *determines* the category (p. 32, emphasis in original). The inversion from property extraction to category definition, which Freudenthal (2002, p. 30) called "structuring 'upside down'," is a key component of the formal-symbolic mathematical register. It represents a student's transition to reasoning with more classical categorization schemes, rather than relying on the prototypes that have been formed through her subjective past experiences, perceptions, and intuitions.

In a later study, Alcock and Simpson (2011) wrote about the extent to which they had observed the classical model of categorization to be in use by students during their transitions to advanced mathematical thinking. They claimed that the use of students' concept images to classify mathematical ideas may be attributed to their natural categorization strategies, rather than the classical archetype; and that "concept images might be similar in structure to natural categories, whereas mathematical judgements need to be based on (or at least accord with) formal, agreed-upon definitions" (p. 93). These authors introduced the construct of "concept consistency," defined as whether a student "understands that there should be a single mechanism for judging whether a mathematical object is a member of a set (whether or not that mechanism is a formal definition)” (p. 94). This construct will be useful in examining my participants’ responses in the analysis, in terms of their use of non-definitional bases for reasoning and justification during the tasks and interviews.

2.3.4 Metaphor and metonymy

Analogical notions such as metaphor and metonymy have traditionally been studied solely as linguistic devices. However, beginning with the work of Lakoff and Johnson (1980), some cognitive linguists began to re-assess these figures of speech for their impacts on human cognition. According to these researchers, metaphorical and metonymic concepts do not only build structure in language, but also in “thoughts, attitudes, and actions” (p. 39). In fact, from their theoretical perspective, metaphors that occur in language are only secondary consequences and “manifestations of these underlying metaphoric thought processes” (Johnson, 1987, p. 69).

Presmeg (1998, p. 26) suggested that “metaphors, metonymies, and the imagery and symbolism which accompany them are essential components in the representation of mathematical constructs for an individual, because they help the individual to make sense of the construct amid the ambiguities inherent in its representation.” In other words, the use of metaphor and metonymy are necessary, built-in mechanisms in human thought processes—used to connect and organize conceptual structures from disparate cognitive domains.

Although Lakoff and Johnson’s original research was focused on natural language cognition, Lakoff and Nuñez (2000) and other researchers in mathematics education (see Dawkins, 2012; Dogan-Dunlap, 2007; Lai, 2013; Pimm, 1981, 1988; Presmeg, 1992, 1998; Sfard, 1994; Zandieh & Knapp, 2006) have incorporated this perspective, illustrating the usefulness of framing metaphor as one cognitive apparatus for abstract mathematical learning. Presmeg (1998, p. 26) distinguished the two related constructs of metaphor and metonymy in the following way:

> In both metaphor and metonymy, a person uses one construct to stand for another. The difference between the two…is roughly that metaphor links one domain of experience with another seemingly disparate domain—and creates meaning from the connection—while metonymy uses one element or salient attribute of a class to stand for another element or the whole class.

Both conceptual structures are discussed in detail below.

2.3.4.1 Metaphor

According to Lakoff and Johnson (1980, p. 5), "the essence of metaphor is understanding and experiencing one kind of thing in terms of another," such as conceiving of an argument metaphorically in terms of war, or time as money. These are each a pair of distinct concepts, but through a mapping of structure and meaning, one is understood through the context of the other. The authors claimed that metaphor is not an extraordinary use of language, but that in fact "our ordinary conceptual system, in terms of which we both think and act, is fundamentally metaphorical in nature" (p. 4). They went on to consider the consequences of this structure:

> The conceptual structure is grounded in physical and cultural experience, as are the conventional metaphors. Meaning, therefore, is never dis-embodied or objective and is always grounded in the acquisition and use of a conceptual system. Moreover, truth is always given relative to a conceptual system and the metaphors that structure it. (Lakoff & Johnson, 1980, p. 197)

Many authors have extended and developed the ideas of Lakoff and Johnson (1980) and Lakoff (1987) with regards to the recognition of metaphor as a structural element of cognition in mathematics. For instance, Pimm (1988) discussed adjectival modifiers for object-nouns as metaphoric in structure. Phrases like “genetic fingerprint” and “electronic mail” were shown to place emphasis on an object’s *function* rather than on the object itself. In this way, another object of a different nature might stand in metaphorically to fulfill the same, or a similar function. He discussed several mathematical concepts that are also structured in a metaphoric way, (e.g., *spherical* triangles or the *complex* plane).

Sfard (1994) expounded on these ideas, examining how mathematical understanding is built up from perceptual experiences through the mechanism of metaphorical mappings. Her insights emerged out of interviews she conducted with three expert mathematicians from distinct fields, and were based on the mathematicians’ self-explanations of their mental activity. Reflecting on Lakoff and Johnson’s (1980) construct of an “image schema”—which she described as a means of organizing and preserving “the essence of our experience” (Sfard, 1994, p. 46) and as “the carrier of a metaphor” (p. 48)—Sfard claimed that the understanding we gain through our experiences and perceptions seems to produce “the primitives from which more advanced meanings are built” (p.46). She portrayed a chain of metaphors linking bodily perception to advanced mathematical constructs:

> In advanced mathematics, at levels far removed from physical reality, it may well be that the immediate source of a basic metaphor is another, lower-level mathematical structure. Even so, and however long the chain of metaphors may be, whatever is going on in our mind is primarily rooted in our body. The intelligibility of abstract objects stems from their being metaphorical reflections of our bodily experience. (Sfard, 1994, p. 47)

In their book, *Where Mathematics Comes From*, Lakoff and Nuñez (2000) systematically applied the perspective of embodied cognition to mathematical thinking, through their development of *Mathematical Idea Analysis*. This methodology provides a means for studying mathematics through what they call "embodied cognitive science." Much like Johnson (1987) and Sfard (1994), they claimed that conceptual metaphors, with origins in our perceptions and experiential understanding of the world around us, also structure our understanding of mathematics. They highlighted three major types of metaphor. These were "grounding metaphors," which establish concepts within embodied schemata; "re-definitional metaphors," which re-orient concepts within broadened mathematical contexts; and "linking metaphors," which connect ideas within mathematics itself.

Although these authors are not mathematicians, and they faced criticism for their ideas in both mathematical and cognitive domains (cf. Presmeg, 2002; Schiralli & Sinclair, 2003; Thomas, 2002; Voorhees, 2004), their ideas nevertheless provide an important framework for studying mathematical cognition, and have now played a role in shaping research in mathematics education over the greater part of the past two decades. Whether or not embodied metaphors do lie at the root of human cognition, there is evidence that metaphors are helpful for learning and teaching mathematics.

Dawkins (2012) illustrated a case in which a metaphor for sequence convergence within a realistic context helped a real analysis student transition toward a property-based definition for the concept. Other findings have provided a useful lens to examine the use of metaphor and analogy in teaching as well. González (2015) used linguistic analysis to investigate "how teachers construct the mathematical classroom register through analogies, particularly when connecting colloquial and mathematics discourses" (p. 81). Similarly, Lai (2013, p. 43) showed that for pre-service teachers, "the development of the capacity to understand the salience of particular metaphors in the prior experience of their students and the capacity to provide teacher explanations that support the transition between less-mathematical and more-mathematical language" is important.

On the other hand, metaphors can pose challenges to students' understanding of new concepts as well. Hazzan (1994) studied the influence of students' previously-held conceptions about the real numbers on their understanding of algebraic group operations. She reported that her participants may have understood group multiplication metaphorically via real number multiplication; and tended to "borrow properties" (p.708) from this particular instantiation of the abstract operation. This, she claimed, "demonstrates how a familiar system influences the understanding of mathematical concepts" (p.709).

The challenges associated with using metaphors in mathematics learning are closely related to those involved in using prototypes and exemplars. Metaphors and prototypes are both used independently for cognition; however, a metaphor may be more or less prototypical "depending on its centrality in a category" Presmeg (1992, p. 597). Pimm (1981) argued that "we need to be aware of metaphors because unexamined ones lead us to assume the identity of elements and processes which will conflict with our past experience" (p. 50). Note the close similarity between this warning and that of Presmeg (1992), mentioned earlier, about the use of prototypes.

2.3.4.2 Metonymy

Lakoff (1987, p. 77) described metonymy as the act of taking "one well-understood or easy-to-perceive aspect of something and [using] it to stand either for the thing as a whole or for some other aspect of part of it." One example he described was when a place stands for an entity, such as "Wall Street is in a panic," or "Washington is talking with Moscow." An example of a different kind is provided by a stereotype standing for a whole class of people—the term *mother* was shown to entail a list of assumed attributes that must be modified by adjectives to form distinct concepts, e.g., *working mother, stepmother, unwed mother*, etc. The *housewife-as-mother* was argued to be a stand in, or "metonymic model" for these various types of mother.

Metonymies are also common in mathematical reasoning. Presmeg (1992, p. 600) claimed that "every time a student of any branch of mathematics uses a diagram in reasoning, there is created 'a situation in which some category or member or submodel is used (often for some limited and immediate purpose) to comprehend the category as a whole'." Pointing out that this was Lakoff's (as cited in Presmeg, 1992) exact definition for the concept of metonymy, she goes on to argue that "metonymic usage occurs because a diagram or image, by its nature, depicts one concrete case." In yet another sense, as Pimm (1988) explained about metaphors, he also noted that some adjectival phrases may act metonymically. He argued that the concept of matrix multiplication is metonymic with regards to "the decision about which properties associated with whole-number multiplication can be seen as a form of multiplication" (p. 33).

Providing a detailed explanation of the notion of metonymy for mathematics education, Zandieh and Knapp (2006) delineated three kinds of metonymies based on two criteria. First, they distinguished between *part-part* and *part-whole* metonymies, the latter being otherwise referred to as a *synechdoche*. A part-part metonymy entails the representation of part of a conceptual or perceptual object by another part. “The strings played superbly” (Pankhurst, as cited in Zandieh & Knapp, 2006) is an example of this kind of metonymy, where the instruments’ strings stand in for the players themselves. This would be the sense in which a place might stand for an entity or institution, as in Lakoff’s first example above.

Zandieh and Knapp (2006) illustrated two types of part-whole metonymies: *individual* and *paradigmatic*. An individual metonymy occurs when the representative part is one piece of an overall cognitive structure, such as in “I’ve got a new set of wheels.” In this case, the wheels of the car are chosen for their underlying properties that we associate with our cultural understanding of cars. The authors pointed out that other portions of our cognitive structure for cars, such as windows, would not communicate the same information. On the other hand, a paradigmatic metonymy occurs when “one element of a class may be taken to stand for the whole class” (Presmeg, as cited in Zandieh & Knapp, 2006), such as drawing a particular isosceles triangle to represent the entire class of such triangles. Their label for this type of metonymy was a reference to Fischbein’s (1987) “paradigmatic models;” which reflects that these structures, acting as either exemplars or prototypes, “provide enough variety of features to be representative of the entire group, yet are simple enough to be easy to use in reasoning” (Zandieh & Knapp, 2006, p. 3).

Zandieh and Knapp (2006) described metonymies as “necessary for efficient communication” (p. 9). However, in agreement with Pimm (1981), Presmeg (1992), and

Hazzan (1994), they point out that such devices may also be a source of obstacles and frustration for students, as they may hide important properties, or accidentally transfer extraneous ones. Zandieh and Knapp (2006) found that students used shortened metonymic phrases such as “the derivative is the tangent line,” or “the derivative is the change,” instead of referring to it as the slope of the tangent line, or the ratio of the change over two coordinates. They showed that there is an individual metonymic “part-whole” relationship between the slope of the tangent line and the line itself that students may apparently misuse as they construct their understanding of the concept of derivative. Similarly, both the value of the derivative and the derivative function itself are correctly referred to as “the derivative” in normal mathematical discourse. This caused difficulties for some of the participants in this study, as they struggled to reconcile the different uses of the term.

Some scholars have lodged critiques against the works of Rosch (cf. Harnad, 2005), Lakoff (see Vervaeke & Kennedy, 2004), and related authors (see. Rakova, 2002; Schiralli & Sinclair, 2003; Thomas, 2002; Voorhees, 2004). Yet the preponderance of studies mentioned above would seem to challenge classical theories of categorization by demonstrating ways of understanding that circumvent strict category boundaries and membership criteria. As evidenced by the discussion above, elaborations on these developments in cognitive linguistics have continued to inform research into mathematics education. However, many questions remain to be investigated.

The linguistic studies examined above relied on basic assessments of response times, typicality ratings, and other quantitative measures involved in researching natural language cognition. It will be worthwhile to examine the reported findings through methodologies appropriate to research in mathematics education. The field of topology is ripe for the investigation of embodied cognition. As a generalization of humans' perceptual experiences of physical space, it is likely that there are many ways in which metonymic exemplars, metaphorical prototypes, and other analogical constructs may contribute to, or obstruct, students' understanding and internalization of topology's central concepts.

2.4 Content-Specific Research

My study focused on undergraduate students' understanding of continuity during their first course in topology. Due to the context of topology, several other set-theoretic concepts were also studied, including the following ideas: *open set, closed set, function, limit point, boundary point,* and *interior point*, as well as set operations like *union, intersection*, and *complement*. In this section, the mathematics education research literature about these content topics is reviewed.

2.4.1 Student understanding of set theory and topology

The field of topology extends set theory. Learning topology demands the successful navigation of mathematical structures that are built up from three levels of the set hierarchy—elements, sets, and set families (sets of sets). To use concepts like limit point, interior point, or boundary point in a topology context, students must reformulate definitions, theorems, and claims that are presented in terms of sets and set families into statements about the elements, or *points* of the topological space. The studies described below show that students have difficulty considering sets as objects and simultaneously as members of a larger collection, or family, of sets.

2.4.1.1 Set theory

Zazkis & Gunn (1997) applied an APOS (Arnon, et al., 2014; Dubinsky, 1991) perspective to students' understanding of set theoretical concepts. They found deficiencies in many of their participants' use and understanding of sets, when considered as elements of superordinate set families. For example, some of their participants had difficulty differentiating elements from subsets within the set $\{5, 7, \{5\}, \{5, 7, \{7\}\}\}$. Bracketed elements of this set were often confused with its subsets. Furthermore, as the authors point out:

> …relying on "everyday life" experiences, many participants interpreted the relation "element in the set" as a transitive one. The difference between the symbols [f] and f, or even [5] and 5, was at times not recognized. (Zazkis & Gunn, 1997)

In discussing their results, Zazkis and Gunn (1997) pointed that students need to encapsulate an object conception for sets: "thinking of the set and treating it as one entity" (p. 18). However, they claimed that "the main difficulty students face is not in considering a set as one conceptual object but in considering a set *as an object and as a collection of objects simultaneously*" (Zazkis & Gunn, 1997, p. 18, italics added).

2.4.1.2 Topology

The field of topology is an example of the axiomatization of a set of concrete phenomena. That is, it is rooted in the abstraction and generalization of physically intuitive concepts, such as continuity and connectedness, which underlie our understanding of the geometric and analytic properties of the real numbers and Euclidean space (Moore, 2008). In this way, the mathematical theory of topology models the physical properties of space that are readily perceived within the human environment. In some ways, attempting to model such an experientially real phenomenon may benefit students' understanding of the axioms (Fischbein, 1987; Tall, 2013); while in other ways it may be an obstacle to developing an accurate formal understanding of the mathematics involved (Alcock & Simpson, 2002; Harel & Tall, 1991; Tall, 2013). These attributes make topology an ideal candidate for study as an intermediary between students' perceptual knowledge and their development of formal schemas.

There have been few studies that directly addressed student understanding in introductory topology. Narli (2010) described fourteen teacher trainees in an introduction to topology class as misunderstanding "abstract mathematical concepts such as sets, set families and combination, intersection in families which provide the foundations of topology" (p. 70). He claimed that many of these participants had difficulties applying topological definitions to specific examples, and that most did not understand topological structure as being defined through set families. It is unclear which factors may have led to these results, however, the relevance of Narli's study is reflected in the author's repeated reference to his participants' difficulties with the set theoretic components of the content they were being taught; in particular, the statement of the axioms via families of sets.

A few studies have been conducted to document specific content that overlaps with topology, such as limits and limit points (Kidron & Tall, 2015; Roh, 2007); open sets (Dogan-Dunlap, 2007); and interior/exterior/boundary points (Lakoff & Nuñez, 2000). Finally, student understanding of continuous functions has been studied in some depth, though not in the context of topology.

2.4.2 *Student understanding of continuous functions*

Continuous functions play an important role in many advanced mathematical domains, especially in the field of topology. Historically, formalizing the property of continuity required a sustained, long-term effort by nineteenth century mathematicians (Harper, 2016; Moore, 2008), requiring them to replace their intuitive, geometric notions of prototypical curves in space with a rigorous, static, and discretized concept (Lakoff & Nuñez, 2000). This occurred during a transition between paradigms for the western scientific tradition at the time, in which a 'pragmatic' theory of knowledge was systematically replaced by an axiomatic-deductive one (Harper, 2016; Job & Schneider, 2014).

While these mathematicians succeeded in their attempts to re-conceptualize and build structure for their 'natural' human conceptions, today's students must revisit those struggles for themselves again and again. By comparing students' recurrent misconceptions with such historical obstacles to understanding, this analysis will explore linguistic, experiential, and epistemological obstacles (Radford, 1997; Sierpinska, 1987, 1994) to explain the challenges observed in my participants' reasoning.

A growing body of research has documented the ways that undergraduate mathematics students reason about continuous functions, as well as the challenges associated with learning this complex idea. Students have been shown to reason with incomplete concept images (Ferrini-Mundy & Graham, 1994; Tall & Vinner, 1981), and to lack coordination between their informal and formal ways of understanding continuous functions (Bezuidenhout, 2001; Ferrini-Mundy & Graham, 1994; Tall & Vinner, 1981; Williams, 1991). In introductory proof courses, undergraduate students have been shown to lack sufficient understanding of continuous functions to establish the validity of mathematical statements or complete proofs about them (Ko & Knuth, 2009). This is significant for mathematics majors because "the domain of continuous functions…is both central and pervasive in undergraduate mathematics" (Ko & Knuth, 2009, p. 69), especially in later courses such as advanced calculus (Parameswaran, 2009; Raman, 2004), real analysis (Alcock, 2014; Nadler, 1994; Raman, 2004) and topology (Croom, 1989; Narli, 2010; Tall & Vinner, 1981).

2.4.2.1 Origins of students' conceptions for continuous functions

For many students, the notion of a continuous function is first introduced in an informal way during their pre-calculus courses (Raman, 2002). This may entail an appeal to the students' perceptual intuitions about unbroken curves, with operational definitions that involve sensori-motor activities, such as: "it is possible to trace the graph of the function between *a* and *b* without lifting the pencil from the paper" (Demana, Waits, & Clemens, as cited in Raman, 2002). Later, in their calculus courses, students will be introduced to the formal limit definition of continuity for real-valued functions (see Figure 2). This definition further requires the concept of the limit of a function, which is defined mathematically through a complex logical implication involving multiple quantifiers and predicates (see Figure 3).

Most students will have explored examples, uses, and consequences of the continuity property throughout their calculus sequence, but many have difficulty interiorizing the processes involved in the formal definition of continuity in the early stages of learning about it (Benzuidenhout, 2001; Ferrini-Mundy & Graham, 1994; Shipman, 2012; Tall & Vinner, 1981; Vinner, 2002). Later, during their courses in real analysis, mathematics majors will be reintroduced to the concept of continuity in a more abstract form that involves functions between general metric spaces. They will encounter proofs that may explicate the conceptual introductions they received for important theorems—such as the Intermediate Value Theorem and the Extreme Value Theorem.

Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say that the **limit of as *x* approaches x_0 is the number *L***, and write

$$\lim_{x \to x_0} f(x) = L,$$

if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon.$$

Figure 2. The definition for the limit of a real-valued function. For brevity, I have excluded the definitions for left-hand and right-hand limits (Thomas, et al., 2005).

A function $y = f(x)$ is **continuous at an interior point *c*** of its domain if

$$\lim_{x \to c} f(x) = f(c).$$

Figure 3. The definition for a continuous function at a point. For brevity, I have excluded the definition for a continuous function at an *endpoint,* which involves the use of left-hand and right-hand limits (Thomas, et al., 2005).

This interaction between the formal theory students learn and the informal conceptions they develop may serve to help some students coordinate the mathematical concept of a continuous function with their perceptually intuitive schemes for connected curves. However, given the limited time that is usually spent, and the brevity of such discussions during the calculus-analysis sequence, it is also likely that salient examples of continuous functions will play a dominant role in the conceptualizations that some students construct about the property of continuity (Brijlall & Maharaj, 2013). This may reinforce their use of unexamined perceptual knowledge as a basis for reasoning in the future.

As the discussion above makes clear, undergraduate students have already reasoned about the continuity property and its consequences in variously formal settings for years before they are introduced to the *topological* definition for continuous functions. By then, many students have formed a substantial concept image for continuous functions, steeped heavily in the context of the real numbers and somewhat more generalized metric spaces (Dreyfus & Eisenberg, 1983; Ferrini-Mundy & Graham, 1991; Tall & Vinner, 1981). Because of these prior experiences, the introduction to the concept of continuous functions for more abstract topological spaces entails a process of reconstruction of the students' conceptual schemes for a collection of interrelated ideas, including: sets and set operations, functions, open sets, and continuity itself.

2.4.2.2 Challenges in students' conceptualization of continuous functions

Research has demonstrated that students' difficulties in reasoning about continuous functions often arise from their misconceptions about functions themselves (Lauten, et al., 1994; Tall, 1991; Vinner, 1992; Vinner & Dreyfus, 1989), or about the concept of function limits (Benzuidenhout, 2001; Cornu, 1991; Roh, 2007, 2008; Sierpinska, 1987; Williams, 1991). They may also struggle with the symbolic aspects of the mathematical register (Benzuidenhout, 2001; Gray & Tall, 1994; Tall, 2013), or their reasoning may be dominated by symbolic procedures without relation to any meaningful semantic structure (Dreyfus & Eisenberg, 1983; Brijlall & Maharaj, 2011). Yet, there is another obstacle that may play a more significant role in the early transition to topology, which is the tendency for students to continue to rely on their perceptual representations and physical intuitions about limits and continuity, without coordinating them with the formal mathematical tools they are presented in class (Brijlall & Maharaj, 2011, 2013; Job & Schneider, 2014; Farmaki & Paschos, 2006; Roh, 2008).

Like many concepts in topology, the mathematical idea of continuity is intricately bound to perceptual, sensori-motor cues that suggest (and are modelled by) the formal concept. There is a growing body of evidence in favor of the claim that, in cognition, such perceptual knowledge is re-presented independently from conceptual understanding (Paivio, 2010). Deductive conceptual systems do not replicate sensori-motor experiences (Piaget & Inhelder, 1948/1967), but are overlaid as theories to explicate (Dawkins, 2015) the more inductive world of physical embodiment and pre-operational intuitions (Lakoff & Nuñez, 2000; Tall, 2013; Piaget & Garcia, 1983/1989).

2.5 Theoretical Frameworks

My research and analyses evolved out of a synthesis of two related theoretical frameworks: 1) the "radical constructivist" (Glasersfeld, 1995) interpretation of Piaget's (1970) genetic epistemology; and 2) Action Process Object Schema (APOS) theory (Arnon, et al., 2014; Dubinsky, 1991), which is a neo-Piagetian framework designed to analyze undergraduate students' mathematical conceptions. While both frameworks are based on the research of Piaget and his colleagues, they differ with respect to their theoretical terms, methodologies, and epistemological perspectives. I chose to adopt APOS theory in my analysis of the participants' understanding of the open set concept, while Glasersfelds' (1995) method of conceptual analysis was a better fit for my analysis of the participants' understanding of the property of continuity.

I adapted the two frameworks above through my choice to prioritize the following activities as focal points for the research:

1) participants' attention to mathematical properties as a basis for reasoning (Dörfler, 1993; Slavit, 1997);

2) participants' use of sensori-motor knowledge (Paivio, 2010; Piaget & Inhelder, 1948/1967), intuitions (Fishbein, 1987), and analogical linguistic structures (Lakoff & Nuñez, 2000; Presmeg, 1992; 1998; Sfard, 1994; Zandieh & Knapp, 2006) as tools for reasoning.

Literature about each of these perspectives will be presented in detail below.

2.5.1 Piaget's genetic epistemology

Jean Piaget conducted a decades long research program (see Piaget, 1970, 1975; Piaget & Garcia, 1983/1989; Piaget & Inhelder, 1948/1967) in which he and his colleagues studied the psychological development of children in many domains of knowledge and conceptual understanding. Piaget (1970) explained his epistemological philosophy in this way:

> Genetic epistemology attempts to explain knowledge, and in particular scientific knowledge, on the basis of its history, its sociogenesis and especially the psychological origins of the notions and operations upon which it is based. These notions and operations are drawn in large part from common sense, so that their origins can shed light on their significance as knowledge of a somewhat higher level. But genetic epistemology also takes into account, wherever possible, formalization—in particular, logical formalizations applied to equilibrated thought structures and in certain cases to transformations from one level to another in the development of thought. (p. 1)

The "equilibrated thought structures," which Piaget alternately referred to as *schemes* or *schemas* (schemata)[3], were theorized to be mechanisms for cognitive

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[3] I will use *schema* to refer to the total, coherent cognitive structure that a person has in her mind for a given concept (Arnon, et al., 2014), while *scheme* will refer to less inclusive cognitive strategies based on previous responses to stimuli (Glasersfeld, 1995). The open set analysis in Chapter 4 will refer exclusively to schemas, while the continuity analysis presented in Chapter 5 will refer exclusively to schemes.

adaptation. When encountering a new stimulus, a thinking subject would attempt to *assimilate* the new information into their schema for some previously-experienced and constructed concept. If the subject is unable to assimilate the new information into an existing schema, she will experience a *perturbation* (or *cognitive disequilibrium*) that motivates the thinking subject to modify (or *accommodate*) an existing schema to resolve the perturbation and return to *equilibrium.*

Von Glasersfeld's (1995) interpretation of Piaget's philosophy, though called 'radical constructivism', was in fact meant to preserve Piaget's original intent through a close analysis of his life's work. Von Glasersfeld (1995) rejected realist critiques of Piaget's work, but qualified his philosophy by saying that it begins from the assumption "that knowledge, no matter how it be defined, is in the heads of persons, and that the thinking subject has no alternative but to construct what he or she knows on the basis of his or her own experience" (p. 1). This viewpoint was "shocking" to positivist and neo-positivist critics (Piaget & Garcia, 1983/1989), because it was often taken as an assertion that "nothing exists outside peoples' heads" (Glasersfeld, 1995). However, the radical constructivist assertion is in fact that thinking individuals have no direct access to any external reality; and they must rely on the media of the senses and learning strategies that originate in biological adaptation.

2.5.2 APOS theory

APOS theory evolved from the works of Jean Piaget as an attempt to mold his notion of "reflective abstraction" to the understanding of mathematics learning at the undergraduate level (Arnon, et al., 2014, Dubinsky, 1991). The acronym stands for *action, process, object,* and *schema*—the main cognitive structures proposed by the theory.

According to Dubinsky & McDonald (2010), an action conception entails the student's ability to transform objects according to prescribed instructions. The activity is "external" to the student's cognitive structures until she can build a mental process representing the action, and perform the action wholly in her mind. *Interiorization* is the APOS mechanism involved in the student's transition toward perceiving an action (or series of actions) as a complete process and dynamic mental image. This results in the student's development of a process conception. A process conception can be *reversed* or *coordinated* with other mental structures to form new processes; or may be *encapsulated* to form an object conception. An object conception is obtained by viewing a process or coordinated processes as a unified entity, which can then be used as a placeholder for other actions and processes to be performed upon. Finally, the schema is simply the collection of all the actions, processes, and objects involved with a mathematical concept, as well as their connections with other schemas for relevant and related concepts.

This is the original basis of APOS theory, which has been developed and extended in various ways over the last several decades. For example, Clark, et al. (1997) incorporated Piaget's original construct of the *triad* model for schema development into APOS theory during a study of calculus students' construction of understanding of the chain rule. An individual's schema for a mathematical topic was theorized to pass through the three stages of *intra-*, *inter-*, and *trans-* development. These stages correspond to the levels of connection and coordination between the interrelated constructs that make up the schema. In the *intra-* stage, these connections are essentially absent, and a student sees only the isolated concepts as specific to the contexts in which they were learned. Once a student begins to relate these ideas to other contexts and create meaningful links between the various properties involved, her conception is classified as belonging to the *inter*- stage. As these ideas are subsumed under an overall framework of understanding, the student notices that these once-disparate ideas are each facets of an interrelated structure, and coordinates them into a coherent schema. This is the *trans*- stage, a "birds-eye" view of a network of mathematical ideas that facilitates compressing the knowledge in the sense of Thurston (1990). A *coherent* schema is indicated by the student's ability to determine the placement of elements with respect to the overall schema.

I chose APOS theory as a framework for describing my participants' mathematical activities during tasks involving the definition and categorization of open sets. The APOS terms and theoretical constructs proved useful for describing activities about open sets because they are defined through processes that can readily be encapsulated to form structural objects, which can then be de-encapsulated to retrieve the original processes. On the other hand, the property of continuity did not lend itself to an APOS description since both continuous and non-continuous functions can be conceptualized as objects first, which either 'carry' the property of continuity or do not. In my judgement as a researcher, an open set can be encapsulated as such, while the property of continuity can only be affixed to the function concept, which is itself amenable to encapsulation.

2.5.3 Conceptual analysis in radical constructivism

Von Glasersfeld (1995) espoused a method for analyzing the mental actions required to understand a concept in a radical constructivist context, which he simply called "conceptual analysis." In the domain of mathematical learning, Cobb and Steffe (1983) adapted conceptual analysis into a specific methodology, known as a *teaching experiment*, which was elaborated by Steffe & Thompson (2000), and has recently been used in a variety of contexts (see Boyce & Norton, 2017; Hunt, Tzur, & Westenskow, 2016; MacDonald, 2015; Nam & Stevens, 2014; Norton, 2008, Ulrich, 2016). While my study was not a teaching experiment, it shared commonalities with the teaching experiment methodology, such as my role as a *teacher-researcher* during the interviews.

Thompson (2000) characterized conceptual analysis as a method for describing "ways of knowing that operationalize what it is students might understand when they know a particular idea in various ways" (p. 307). The goal of conceptual analysis, as stated by von Glasersfeld (1995) was to answer the question "what mental operations must be carried out to see the presented situation in the particular way one is seeing it?" (p.78). In the context of children's mathematical language and actions, Steffe and Thompson (2000) pointed out that "conceptual analysis concerns the mental operations of children necessary to produce the children's observed mathematical activity" (p.181). Analogously, I chose von Glasersfeld's (1995) method of conceptual analysis as a way of discerning the mental operations involved in producing my participants' observed mathematical activity with respect to the property of continuity, but in the context of undergraduate topology.

2.5.4 Property-oriented frameworks

While researchers in undergraduate mathematics education have documented many of the cognitive processes involved in building students' mathematical conceptions, I claim there is a need for greater emphasis on student understanding of the underlying properties of those mathematical concepts to which the current literature refers. Students' interactions with mathematical properties may determine the forms of their conceptions within the accommodated schemas and related concept images formed around new mathematical content. (Dörfler, 1993; Slavit, 1997). This is significant considering the central role that students' interpretations of mathematical properties can play in understanding formal and, especially, axiomatic systems (Freudenthal, 2002; Garcia & Piaget, 1983/1989; Tall, 2013).

Dörfler (1993) argued that mathematical objects and their mental representations are invariably linked to some set of properties:

> …my subjective introspection never permitted me to find or trace something like a mental object for, say, the number 5. What invariably comes to my mind are certain patterns of dots of other units, a pentagon, the symbol 5 or V, relation like 5+5=10, 5*5=25, sentences like five is prime, five is odd, 5/30, etc., etc. But nowhere in my thinking I ever could [*sic*] find something object-like that behaved like the number 5 as a mathematical object does. But nevertheless I deem myself able to talk about the number "five" without having distinctly available for my thinking a mental object which I could designate as the mental object "5". (p.146-7)

Dörfler (1993) was expressing one of the fundamental challenges and benefits of mathematics—its abstract nature and severance from the sensory information of the physical world. The concept of the number five can be a property of physical or mental objects, and it is also a mental object with intrinsic properties of its own. Mathematical theory is built around conceptual objects like numbers, which seem to have only indirect relationships with the physical world they are intended to describe. The disconnection from the ontological roots of mathematics, or *anontologization* (Freudenthal, 1973), is especially pronounced in axiomatic fields in which the abstracted properties are even further removed from human experience.

From a radical constructivist perspective, it is important to emphasize that when I speak of the "physical world," I am speaking of individual responses and equilibrations to external stimuli. As a researcher, I can only form consistent models of my participants' conceptions, which in turn are only "experientially adequate" (Glasersfeld, 2007, p. 9) as models for that account for those stimuli. Therefore, my choice to study participants' attention to properties involves building a model that fits their particular experiences, rather than a model that addresses an underlying reality (outside of the world of formal mathematics).

2.5.4.1 Properties in formal and axiomatic systems

One challenge in building axiomatic understanding is that it involves a fundamental transition in the way mathematical properties are recognized and used. Freudenthal (2002, p. 94) elaborated on this claim in a discussion about mathematical formalization:

> It is not merely the switch from measuring to constructing. It is a kind of jump, which I have called change of perspective - an important and indispensable activity, characteristic of a mathematical attitude - in this particular case, from measuring a given thing to making a thing with a given measure and, more generally, from examining a given thing and stating its properties to making a thing with prescribed properties (and of course if need be, the other way around).

In the transition from informal to formal ways of understanding, the properties of mathematical objects gain importance as they shift from empirical descriptions to *a priori* characterizations of the phenomena they once described (Freudenthal, 2002; Tall, 2013). Understanding in axiomatic contexts then demands a complete reversal of the relationship between properties and the mathematical objects they define (Freudenthal, 2002; Piaget & Garcia, 1983/1989). That is, relationships between the properties of mathematical objects become the foundation of the axiomatic system, which then defines the objects within it. Thus, mental representations of mathematical properties are a vital research focus for the exploration of students' transitions to axiomatic understanding.

Dörfler (1993) argued that what is useful and important for formal mathematical understanding are precisely the properties attributed to mathematical objects:

> …it appears we can get hold cognitively only of specific properties of those objects but the objects (they being abstract, mathematical or mental) themselves elude our awareness or consciousness. But do we need the objects? All the mathematical reasoning and arguments are exclusively concerned with mathematical properties and relations which, as it is, are mostly formulated as being attached to objects. Might be, it is rather a matter of convenient expression and communication that one uses a language with objects as carriers of the properties and relations. It for sure is cumbersome to address properties directly. (p.150)

Similarly, Tall (2013) highlighted the importance of mathematical properties, outlining the following actions as the defining characteristics for each stage in the development of abstract ideas:

1. *Recognition* of properties of thinkable concepts.
2. *Description* of properties in the given context.
3. *Definition* of properties that are now used as a basis for identification and construction of thinkable concepts.
4. *Deduction* of properties from definitions using specified methods of proof (Euclidean, algebraic, and formal) to build up integrated knowledge structures. (p. 153)

Given the central role of mathematical properties in the formalization and axiomatization of mathematics, there is a need for research that targets those properties and their relationships within the larger conceptual systems in which they are defined.

2.5.4.2 Attending to properties through reification theories

Reification theories have consistently focused on *encapsulation* (Arnon, et al., 2014) or *reification* (Sfard, 1994) of objects from *interiorized* processes (Arnon,, et al., 2014). There has been a strong emphasis on transitions between these “operational” and “structural” conceptions (Sfard, 1994), as well as the qualitative difference between them. These transitions are generally attributed to the Piagetian concept of “reflective abstraction” (Glasersfeld, 1995; Piaget, 1970), which has proven valuable to education researchers since its introduction.

While this theoretical emphasis has benefited research into students’ conceptions of certain mathematical objects, it may lack the appropriate focus to build theory around mathematical conceptions of properties and attributes attached to the interiorized processes and encapsulated objects represented in students’ minds. Alternatively, Slavit (1997) argued for a “property-oriented view” (p. 263) of students’ understanding of functions, blending with Sfard’s (1994) operational-structural perspective to “discuss how a student can reify the notion of function as a mathematical object that possesses or does not possess various functional properties” (Slavit, 1997, p. 263).

Elaborating the perspective of Slavit (1997, 2006), this research establishes a scheme for describing the structure of participants’ concept images for continuous functions in terms of abstracted mathematical properties, non-mathematical attributes, and students’ mental actions upon them. I found cases for which a property-based analysis could provide detailed information about participants’ cognitive structures.

2.5.5 Experiential and perceptual frameworks

The mathematical concept of a continuous function formalizes an essential property of the human perception of space, which involves the sensori-motor experiences of uninterrupted motion and connected length. I will argue that a student's mental conception of continuous functions is intricately bound to those experiences, and that successful students' advantage may lie in their ability to coordinate their formal mathematical understanding with other mental schemas based in more informal, perceptual, and intuitive ways of understanding.

2.5.5.1 <u>Perceptual and conceptual representations of space</u>

Piaget and Inhelder (1948/1967) investigated children's conceptions of physical space through the lens of genetic epistemology. One of the major theses of this work, which became known as the "topological primacy thesis", was that children developed the ability to differentiate "topological forms" prior to "Euclidean forms." In other words, the most discernible attributes of a spatial configuration were not related to geometric measurements of distance, angle, dimension, etc. Instead, children were said first to have the capacity to recognize characteristics that the authors called "topological," such as proximity, enclosure, boundary, and continuity.

The topological primacy thesis has been criticized on a number of fronts, including: mathematical critiques (Kapadia, 1979), methodological critiques (Darke, 1982; Fischer, 1965; Page, 1959; Weinzweig, 1978), and contrary findings based on replication experiments (Bass, 1975; Lovell, 1959; Martin, 1976; Miller & Baillargeon, 1990; Pinard & Laurendeau, 1966). However, many of the relevant findings in Piaget and Inhelder's (1948/1967) work remain unaffected by such criticisms, reinforcing the larger body of theory put forward by Piaget and his colleagues. Therefore, I incorporated relevant results from their work whenever they were well supported by similar research from other theoretical traditions (see subsections 2.5.5.3—2.5.5.2 below).

Specifically, I adopted Piaget & Inhelder's (1948/1967) distinction between a child's perceptions and mental representations of physical space. These scholars examined the development of *conceptual* or "representational" space at the operational levels of understanding, in contrast to the child's initial, more primitive construction of *perceptual* or "sensori-motor" space. The authors explained that "the evolution of spatial relations proceeds at two different levels. It is a process which takes place at the perceptual level and at the level of thought or imagination" (Piaget & Inhelder, 1948/1967, p. 3).

Piaget and Inhelder (1948/1967) pointed out that the dichotomous nature of humans' conceptions of space can complicate the study of the psychological roots behind spatial concepts. They cautioned that conceptual representations and geometrical ideas are more than "a mere copy of existing sensori-motor constructs." Rather, these authors argued that conceptual space and its perceptual counterpart influence the development of each other in a recursive and dialectic manner, as they explain: "Thus thought has the task of reproducing at its own level (of representation as distinct from direct perception) everything that perception has so far achieved within the limited field of direct contact with the object" (p. 13).

And on the other hand, concepts and representations are projected on to our perceptual knowledge of the world:

> …during the development of representational space, representational activity is, in a manner of speaking, reflected or projected back on to perceptual activity. Thus,

> beginning with the stage at which representation can arrange all spatial figures in co-ordinate systems…perception itself begins to locate the partial configurations it has achieved within such system, whereas formerly it was content with a far more limited degree of structurization. (p. 4)

Thus, upon reaching the stage of formal operations, a child has constructed two versions of their schema for psychological space—the schema they originally developed out of their sensori-motor experiences, and a more recently developed schema formed by the mathematical (geometric or analytic) representations they overlay onto perceptual space. These conceptions may be coordinated to varying degrees. If they are successful at accommodating the representational schema, then the perceptual schema may be subsumed by the child's conceptual representations (Piaget & Inhelder, 1948/1967).

My analysis was conducted with a consideration of the distinction between perceptual and conceptual space in the participants' cognitive structure. Although Piaget and Inhelder (1948/1967) examined early cognitive developments in children, these developments are the building blocks for structures involved in future adult learning and understanding as well.

2.5.5.2 Dual coding theory

Dual coding theory (DCT) was developed by Paivio (1986) in response to "a long-standing dispute concerning the nature of the mental representations that mediate perception, comprehension, and performance in cognitive tasks" (Paivio, 2010, p. 205). DCT contrasts with the standard approach to the so-called "coding" of mental representations, in which "a single, abstract form of representation underlies language and other cognitive skills" (p. 205). Instead, DCT entails the coordination of two independent systems of representation, as Paivio (2010, p. 207) explained:

> The basic assumption of dual coding theory is that all cognition involves the activity of two functionally independent but interconnected multimodal systems, an internalized nonverbal system that directly represents the perceptual properties and affordances of nonverbal objects and events, and an internalized verbal system that deals directly with linguistic stimuli and responses. The systems are made up of "mental" representational units, structures, and dynamic processes that are learned in cultural contexts, given perceptual, memory, and other innate capacities that resulted from biological evolution.

In other words, humans represent knowledge both linguistic and nonlinguistic ways which influence one another, but may also act independently. The author emphasized this independence by claiming that "verbal and nonverbal representations can be activated separately or together," and "different sensorimotor modalities within nonverbal or verbal representations (e.g., auditory and visual language systems) can function independently" (Paivio, 2010, p. 208). He called objects within the verbal representation system *logogens*, which he explained are "typically organized sequentially into higher order structures" (p. 208). Objects within the nonverbal system were called *imagens*, which instead "combine into synchronous ensembles" (p. 208). The differences between the two systems are similar to Piaget & Inhelder's (1948/1967) distinction between perceptual and conceptual space; and the functional independence of the two systems relates to their sentiment that conceptual representations of space are more than "a mere copy of existing sensori-motor constructs."

Brijlall & Maharaj (2011, 2013) incorporated DCT into their study of mathematics students' learning about continuous functions. They applied multiple theoretical perspectives to account for the different "worlds of mathematics" (Tall, 2013) that students inhabit. In this context, DCT was used to examine the embodied "world of continuous function" (Brijlall & Maharaj, 2011, p. 655). They also applied the APOS framework in combination to the formal and symbolic worlds. Similarly, in my analysis I interpreted the participants' responses as originating in two representation systems, the nonverbal, perceptual system, or embodied world; and the multi-modal linguistic system, analogous to the paired formal-symbolic worlds to which Tall (2013) originally referred.

2.5.5.3 Epistemological obstacles involved in the continuity concept

The historical challenges faced by mathematicians during the formalization of the function, limit, and continuity concepts (Harper, 2016; Moore, 2008) may point to several *epistemological obstacles* (Radford, 1997; Sierpinska, 1987, 1994) involved in building abstract conceptions about perceptual knowledge. Brousseau adapted this idea from Bachelard (as cited in Sierpinska, 1994), which were defined to be "ways of understanding based on some unconscious, culturally acquired schemes of thought and unquestioned beliefs about the nature of mathematics and fundamental categories such as number, space, cause, chance, infinity…inadequate with respect to the present day theory…" (Sierpinska, 1994). She pointed out that, "epistemological obstacles are not obstacles to the 'right' or 'correct' understanding; they are obstacles to some change in the frame of mind" (p. 120). Epistemological obstacles are not individualized, but shared with historical individuals and modern society. They "grow on the soil of complexive, childish, thinking…But the fertilizers (the challenges that make them grow) come from the surrounding culture, from the implicit and explicit ways in which the child is socialized" (p. 120).

Sierpinska (1987) documented a collection of epistemological obstacles about function limits, which concerned students' geometric/numerical, static/kinetic, and heuristic/rigoristic ways of conceptualizing. More recently, applying Sierpinska's (1987) approach to epistemological obstacles to the teaching and learning of calculus, Job and Schneider (2014) established that an implicit belief in empirical positivism is an epistemological obstacle to students' learning about the limits of real functions. They referred to "the development of calculus as an epistemological transition between two types of praxeologies, pragmatic and deductive"; where a praxeology is "an anthropological and epistemological model of knowledge" (p. 635). Job and Schneider (2014) explained the difference based on the status of conceptual definitions in each:

> In pragmatic praxeologies, a concept is, in the first place, an instrumental model of an object, whatever its membership, mathematical or not, which requires some kind of detachment between the model and the modelled objects. In deductive praxeologies a definition is chosen according to its ability to build a deductive architecture which is coherent with fewer concepts that are the most general possible. (p. 637-638)

In other words, the calculus-analysis sequence requires a shift in students' basic understanding of the meaning of concepts and what constitutes mathematical reality. The authors argued that students need guidance through this transition, to learn to coordinate these two incompatible perspectives.

Note the similarities between this discussion and that of the previous two sections—concerning the construction of perceptual and conceptual space, and the "dual coding" of experience into verbal and nonverbal representation systems. These theoretical distinctions are related in the sense that a pragmatic praxeology involves the informal modeling of a perceptual or experienced object, while a deductive praxeology builds an explicating (Dawkins, 2015) mathematical structure that may subsume the previous pragmatic model. A major theme of the following analysis will be the interplay of the participants' *pragmatic/perceptual* and *deductive/conceptual* ways of knowing as they reasoned about continuity in new contexts.

2.5.5.4 Analogical reasoning and embodied schemas

Connected to the themes described above are the semantic challenges that students face when learning about continuity. Reasoning from the properties and attributes of metaphors, visual representations, prototypes, or common examples may provide organization to a student's conceptual understanding (Dawkins, 2012; Parameswaran, 2010; Presmeg, 1998, 1992; Pinto & Tall, 2002; Pirie & Kieren, 1989; Sfard, 1994; Sierpinska, 1994). On the other hand, semantic reasoning may also obstruct the formalization of her reasoning in various ways (Alcock, 2002; Hazzan, 1994; Zandieh & Knapp, 2006). I will refer to linguistic elements like metaphor, metonymy, and representations as *analogical tools* for the organization of conceptual knowledge.

According to Sierpinska (1994), metaphors, metonymies, analogies, models, and exemplars are all tools that can be used to overcome epistemological gaps between our perceptual knowledge of the world and our conceptual understanding of it.

> Metaphors generate novel structurings of our experience in a way not fully anticipated by our available systems of concepts…So understood, metaphors may be seen as grounding the concepts that we then use to speak determinately of as objects. The primary role of metaphor is thus to establish those structures we later articulate by means of fixed, determinate concepts (and systems of concepts). (Johnson, 1980, as cited in Sierpinska, 1994, p. 97)

Moreover, as suggested by embodied cognition theorists (see Johnson, 1987; Lakoff, 1987; Lakoff & Nuñez, 2000; Piaget & Garcia, 1983/1989), metaphorical structures can act as bases for understanding the world around us, and especially highly abstract concepts, by mapping them onto our existing re-presentations of reality.

> Thus, by structuring, ordering our experience and making it 'fit in with' the existing mental structures, a metaphor is a basis for understanding. An act of understanding based on a new metaphor is a creative act of understanding insofar as it 'causes a reorganization of our conceptual frameworks'. Moreover, an act of understanding based on a metaphor plays a crucial role in the development of our thinking: it prepares the ground for the formation of a concept. (Sierpinska, 1994, p. 97).

In this analysis, the use of any of the analogical tools listed above for the purpose of structuring new mathematical understandings will be considered a creative act of understanding with consequences for the interpretation of the participants' underlying thought processes.

2.5.5.5 Role of intuitions and perceptual knowledge

Finally, an important distinction is found in Fischbein's (1987) definition for intuitive knowledge as opposed to the knowledge derived from the sensori-motor experience; i.e., *intuition* versus *perception*. He defines intuitive knowledge as "immediate knowledge; that is, a form of cognition which seems to present itself to a person as being self-evident" (p. 6). By "self-evident" he means that they are "cognitions which are directly grasped without, or prior to, any need for explicit justification or interpretation" (p. 3). However, Fischbein (1987) notes that "perception is also an immediate cognition"; and so, he differentiates the two notions by further requiring that intuition "exceed the observable facts" of the situation. That is, "an intuition is a theory, it implies an extrapolation beyond the directly accessible information" (Fischbein, 1987, p. 13).

Morena-Armella (2014) discussed the central tension in calculus education between the intuitive and the formal; and talked about the need to develop meaning around the existence of mathematical objects, such as continuous functions. So, in the sense of Fischbein (1987), recognizing the connectedness of a graphical representation of a function would be considered a perception. The knowledge is immediate within the visual representation system of the individual. However, for a student to declare the function to be continuous on this basis would constitute an intuition, since this would require some past experience to connect the mathematical notion of continuity to the perceptual awareness of its graphical representation. She would be extrapolating beyond the perceptual information provided by the graph. I will refer to intuitions based on perceptual knowledge as *perceptual intuitions*, while intuitions based on prior mathematical experiences will be called *conceptual intuitions.*

2.6 Conceptual Framework

The term "conceptual framework" refers to the theoretical, hypothesis-rich perspective I developed over three semesters of preliminary research leading up to the main study (see Chapter 3). This is the primary lens for the analysis of the data. In turn, this conceptual framework incorporates constructs and perspectives from several, broad-scoped theoretical frameworks, which collectively served as a frame of reference for the analysis. The various perspectives provided by these frameworks guided the construction of the protocols and interview tasks, as well as the coding paradigm that I used to interpret the collected data (see Chapter 4, Section 3).

2.6.1 Description of the conceptual framework

The conceptual framework for this study is displayed in Figure 4. This diagram represents the processes of abstraction (arrows) of mathematical properties and non-mathematical attributes (center circle) involved in a student's schema for a concept. Through the four mental operations I examined (pie wedges), the student may project those properties and attributes as a *reflected abstraction* (Glasersfeld, 1995; Piaget, 1970) to form distinct cognitive structures (outer ellipses). I will also demonstrate the participants' construction of that structure through other forms of abstraction, acting in coordination with their reflected abstractions. I will refer to these collectively as secondary abstractions. This diagram is explained further in Table 1, which delineates the varied forms of abstraction each arrow may theoretically represent and the type of property that may be abstracted.

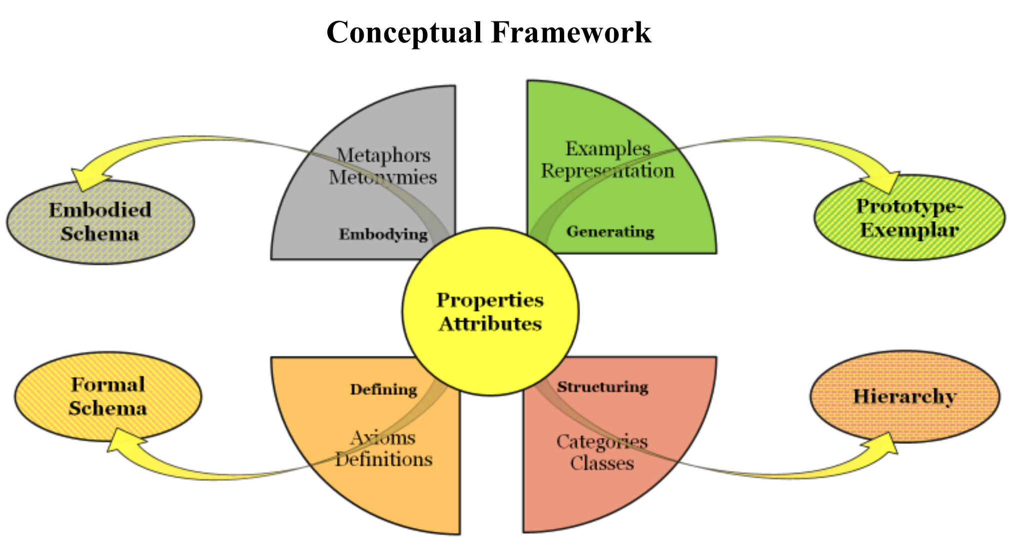


Figure 4. Diagram of the conceptual framework used in this study.

Table 1: Conceptual framework

Conceptual Structure	Property or Attribute	Modes of Abstraction	Product of Abstraction
Axiom Definition	Symbolic-Syntactic Form & Process	Symbolic	Formal Schema
Class Category	Rules & Criteria	Classification, Categorization & Hierarchization	Class Hierarchy
Example Representation	Intra-Class Feature Similarity	De-Contextualization & Modeling	Prototype-Exemplar
Metaphor Metonymy	Inter-Class Relational Similarity	Cross-Contextualization & Empirical Abstraction	Embodied Schema

2.6.2 Properties and attributes in the conceptual framework

In the following discussion, I will distinguish mathematical *properties* and non-mathematical *attributes*, using the italicized terms for brevity. The choice of these words is intended to reflect a difference between the idealized *properties* belonging to a mathematical object (which are precisely specified by the object's definition or a logical derivation thereof); and the non-mathematical characteristics *attributed* to a concept within an individual's personal set of meanings.

The attributes within a learner's conception may not reflect the properties of its associated concept. For example, the continuity of a function was frequently conceptualized by my participants as a connected two-dimensional graph. Such a representation is likely to be an attribute of most students' continuity checking schemes, but it may often have little connection within their cognitive schemas to the mathematical process which defines the property of continuity. In many cases, my participants relied on non-mathematical attributes of their conceptions, based on embodied metaphors or prototypical comparisons to make sense of abstract concepts during the tasks.

While it may be desirable for a student's conception of a mathematical object to develop towards a formal-symbolic understanding of the object's properties, there is no evidence that the actual mental objects created by a student will remain fixed to its categorical structure. It may be the case that reified cognitive objects vary in structure from student to student, embedded in the particularities of each one's concept image. Thus, to gain a broad perspective on my participants' responses, I looked for both the properties they attended to mathematically, and the additional attributes they assigned to their conceptions. Both are important aspects of interpreting the activity of participants and modelling the mental operations involved in that activity, which is the basis of the type of conceptual analysis (Glasersfeld, 1995) I performed in this research.

2.6.3 Mathematical properties in the concept image

The evidence I present will build a case for the idea that a student's mental representations for definitions, examples, and categories can be characterized by the collections of properties she attends to in their construction. To wit, formal definitions are collections of specified properties that are synthesized into a conceptual unit through the process of reification. Examples are instantiations of an individual property or a collection of properties, allowing students to deductively manipulate otherwise abstract ideas. Finally, categorization schemes use properties to determine their hierarchical structure—a category is an equivalence class of objects whose criteria are based on that objects' exhibition of a descending chain of properties, ranging from the general to the specific.

2.6.3.1 Definitions

In the classical categorical hierarchy (Rosch, 1973), mathematical objects are simply collections of properties taken together; and a definition is the formal attribution of a name to a given collection of such properties, usually in the context of a previously established setting (e.g., a given set of axioms). In axiomatic systems, definitions have the role of collecting various mathematical properties into a single formal construct. The resulting interactions of those properties lead to further, emergent properties that are interesting to the formal mathematician.

Thus, formal definitions and the properties they consist of are aspects of the same phenomenon. I considered the participants' personal concept definitions (Tall & Vinner, 1981) through this lens, allowing me to operationalize their definitions as a list of properties they expressed or inferred through their mathematical activity. Note that these properties may include or involve mathematical processes or procedures; and this is a characterization of definitions rather than the objects the define.

2.6.3.2 Examples and Prototypes

Another illustration of the importance of focusing on properties is in students' generation and use of examples. Examples provide not only a means to connect to a learner's personal knowledge structure, but also a means to differentiate between many similar constructs within it.

A mathematical construct can be seen as an example of some operation or structure only when a student can explicitly reference it as an example of some reified *property* of that construct (Sfard, 1994). For example, the number four can be representative of many different properties, including the property of being an even number, a positive integer, or a perfect square. While the number four could never have been such a representative prior to the encapsulation of these formal properties, it is more significant to note that formal conceptions of those properties required a rich example space of such representative numbers (Goldenberg & Mason, 2008 Parameswaran, 2010).

The discussion above highlights a dual relationship between "examples" (objects with a given characteristic) and "properties" characteristics of a given object). A property can be seen to arise as a sort of "subspace" of one's example space for a given concept that includes examples of mathematical objects for which each property holds (ibid). Other than a property's formal definition and some related theorems, the examples explored by a student are the only source of conceptual intuition at her disposal.

2.6.3.3 Categories and Hierarchies

Learning mathematics, particularly advanced mathematics, requires the classification and categorization of conceptual objects based on their properties. The way students perform the mental action of categorizing such objects will influence how they adapt to novel mathematical environments and solve problems within them. Categories, which are collections of conceptual objects with shared properties, store information about their members. The way a mathematical object behaves and interacts with other objects is determined by its placement in an overall categorical hierarchy of concepts, also referred to as a conceptual taxonomy.

Inappropriately categorizing a phenomenon may hamper a student's ability to justify, prove, and solve problems with it. Improperly naming an object as a member of a category may imbue that object with properties it does not have, or occlude certain properties that it should have. On the other hand, improperly delineating a category's properties may admit membership to objects that don't belong in the category, or deny membership to some that do. Students may then go on to make mathematical decisions based on an object's assumed membership in a category, rather than logical deduction from its definition. Thus, students' approaches to determining a concept's membership within a category, and within the categorical structure that encompasses it, may be an important factor in how students make sense of mathematical ideas.

2.6.4 Non-mathematical properties in the concept image

The radical constructivist perspective emphasizes the mental operations involved in students' mathematical activities, but does not ignore the "experiential elements that may serve as its raw material" (Glasersfeld, 2007, p. 214). In this context, mathematics is seen as being embedded in the world of sensory perceptions, despite consistently pushing toward abstraction. Sensory information from physical objects or their representations play an important role in how abstraction occurs, as well as which forms of abstraction might be observed. Glasersfeld (2007) explains that mathematical ideas arise out of sensori-motor experiences:

> …I have argued that common non-mathematical activities, such as isolating objects in the visual or tactual field, coordinating operations while they are being carried out, and generating a line by a continuous uniform movement, are the experiential raw material that provides the thinking subject with opportunities to abstract elementary mathematical concepts…some (and perhaps all) of the indispensable elements in mathematical thinking are conceptual constructs that were abstracted from operations carried out with sensory material, operations that are involved in segmenting and ordering experience long before we enter the realm of mathematics (p. 218-221).

Therefore, the participants' visual representations, perceptual metaphors, physio-spatial models, and embodied schemas were all considered to be theoretically important aspect of their concept images for my study.

3. METHODOLOGY FOR THE MAIN STUDY

3.1 Introduction

What follows is a detailed account of the procedures used in the main study of my dissertation research, which took place during the Fall semester of 2015 at Texas State University. The sampling procedures, interview protocols/tasks, and conceptual framework were developed through three semesters of preliminary studies and a textbook analysis (see Appendix E). This chapter will introduce the research design, setting and participants, as well as the data collection and analysis procedures for the main study. Contextual data from the participating class and a series of individual interviews with some of its students were used to answer research questions related to how collegiate-level mathematics students who were enrolled in topology courses attempted to reconstruct (e.g. assimilate or accommodate) their understanding about open sets, closed sets, and continuous functions to reflect an axiomatic level of understanding.

3.2 Research Questions

My research questions were developed through three semesters of preliminary research (see Appendix E), in which I used the "constant comparative method" (Strauss & Corbin, 1997) to build theoretical categories for exploration, specify my research goals, and establish boundaries for my inquiry and cases (Stake, 2000; Yin, 2003). Ultimately, the research questions that emerged for the main study (discussed in this chapter) took the following form.

During an undergraduate, introductory topology course:

1) What distinctions and comparisons can be made between the ways that students manage their transition to an axiomatic understanding of continuous functions?

2) What obstacles do the students face during this transition?

I answered these questions through an analysis of my participants' mathematical activities during tasks that addressed the following mathematical content areas: 1) sets and set operations; 2) open and closed sets; 3) interior, limit, and boundary points; and 4) functions and continuous functions. The participants explored tasks that dealt with each of these content areas while I observed their mathematical activity and guided them to establish a consistent and coherent record of their understanding of the task situation. I then retrospectively categorized the participants' actions according to the five sense-making activities established in the conceptual framework in Chapter 2: *defining*, *structuring*, *generating*, and *embodying*; as well as the *abstraction of properties* through which those activities can modify a participant's scheme (or schema).

3.3 Coding Paradigm

The primary research questions were designed to permit flexibility in my process of building a theory around my participants' mental actions, but limit the scope of my inquiry to a manageable size. To operationalize my conceptual framework, I established its five categories as coding themes and wrote three guiding questions for each (see Table 2). These fifteen questions will be referred to as the *coding paradigm* (Strauss & Corbin, 1998) for the analysis. The new data were then collected, coded, and analyzed through these questions on an episodic basis (Charmaz, 2010).

Table 2: Coding paradigm

(i) *Definitions*

a. What defining activities can be observed by the participants?
b. What differences exist between participants' personal concept definitions and the formal concept definition?
c. How do these activities and differences affect the participants' transitions to an axiomatic understanding of continuous functions?

(ii) *Examples*

a. What examples/prototypes do the participants generate?
b. How do the participants' example spaces interact with their personal concept definitions and the formal concept definition?
c. How do these examples and their interactions affect participants' transitions to an axiomatic understanding of continuous functions?

(iii) *Metaphors/Metonymies*

a. What metaphors/metonymies do the participants use for understanding? Are they embodied or abstract?
b. How do these metaphors/metonymies interact with the participants' example spaces and personal concept definitions?
c. How do these metaphors/metonymies and their interactions affect the participants' transitions to an axiomatic understanding of continuous functions?

(iv) *Categories*

a. Which elements of the participants' concept images are least/most central to the taxonomic structure of their understanding?
b. How do the participants categorize elements in their concept images (e.g. rules, exemplars, prototypes, metaphors/metonymies)?
c. How do these categorization schemes affect their transitions to an axiomatic understanding of continuous functions?

(v) *Abstraction/Instantiation*

a. What acts of abstraction/instantiation do participants use to reorganize their concept images and definitions?
b. Which elements of the participants' concept images are the objects of these acts of abstraction/instantiation?
c. How do these acts of abstraction/instantiation affect participants' transitions to axiomatic understanding?

For example, during the open set analysis reported in Chapter 4, I looked at the defining activities of the participants (i-a) during individual tasks, examining how these activities aligned with the formal concept definitions for open sets, limit points, interior points, and boundary points (i-b). In Chapter 5 I explored interpretive connections between the participants' acts of understanding and theoretical sub-elements of their overall concept images (iv-a); such as example spaces (ii-b), visual prototypes (ii-a), metaphors (iii-a, iii-b) and paradigmatic models (iv-b).

3.4 Research Design

This research adhered to a qualitative (Strauss & Corbin, 1998; Denzin & Lincoln, 2008); case-study design (Ragin, 2004; Stake, 2000). Tasks, clinical interview protocols (Clement, 2000; Garfield, 1981), and a coding paradigm (Strauss & Corbin, 1998) were developed through three semesters of preliminary research (see Appendix E) to answer the research questions. The interviews were designed to elicit evidence of the participants' ways of assimilating and/or accommodating novel topological spaces into their schemes (schema) for continuous functions. The coding paradigm in Table 2 then acted as a framework for building cases to profile participants' ways of abstracting mathematical properties in the new topologies. The analyses were reported by episodic cases: defined as individual episodes of the participants' goal-oriented reasoning activity in the context of the tasks. For clarity and context, the analyses are presented in order of the participants' responses to complete tasks. However, the unit of analysis for the cases was considered to be the reasoning activities themselves and not the participants as individuals.

3.4.1 Qualitative versus quantitative inquiry

Qualitative research involves a creative search for patterns within specific instances, which allows for new meanings to be built out of observed experience. Strauss and Corbin (1998) described qualitative research as "a nonmathematical process of interpretation, carried out for the purpose of discovering concepts and relationships in raw data and then organizing these into a theoretical explanatory scheme" (p. 11). They point out that the purpose of research activities manifest in different ways for qualitative versus quantitative methods. One of the key differences between the two methods relates to the use of inductive reasoning in qualitative research, as opposed to the strict deductive reasoning required in quantitative research. Rather than providing statistical support for specific answers to strictly worded, binary research questions; qualitative research uses induction to build explanatory schemes out of empirical data. Once developed, these explanatory theories may then be tested through quantitative means.

Qualitative methods pursue different research goals than quantitative methods. Throughout the social sciences, qualitative theorists have highlighted the benefits of aiming for interpretation over inference, extrapolation over generalization, and meaning-making over truth-discovering (Patton, 2002). Qualitative methods can be used to obtain the intricate details about phenomena such as feelings, thought processes, and emotions that are difficult to extract or learn about through more conventional research methods" (Strauss & Corbin, 1998, p. 11). Moreover, such approaches can be used to build theory "when partial or inadequate theories exist for certain populations and samples or existing theories do not adequately capture the complexity of the problem we are examining" (Creswell, 2013, p. 48).

My research involved the mental processes and conceptions of students engaged in learning an abstract mathematical domain which has not been extensively studied. To interpret the unobservable thoughts and representations of my participants is a challenge, subject to a complex array of factors: psychological, perceptual, mathematical, logical.

The differences between qualitative and quantitative research methods have led to the development of alternative criteria for evaluating qualitative methods. Lincoln and Guba (1986) offered a series of criteria as parallels to traditional evaluative criteria in the social sciences, suggesting “credibility as an analog to internal validity, transferability as an analog to external validity, dependability as an analog to reliability, and confirmability as an analog to objectivity” (p. 76-77, as cited in Patton, 2002, p. 546). These criteria are said to address the research effort’s “trustworthiness,” as a parallel to scientific “rigor” (p. 546). Criteria such as these fit well within the constructivist tradition, in which researchers are careful to limit their epistemological claims and “replace the usual positivist criteria” (Denzin & Lincoln, 2000, p. 21) for the interpretation and evaluation of knowledge.

3.4.2 Case-oriented research

A common form of qualitative research is the case study, in which bounded systems and their specific contexts are described in detail and often compared with other systems in similar contexts (Stake, 2000). My study investigated how altering the topological context of a well-known mathematical situation could affect the participants’ reconstruction of her continuity schema. Such a study is well suited to case-oriented research because, as Yin (2003) pointed out, case study research is not only useful to answer “how” and “why” questions, but also for studying contextual conditions when they are believed to be relevant to the phenomenon and when “the boundaries are not clear between the phenomenon and context” (Baxter & Jack, 2008). Without making any assumptions about the participants’ reasoning, the case-study was a useful

In fact, a case-oriented research design is not only useful, but necessary to understanding certain kinds of phenomena. The practice of building cases permits a researcher to develop theory in a manner that would create practical difficulties for other approaches. Ragin (2004) outlined several difficulties for variable-oriented approaches that may be addressed by case-oriented research. These included the initial constitution of cases to study, the "analysis of multiple and conjunctural causes," and the treatment of non-conforming cases.

The goal of my study was to observe and describe student reasoning, and to document both successful activities and the obstacles involved in transitioning to axiomatic understanding. There were potentially many distinct mathematical approaches to the content that might lead a student to success, as well as many obstructions to it. Moreover, multiple factors could explain any of my observations. Therefore, a case-study design was useful for determining which themes would be useful to explore, and for organizing the results. My constitution and description of the cases presented were grounded by my observations and interpreted through the lens provided by my coding paradigm.

3.4.2.1 Constituting cases and relevant populations

Ragin (2004) explains that "the bulk of a qualitative research effort is often directed toward constituting 'the cases' in the investigation and sharpening the concepts appropriate for the cases selected" (p. 127). These cases,

> …usually are not predetermined, nor are they 'given' at the outset of an investigation. Instead, they often coalesce in the course of the research through a systematic dialogue of ideas and evidence…In many qualitatively oriented studies, the conclusion of this process

> of 'casing' (Ragin, 1992) may be the primary and most important finding of the investigation. (p. 127)

Case studies allow researchers to take the essential first step of describing the relevant populations for new areas of research. As Ragin (2004) explains, "the entire process of case-oriented research—learning more about a case to see what lessons it has to offer—can be seen, in part, as an effort to specify the population or populations that are relevant to the case" (Ragin, 2004, p. 133). This goal reflects the purpose of this study—to investigate the population of diverse approaches to abstract mathematical understanding, examine cases of student reasoning indicative of each approach, and describe the essential qualities that make these cases worthy of study.

3.4.2.2 Building theory with cases

Case studies are useful tools for the origination and development of research theory (Ragin, 2006; Walton, 1992). Per Walton (1992), "…cases come wrapped in theories. They are cases because they embody causal processes operating in microcosm" (p.121). These processes are recognized by researchers as operational examples of evolving typological distinctions, "drawn from a fund of prior general knowledge" (p.125). He argues that to label an event as a 'case' necessarily asserts membership in some theoretical family, or categorical level. "Implicit in the idea of the case is a claim" (p. 122).

Therefore, a case analysis of my participants' types of reasoning was a way to develop conceptual links and begin to build an explanatory theory. Walton (1992, p 92) offers case-oriented research as the best way to build theory:

> The processes of coming to grips with a particular empirical instance, of reflecting on what it is a case of, and contrasting it with other case models, are all practical steps

> toward constructing theoretical interpretations. And it is for that reason, paradoxically, that case studies are likely to produce the best theory.

3.4.3 Diversity of methodological and theoretical perspectives

Qualitative research designs embrace a diverse array of potential research methods and theoretical perspectives to deeply analyze phenomena from many directions. The combination of these approaches can provide a more robust understanding of the phenomena being studied, and ensure the validity of the research through triangulation (Denzin & Lincoln, 2008) and the provision of multiple sources of evidence for confirmability of the findings.

3.4.3.1 Multi-modal approaches in qualitative research

I followed a multi-modal approach to research that has been embraced by qualitative researchers. Denzin and Lincoln (2008) described a useful analogy for the qualitative researcher as "bricoleur," a French term for one who crafts with whatever tools are available:

> The qualitative researcher as *bricoleur*, or maker of quilts, uses the aesthetic and material tools of his or her craft, deploying whatever strategies, methods, and empirical materials are at hand (Becker, 1998, p. 2). If the researcher needs to invent, or piece together, new tools or techniques, he or she will do so. Choices regarding which interpretive practices to employ are not necessarily made in advance. (p. 5)

Flick (2002, p. 229) also condoned such a multimethod approach in his claim that "the combination of multiple methodological practices, empirical materials, perspectives, and observers in a single study is best understood, then, as a strategy that adds rigor, breadth, complexity, richness, and depth to any inquiry." In this way, "triangulation" of the data serves as an "alternative to validation" (Flick, 2002, p. 227) by providing a multidimensional perspective. Denzin and Lincoln (2008) explained the notion of triangulation:

> Viewed as a crystalline form, as a montage, or as a creative performance around a central them, triangulation as a form of, or alternative to, validity thus can be extended. Triangulation is the simultaneous display of multiple, refracted realities. Each of the metaphors 'works' to create simultaneity rather than the sequential or linear. Readers and audiences are invited to explore competing visions of the context, to become immersed in and merge with new realities to comprehend." (p. 8)

3.4.3.2 Sensitizing concepts in qualitative research

The sociologist Herbert Blumer (1954) delineated two types of concepts:

> A definitive concept refers precisely to what is common to a class of objects, by the aid of a clear definition in terms of attributes or fixed bench marks…A sensitizing concept lacks such specification of attributes or bench marks and consequently it does not enable the user to move directly to the instance and its relevant content. Instead, it gives the user a general sense of reference and guidance in approaching empirical instances. (p. 7)

Definitive concepts can be related to the role of classification criteria within the 'classical theory' of cognitive categorization, while sensitizing concepts reflect the more 'natural categories' of Rosch (1973). Blumer (1954) pointed out these two types of concepts operate in different ways in our language and cognitive awareness:

> Whereas definitive concepts provide prescriptions of what to see, sensitizing concepts merely suggest directions along which to look. The hundreds of our concepts—like culture, institutions, social structure, mores, and personality—are not definitive concepts but are sensitizing in nature. They lack precise reference and have no bench marks which allow a clean-cut identification of a specific instance and of its content. Instead, they rest on a general sense of what is relevant.

For my main analysis, I considered five “sensitizing concepts,” expressed as themes within the coding paradigm: *definition, example, metaphor*, *categorization,* and *abstraction*. While these general terms may be defined explicitly, it is not as simple to precisely define how a participant may have been reasoning with these cognitive structures during an interview. For instance, to define a concept describing a participant’s use of a definition, example, or metaphor, the researcher must perform a qualitative act of interpreting her verbal, written, and gestural responses to the task. There are no clear “bench marks” to serve as criteria for stating definitively how the participant reasoned about the task. While a researcher may gather evidence in support of a particular assumption, there is no way to be sure whether the participant based her responses on her personal concept definition, or by analyzing the properties of a recalled example, or by comparison with some real-world perceptual metaphor.

Therefore, the five concepts listed above could not be relied upon as “definitive concepts” in this research situation. Rather, these five concepts acted as focal points from which to interrogate and interpret my data, which guided my analysis and continued to provoke new questions. For this reason, I describe these terms as sensitizing concepts within the analysis.

Analogously to the notion of sensitizing concepts, it is also possible to consider entire theoretical perspectives as sensitizing agents rather than definitive frameworks for conceptualizing data. In the social sciences, such perspectives can serve as heuristic devices, or “bridge hypotheses” (Kelle, 2007, p. 208) during the process of establishing empirically grounded categories. Kelle (2007, p. 208) explains:

> Thus one can apply abstract theoretical categories with a general scope (which refer to various kinds of phenomena) but with limited empirical content (like identity or social

> role) as heuristic devices to develop empirically grounded categories with a limited scope and high empirical content…A variety of concepts coming from differing theoretical approaches in sociology and social psychology can be used in such a way.

However, citing the grounded theory tradition, Kelle (2007, p. 209) explains the dangers of fixating on a single (definitive) theoretical framework (italics are added for emphasis):

> In this manner, a wide array of sensitizing categories from different theoretical traditions can be used to develop empirically grounded categories. *A strategy of coding which uses different and even competing theoretical perspectives may often be superior to a strategy which remains restricted to a limited number of pet concepts*. Furthermore, analysts should always ask themselves whether the chosen heuristic categories lead to the exclusion of certain processes and events from being analyzed and coded…

In consideration of these issues, I chose to acknowledge my continuing awareness of a broad array of theoretical perspectives during my analysis. Rather than attempting to neutralize the influence of my previous theoretical knowledge, both within and outside of the field of mathematics education, I embraced multiple contrasting conceptual frameworks as sensitizing perspectives to provide focal points for my investigations. Frameworks such as APOS theory (Arnon, et al., 2014), "defining as a mathematical activity" (Zandieh & Rasmussen, 2010), "natural language categories" (Rosch, 1973), and "embodied mathematics" (Lakoff & Nuñez, 2000) served as lenses through which to observe and code my data in diverse ways and along different conceptual dimensions. Many of these frameworks were employed tentatively and provisionally as heuristic tools for making sense of the data as my empirically grounded categories began to emerge.

3.5 Data Collection

Data from the initial three semesters of preliminary research were used to develop criteria for the selection of the final participants, and to validate the tasks and lines of questioning for the clinical interview protocols. The culmination of these efforts resulted in a case analysis based on the *triangulation* (Denzin & Lincoln, 2008) of multiple, overlapping sources of data from the class.

3.5.1 Triangulation

Denzin and Lincoln (2008) define "triangulation" as "a process of using multiple perceptions to clarify meaning, verifying the repeatability of an observation or interpretation" (p. 133). However, these authors also acknowledge that, from a constructivist standpoint, "no observations or interpretations are perfectly repeatable" (p. 133). Thus, they emphasize the need to identify alternative ways to perceive each case; and value a "diversity of perception, even the multiple realities within which people live. Triangulation helps to identify different realities" (p. 133).

In reference to evaluative criteria for qualitative research, the authors agreed with Flick (2002) that "triangulation is not a tool or a strategy of validation, but an alternative to validation" (p. 227). They argued that a diversity of methodological practices, empirical data, and sources of perspective "adds rigor, breadth, complexity, richness, and depth to any inquiry" (Denzin & Lincoln, 2008, p. 7).

To triangulate my own data, I collected copies of the quizzes, tests, and extra credit assignments from each of the participants (not just the interviewees). Participants were also given a preliminary knowledge assessment to obtain information about their formal-symbolic, and conceptual fluency with pre-topological ideas. Combined, these assessments and classwork offered a perspective on the participants' conceptualization of the focal topics for the study. Furthermore, classroom observations and consultations with the participating professor ensured that the data collection instruments remained true to the shared curricular space of the class. Finally, the interview process itself was designed with the goal of triangulating the student's perspectives with respect to the textbook, the teacher-researcher, and the student's prior mathematical activities and utterances. These interviews were recorded in three formats, with audio and visual data, as well as the students' written work to compare. In this way, I ensured that the interpretations I made as a researcher were informed by multiple perspectives and sources of empirical evidence.

3.5.2 Data collection procedures

Data collection for this study involved five primary data sources: (a) a variety of topology textbooks; (b) the course context, classroom observations and field notes; (c) in-class assignments; (d) research assessments; and (e) three task-based clinical interviews for seven students. The responses of six of the seven interviewed participants were then used to create case analyses that will be reported in the results. Data collected from each of these sources were meant to be complementary, as a way of triangulating my data. Table 3 describes the use of these data sources over the course of the preliminary and main studies.

Table 3: Data sources

Course/ Semester	Textbooks/ Resources	Interviews	Field Notes	In-Class Assignments	Research Assessments
Textbook Analysis Summer 2014	12 textbooks representing various approaches	N/A	N/A	N/A	N/A
Undergraduate Real Analysis Spring 2014	Lay (2014) Class notes posted by professor	Single interview with 11 participants	N/A	N/A	N/A
Undergraduate Topology Fall 2014	Croom (1989)	Single interview with 14 participants	Regular classroom observations and field notes	Provided by instructor and discussed in interviews	N/A
Graduate Topology Spring 2015	Class notes provided by professor	Single interview with 12 participants	Regular classroom observations and field notes	Provided by instructor and discussed in interviews	N/A
MAIN STUDY Undergraduate Topology Fall 2015	Croom (1989) Class notes posted for students	Three interviews each with 7 participants (6 reported)	Recorded class observations and field notes	Copies of completed, homework, quizzes and tests for all 39 participants	Diagnostic survey Prelim. assessment Alternative assessments for 32 non-interviewed participants

3.5.2.1 Textbook Analysis

The textbook analysis (see Appendix E) was performed on twelve commonly-used topology textbooks and was described in the Textbook Analysis section of this chapter. Croom (1989) was adopted as the class textbook during the fourth semester of data collection, and was used in the design of the interview tasks and the interpretation of the transcripts. Participants were also permitted to use this as a resource during the interviews if they were unable to complete a given task without it.

3.5.2.2 Research assessments

Prior to my selection of participants for the interviews, thirty-nine students who signed consent forms agreed to fill out a demographic survey (see Appendix D) that assessed their previous aptitude and affective response to other specific mathematics courses. Those participants also responded to a knowledge assessment (see Appendix C) concerning their understanding of the symbolic notation, definitions, and logic of their pre-topological mathematics courses.

These documents were used to screen the participant pool for students that I considered to have a high potential for theoretical insight. I looked for students with a reasonable (not perfect) understanding of the definitions, formal mathematical structure, and symbolic reasoning of pre-topological mathematics. I also looked for students with positive attitudes about mathematics, and who seemed inclined to work diligently to learn topology. Those participants who were not selected to be interviewed were given the opportunity to respond to the interview tasks as extra credit homework assignments, which were used as supporting documents for the validation of the interview findings.

3.5.2.2.1 Knowledge assessment

A preliminary knowledge assessment was administered as homework to each of the thirty-nine consenting students, who were asked to answer the questions on their own, without the use of a textbook or other resources. They were informed that they would receive extra credit for completion of the assignment, and not for the validity of their work. This assessment was used to gather information about the participants': 1) prior understanding of relevant concepts, 2) fluency with symbolic notation, and 3) ability to comprehend and produce formal proofs.

3.5.2.2.2 Demographic survey

Participating students also filled out a survey concerning demographic information, including: 1) the previous mathematics courses they had taken and their final grades, 2) their affective perceptions of the class and mathematics in general, and 3) their preferences for the use of visualizations, examples, and definitions when learning new mathematical content.

3.5.2.3 Classroom observations and field notes

During the main study, I observed twenty classroom periods in their entirety, in which I recorded and took notes on the class lectures in detail. I noted the mathematical content of each lecture and the professor's presentation of it, with special attention to the figures, symbols, representations, clarifying examples and heuristic or semantic devices that might have been used or discussed in the lecture to convey meaning. I also took note of the students in each lecture, including their questions and instances of confusion, I also recorded assigned homework and quizzes, relevant announcements and the numbers of students attending each class. I used all these observations as a baseline for understanding the content of the course as it had been presented to the participants.

The field notes from the classroom observations were used to validate both the task designs and participant responses. During the main study, approximately twenty-five hours of field notes were written with the Livescribe Echo Smartpen, described in the Interview section above. The use of this technology allowed me to review details of the classes to recall specific forms of definitions, theorems, proofs, or explanations used by the professors throughout each semester. This was useful for two purposes: 1) for designing interview tasks that would be relevant to the participants' level of understanding, and 2) for comparing the professors' presentation of topics to my participants' thinking about those topics during the interviews. The definitions, theorems, proof strategies, and notational conventions used during the class were a part of the classroom's shared realm of accessible knowledge, and were therefore important as a means for communicating with my participants and interpreting their responses.

3.5.2.4 In-class assignments

I was given access to ungraded copies of thirty-nine participants' quizzes, exams and extra credit assignments. I used these documents to screen the participant pool for the seven final interview subjects, and to triangulate my classroom observations with another measure of the students' level of understanding of the material.

3.5.2.5 Clinical Interviews

Thirty-five hours of video and digital pen recordings were made for the main study, from which the analysis presented below is derived. Interview protocols for the three interviews can be found in Appendices A and B. A Livescribe Echo Smartpen was used throughout these interviews to capture the participants' written work along with their accompanying verbal remarks. Digital pens such as these contain an audio microphone, embedded computer and an infra-red camera affixed next to a ballpoint stylus. When used with specially-made dot-paper, it records what is written with it and synchronizes those notes with the accompanying audio it has recorded. This *pencast* can then be uploaded to a computer for analysis, review, or presentation; and, users may replay portions of the recording by tapping on the notes that correspond to the time the recording was made. During my analysis, this allowed me to align each student's written work with her or his spoken comments and easily locate moments within the interview.

Interviews for the second, third, and fourth semesters were transcribed from the pencasts, and coded as discussed in the Analysis Procedures section.

For the main study interviews, I wished to capture conversational aspects of the participants' responses, such as physical gestures and facial expressions. For this reason, these interviews were video recorded in addition to the digital pen recordings that were made. After transcribing the digital pen notes and audio recordings, the videos were watched simultaneously with the pencasts as a way of checking the accuracy of the transcriptions and providing greater depth to the analytical process. Notes were added to the transcripts whenever the video offered additional information that could not be inferred from the pencasts alone.

3.5.2.5.1 Task Sequences

To take advantage of the three interviews I had with each participant, I established two sequences of tasks within the interview protocols that could be taken together analytically. The content matter for these sequences was clustered around two topics: *open sets* and *continuity*. In both cases, the tasks were designed to become progressively more abstract during the three interviews. The *open sets* sequence began with a task concerning the visualization of open balls in a three-dimensional Euclidean space, progressing to proof tasks involving open and closed sets in the plane, and ending with a task to identify open sets and their derivatives in a new topology. The *continuity* sequence began with a task involving epsilon-delta or limit proofs, then progressed to a task that concerned a function with a Cartesian product for its domain that involved two distinct topologies. The final task in this sequence asked students to investigate the properties of three successive functions, each representationally similar to the last, but with distinct properties and topologies.

The culminations of the open set and continuity task sequences form the subject matter for the two chapters of analysis and results to follow. The first sequence led to an APOS analysis (Arnon, et al., 2014) of the open set concept (see Chapter 4) and the second sequence led to a conceptual analysis (Glasersfeld, 1995) of the participants' understanding of continuous functions (see Chapter 5).

3.5.2.5.2 Role of the interviewer

I approached my role in the interviews as a "teacher-researcher" (Steffe & Thompson, 2000), participating with my participants as they reconstructed their understanding via the tasks. As the "more knowledgeable other" (Vygotsky, 1978) in the task situations, I attempted to guide my participants to establish a coherent record of their conceptions at the time of the interview. For example, I would ask for clarification on unclear claims, arguments, or illustrations they made; point out contradictions in their reasoning in relation to previous statements or task solutions; and prompt them with advice or the textbook if they could not proceed without help. The purpose of this interaction was to observe the participants as they were in the process of constructing meaning about the task content, at whatever level of understanding they indicated having.

3.5.2.5.3 First interview procedures

Nine participants were selected to take part in the first interview in the two weeks following the first exam. Each interview was approximately ninety minutes long, and consisted of two parts: a semi-structured series of questions and a set of proof/justification tasks (see Appendix A). The questions in the first part of the interview dealt with the participants' conceptions of the following constructs:

- functions,
- limits of functions,
- continuous functions in real number and metric space contexts,
- open and closed balls in real number and metric space contexts, and
- boundary and interior points in a metric space.

My goal as the teacher-researcher in this process was to elicit the state of the students' conceptual structure for these concepts as they existed at that moment, somewhat early in the class curriculum, to see how their prior conceptions played a role in initially grasping this new context. Participants were asked to give definitions and examples of the concepts and were queried about the relationship between the examples and the definitions they had provided. Each participant's responses were validated as representing the student's understanding of the concepts—they were never corrected or given explicit affirmation that their mathematical activities were formally correct, unless it was deemed necessary for the continuation of the task. However, if any of a participant's statements were in contradiction with another, or if their definition and examples were not aligned in some way, I would seek clarification with the goal of eliciting a coherent conceptual structure, declared with conviction by the student.

Once the participants felt they had clearly elaborated their conceptions about each of the content topics, they were given three multi-part proof/justification tasks involving the following activities:

- identifying continuous functions in real number contexts;

- using continuous functions in real number and metric space contexts; and
- identifying boundaries, interiors, and open/closed sets in metric space contexts.

As the participants worked through these tasks, I remained an active participant in my role as a teacher-researcher, asking clarifying questions and guiding students to notice and resolve contradictions in their reasoning, examples, or use of definitions. If a participant became stuck or was unable to continue a task, I offered guidance to help get them back on track, which included explanations of concepts or definitions, or by suggesting and permitting the use of the student's class notes or textbook as an aid. When necessary, the parameters of a task were altered so that the setting was more concrete or familiar to the student.

3.5.2.5.4 Second and third interview procedures

Seven participants were selected to take part in the second and third interviews, which took place after the second exam and in the weeks leading up to the final exam. The goal of both interviews was to elicit the participants' understanding of various concepts within non-standard topological contexts. These interviews were exclusively devoted to proof/justification tasks (see Appendix B) and addressed the following topics:

- open/closed sets in the Euclidean plane;
- boundary, interior and, open/closed sets in the lower half-open interval topology;
- continuous functions from the real number line to the plane, with the standard and lower half-open interval topologies; and
- the relationship between continuous functions and connected subsets.

Due to differences in pace and conceptual direction between the individual interviews of the seven students, some of the topics were either addressed during the second or third interview for the different participants. Nevertheless, the order and presentation of the questions remained the same. They are presented as a single document in Appendix B.

As in the first interview, I remained an active participant in my role as a teacher-researcher during these interviews. I continued to ask clarifying questions and guided students to resolve contradictions that arose, urging them to complete the task to an extent that would satisfy the professor on a class assignment, quiz or test. If a participant was unable to complete a task, I continued to provide explanations of concepts or definitions, or suggested the use of their class notes or textbook, on a limited basis, to facilitate the participant's progress.

Again, the parameters of a task were sometimes modified so that the setting was more concrete or familiar to a student if he or she was unable to complete it. For instance, when a participant was unable to reason effectively about open balls in two- or three-dimensions, I asked them to modify the problem and reason in a lower dimensional setting. If a student was unable to remember a definition or theorem after several attempts, I asked them to use their textbook or notes as a guide. My goal was to match the level of difficulty of the task with each participant's apparent level of understanding by increasingly aiding their progress only when necessary to continue the task.

3.6 Setting and Participants

To answer my research questions, I used the criteria developed during the preliminary studies to select theoretically interesting participants for a series of task-based interviews. I then used data from these interviews to build cases of student reasoning around the categories in the coding paradigm.

3.6.1 Class and participants

The cases of student reasoning that will be presented for the main study occurred during the Fall 2015 semester, and were drawn from students enrolled in Math 4330: General Topology, taught by Dr. S., a full professor at Texas State University. The professor chose Croom (1989) as the guiding textbook for the course, organizing his curriculum around the first six chapters. As such, he took an analytic (metric) approach to the subject, again as described in the textbook analysis (see Appendix E).

Nine students were chosen out of forty-two (thirty-nine agreed to participate) students enrolled in Math 4330 based on: researcher and instructor observations, a written pre-assessment (see Appendix C) and diagnostic survey (see Appendix D), and a review of their work on two quizzes and the first exam. The chosen participants were not intended to be cases for investigation in and of themselves; rather, they were likely candidates for exhibiting theoretically interesting cases of student reasoning. Overall, I sought participants that had a strong understanding of the syntactic and symbolic structure of the mathematical content in the class; but who still indicated that they were having difficulty in the class. In addition to these criteria, I selected a group of students with a diverse range of reasoning styles, to look for multiple and conjunctural causes for commonly elicited behaviors.

3.6.2 Sampling procedures

I screened a pool of thirty-nine participants and progressively narrowed the interview participants down to six. I did this to investigate the conceptions of motivated and communicative students who performed well in the class, but who still found the material challenging. I sought participants who might experience significant cognitive disequilibrium when confronted with new topological situations, but who had tools to work through their confusion and potentially overcome the perturbation. I was not interested in studying students who had not acquired a basic understanding of the fundamental concepts in the class. I was also not interested in studying students who found the material easy, and who did not experience many challenges to their cognitive equilibria.

My method of sampling followed the example of many grounded theory researchers, who “deliberately seek participants who have had particular responses to experiences, or in whom particular concepts appear significant” (Morse, 2007/2010, p. 240). Dey (2007/2010) considered such sampling practices to be the next step for grounded data collection after substantial theory has already emerged (see Appendix E for preliminary studies). He explained that, while theoretical sampling works as a tool for “theoretical exploration,” it is not a tool for “confirmation” or “investigating cases” (p. 186). For these research goals, Dey (2007/2010) claimed that a more nuanced approach to sampling is required, with greater specificity: “Once a theory is ‘up and running’, it is possible to be highly focused and selective in producing further data relevant to the elaboration or refinement of existing categories” (p. 186).

This “highly focused and selective” procedure reflects my sampling method for the main study because my goal was to elaborate the categories within my conceptual framework with specific themes applied to open sets and continuous functions in topological contexts. To do so in the context of a relatively small participant pool required that I choose participants with particular characteristics to extract as much useful data as possible in a limited number of interviews. Therefore, I looked for motivated students with 1) strong communication skills, 2) significant fluency with the symbolic notation and terminology of the class, and 3) a moderate level of theoretical understanding in the class.

To assess the communication skills, symbolic fluency, terminology, and theoretical understanding of the candidates, I reviewed and analyze their preliminary research assessments and copies of the first exams for the entire class. The knowledge assessment and first exam review allowed me to assess the participant pool through a number of theoretical perspectives in the mathematics education literature (see Figure 5) which are discussed in the literature review (Chapter 2). Not each of the perspectives were used for or applied to every participant. Rather, the perspectives outlined in Figure 5 were used as guidelines for selecting participants who had: 1) interiorized personal concept definitions that aligned in some ways with the formal concept definitions; 2) some coherence between their personal concept definitions and other elements of their concept images; 3) a well-populated example space for most of the content areas; 4) semantic mental representations for some or all of the content areas; and 5) a diverse set of tools for representing essential concepts in the class.

Theoretical Guidelines for Participant Selection

1) The candidate's alignment of personal concept definitions with the operational formal definitions (those used by the professor and/or textbook of the class) for the concepts associated with: real-valued functions (especially images, pre-images, and continuity); limits of sequences and function; set theoretical concepts; and open, closed sets/metric balls.
2) The candidate's DMA stage of formalization for the concept of continuous real-valued functions. (Dawkins, 2012; Zandieh & Rasmussen, 2010).
3) The structure and extent of the candidate's example space for each of the mathematical ideas above (Goldenberg & Mason, 2008; Mason & Watson, 2005; Sinclair, et al., 2011).
4) The central metaphor expressed by the candidate for the concept images for functions, limits, and continuity. (Lakoff, 1987; Lakoff & Nuñez, 2000).
5) The levels of categorization and basis for reasoning (e.g. exemplars, definitions, prototypes, metaphors) exhibited by the candidate. (Alcock & Simson, 2002; 2011; Hampton, 2003; Lakoff, 1987).

Figure 5. Theoretical research included in the selection guidelines for the main study.

Based on these selection guidelines, I chose nine students to participate in the first round of clinical interviews in the weeks following the first exam. Each of the nine selected participants took part in a one-on-one task-based interview lasting approximately ninety minutes, which included a series of questions and proof-related tasks (see Appendix A). From the nine students who were initially interviewed, two were excluded from further participation because their characteristics were judged to overlap with those of other participants, and their inclusion was considered redundant. Therefore, seven of the original nine participants took part in the last two interviews—one after the second exam for the class, and one in the weeks leading up to the final exam. These interviews were exclusively devoted to proof/justification tasks (see Appendix B). These interviews were intended to address the following topics:

- open/closed sets in the Euclidean plane;

- boundary, interior and, open/closed sets in the lower half-open interval topology;
- continuous functions from the real number line to the plane, with the standard and lower half-open interval topologies; and
- the relationship between continuous functions and connected subsets.

Although seven students participated in all three interviews, only six are reported in this analysis. As the focus of the study was on making comparisons between the ways that students manage their transition to topology, I wished to observe students with at least a basic understanding of the definitions, examples, and proof methods relevant to the course. I chose to selectively sample from these students to avoid studying issues that are common to all mathematical learning, but rather gain insight into the unique challenges posed by this transition. In this sense, one of the original seven participants was judged to have achieved an insufficient level of understanding by the end of the semester to warrant inclusion in the final case analyses. The remaining six participants' responses were used as the basis for the development of case analyses illustrating various ways of thinking about mathematical properties in the transition to axiomatic understanding.

3.6.3 Participant profiles

Based on various data sources, I established profiles of the six interview participants that are reported in the analyses and results. These profiles are intended to give the reader a basic introduction to the reasoning styles and sense-making preferences of the participants based on the researcher's interpretations. The profiles are not meant to be unique descriptions for the students, nor exhaustive in terms of relevant factors for the analysis. The information provided below is based on the Preliminary Assessment and Diagnostic Survey turned in by the participating students (see Appendices C and D). All of the participants were senior-level undergraduate students majoring in mathematics (Nolan was double majoring in mathematics and physics).

3.6.3.1 Saul

Saul had A's in all his undergraduate mathematics courses. His one B grade was in the Introduction to Advanced Mathematics course, which he said was class he enjoyed the most because of what he learned despite its difficulty. His least favorite class was Differential Equations. He demonstrated fluency with the relevant terms and definitions for the topology course, even though he had not yet taken an analysis class at the beginning of the semester of the study. He could interpret formal-symbolic notation well, and his responses reflected an accurate understanding of the definitions. He claimed to frequently use and appreciate visualizations; and said that it was fairly difficult to memorize definitions. He said that he only occasionally adhered to the definition when proving mathematical statements. His definition for a continuous function reflected the properties of both a connected graph, and those associated with the limit definition. His representation for an 'image' was a complete graph and his representation for a 'pre-image' was accurate.

3.6.3.2 Wayne

Wayne's grades ranged from A's to C's in his undergraduate mathematics courses. He had done well in Differential Equations and Analysis I; but had apparently had difficulties in his advanced calculus courses. His favorite class was Differential Equations, while his least favorite was Calculus II. He demonstrated some fluency with the formal-symbolic notation and his responses reflected an fairly accurate understanding of the definitions. He claimed not to use or appreciate visualizations; and said that he had difficulty memorizing definitions and only occasionally adhered to the definition when proving mathematical statements. He did not respond to the question about his definition for a continuous, real-valued function. His representation for an 'image' was a complete graph but he did not attempt to draw the 'pre-image'.

3.6.3.3 Amy

Amy's grades in her undergraduate mathematics courses were mostly A's, but she did have a C in Calculus III and a D in Differential Equations. She had taken both Analysis I and Analysis II and done well; but claimed to have had difficulties with the professor's teaching style in her Differential Equations class. She seemed to demonstrate a syntactic understanding of the formal-symbolic notation and her responses reflected an apparently rote understanding of the definitions. She claimed to frequently use and appreciate visualizations; and said that she had little trouble memorizing definitions. She claimed to always adhere to the definition when proving mathematical statements. Her response to the question about her definition for a continuous, real-valued function was simply that the function was "continuous at each point in the domain." Her representation for both an 'image' and a 'pre-image' were complete graphs of the functions.

3.6.3.4 Gavin

Gavin had received all A's and B's in his undergraduate mathematics courses, except in Calculus III. He did not indicate which of his classes were his most or least favorite, but did well in Differential Equations and Analysis I. He seemed fluent in formal-symbolic notation and the terms for the course, and his continuity definition was a highly accurate version of the epsilon-delta definition of continuity for real-valued functions. He claimed to appreciate visualizations, and said that he usually adheres to formal definitions when proving mathematical statements. He provided no definition for 'pre-image' and drew a complete graph for both the 'image' and 'pre-image' question.

3.6.3.5 Nolan

Nolan had all A's except for one B in his Calculus II class. He was concurrently enrolled in Analysis I during the semester of the study, and claimed to have "loved every math class" he had taken. His fluency with symbolic-notation was apparent, and he seemed to understand the terms, definitions, and set-theoretic concepts well. He claimed to appreciate visualizations and use them whenever he could "imagine such an illustration." He said he always relied on the formal definition as it was "the best place to start," and said it was fairly easy to memorize definitions because understanding leads to memorization. His definition for the continuity of real-valued functions was an incomplete version of the limit definition from calculus, without quantifiers or predicate statements. His 'image' drawing was a complete graph, while his 'pre-image' drawing was accurate.

3.6.3.6 Maren

Maren had both A's and B's throughout her undergraduate mathematics courses and was concurrently enrolled in Calculus III during the semester of the study. She had taken Analysis I, but not Analysis II. She stated that her favorite classes were Differential Equations and Analysis I, and that her least favorite was Modern Geometry. She said that she usually appreciates and uses visualizations, and only occasionally adheres strictly to the formal definition when proving mathematical statements. She said it was fairly difficult for her memorize definitions, and her knowledge of the definitions in the preliminary assessment was not apparently strong. Her definition for continuous real-valued functions was inaccurate and both of her drawings for ‘image’ and ‘pre-image’ were complete graphs.

3.7 Data Analysis

Reporting on the main study took the form of two multiple-case analyses that focused on the forms of student reasoning that had been anticipated by previous iterations of the grounded theory coding cycle (described in Appendix E). This section will describe the methods used to code approximately thirty-five hours of transcriptions from digital pen, audio, and video recordings of the interviews, and the resulting theory that emerged.

My process of interpreting and constructing an integrated theory through these coded transcripts is presented through a systematic narrative of the cases of student reasoning and acts of understanding that I observed during the interviews. From the data presented, I built cases to examine the five central themes of the coding paradigm (see Section 3.3); as well as the relationships between those themes (established by the secondary research questions within the coding paradigm). These themes were then synthesized into two reports with distinct but complementary, analytical frameworks which are described in the next section.

3.7.1 Analytical frameworks

The main study’s results are reported as an interwoven collection of interpretive accounts of my participants’ mathematical activities during both the tasks and semi-structured portions of the interviews. To interpret those activities, I selected conceptual tools from a multidisciplinary array of theoretical frameworks, each with its own advantages and limitations. I chose to look at the data through these different theoretical lenses, depending on how well their individual perspectives could illuminate specific episodes from the interviews. Two overarching analytical frameworks were used to unite these theoretical constructs: an APOS analysis (Arnon, et al., 2014; Dubinsky, 1991) of participants' *open set* schemas; and a radical constructivist conceptual analysis (Glasersfeld, 1995) of the participants' *continuous function* schemas (see Chapter 2 for theoretical framework).

The results in Chapters 4 and 5 are presented as a "quilt" or "montage" (Denzin & Lincoln, 2008, p. 7) of interpretive accounts that cover two distinct concepts (open sets and continuous functions) from distinct but related analytical frameworks (APOS and conceptual analysis), consisting of episodes of my participants' mathematical activities and my interpretations of them through the theoretical lenses described in this section. The various components of the framework supported each other by broadening the perspective on the participants' ways of connecting their schemas for interrelated ideas; and by providing multiple models for the basis of the participants' reasoning within each of their individual schemas.

For example, the APOS analysis of the open set concept (see Chapter 4) informed the conceptual analysis of the continuous function concept (see Chapter 5) by providing a specific structure for understanding how the participants' thought about the underlying construct used to build continuous functions. With that understanding, I was then able to investigate whether the participants were conceptualizing ideas about continuity via metaphors, metonymies, prototypes, exemplar sets, embodied schemas, or through formal logic and mathematical definitions. This interpretive triangulation allowed my analysis to take shape as a rich description of viable perspectives on student thinking in topology.

I incorporated a blend of multiple theoretical frameworks to reflect each of the five components of the coding paradigm: definitions, examples, categories, metaphors/metonymies, and abstraction. Each of these theoretical constructs was operationalized as a collection of specific forms of communication or mathematical activity that could be observed and interpreted through interactions with the participants, as described in the conceptual framework (see Chapter 2). Transcriptions of these activities were then coded with categories derived from my two theoretical and research programs: 1) radical constructivism and APOS Theory; 2) linguistic theories of categorization and embodied cognition. These are described below as theories that informed this analysis.

3.7.2 Overview of analyses

I report my findings in Chapters 5 and 6 as two individual analyses, based on different theoretical considerations and mathematical content. These were: 1) an APOS analysis of open sets, and 2) a conceptual analysis of continuous functions. Both perspectives are based on a constructivist epistemology and research philosophy. For clarity and brevity, I will distinguish my analyses by content topic rather than methodology. The analysis reported in Chapter 4 will be called the ‘open set analysis’, and the analysis in Chapter 5 will be referred to as the ‘continuity analysis’.

3.7.2.1 <u>APOS analysis of open sets</u>

Chapter 4 is an APOS analysis of my participants’ conceptions about the open set concept and its related notions. I used APOS theoretical constructs (see Chapter 2) as a basic vocabulary for exploring two further dimensions of possible abstractions in my participants' reasoning: the multiple processes that may be encapsulated into a mental object for open sets, and the three levels of set abstraction (i.e., points, sets, or families) on which those processes may operate. This analysis reviewed participants' responses only to Task 3.1(B), but was informed by other tasks in the *open set* task sequence.

My analysis of the participants' open set conceptions focused on two forms of abstraction, which acted as theoretical dimensions of variation between and within the participants' defining and structuring activities. I referred to these forms of abstraction as: 1) *reflective abstraction* (Arnon, et al., 2014; Piaget, 1970) and 2) *predicate level abstraction.* They were found to influence the participants' conceptions for the open set concept, both individually and in coordination with one another.

I emphasize that the cognitive transformations involved in APOS theory are forms of abstraction. Within the APOS framework, the interiorization of actions into mental processes, and the encapsulation of those processes into mental objects, are considered forms of Piagetian *reflective abstraction* (Arnon, et al., 2014; Ferrari, 2003), in which an activity or mental operation is reflected from a "lower cognitive level or stage to a higher one" (Arnon, et al., 2014, p. 6). The reflected operation is then reconstructed and reorganized in a way that results in "the operations themselves becoming content to which new operations can be applied" (p. 6). I will continue to refer to these mechanisms for abstraction through the language of APOS theory.

However, I also differentiated the APOS mechanisms within the emergent theory according to their predicate levels, which is a different form of abstraction. Considering points, collections of points (sets), and *collections of collections* of points (set families) suggests a categorical hierarchy. Each successive layer of such a hierarchy is increasingly abstract, in terms of decontextualization (Ferrari, 2003; Hershkowitz, Schwarz, & Dreyfus, 2001), since its criteria for membership is based on fewer properties than those layers it subsumes (Jacobs, 2004; Sierpinska, 1994). That is, a set may contain many points with different properties (e.g., limit, boundary, or interior points of the set), which are not differentiated through set membership. Similarly, many sets with different properties (e.g., open, closed, connected, or dense) may be members of a single family of sets. Therefore, shifting reasoning from lower predicate levels (e.g., points and sets) to higher predicate levels (e.g., sets and families), was considered one form of abstraction involved in the defining and structuring activities of my participants.

3.7.2.2 Conceptual analysis of continuity

Chapter 5 is a radical constructivist conceptual analysis (Glasersfeld, 1995) of my participants' conceptions about continuous functions, which I will refer to as the *continuity analysis*. It is presented as a retrospective analysis of three continuity tasks embedded in the protocols of a sequence of clinical interviews (Clement, 2000; Garfield, 1981; Piaget, 1970). I used this framework to explore the coordination of my participants' spatial and mathematical intuitions (Fischbein, 1987) with their personal concept definitions for a continuous function (at a point and/or on its domain).

Building on von Glasersfeld's (1995) interpretations of Piaget's theory of genetic epistemology, I modeled the participants' cognitive structures from their own perspectives. The focus of the continuity analysis was to model the sources of my participants' reasoning strategies as they developed and reconstructed their schemes for continuous functions—from their previous conceptions about the continuity of certain real-valued functions (given the standard topology), to new and more general notions of continuity between functions endowed with novel topologies. This focus evolved out of my research interest in exploring the challenges that introductory topology students may face in coordinating their past and present conceptions for a highly experiential and perceptual construct like continuity.

Through this analysis, I established independent cases around the various ways the participants coordinated: 1) their prior mathematical understanding of continuous, real-valued functions; 2) their perceptual experiences with the informal notion of continuity; and 3) the new, abstract conceptual structures that were presented to them during the participating class. I established a distinction between the formal accommodations the participants made to solve the task problem, and the various accommodations they made to explicate their embodied conceptions, and reconcile the perceptual discrepancy involved in the task situation.

3.7.3 Analysis procedures and coding themes

With each analysis described above, my primary goal was to explicate dimensions of my participants' use of properties (mathematical and non-mathematical) as they reasoned within a theoretical environment of an increasingly abstract nature. Thus, in terms of Strauss & Corbin's (1998) coding methodology (see Appendix E), the "central category" for my overall analysis is described as *property-use in abstract reasoning*. Specific coding themes used as sub-categories included the five categories of the conceptual framework, which involved the participants' *defining, structuring, generating, embodying,* and *abstracting* activities during the tasks.

Each of these themes were used as lenses for coding the transcripts 'episodically'. For the purposes of coding, an episode consisted of a 'train of thought', in which a participant sought and achieved some short-term or long-term goal within the larger task structure. Episodes could be of any duration, and were chosen for analysis because they were exemplars of common or distinctive behaviors among the different participants' responses to similar tasks or situations. Each episode was also coded within the individual framework chosen for that chapter's analysis. Episodes were screened multiple times, with different thematic lenses employed each time. The most theoretically insightful episodes were presented from what I judged to be the most insightful theoretical perspectives.

For the open set analysis, in addition to the coding themes above episodes reviewed were also screened with three broad coding categories: 1) APOS *structures* (e.g., action, process, object, schema) and *mechanisms* (e.g., interiorization, encapsulation, de-encapsulation, coordination); 2) *conceptual approach* to defining the open set idea; and 3) the *predicate level* (i.e., points, sets, or families) at which reasoning about the conceptual approach takes place. For the continuity analysis, additional coding categories included: 1) *perceptual representations* used by the participants, and 2) *accommodations* used to *formalize* or *explicate* the participants' understanding.

Finally, the fractured perspectives gathered through these different analytical lenses were re-assembled into coherent and plausible explanations for the mathematical activities of my participants, using theoretical codes from a variety of research fields. Themes were integrated where possible; but discrepant cases were examined with interest. In this way, I offer a rich narrative that compares cases along many dimensions, but highlights contradictory findings as well (see Chapter 6). This provides the space for more dialectic interpretations of my findings, and leaves many questions open for further study.

4. APOS ANALYSIS OF THE OPEN SET CONCEPT

4.1 Introduction to the Open Set Analysis

The analysis reported in this chapter was part of a larger study designed to answer the following research questions concerning students in an introductory topology class:

1) What distinctions and comparisons can be made between the ways that students manage their transition to an axiomatic understanding of continuous functions?
2) What obstacles do students face during this transition?

In topology, the continuous functions are a class of functions with a complex defining property, which is canonically expressed using the mechanism of the *open set* concept (explained below). To answer my research questions, it was necessary to explore my participants' ways of thinking about open sets as a vantage point from which to investigate their understanding of continuity. To accomplish this, I chose to conduct a modified APOS analysis (Arnon, et al., 2014) of the open set concept. My analytical methodology was modified from the standard APOS methodology by focusing on the multiplicity of interrelated processes that were used to encapsulate the open set, and by emphasizing the differences in logical format between those processes. I referred to this as the *open set analysis*.

I will answer the first of my research questions through the open set analysis by providing evidence to distinguish between the defining and structuring activities I observed in my participants' responses as they tried to make sense of the open set concept in a novel topological context. As an answer to the second research question above, I will show how various didactic, logical, and epistemological obstacles constrained my participants' attempts to build abstract formal schemas and conceptual hierarchies around their diverse conceptions for open sets.

4.2 Purpose of the Open Set Analysis

The purpose of the open set analysis was to examine factors related to two forms of abstraction that were observed to influence my participants' reasoning about the open set concept. These were: 1) the potential to encapsulate multiple *processes* into separate object conceptions within a participant's personal conceptual schema for open sets; and, 2) the existence of multiple set-theoretic category levels on which those processes could be completed (i.e., the logical *predicate levels* of points, sets, or set families).

Combinations of these two factors were observed during the participants' reasoning about a proof task that involved a topology with which they were unfamiliar. For some of the participants, their reliance on familiar instantiations of the open set concept was found to create both obstacles and affordances for reasoning about open sets and continuous functions in abstract settings. Meanwhile, some participants' use of the axiomatic definition of open set seemed to reveal a more integrated and holistic conceptual understanding of the concept. Below, I provide four distinct, grounded genetic decompositions of the open set concept, varying along the theoretical dimensions of processes and predicate levels. These will then be used to develop a decomposition of the developmental stages of the participants' overall schemas for open sets.

4.3 Mathematical Background and Tasks for the Open Set Analysis

The axioms of topology define a special family of subsets in a topological space, the members of which are called the *open sets*. The concept of the open set is the fundamental basis for defining every other idea in general topology, including the important notion of a continuous function. Every definition presented to my participants during the semester-long study (both in the classroom and textbook), either directly referred to open sets or indirectly did so through a chain of linked definitions. Due to this concept's central position within the axiomatic structure of topology, I began my analysis with an examination of the ways my participants understood and reasoned with the open set concept. I examined the ways that they personally defined open sets, how they used open sets in reasoning and proof, and what sort of attributes they assigned to open sets in their minds, whether consciously or unconsciously.

Specifically, I report how some of my participants approached the open set concept while reasoning about the open interval $(0,1)$ in the context of the 'lower limit topology'. In this topology, the right half-open intervals $[x, y)$ constitute the topology's basis, altering many well-known properties of the real numbers. Although this is a commonly studied topology, the participants had few experiences with it during their class. For their third interview, in the week leading up to their final exam, the participants were presented with the tasks in Figure 6. Task 3.1(A) is provided for context, because the participants may have been influenced by recently considering the right-half-open interval $[0,1)$ during their work on that task. The main section of the results for this chapter will focus exclusively on the participants' responses to Task 3.1(B); and I will highlight a special case of reasoning at the end of this chapter, in which Gavin reasons about Task 3.1(A).

3.1(A) Find the interior and boundary of [0,1) in the half-open interval topology.

3.1(B) Prove that (0,1) is an open set in the half-open interval topology.

Figure 6. Two tasks from the third interview. Task 3.1(B) is the focus of the results and analysis in this section. Note: through classroom observations, I was aware that the students and professor of the participating class referred to the lower limit topology as the "half-open interval topology," resulting in my use of the phrase in the task prompts.

4.4 Analytical Framework for the Open Set Analysis

In this chapter I report on an APOS analysis of my participants' conceptions of the open set concept in abstract contexts ("abstract" in contrast with their prior real number instantiations of the open set concept). This analysis resulted in four major *genetic decompositions* (Arnon, et al., 2014, p. 27) based on the distinct actions and processes my participants used to develop their mental schemas about open sets and related ideas. I made no attempt to construct a "preliminary genetic decomposition" (Arnon, et al., 2014) of the open set concept. I chose to enter my analysis of the participants' activities with as few preconceptions as possible based on my own understanding.

In Section 4.6, I present the resulting genetic decompositions as conceptual approaches for defining the open set concept. I will describe variations I observed within each decomposition, based on the predicate level (points, sets, or families) attended to via the approach. The APOS theoretical construct of *schema* will play a central role in the open set analysis, which will be distinguished Finally, I will show how differentiations between these defining approaches and accompanying predicate levels combined to influence which stage of schema development seemed to be achieved by my individual participants as they built structures for their understanding.

4.5 Overview of Results for the Open Set Analysis

In this section, I provide a summary and overview of the analytical themes for the open set analysis, which were centered around two distinct but related forms of abstraction involved in my participants' defining and structuring activities. These included *reflective abstractions* directed toward actions and processes, and the *predicate levels* at which they occurred. A combination of factors involving these two types of abstraction played a central role in my participants' reasoning.

4.5.1 *Alternative processes for encapsulation*

The first form of abstraction I report involves the existence of multiple, distinct mental actions that my students must interiorize and then encapsulate as they rebuild their open set conceptions in terms of the topology axioms. I will demonstrate how my participants made use of three separate formulations of the open set concept as they defined and reasoned about open sets while responding to Task 3.1(B) in Figure 6. Each of these logical approaches to defining the term can be identified with one of three characterizations of open sets described in the textbook for the participating class (Croom, 1989):

1) the *axiomatic* definition of open set as a member of the topological family of sets (p. 99);
2) Theorem 4.3(2), which states that a subset A is open if and only if it is equal to its *interior* (p. 103); and
3) the definition of a closed set as the *complement* of an open set (p. 100).

These three ways to define the open set concept were presented in both the textbook and during observed classroom instruction. I made a distinction between the participants' personal conceptual approaches and the underlying mathematical characterization on which the approach was based. The conceptual approaches helped to build a participant's conception of open sets, while the mathematical characterizations define and elaborate the concept itself within formal mathematical theory. I interpreted the ongoing development of my participants' conceptions based on these characterizations, which I classified as the participants' *defining* activity (see conceptual framework, Chapter 2).

The conceptual approaches I observed for defining the open set were referred to as: 1) the *axiomatic approach*, 2) the *interior approach*, and 3) the *complement approach*. The use of each of these approaches to build a mental conception of abstract open sets requires the encapsulation of specific processes for determining whether a given set should be considered open. I will show how this multiplicity of processes influenced the development and coherence of my participants' abstract open set schemas. In Table 4, the three conceptual approaches are outlined regarding the mathematical characterizations they are based on and the cognitive actions and processes they require.

In the analysis below, I will describe the distinct actions and processes my participants seemed to have developed to form an open set object conception around their personal concept definitions. The participants' conceptions will be shown to vary according to the mathematical concepts they were based upon. Since each approach could potentially be used to define the concept of an open set, they are all logical ways to think about open sets in new situations. Participants were observed to vary and adapt their approaches to defining the open sets as they deemed necessary. Some approaches permitted more affordances for appropriate reasoning than others; but each approach also created its own specific obstacles, encountered by multiple participants throughout the study.

In Table 4 below, the three conceptual approaches presented in the open set analysis are summarized, along with the mathematical characterizations that serve as the basis of the approaches.

Table 4: Conceptual approaches to the open set

Conceptual Approach	Mathematical Basis	Mental actions required to show a subset $U \subset X$ is an open set in a given topology
Axiomatic	Definition of open set (Croom, 1989, p. 99)	Show that U belongs to the topology, i.e., U is: (i) $\varnothing$ or X (ii) a union of a family of open sets, or (iii) a intersection of finite family of open sets
Complement	Definition of closed set (Croom, 1989, p. 101)	Show that the complement $X \setminus U \subset X$ is a closed set, i.e., every point $x \in X \setminus U$ is a limit point, so that for each neighborhood V of x, $V \cap \left(X \setminus U \right) \setminus \{x\} \neq \varnothing$
Interior	Theorem 4.3(2) (Croom, 1989, p. 103)	Show that every point $y \in U$ is an interior point, i.e., there is a neighborhood W of y such that $y \in W \subset U$

4.5.2 Predicate levels for the approaches

I will show how these three approaches for defining open sets also varied in a different sense. I found that predicate levels represented a second form of abstraction that had influenced the development of my participants' open set schemas, from the different categorical levels used in the logical expression of the definitions, theorems, and axioms the participants had studied. In contrast to first-order predicate statements about the elements (points) of a space, a mathematical statement about a *set* of elements requires abstraction over second-order predicates; while statements about *set families* (sets of sets) represent abstraction over third-order predicates, and so on (Kleene, 1967). My use of the term ‘predicate level’ was in reference to the study of these higher-order predicate statements in logical theory.

I will show how some of the differences I observed between my participants’ use of the three approaches (see Section 4.6) may be attributed to the predicate levels required for their use. For example, the axiomatic approach requires a student to reason with third-order logical statements involving the unions and intersections of set families. Therefore, this approach provided little benefit for participants attempting to reason at lower predicate levels (i.e., with points or sets). On the other hand, the complement and interior approaches are both stated in second-order terms involving sets rather than set families, which provided participants with a less complex logical structure to consider. But these two approaches also differ, in that the interior approach offers students a salient first-order proof strategy involving the set’s interior points. While the complement approach may be considered at the point level as well, this can only be achieved through coordination with a second theorem from Croom (1989, p. 101), which states that a closed set contains all its limit points. The compound requirement to use this theorem along with the definition of closed set may have obscured participants’ opportunity to shift their reasoning from the level of sets to that of points when using the complement approach.

Table 5: Predicate levels observed for the conceptual approaches

	Point-Level	*Set Level*	*Family Level*
Axiomatic	Set is a union or finite intersection of open sets constructed around individual points	Set is a union or finite intersection of open sets constructed from a collection of sets	Set is a recognized member of the topology
Complement	Set contains all its limit points	Set is the complement of a recognized closed set	N/A
Interior	Set consists precisely of its interior points	Set is recognized as its own interior	N/A

4.6 Open Set Analysis

I will present cases of reasoning observed during work on Task 3.1(B) by each of the participants. Gavin's response to part of Task 3.1(A) will also be presented as a highlighted case to provide specific points of evidence targeting claims I will make in the analysis. Symbolic notation has been added to the transcript data to clarify the meanings of ambiguous verbal statements, such as the names of open, closed, or half-open intervals, etc. Numbers that are used in the context of the real number line are written as Arabic numerals; but outside of this context, they are written out in words. Short pauses are notated with ellipses…while longer pauses (>10 seconds) are notated with double ellipses……Deleted remarks are indicated with a bracketed ellipse […]. Pseudonyms were used throughout.

4.6.1 Evidence for the axiomatic approach

Croom (1989, p. 99) defined an open set to be any member of a "topology," which is a family of sets for which three defining axioms hold. These axioms require that the following sets are considered "open" (members of the topology): 1) the union of any family of open sets, 2) the intersection of any finite family of open sets, and 3) the entire space and the empty set. The third axiom is the trivial condition for the existence of a topology but the other two impose significant structure on such a family of sets. To encapsulate the *axiomatic* open set conception, a learner must first interiorize a process for determining whether a given set may be constituted as an arbitrary union or finite intersection of open sets (typically of members of a ‘basis’ for the topology being considered). This process must then be conceived by the learner as a single entity—a conceptually unified family of sets for which the relationships defined by the axioms hold—to be encapsulated as an *axiomatic* open set object.

Although the axiomatic definition was the first topological formulation for open set provided in the textbook (Croom, 1989), only two of the five participants that are reported in this section attempted to use the definition on their own; while a third student used the definition successfully, but only after consulting his textbook. Other students made limited references to open sets as the unions or intersections of “basis” elements in other contexts, but only two students used the approach effectively, and only one of them without any aid.

4.6.1.1 <u>Saul’s use of the axiomatic approach</u>

Saul was the most successful participant in terms of using the axiomatic approach, which he did exclusively on the second predicate level of sets. To prove that $(0,1)$ is an open set in the lower limit topology, Saul immediately began to reason in terms of the unions of basis elements (i.e., right half-open intervals):

> *Saul:* Right, um……Well, I would need to be able to construct that interval using half open intervals…
>
> *Interviewer:* What do you mean by construct?

> *Saul:* Uh, show that it's a, uh, a union of half-open intervals. Because if that's true, and the half-open intervals are open, then any union of open intervals is…I guess, by definition, an open interval in the topology.
>
> *Interviewer:* Or, an open set, right?
>
> *Saul:* Open set!

Although he accidentally used the term "open interval," as the researcher it was clear that he intended "open *set*" based on previous discussions during the interview. For example, earlier he had stated that he continued to make this mistake because of his prior experiences with the real numbers, and his need to visualize the definitions: "part of it is I have to picture it in my head, and I'm picturing the reals, in order to have something to grasp. But right, a *set*." In this case, Saul managed to reason appropriately with the open set concept in the lower limit topology, even though his terminology reflected the standard topology within his conception of the idea.

Regardless of issues with his terminology, Saul managed to complete the task based on his prior understanding from his real analysis classes. When asked how he might construct the interval $(0,1)$ out of right half-open intervals, he responded by inscribing the following:

$$\bigcup_{n=2}^{\infty}\left[\frac{1}{n},1\right)$$

This formula relies heavily on Saul's understanding of nested intervals, infinite unions and convergent sequences of endpoints, which he had indicated he learned about in his real analysis classes. His prior knowledge seemed readily accessible for this purpose, and he used it to effectively complete the task.

4.6.1.2 Amy's use of the axiomatic approach

Amy's initial attempt to reason about Task 3.1(B) was also based on the axiomatic definition for open sets, although she apparently lacked an interiorized process conception for infinite unions. In our first interview, Amy had expressed that she would not feel satisfied with any proof unless it was based explicitly on the formal definition. Therefore, it seemed that her motivation for using the axiomatic approach was not based on any perceived affordance or economy of thought. Rather, she stated that to use an alternative approach would be "kind of like cheating." For instance, although she expressed an awareness of the interior approach, she claimed that "you're not supposed to know that yet; or, you should be able to do it just like this [using the formal definition] first."

During Amy's attempt to follow the *axiomatic* approach, she accurately named the processes involved in the axiomatic definition for open set:

> *Interviewer:* Do you know the definition [of 'open set in a topology']?
>
> *Amy:* Yeah, the empty set and then the set that you're talking about, like for instance $(0,1)$. Um, those would have to be, show that they're open. And then show that, like…an arbitrary union of open sets is an open set. And then, the third one, a finite intersection is open.

Amy demonstrated her awareness that using the axiomatic definition would require a set of actions to "show that" certain types of sets were open. However, she was unable to use this definition effectively to complete the task. Despite her knowledge of the definition's form, she did not demonstrate fluency in the required manipulation of unions and intersections of set families to construct the interval $(0,1)$ in the lower limit topology:

Interviewer: So is there a way you can express $(0,1)$ as an arbitrary union or a finite intersection of sets that look like this [indicates $[a,b)$]?

Amy: ……Hmm…well……the union of, because I'm thinking about like, you know like, one right here, one right here, and then you keep- [indicates a progression of sets inward towards the center], it's always going to look like that, no matter what, or-

Amy seemed to be considering a descending sequence of nested intervals here, and could not find any means to construct an open interval in this way. She seemed to concentrate on finding a strictly increasing sequence of left-hand endpoints, without considering the more appropriate notion of a convergent, decreasing sequence of minima. Based on her drawing it is also possible that, although she used the term *union*, she was erroneously imagining a nested *intersection* instead. While finite intersections are one of the conditions in the axiomatic definition of *open set*, it is not possible to construct an open interval from a finite intersection of intervals with closed endpoints.

When she seemed to have become stuck in her attempt to use the *axiomatic* approach, I suggested that she consider unions in addition to intersections and she made another attempt. However, this time she seemed to restrict her attention to unions of disjoint intervals:

Interviewer: For the union, you'd get the biggest set you could make, right?

Amy: Right. So, like even if it's just one here and then like, one here, you're always going to have closed and open. But, we consider *that* open [indicating one of the half-open intervals she had drawn], that's what confuses me…

If Amy had considered a family of non-disjoint intervals, she may have found a way to accomplish the task. In this case, she could also have chosen a family of disjoint intervals to form the necessary union and generate the interval $(0,1)$. She seemed not to have interiorized the actions involved in forming an open set from an infinite union of closed sets. Therefore, without an interiorized process conception to work from, she was ultimately unable to continue further with the *axiomatic* approach or construct the appropriate union of basis elements to form the interval $(0,1)$.

4.6.1.3 Gavin's use of the axiomatic approach

Another student, Gavin, used the *axiomatic* approach, but only after attempting and being unable to complete the task in any other way. He recalled the interior approach for formulating the *open set* concept, but none others. Expressing his preference for reasoning with the *interior* approach, he said "I honestly don't remember the other [definitions], because I used the interior and the closure…" However, he was unable to use the *interior* approach to complete the task (see Section 4.6.5.4). Since he was apparently unable to continue, I prompted Gavin to use his textbook (Croom, 1989) as an aid, where he found two versions of the axiomatic definition for open sets—one for general topological spaces (p. 99), and an earlier version that was introduced for metric spaces (p. 64). He seemed to prefer the metric space version, which is identical to the topological version, except that open sets are defined with a basis of *open balls*, rather than basic open sets.

Gavin's responses to other tasks (see Section 0) had demonstrated that his mental conception for the open set construct was strongly tied to Euclidean spaces and his prior understanding from real analysis. So, it was unsurprising that he adopted the language of the metric space definition for the remainder of this task. However, there is potential for a significant logical obstacle in making this choice since the real number line with the lower limit topology is not a metrizable space. That is, the lower limit topology cannot be generated by any metric; and therefore, reasoning that relies on open balls, whose radii are necessarily defined by a metric, could have proven difficult or impossible.

This obstacle may have contributed to Gavin's difficulty in applying the definition. While he was able to use the notion of an open ball as a mental prototype for the open set concept, the Euclidean properties of the open ball continued to interfere with his reasoning about open sets in the lower limit topology.

> *Gavin:* Yeah, like with this, it could be the union of open balls around all of these points, you know, from 0 to 1…
>
> *Interviewer:* And now in this case we're in this [lower limit] topology right?
>
> *Gavin:* Yeah, that's the only, um, so I guess the part I don't know is if it has to include *a*, or this lower limit, to still be open.

He struggled to understand the role of the lower limit, or minimum, in intervals of the form $[a,b)$, and could not relate these intervals to his notion of an open ball in a Euclidean space. He remained fixated on the interval's symbolic form and didn't relate the "open balls" in the definition to the basis elements $[a,b)$ of the topology, so I directed his attention back to the definition he had just read:

> *Interviewer:* Can you think of a way to write that set [$(0,1)$] as a union of these half-open intervals?
>
> *Gavin:* No, so it wouldn't be-, oh okay, since it's not [open], since I can't, since there's no union of $[a,b)$ that equals $(0,1)$.

$$\bigcup_{i \in I} [a,b)_i \neq (0,1)$$

It wasn’t immediately obvious to Gavin how he could construct $(0,1)$ out of a union of half-open intervals, and I prompted him again to consider the use of an infinite union.

> *Interviewer:* So what if it was…an infinite union, those are allowed right?
>
> *Gavin:* […] I guess if the…if this lower [limit], if this *a* approached 0, and you kept taking the union of those as it gets closer and closer and closer…
>
> $$a \to 0$$
>
> *Interviewer:* You think that might work? Like what would be a sequence like that?
>
> *Gavin:* Um, 0…1 over epsilon…so, you have epsilon, and then you just have 1, and then…if epsilon keeps approaching infinity, then it’s just going to get closer and closer to 0
>
> $$\bigcup \left[0 + \tfrac{1}{\varepsilon}, 1\right) \quad \varepsilon = (1, 2, \dots, \infty)$$
>
> *Interviewer:* […] Does that seem to convince you that it would be open? Or do you think there’s something that you’re missing there?
>
> *Gavin:* I think if it was an infinite union then it would be open.

Although it did not occur to him without prompting, Gavin was able to demonstrate an understanding of the task. Once he was reminded to consider the unions of infinite families of sets, he was promptly able to leverage his prior understanding from real analysis to construct a sequence of nested half-open intervals that satisfied the definition. As Saul had done, he used an *axiomatic* process to build a family of sets, the union of which equaled the original interval. It is likely that Gavin had developed his procedural knowledge in his previous experiences with real analysis. His fluency with these formal techniques from analysis, but hesitation in using them, may indicate that he wasn't aware that he could use infinite unions in this task setting. In fact, Croom's (1989) axiomatic definition of a topology is stated so as only to *implicitly* allow infinite unions: "The union of *any* family of members of $\mathcal{T}$…" [emphasis added]. It is plausible that students might overlook that such a family could be infinite. Similarly, none of the other textbooks analyzed for this study made any explicit reference to infinite unions in the axiomatic definition of open sets and topologies. Unions of infinite set families are used extensively in reasoning and proving in topology, but Gavin and Amy's cases demonstrate that some students may struggle with conceptualizing or using such unions.

4.6.2 Discussion of the axiomatic approach

The *axiomatic* conceptual approach is based on the central organizing structure of the field: the axioms of topology. Both the textbook and the observed classroom instruction emphasized the axioms, and employed them as the basis for defining other concepts, such as the interior, closure, or boundary. Therefore, it was likely a goal of the author and professor for the students to develop facility with the axioms, and to understand other conceptual approaches in terms of the axioms. However, Saul was the only participant who was successful in using the *axiomatic* approach without significant prompting. Amy and Gavin made explicit attempts to use this approach with varying levels of support and success; and further on, Wayne's use of the *interior* approach (see Section 4.6.5.3) will reveal his implicit use of the *axiomatic* approach as well. Whether implicitly or explicitly, the *axiomatic* open set object conception was observed to take on three different forms, which I categorized by the separate predicate levels at which their underlying processes were initiated. (See Figure 7).

4.6.2.1 Pointwise axiomatic approach

In his attempt to apply the textbook's axiomatic definition of open set, Gavin initially referred to a process for constructing a union of open sets involving "…the union of open balls around all of these points, you know, from 0 to 1." This statement indicated the use of one variation of the *axiomatic* approach: to construct a union from a family of open sets, in which each open set corresponds to an individual point in the set. I will refer to this as the *pointwise axiomatic* open set conception. Although he did not continue with this line of reasoning, Gavin stated that he could imagine finding an "open ball" around each point in the interval, and that the union of these open balls would constitute the set. This entails imagining a series of mental actions which involve identifying an open set that contains each point, and which does not intersect the complement of the set. By running through these actions in his mind, Gavin seemed to have interiorized this repeated action into a process conception (see Section 4.8.1.2 for a full genetic decomposition of this approach). Had he indicated that the union of those open sets would be an open set, Gavin would have demonstrated that he had encapsulated the *pointwise axiomatic* open set into an object conception. In the next section, I will contrast this first-predicate level variation of the *axiomatic* approach with other cases in which the family of open sets was chosen at the second predicate level of sets instead.

4.6.2.2 Setwise axiomatic approach

Saul appeared to follow the *axiomatic* approach when he invoked a nested family of intervals $\bigcup_{n=2}^{\infty}\left[\frac{1}{n},1\right)$ whose union was $(0,1)$. This response is representative of the second variation on the axiomatic approach I observed, in which the union is constructed out of a family chosen at the second predicate level of sets, rather than the first predicate level of points. Therefore, I will refer to this as the *setwise axiomatic* approach. Saul's statement that he "would need to be able to construct that interval using half open intervals…" indicates that he could imagine building an infinite sequence of nested intervals, and that he recognized that the union of these intervals would constitute the interval in question. These actions that Saul imagined performing were quite different than those imagined by Gavin above for the *pointwise axiomatic* approach. For Saul, the intervals were chosen specifically for their ability to cover the interval $(0,1)$, without reference to each of its points. He called on his prior schemas for sequences and their limits to construct a family of sets; and expressing this family as a union completed his line of reasoning since "any union of open intervals is…I guess, by definition, an open interval in the topology." These statements indicate that Saul had encapsulated a process conception for open sets involving the union of a setwise constructed family of sets.

While Saul could use his conception without the need to consider each individual point in the set, he was significantly aided by the relatively sophisticated mathematical technique described above, which he recalled from his real analysis classes. It seems unlikely that the setwise axiomatic approach would be successful for any student without some such conceptual tool. For example, Amy was unable to complete the task through the axiomatic approach without having developed this fluency with sets and set operations. Meanwhile, Gavin implicitly adopted this line of reasoning when he was asked to write $(0,1)$ as a union of open sets. Despite having used other approaches before, his written response indicated his use of the *setwise axiomatic* approach when he inscribed a union of a nested sequence of intervals: $\bigcup\left[0+\frac{1}{\varepsilon},1\right) \;\; \varepsilon=(2,\ldots,\infty)$.

Predicate Level		Axiomatic Approaches		
		Pointwise Axiomatic	*Setwise Axiomatic*	*Familywise Axiomatic*
	Family	*Object:* open set as the union of a family of sets constructed pointwise $\bigcup_{x\in O}[x,1)$	*Object:* open set as the union of a family of sets constructed setwise $\bigcup_{n\in\mathbb{Z}^{+}}[\frac{1}{n},1)$	*Object:* open set as a recognized member of the family of the topology $[0,1)\in\mathfrak{T}_{\mathrm{L}}$
	Set	*Process:* Construct a family of basic open subsets around each point in the set $\{[x,1)\}_{x\in O}$	*Process:* Construct a family of basic open sets by defining criteria, e.g., a convergent sequence of left-hand endpoints $\{[\frac{1}{n},1)\}_{n\in\mathbb{Z}^{+}}$	N/A
	Point	*Action:* Identify a basic open subset around individual points $[x,1)\subset(0,1)$	N/A	N/A

Figure 7. Axiomatic approaches to the open set concept. Alternative predicate levels may serve as the basis for a student's axiomatic approach to defining an open set. Arrows indicate pathways for abstraction to the next predicate level, which includes the interiorization or encapsulation of the action or process involved at the lower predicate level.

4.6.2.3 Familywise axiomatic approach

Figure 7 also references a *familywise axiomatic approach*, which was evident whenever a student recognized a set as being a member of the set family comprising the topology through other factors. This recognition may have been based on empirical factors like its representational form (e.g., half-open intervals) or a student's prior knowledge about the topology. I consider this conception to be initiated at the family level since the open set is recognized as a member of the topology set family, rather than as a union or intersection of a sub-family of its members. It was evident that this was the least useful of the axiomatic approaches, as it required participants to recall and recognize objects based on non-mathematical criteria. This approach was demonstrated whenever a participant denoted a set as being open automatically, based on its notational or mathematical form, or other non-mathematical factors.

4.6.3 Evidence for the complement approach

In Croom (1989, p. 100), as in many topology textbooks, a subset of a topological space is defined to be closed when its complement in the space is open. To show that a given subset *O* is open in a topological space *X*, a student might attempt to show that its complement $X \setminus O$ is closed using some other strategy, such as a theorem from Croom (1989, p. 104) stating that a closed set is equal to its closure. Then, by the definition of closed set, the set *O* must be open. On the other hand, by contraposition, a set $O \subset X$ can be shown *not* to be open by proving that its complement $X \setminus O$ is *not* closed. The logical complexity of the complement approach may explain why only one participant attempted to use this definition to verify that $(0,1)$ is an open set.

To verify that a given set is open with the *complement* approach will require a participant to coordinate multiple process conceptions resulting in three mathematical activities: 1) identifying the complement of a set, 2) identifying individual limit points of a set, and 3) verifying that a set contains all its limit points. Considering that two of these activities involve reasoning with and about the set's limit points, it would be difficult for a student to use this approach without reasoning on the first predicate level to consider individual points of $(0,1)$ and its complement. This difficulty was evident in Nolan's initial attempts to use the *complement* approach.

4.6.3.1 Nolan's first attempt to use the complement approach

Nolan was the only student who attempted to use the *complement* approach to defining an open set. Although he was unsuccessful at first, he would later revisit this approach and successfully arrive at a solution. During the task, Nolan began to use the *complement* approach, but initially mistook its underlying process and tried to verify that the complement of the interval $(0,1)$ was the union of two open (rather than closed) rays:

$$\mathbb{R}-(0,1)=(-\infty,0] \leftarrow \text{open}$$
$$\cup$$
$$\text{open} \rightarrow [1,\infty)$$

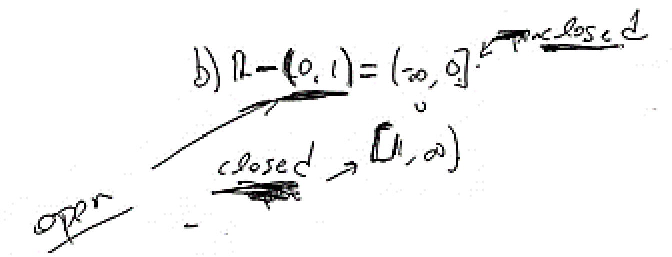
He quickly recognized the issue, saying "ah, I just got that backwards didn't I? I just found that its complement was a union of open sets, meaning that it's closed…" Since he did not seem to attend to individual points in his reasoning, I concluded that he was operating with the *complement* approach on the set level, for which he relied on unexamined assumptions about sets of the form $(-\infty,a]$ and $(b,\infty]$. After recognizing his initial error, he was still unable to use the processes involved in the necessary theorem on limit points and closed sets, and seemed unsure how to go about showing that a given set was closed. His reasoning seemed to remain focused on the *sets* in question, the intervals $(-\infty,0]$ and $[1,\infty)$, rather than on the underlying notion of limit points that would have provided him with a concrete, first-order proof strategy.

Despite catching his mistake quickly, Nolan did not try again to use his initial approach. He was apparently unsure about his understanding of the complement definition for open sets, or did not have a strategy for showing that the complement was a *closed set* instead. Rather, he switched tactics, opting to use the *interior* approach instead (see Section 4.6.5.1).

4.6.3.2 Nolan's second attempt to use the complement approach

Although he had already solved the task using the *interior* approach, later in the interview Nolan revisited the complement approach, and went on to leverage his successful strategy from the *interior* approach to determine whether a set was open.

Interviewer: You'd want to show that these two sets are closed sets right?



Nolan: Mhm…I know $[1,\infty)$ is closed, right? This is clopen…

Interviewer: Okay, so it's closed because…

Nolan: Uh, it contains its limit points. Uh, 1 is a limit point of it, right?

Interviewer: And it contains all the rest of them?

Nolan: Yeah, everything, yeah. Um, so this guy…uh, [indicates $(-\infty,0]$]…I think it's closed as well, for the same reason…uh, yeah. Because 0 is a limit point of it.

Interviewer: 0 is a limit point of $(-\infty,0]$?

Nolan: Yeah, I think so…No! Because here's an open set that intersects it that contains only 0 [indicates $[0,1)$]. Um…so 0 is not a limit point.

Interviewer: Ok, but does $(-\infty,0]$ contain all of its limit points?

Nolan: Right, mhm, right. It can contain a point that's not its limit point and still be closed. But…so the question would be like, is 1 a limit point of it [the interval $(-\infty, 0]$]. And for the same reason, no, it's not. So, then is like, -1 a limit point of it? Yes. But everything [between] -1 and 0 would also be a limit point of it, which will all be contained in that set. So I guess I could just do an open set around it. So now I'm pretty convinced that that's closed as well then.

Nolan experimented with several points that were outside and inside of the interval $(-\infty, 0]$, testing his reasoning against the rules of the topology. Nolan's activity was different than I had observed in his first attempt to use the *complement* approach. In his first attempt, he had confined his thinking to the second predicate level of sets. Thus, he had looked to non-mathematical properties as his classification criteria for the open set construct. For example, without explanation or justification, he had labelled both $[1, \infty)$ and $(-\infty, 0]$ as open sets because of their superficial similarities to the form of the generic right half-open interval $[a, b)$. This is an example of the *familywise axiomatic* open set conception because Nolan had declared the sets open on the basis of family membership. He did not de-encapsulate his object conception into its underlying processes, but rather made *empirical* assumptions about the intervals' membership status in the topology at hand.

However, on the second attempt, working with point-level reasoning, Nolan clarified his understanding of the situation by examining the technical details of the theory at individually instantiated limit points. With this new perspective, he was able to conclude that $(0,1)$ must be an open set since its complement contained all its limit points, and was therefore a closed set.

4.6.4 Discussion of the complement approach

When he successfully reconsidered his use of the *complement* approach, Nolan's solution to the task required him to reason at the first predicate level to coordinate processes involved with limit points. His success with this approach (and his difficulty at the higher predicate level of sets) is evidence that first-order reasoning was more accessible to Nolan by providing him a clear criterion for judging whether a set is closed or not—i.e., whether the set contained all of its limit points.

At first, Nolan struggled to determine whether the complement of $(0,1)$ was a closed set in the lower limit topology. Specifically, this required a process for determining whether or not the intervals $(-\infty,0]$ and $[1,\infty)$ were closed. But without attending to the notion of limit points, he would need some pre-encapsulated conception of what constitutes a closed set in the lower limit topology to make any such determination. In the usual topology on the real numbers, Nolan could have recalled that closed rays are closed sets. However, the lack of such a conceptual tool for the lower limit topology may have led to his initial inability to complete the task. Similar observations were made in interviews with other participants as well, in which their reasoning was based on symbolic-empirical factors (e.g., the notational form of the set).

Later in the interview, by attending to the first predicate level, Nolan was able to recall and make use of the idea of limit points, and therefore use the *complement* approach more effectively. He could not have effectively used the *complement* approach as long as he was concentrating on the second predicate level of sets. But, by focusing instead on the first predicate level of points, he tapped into a proof strategy involving limit points that he had not previously considered, opening new mechanisms for verifying the properties of closed sets. By determining that the intervals $(-\infty,0]$ and $[1,\infty)$ contain all their limit points, Nolan identified those sets and their union as closed by referring (indirectly) to Theorem 4.4(2) from his textbook (Croom, 1989, p. 104). Finally, he then verified that the original interval $(0,1)$ was an open set based on the definition of closed set (Croom, 1989, p. 100).

It is possible that Nolan's *closed set* conception developed directly from the point-level process for checking a set's limit points; or that this process formed the basis for an intermediate object conception of the *derived set*. The derived set represents the collection of all the limit points of a given set. The *closed set* would then have been encapsulated as a set that contains its derived set. This is the presentation in the textbook, requiring a student to de-encapsulate the notion of derived set to consider a set's limit points. It is unclear from the data whether the derived set was an integral part of his conception for closed sets, but Nolan's spoken responses did not refer to the derived set. Rather, he claimed directly that a set was closed because "it contains its limit points," indicating that he may have bypassed the derived set conception (see Section 4.8.2.2 for a genetic decomposition). It is also unclear whether Nolan had encapsulated a *limit point* object conception independently as a point-level object, before using the *limit point* process to build the set-level *closed set* object conception. Figure 8 outlines the *complement* approach and its predicate level factors, including several possible theoretical paths that were not supported directly by evidence in this study. Nolan's defining activity led him to encapsulate the closed set object directly from his object conception of the limit point without referencing the derived set.

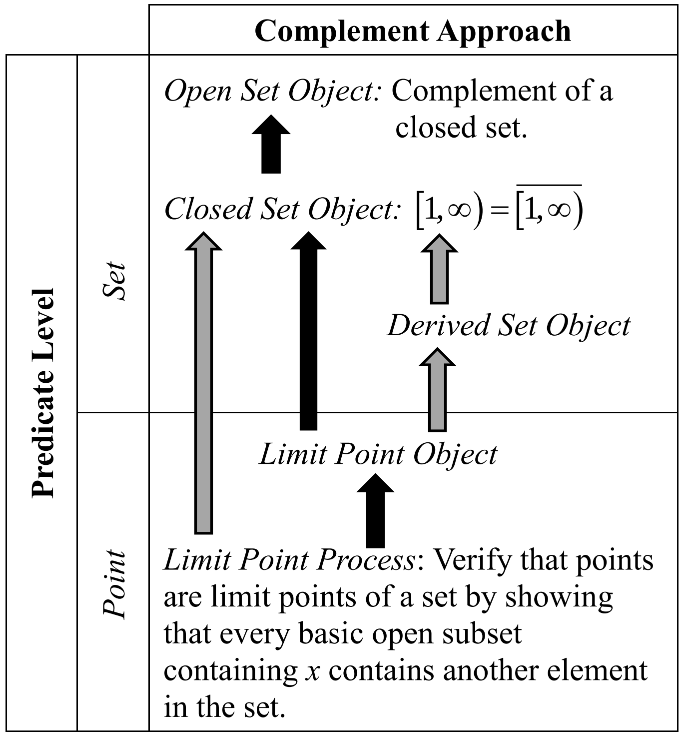


Figure 8. The *complement* approach to the open set concept. This approach allows students to encapsulate the *limit point* conception at the point level, and the *closed set* object at the set-level, but with point-level processes. The *derived set* object may be an intermediary between these conceptions; and an non-encapsulated *limit point* process may be used to form a closed set object conception. Black arrows indicate the logical path followed by Nolan, while grey arrows indicate theoretical pathways that were not observed in the data.

4.6.5 Evidence for the interior approach

The most commonly observed approach to defining open sets was using Theorem 4.3(2) in Croom (1989, p. 103), stating that any open set is equal to its subset of interior points. This approach allowed participants to express their proof strategies directly in terms of a set's interior points, which provided a potential proof strategy. To encapsulate the interior approach to the open set object, a student must first interiorize a process for determining if any point x in a set O is an interior point; i.e., if there exists an open set U such that $x \in U \subset O$.

The *interior point* process conception can then be encapsulated at two different predicate levels. As with the *limit point* conception, a student may develop an object conception of the *interior point* concept itself, encapsulating at the first predicate level of points in the space. Independently, a student may encapsulate the *interior open set* object at the second predicate level of sets, as a set that equals its own interior.

4.6.5.1 Nolan's use of the interior approach

When Nolan tried to use the complement approach, as described above, he attempted to prove that the complement of the interval $(0,1)$ is an open, rather than a closed, set. This mistake temporarily de-railed his attempt to use the complement approach; however, there was another error in his work that was potentially more significant. Nolan had inaccurately labelled the interval $(-\infty,0]$ as an open set in the lower limit topology. When queried about his labelling, he was quick to realize the mistake and offered the counter-claim that the interval "is not open, because, um…it *doesn't equal its interior*."

By using the counter-warrant that "it doesn't equal its interior" against his previous claim that $(-\infty,0]$ is an open set, Nolan signaled a transition from using the criteria for the complement approach to those of the interior approach. It is unclear what may have prompted this switch in the basis for Nolan's reasoning; however, it is noteworthy that his shift in reasoning strategies occurred within the context of a single task, rather than in distinct or sequential episodes. In other words, at one step within the process of using the complement approach to determine whether a particular set was open, Nolan had adopted the interior approach to check for the same property in a different set.

It is possible that by shifting his attention to a non-basis interval (i.e., the *left*-half open interval $(-\infty, 0]$), Nolan recognized the intellectual need to alter the process he was using to check for the open set property. He may have noted that without attending to the first-order points of the space, he would be unable to effectively use the complement approach, which is couched in second-order language. He did not say very much about his reason for switching to the interior approach: "I think sometimes the whole complement route can be really straightforward, sometimes it's not, and I started confusing myself…So, let's see if it's equal to its interior."

In using the interior approach, Nolan first struggled to recall the correct quantifier in the definition for interior point, mistakenly reasoning with the universal quantifier, 'for all,' rather than the existential quantifier, 'there exists.' This discussion is excluded here, as it does not directly pertain to this analysis; however, in the second section of these results (Chapter 5), this issue will be addressed. After he had corrected his definition, he went on to find a proof strategy involving the interior approach:

> *Nolan:*…so *if there exists* an open set containing the point, contained in the space…
>
> *Interviewer:* […] So, if you just pick an *x* out of this interval [indicates $(0,1)$], some number out of that interval, can you find an open set around it and stay in $(0,1)$?
>
> *Nolan:* Yeah, like $1/2$ …[writes $[1/2, 1)$]. So I mean, any point in here [indicates $(0,1)$] is an interior point of this set. So, it contains all of its interior points…
>
> *Interviewer:* And I guess, I mean you could just generalize that for any [point], right?
>
> *Nolan:* Yeah, I could do like, $[1/n, 1)$.

By constructing a basic open set $[1/n, 1)$ around every point in the open interval $(0,1)$, Nolan had demonstrated an *interior* (and interiorized) process, which he actively imagined applying to every point in the interval. Although this generalization of his previous response seemed at first to be formed superficially based on the fractional form of the left-hand endpoint ($1/2 \to 1/n$), further discussion revealed that he was considering something more like the Archimedean Principle in real analysis, allowing him to find a rational number between 0 and any arbitrary x in $(0,1)$. This is further supported by the form of the basis element he chose in his next response $[x,1)$, which ranges over the entire interval rather than a nested sequence of endpoints. This highlights a comparison between the the point-level *interior* process that Nolan was operating with and the *pointwise axiomatic* process that Wayne and Gavin had both briefly examined—both processes involve constructing a family of open sets in a pointwise manner (see Section 4.6.6 for discussion).

To complete the task, I asked Nolan to find a general statement that would show that an arbitrary point in $(0,1)$ was an interior point. He provided the following accurate justification that $(0,1)$ is contained in its interior:

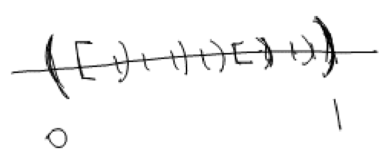
$$\forall\, x \in (0,1), \quad [x,1) \subset (0,1). \qquad \therefore\ (0,1) \subset \mathrm{int}(0,1).$$

Nolan had found a key idea for the proof, and demonstrated an encapsulated *interior* open set by imagining a point-by-point construction of a family of open sets. He did not refer to a union of these sets, which would have indicated his encapsulation of the *pointwise axiomatic open set* object. Instead he established that the set was equal to its interior, and took the family of open sets he had created as fulfillment of the criteria for the *interior* open set object.

4.6.5.2 Amy's use of the interior approach

As discussed above, Amy first attempted to use the axiomatic definition to reason about the task. But when she was unable to complete the task with this approach, I asked her if she could remember any other way of formulating the open set concept. After some time, she adopted the interior approach, and began to reason about points in the interior of $(0,1)$. She described the need to show that every point was an interior point:

> *Interviewer:* So can you show that no matter what number I pick between 0 and 1, not including 0 or 1, you can find […] a half-open interval around it that stays in the set?
>
> *Amy:* …Yeah, I could pick one, but…Oh wait, wait. For every, okay…because these [indicates the right-hand endpoints] are not included.
>
> *Interviewer:* Right.
>
> *Amy:* So for every single point in here, you're asking if I could pick, for every one, like say right here [draws a point in her diagram between 0 and 1], if I could do that to where it's still contained in here [the interval $[0,1)$]?



> *Interviewer:* Mhm.
>
> *Amy:* Yes. Because, uh, this could be like, uh, .00001, and I put it right there, right? And then I can go all the way to right before 1. All the way, it will be all open, and it will still be inside there […] Right, yeah because it could work for every single one, no matter how small or like, until you get here [indicates 1], or close to whatever, as long as you don't get over here [indicates to the right of 1]. So then, yes.

Interviewer: So you believe that this is an open set, even though we're in this [the lower limit] topology?

Amy: Yeah. Yes.

Similar to Nolan, Amy had used the point-level *interior* process in a way that could have afforded her a path to the *pointwise axiomatic* conception (by taking the union of her family of open sets). However, there was no indication that she had related these two process conceptions in her mind.

4.6.5.3 Wayne's use of the interior approach.

Wayne immediately sought to prove that the open interval $(0,1)$ was an open set by showing that it was equal to its interior. He had been easily able to do this in a previous task regarding the half-open interval $[0,1)$. However, when confronted with the open interval $(0,1)$, he seemed to be unsure of how to reconstruct the same argument.

Interviewer: [...] So in order to show that [the open interval $(0,1)$] is open, what would you have to show?

Wayne: Um…you could try to prove that the interior of A equals A [writes $\operatorname{int} A = A$].

$$\operatorname{int} A = A$$

Interviewer: Okay.

Wayne: Um…that's what I was thinking of right then, but I don't know if that would necessarily work in this topology, or using this topology……

Interviewer: So, I guess, in the same way you did [on the previous task], you need to find out how to express the-

Wayne: Express the interior of A, mhm. That's what I was just thinking, so…I guess, would you say…I know $[0,1)$ is open…um…

$$[0,1)$$

Interviewer: Mhm, and so, and $[0,1)$ you showed was equal to its own interior, so I wonder if you could just modify that argument with $(0,1)$?

Wayne: So, I guess……But I mean, I guess the question is, would they necessarily be synonymous? In the half-open interval topology?

It is unclear what he meant by "synonymous," but I interpreted his question to mean that he was unsure if his proof strategy from the earlier task would work in the same way for this task. Previously, Wayne had shown that for every element in the half-open interval $[0,1)$, there exists a basis element of the form $[a,b)$ that both contained the element and was itself contained in $[0,1)$. By that argument, $[0,1)$ was equal to its interior and so an open set. This argument needs little modification to work with the open interval $(0,1)$; however, Wayne seemed to struggle with reconciling the theorem he had used to this new set. The exclusion of 0 from the half-open interval $[0,1)$ rendered a basic open set into a non-basis element, $(0,1)$, and he seemed to be challenged by this difference, thereby obstructing his attempt to apply the same, previously-successful strategy.

Noticing that Wayne was attending only to the set level definitions in his reasoning, I asked him to examine an arbitrary point in the open interval, as he had done earlier.

Interviewer: […] maybe if we just let a point, let *x* be in there [$(0,1)$], could you make an argument about *x* being an interior point?

Wayne: Let *x* be in $(0,1)$…

Let $X \in (0,1)$. Then $X \in [0,1)$, which is open in the H topology. Therefore $(0,1)$ is open.

As Gavin had done, Wayne used the implicit assumption that the subset of an open set is necessarily open as well. When I asked him to justify his response, he quickly recognized his error and retracted this claim. Similar to Gavin, Wayne's reasoning remained on the second predicate level as he searched for patterns in the interactions of sets with their subsets. This interfered with his ability to use the interior theorem effectively by examining the set's interior points.

I redirected Wayne to the definition of interior by focusing his attention on the specific point ½ within the interval, and asking if he could find a basic open set around it that was contained in $(0,1)$. By instantiating an arbitrary point x as the *specified* point ½, he seemed able to reason concretely about the properties needed to solve the problem:

> *Wayne:* Right, yeah. Um, so you could say, let x be in $(0,1)$, and then, […] but x could be in, like $[\frac{1}{2},1)$, or $[\frac{1}{4},1)$.

By concretely discovering the key idea for a proof, Wayne was then easily able to generalize his understanding to arbitrary points in the interval, invoking the generalized basis element $[x,1)$. He stated his belief that he could find an interval of this form that would satisfy the definition for every point in $(0,1)$. However, rather than taking as given that intervals of the form $[x,1)$ are open sets (as members of the basis for the topology), he warranted the claim:

> let $x \in (0,1)$. Then $x \in [x,1)$. Which is open in the H topology since it is a union of basis. Therefore $(0,1)$ is open.

Although Wayne had successfully shifted his focus to the first predicate level of points and satisfied the criteria of the interior approach, he nevertheless concluded his proof by claiming that the $[x,1)$ was a "union of basis" [elements], which was not the approach he had used just minutes earlier while attempting to classify the open interval $(0,1)$ as an open set. This was significant because he had started the task by claiming that each of its members were interior points, but appealed to the axiomatic definition when discussing the members of the topology's basis.

To explore this, I asked Wayne if he could imagine a way to write $(0,1)$ as a union of basis elements and he produced the following:

$$\bigcup_{n=1}^{\infty} [1/n, 1) = (0,1)$$

Interviewer: Yeah, that would have been a different way to tackle it right?

Wayne: Would have been a lot easier! There goes my brain!

Wayne was quickly and easily able to write the open interval as a union of basis elements by invoking a nested sequence of half-open intervals, with endpoints $1/n$ that converges to 0. It is likely that his *setwise axiomatic* process had been primed by having recently reasoned about the definition for interior point in a prior task; however, his use of a nested sequence of sets such as this had not been observed prior to this response.

4.6.5.4 Gavin's use of the interior approach

Gavin's initial response to the task was to attempt to use the interior approach to determine whether $(0,1)$ was an open set in the lower limit topology. He seemed to focus his attention on the most salient difference between the open interval $(0,1)$ and the half-open interval $[0,1)$--the inclusion or exclusion of 0 from the set. He may have been primed to consider this basic open set by the previous task, in which he had been asked to find the interior and boundary of $[0,1)$ in the lower limit topology; or because $[0,1)$ is a basis element in the lower limit topology.

> *Interviewer:* So what about this, can you prove that the open interval from zero to one $(0,1)$ is open in [the lower limit] topology?
>
> *Gavin:* Yeah…see this is one of the things I had an issue with, going from one topology to another, because, I don't understand why, like, I don't know if it would be any different than just $(0,1)$ also in this [the standard] topology, in like, the half-open interval topology. But then it doesn't include that lower limit point…[writes $0 \notin (0,1)$].
>
> $$0 \notin (0,1)$$
>
> *Gavin:* …Um, so for the conditions of this, would it not be an open set because it doesn't-, because 0 is not included in $(0,1)$?

Gavin believed $(0,1)$ was not an open set, but his mathematical activity remained focused on comparing two sets, rather than engaging in a process surrounding the set's individual points. Although his comparison of the two sets did highlight the particular point 0, and he did call this point a "limit point," he never de-encapsulated his *limit point* conception into a useful process. Without a pointwise process to structure his reasoning he was unable to make any sense of the implications of excluding 0, or see how its exclusion affected the set's properties. When he became stuck, I steered him back toward his use of the definitions involved:

> *Interviewer:* So…I suppose the definition-, I mean what's the definition of an open set?
>
> *Gavin:* So if we were using that it's equal to its interior…so if we called this *B*, then *B* would have to be equal to the interior of *B*, and…[writes $B = (0,1)$; $B = \operatorname{int} B$].

$$B = (0,1)$$
$$B = \text{int}\, B$$

Interviewer: [...] So, [to show that $(0,1)$ was not open] you'd have to find a point in $(0,1)$ that is not-

Gavin: That's not in the interior. So I guess there's nothing that's not in the interior of B...this just seems like this would be-, because I guess, $(0,1)$ is a subset of $[0,1)$.

$$(0,1) \subset [0,1)$$

So if we were calling that A earlier and B... uh, the interior of B is a subset of A, but......[writes $\text{int}\, B \subset A$]

$$\text{int}\, B \subset A$$

Dissatisfied with the outcome of his reasoning at the point level, Gavin attempted again to justify his belief that $(0,1)$ is not an open set at the higher predicate level of sets. It is unclear if he was attempting to make use of a partially recalled theorem, such as Theorem 4.3(*3*) in Croom (1989, p. 103), which states that for any subsets A, B of a topological space X: If $A \subset B$ then $\text{int}\, A \subset \text{int}\, B$. It is also possible that his intuitive understanding of the concept of interior was heavily involved with the idea of containment. This may have led him to temporarily consider a subset of an open set to be "interior" to it in some respect, and therefore open as well. Regardless, Gavin promptly recognized the error in his logic when prompted:

Interviewer: Can you think of other examples where you've got a subset of an open set that's not open?

Gavin: Yeah, like if you had, um... $[0,1]$, and that could be a subset of negative one to [two]...

$$[0,1] \subset (-1,2)$$

> *Gavin:* Yeah, but that's why, like I was just trying to think of, um……I guess it would be open. There's no, I can't think of a…

Gavin tentatively agreed that $(0,1)$ is an open set, but he was not able to gain full conviction. I asked him if he could remember another formulation of the open set concept, which he could not. This was when he consulted his textbook and began a discussion about the axiomatic approach to complete the task, as described in Section 4.6.1.3.

4.6.6 Discussion of the interior approach.

Like the *complement* approach, the *interior* approach provided participants with an accessible proof strategy involving a set's interior points, allowing them to reason at the first predicate level. Four of the participants attempted to use this reasoning to justify that individual sets were open, despite the fact that this approach makes no direct appeal to the formal definition of an open set.

Gavin's use of the *interior* approach remained focused on the set level, preventing him from unpacking any sort of point-level process for defining an open set. He continued to attempt to use set-level comparisons as criteria for determining an open set, first comparing $(0,1)$ to $[0,1)$; and then seeking a general rule about the interiors of sets, (e.g., $(0,1) \subset [0,1)$ implying that $(0,1)$ is open).

Meanwhile, Nolan and Wayne were both able to use the *interior* approach effectively, each noting that he could find a basic open set around every point in $(0,1)$, establishing the interval as an open set based on that justification. This entails the use of a logical implication, provided by Theorem 4.3(2) in Croom (1989, p. 103), to tie the point-level process to a set-level object. Once Nolan was able to recall an accurate definition for an interior point, he quickly focused on the first predicate level of points and easily completed the task. Similarly, Wayne was not able to work effectively with the *interior* approach until he was prompted to focus on individually specified points of the set in question, rather than the set in its entirety. When he was first asked to focus on an individual point x to show that $(0,1)$ was an open set, Wayne continued to think in terms of sets and incorrectly warranted that $x \in (0,1) \subset [0,1)$ and that $(0,1)$ was therefore open. However, in a later exchange he was able to imagine performing actions on specific points, and was able to recall a mechanism for checking the interiority of a point in the set. Allowing this process conception to operate on any arbitrary point in the set, he then demonstrated his encapsulation of the *interior open set* when he declared that $(0,1)$ was an open set, and warranted that for any point x in the interval, $x \in [x,1) \subset (0,1)$. This indicated his conception of open sets as those that contain only interior points. Based on this, I infer that his *interior* object conception for open sets was built from processes that took place on the point-level.

During Amy's attempt to use the interior approach, she also appeared to use a generic prototype of a basic open set in the lower limit topology, (something like Nolan or Wayne's use of $[x,1)$ for $0 < x < 1$); thereby establishing an *interior* process for finding open sets around each point in the open interval $(0,1)$. Although she was less explicit in her response, she seemed to demonstrate an encapsulated *interior* open set object by expressing that the set was open due to the existence of one of these basic open sets around every point in the interval. Similar to what Nolan and Wayne had done, she constructed her *interior* open set object conception based on the entailments afforded by point-level processes involved in the definition of interior points. Figure 9 displays several possible conceptual paths to encapsulate the *interior open set*.

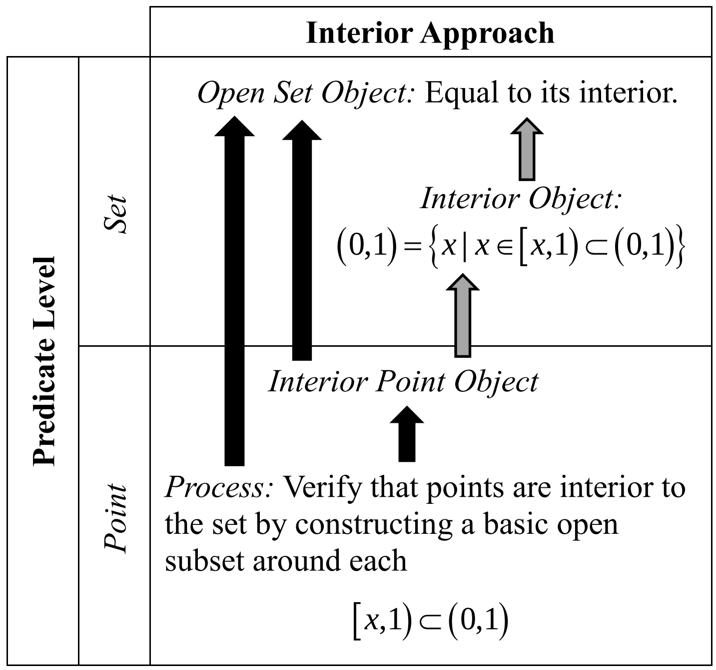


Figure 9. The *interior* approach to the open set concept. As with the *complement* approach and the *limit point* conception, the *interior* approach allows students to encapsulate the *interior point* conception at the point level; as well as the *open set* object at the set-level, via point-level processes. A student may also encapsulate the *interior set* object conception as an intermediate step (not observed). The arrows indicate abstraction to an object conception through the encapsulation of the processes below it. Grey arrows are theoretical paths that were not supported by data.

It is significant to note that through their use of the *interior* approach, each of these students had essentially constructed the family of open sets that they would have needed to succeed in using the axiomatic approach. This is because, as noted, the *interior open set* conception is closely aligned with the *pointwise axiomatic open set* conception. They both constitute open set families on the first predicate level of points, the difference being that an *axiomatic* conception requires the encapsulation of the union of those family members, while the *interior* conception requires only the logical implication that a set is open whenever it is contained in its interior.

For example, Amy was able to imagine the process of placing a basic open set around each point in $(0,1)$, represented by the series of right half-open intervals she placed within her drawing of the open interval $(0,1)$. The collection of open intervals she had imagined was, in fact, a family of open intervals for which the union would be the whole set. Recognizing this could have allowed Amy to classify $(0,1)$ as a member of the family of sets that comprise the lower limit topology, and approach the task in the axiomatic way she had begun. However, she did not explicitly state the equivalence of her solution with the logical form of the axiomatic definition, and did not re-examine her initial approach during the remainder of the interview. Satisfied with her *interior* approach reasoning, there was no indication that Amy recognized that a union of the set family she had built would satisfy the *axiomatic* definition she had originally provided. This may have been because she had not encapsulated the open set conception via the properties and processes involved with the *axiomatic* definition, although she had evidently encapsulated the notion in a different fashion—as equal to its collection of interior points.

As with the *complement* approach, the *interior* approach may result in a students' encapsulation of the *interior point* as an object on the first predicate level; while the same process can be encapsulated to form an *interior open set* object through the student's identification of the open set with its interior.

4.7 Highlighted Case: Gavin's Definition Accommodation

So far in this chapter, evidence has been presented that students employ several distinct and valid strategies for constructing their personal concept definitions for open sets. Benefits and challenges for each strategy were also demonstrated. These were shown to stem from the interactions of these different approaches to the definition, as well as the different predicate levels that reasoning with each strategy involved. In this section, I highlight one case involving Gavin, who was able to accurately reason about a series of linked concepts until he reached the axiomatic definition of the fundamental concept of the *open set*. This was the moment of disequilibrium that prompted Gavin to switch his basis of reasoning from the topological (axiomatic) structure to his mathematical intuitions. At that point, Gavin altered his personal concept definition for open sets, relying on another version of the definition with relevance only to metric spaces.

Due to the axiomatic structure of topology, the concepts presented to the participants throughout the semester were developed through multiple layers of definitions, resting on progressively more fundamental concepts that were ultimately built-up from the topology axioms. For example, the concepts of limit points, interior points, closure, and boundary are all defined in terms of *open sets*, which are defined in terms of the topology axioms. Interactions between these compound terms often lead to chains of definitions, providing options for students to reason with these theoretically-derived structures, rather than directly from the axioms themselves. Below, I present Gavin's defining activities while reasoning about notion of a boundary in the lower limit topology.

Gavin examined a sequence of successive definitions for each of the following terms: boundary point, closure, derived set, limit point; and finally, for the axiomatic notion of an open set. Despite his familiarity with the formal definitions for each of these terms, he was ultimately unable to use the definition for open set to reason accurately about the task. As the following analysis will demonstrate, informal aspects of Gavin's concept image seemed to dominate his mental operations during this discussion, such as the relationship between his spatial intuitions, linguistic associations, and past mathematical experiences. He eventually modified his initial, correct definition to fit his informal conception for the limit point concept, by recalling a concept definition for limit points in metric spaces. Falling back on this familiar definition was a means for him to re-equilibrate his mental conception for open sets, but was in fact a shift away from an accurate understanding.

4.7.1 *Highlighted case analysis*

The following excerpts occurred after Gavin had asserted that 0 was in the boundary of $[0,1)$ in the lower limit topology. I asked him about his definition for boundary to help determine the structure of his thinking.

> *Interviewer:* Okay, and then what about the 'boundary'? What's your definition for that?
>
> *Gavin:* Uh, the boundary is…it's the points that belong to both, um, *A* closure and, if the space is *X*, the complement's closure.

$\text{bdy}A \in \overline{A} \quad \text{bdy}A \in \overline{X \setminus A}$ $\qquad$ $\text{bdy}A \in \overline{A} \quad \text{bdy}A \in \overline{X \setminus A}$

Although he did not use the intersection symbol and appeared to misappropriate the symbol for 'element of' rather than 'subset of', Gavin did accurately indicate that the boundary of a set belongs to the closures of both the set and its complement. Noticing that he did not define the term ‘closure’, I questioned him further:

> *Interviewer:* Okay, so then what’s ‘closure’?
>
> *Gavin:* Uh, the closure is, so if it’s the closure of *A*, it’s equal to *A* union the derived set of *A* […] The set of all its limit points.

$\bar{A} = A \cup A'$ $\qquad\qquad$ $\overline{A} = A \cup A'$

Having broached the subject of limit points within the definition chain, I tried to find out how Gavin might perceive 0 to be a limit point of the complement of $[0,1)$:

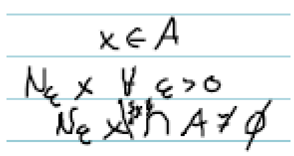
> *Interviewer:* Okay, so then when I look at 0…so 0 is in the set, so it’s clearly in the closure of *A*…
>
> *Gavin:* Mhm.
>
> *Interviewer:* It’s not in the complement…but then you’re saying it *is* in the derived set of the complement?
>
> *Gavin:* Yeah, because it would be, so for that, the complement would be…[writing], so, yeah, because the closure of this would include 0.

$(-\infty, 0) \cup [1 \ \infty)$ $\qquad\qquad$ $(-\infty, 0) \cup [1, \infty)$

> *Interviewer:* Okay, so that means it’s a limit point of the complement, 0 is a limit point of this union?
>
> *Gavin:* Yes.

Gavin had classified 0 as a limit point of the complement of the interval $[0,1)$ in the lower limit topology. Specifically, he had classified 0 as a limit point of the open ray $(-\infty, 0)$. While this would be true in the standard topology on the real numbers, it is incorrect in the lower-limit topology. Although he responded affirmatively when I asked him if 0 was a limit point of the complement, his previous statement reflected that his attention was focused on the belief that "the closure of this would include 0." He did not seem to be de-encapsulating the processes that go into the definition of a set's closure.

This prompted me to clarify what his definition of a limit point was, which he then defined accurately in a metric context. (He did mistakenly require that the limit point should belong to the set in question; but this seemed to be a superficial error since he had just classified 0 as a limit point for a set to which it did not belong. He had no trouble correcting the statement a short while later).

> *Interviewer:* Okay, so what's the definition of 'limit point'?
>
> *Gavin:* Limit point is a point, uh, x that belongs to A, where x, where for some neighborhood around x, that neighborhood intersects A, the interse-, well, some neighborhood of x intersect A is non-empty, so it intersects A at a point other than itself.
>
> 
>
> $$x \in A$$
> $$N_\varepsilon x \quad \forall \varepsilon > 0$$
> $$N_\varepsilon x \setminus \{x\} \cap A \neq \varnothing$$
>
> *Interviewer:* So, for any neighborhood?
>
> *Gavin:* Uh, for some epsilon neighborhood greater than 0.
>
> *Interviewer:* For just some neighborhood?
>
> *Gavin:* Yeah, or I guess it should be for all if it's a limit point, because otherwise it could be outside of the-, uh, it could be well outside of the set and you could make a big enough epsilon to still [touch it], but-
>
> *Interviewer:* Okay, so it's got to be any epsilon neighborhood, or any open neighborhood of the point. So, the point doesn't have to be in A?
>
> *Gavin:* Mm-mm.
>
> *Interviewer:* Because you were saying that 0 would be a limit point of this [the complement], right?
>
> *Gavin:* Mhm.

Interviewer: So you want that every possible open set that contains 0, to also contain-

Gavin: Some point in *A* other than itself.

Having established Gavin's correct formal definitions for the ideas of 'boundary,' 'closure,' and 'limit point,' I attempted to determine why he still considered 0 to be a limit point of the complement of $[0,1)$. In the lower limit topology, this interval itself serves as an example of an open set that contains the element, and yet contains no elements from complement of the set, thereby disqualifying 0 as a limit point of $(-\infty,0)\cup[1,\infty)$. While he did indicate his understanding that the underlying concept of an open set is altered in the lower limit topology he seemed to ignore the consequences of this modified basis as he generated examples of open sets containing the element 0:

Interviewer: Okay…So, this basis gives us different open sets than we're used to?

Gavin: Yeah.

Interviewer: So, is there *no* open set around 0 that doesn't contain an element from the complement?

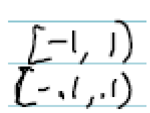
Gavin: An open set around 0……

Interviewer: So, what are some examples of open sets that contain 0 in this topology?

Gavin: Uh, well $[0,1)$ is one of the ones we used. Um, you could also use like, $[-1,1)$, that would contain 0.

Interviewer: Mhm, and that would definitely contain another point besides…in the set.

Gavin: [writing]…and that contains 0…



$[-1,1)$

$[-.1,.1)$

Interviewer: Mhm, so you can get closer and closer to it…

Gavin's examples of open sets were all elements of the lower limit topology's basis—right half-open intervals. Yet, the way that he seemed to choose the latter two sets, with both endpoints converging to 0, is reminiscent of how he might have thought about the same problem in the standard topology—with open intervals as the basis. In real analysis courses, this method of 'homing-in' on a limit point with a sequence of successively smaller open intervals is a common informal strategy for justifying that a number is a limit point of some interval or set (Lay, 2014). In this case Gavin seemed to justify that any open set containing 0 will also contain elements of both $[0,1)$ and its complement. However, his first example of $[0,1)$ itself eliminates the need for any such analysis. Gavin failed to notice that this was an immediate counterexample to his claim that 0 is a limit point of the interval's complement.

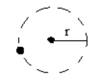
After allowing him to explore other examples of basic open sets that did not violate his claim, I challenged him directly about this problematic example. He seemed to have difficulty considering the question; and after wrestling with the inconsistency, reconciled his understanding by modifying his original definition for the concept of limit point:

> *Interviewer:*...um, now what about $[0,1)$? You mentioned that one.
>
> *Gavin:* …But for $[0,1)$…if its-, so if this is the part of the complement……but that's…does that count as…I don't think that counts as a neighborhood *around* 0, because there's not-, I guess you should say a neighborhood centered-, an epsilon neighborhood centered at *x*?
>
> *Interviewer:* Ah, okay. So you want it to be, you don't want it to be a point on, sort of, the edge?

Gavin: Like the, yeah, like the edge of the neighborhood I guess. So, I want epsilon neighborhoods centered at *x*. So that the *x* is always in the middle of that neighborhood, so the distance on both sides would be the same, *r* or whatever it is…[writing, modifies earlier definition].

$x \in A$
$N_\varepsilon x \ \forall \varepsilon > 0$ centered at x
$N_\varepsilon x \setminus \{x\} \cap A \neq \phi$

$x \in A$
$N_\varepsilon x \quad \forall \varepsilon > 0$ centered at x
$N_\varepsilon x \setminus \{x\} \cap A \neq \varnothing$



Gavin seemed to be reacting to the term "around" within his personal concept definition for limit point. He accepted the fact that $[0,1)$ is an open set in the lower limit topology, but did not consider it an open set "*around* 0." So, despite his understanding that $[0,1)$ is an open set in the lower-limit topology that contains the point 0 and nothing from this interval's complement; and despite recalling the correct formal definition for the limit point concept, Gavin did not accept the conclusion that 0 could not be a limit point of $(-\infty,0)\cup[1,\infty)$. Instead, he fell back on an earlier version of this definition for open sets in a metric space, which had not been introduced in the class text (Croom, 1989). The metric definition seemed to have helped Gavin to reconcile his spatially-oriented concept image for an open set, which relied on some distance *r* between all points and the "edge" of the set, with his need to reason from the formal definition of each concept. The use of his metric conception prevented him from accurately determining the boundary of the open set $[0,1)$, which is empty in the context of the lower limit topology. Gavin's reliance on a distance function for his open set conception is inadequate for the lower limit topology, because it is a non-metrizable space.

4.7.2 Highlighted case conclusions

In Gavin's exploration of the boundary point concept, he explicitly reorganized his personal concept definition for limit points to fit the expectations afforded by other aspects of his concept image. Specifically, he redefined the concept of a limit point to exclude certain open sets from the definition. In a topological setting, a point x is a limit point of a set A if every open set containing x also contains a distinct point from A. Gavin modified this definition to require that all such open sets be "centered" at the point. For the lower limit topology, Gavin's modification removed an important open set $[0,1)$ from his consideration, which was precisely the open set required for him to answer the task correctly.

I claim that Gavin had developed a conception of open sets that was heavily influenced by his experiences with the real numbers in his previous calculus and real analysis courses. His previous understanding then acted as the basis for his abstraction during his defining and structuring activities about the boundary concept. Rather than successfully accommodating his boundary schema, he returned to a previous, less abstract version of the idea. This prevented him from constructing a formal schema for the boundary, or completing his conceptual hierarchy for the definitions that relate to the boundary.

Like Gavin, my other participants often reasoned with complicated definitions without breaking the definitions down all the way to their axiomatic components. In many cases, when I asked them to formally deconstruct their definitions, my participants could do so only in limited ways. At some point, like Gavin, they would often switch to a reliance on informal elements of their concept images. In Chapter 5, I will present a variety of cases where definition-based reasoning competed with intuitive and perceptual reasoning strategies used by my participants.

4.8 Results and Conclusions from the Open Set Analysis

In this analysis, I have categorized responses to Task 3.1(B) (see Figure 6) according to the mathematical characterizations for the open set concept that were referred to and interpreted by my participants during their mathematical activity. These categories were called the *axiomatic, complement*, and *interior* approaches. I also categorized each of these approaches in terms of variations in the logical predicate levels (e.g., points, sets, or families) accessed by the participants during their mathematical structuring activities (classification and categorization).

The analytical categories of *approach* and *predicate level* represent the formal-syntactic half of my conceptual framework (see 2.6.4), which includes the participants' defining and structuring activities. The participants were shown to abstract properties in different ways through these activities. Each participant was observed to define the open set concept with one or more of the three approaches, while attending to one or more predicate levels when classifying mathematical objects (such as whether a given set was open). As described in the analysis above, combinations of these two dimensions of categories presented different affordances for reasoning and potential obstacles to understanding.

I will present four genetic decompositions (Arnon, et al., 2014) of the open set concept reflecting the two dimensions of variation described above. These will be accompanied by mathematical analyses of each approach based on a textbook analysis, classroom observations and my own experience with the subject. A mathematical analysis is only a preliminary decomposition for a concept (Arnon, et al., 2014). While the mathematical analyses informed my interpretation of the results, the genetic decompositions I present are thoroughly grounded in the data of my participants' responses, which often differed from the formal mathematical theory in significant ways. I will demonstrate how my participants seemed to interpret each approach they examined, as well as some of the consequences of their reasoning in those ways. Finally, by synthesizing these distinct formulations of the open set conception, I will address the participants' development stages (Arnon, et al., 2014; Baker, et al., 2000) of the open set schema, to generate a grounded and unified genetic decomposition for the concept.

I present two variations on a genetic decomposition for the axiomatic open set conception, as well as one each for the complement and interior open set conceptions. To differentiate the axiomatic approaches, I will present genetic decompositions for the set operations of *union* and *intersection* and three other sub-processes based on what I called a *neighborhood action*. In support of the latter two approaches, I will also present genetic decompositions for the *interior point* and *limit point* conceptions, since the decompositions for their related *open set* conceptions require that these objects be de-encapsulated into their underlying processes and coordinated with other processes in the development of those schemas.

4.8.1 Axiomatic open set conceptions

I observed two logical starting points for my participants' axiomatic approaches to the open set concept. I called these the *pointwise* and *setwise axiomatic* conceptions, reflecting the predicate levels that formed the basis for choosing a family of sets whose union equaled the whole set. I traced most of my participants' axiomatic open set schemas through operations on the second predicate level of sets. Therefore, the *setwise axiomatic* open set decomposition is more strongly supported by the evidence than the *pointwise axiomatic* open set decomposition.

4.8.1.1 Mathematical analysis of the axiomatic approaches to open sets

Here, I will present a mathematical description of the actions I interpreted in my participants' responses. The actual activities of my participants were not always aligned with these descriptions. However, retrospective mathematical descriptions are provided here, since my interpretations of their actions rested on the formal mathematical theory presented to them in the textbook and class.

Within the formal mathematical theory (Croom, 1989), the construction of the axiomatic open set concept originates with three sets of mathematical actions. The first two are potential mechanisms for constructing a family of sets, and the third involves set operations to combine that family into a unified mathematical object:

- verifying that subsets (neighborhoods of points) are contained within a set,
- constructing a nested sequence of intervals that covers a set, and
- identifying an infinite union or finite intersection of sets.

The first of these actions involves identifying a subset X of a superset Y so that it contains a given point $x \in X \subset Y$ (see Figure 10). Identifying such a subset X for arbitrary points $x \in Y$ (where X is an open set in the topology given to Y) is an important conceptual tool in the field of topology. It is used for defining the concepts of 'interior point', 'limit point', and 'open cover' of a set. I will refer to this as the *neighborhood* action or process, depending on the development of a student's personal conception. The next action is constructing a sequence of intervals to cover a set. This makes use of the ideas of sequences and sequence limits and allows a student to reason with a family of intervals by conceiving of the limit of a sequence of their endpoints. I will refer to this as the *nested intervals* action or process.

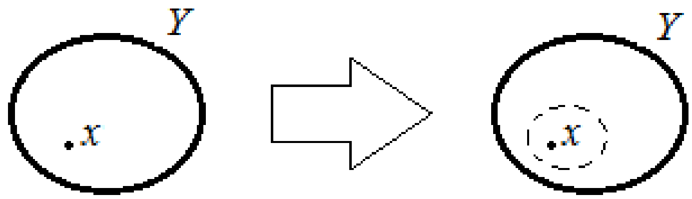


Figure 10. The *neighborhood* action. The dashed ellipse represents an open set around the point *x*, contained in *Y*.

During the *pointwise axiomatic* approach to defining the open set, the neighborhood process can be used to establish a 'cover' of a set *Y* by open sets X_α; that is, a family of open subsets $\{X_\alpha\}$ whose union is equal to *Y*. The existence of such an *open cover* of *Y* establishes that *Y* is itself an open set by applying the set operations to $\{X_\alpha\}$. In other words, if *Y* can be shown to be equivalent to a union of open sets $Y = \bigcup X_\alpha$, or an intersection of some finite collection of open sets $Y = \bigcap X_i$, then *Y* belongs to the topological family of sets via the axioms that constitute the definition of an open set (Croom, 1989, p. 99). Thus, when the *neighborhood* process is coordinated with set operations, one can build a personal concept definition of open set (or identify a given set as open), directly through the axiomatic definition (see Figure 11). In later sections (4.8.2.1 and 4.8.3.1), I examine how the *neighborhood* process is also used within the interior and complement approaches.

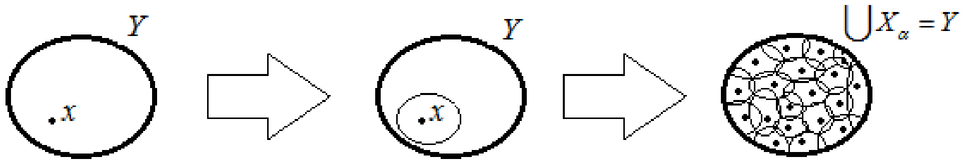


Figure 11. The neighborhood process coordinated with the union operation. Light-bordered ellipses represent neighborhoods around the point *x* (middle) and each point in *Y* (right).

During the *setwise axiomatic* approach, the *neighborhood* action is unnecessary because the student constructs a family of nested intervals, which is used to cover the set in question without addressing each of its individual points. By coordinating the *nested intervals* action with the set operations, the family of intervals can be identified with the whole set to directly satisfy the axiomatic definition of an open set (see Figure 12).

$$\bigcup_{n=2}^{\infty}\left[\frac{1}{n},1\right)=(0,1)$$

Figure 12. Saul's use of the *setwise axiomatic* approach

The mathematical pathways described above may be represented and interpreted in many ways by an individual student. The student's actions, processes and resulting object conception may appear to be more or less developed during any instance of her mathematical activity, leading to potential differences in the developmental stages of each student's open set schema. The following genetic decompositions express an idealized model of the processes and structures required for a "generic student" (Arnon, et al. 2014) to develop an open set object conception through the *setwise axiomatic* and *pointwise axiomatic* approaches, as described above.

4.8.1.2 A genetic decomposition for the pointwise axiomatic open set

The construction of the *pointwise axiomatic* open set conception begins with actions on a set, originating in the student's schema for sets. These actions include: 1) set operations, such as identifying the unions and intersections of (possibly infinite) families of sets; and 2) applying the neighborhood process (see previous section) to points in a set.

A generic student will first encounter the actions of identifying pairwise unions and intersections of finite sets in her early mathematics courses. Further on she may identify unions and intersections of larger (still finite) families of finite sets. By reflecting on these experiences, she will interiorize the set operations into mental processes, capable of being carried out in her imagination alone. Subsequently, by performing actions and processes on the resulting unions or intersections, the student will then encapsulate the set operations for finite sets and finite families of sets. Eventually the student will have experiences operating on infinite sets as well, such as the natural or real numbers; and she will perform operations on infinite families of sets, such as the power set for the natural numbers, etc.

This will require the coordination of her set operations schema with her schema for infinite sets (Dubinsky, et al., 2005) in two ways. First, she will need to assimilate the notion of countable infinity into her set operation schema by de-encapsulating the individual processes into their underlying actions. This will permit her to imagine the construction of a new set through the addition or deletion of infinitely many elements between sets in the family. Later she may assimilate the notion of uncountable infinity as well. Secondly, she will imagine building unions or intersections out of infinite families of sets (with any cardinality). The student may then encapsulate the infinite union (or finite intersection) process for (finite or infinite) sets, by conceiving of an infinite number of those sets combined into a single entity, which itself may be subjected to further set operations, and so on.

Next, as the neighborhood action (see Section 4.8.1.1) is performed in the context of different topologies, the student reflects on it and begins to perceive it as a dynamic process that can mentally be performed on infinitely many points at once. This is the beginning of the interiorization of the neighborhood action, as the student constructs the mental structure necessary to imagine finding an open set around all of a set's points. By interiorizing the neighborhood action into a mental process, the student is then capable of constructing a family of neighborhoods that may cover a set. The student may then perform the actions associated with the set operations on this family of neighborhoods, preparing the way for the open set process.

Finally, by de-encapsulating and coordinating the two processes above, the student will be able to construct a new process for identifying open sets. An individual with a process conception for the *pointwise axiomatic* open set will be able to think of an open set in terms of families of neighborhoods, without the need to explicitly verify that each point is contained in a neighborhood. Evidence for a process conception for the pointwise axiomatic approach would include the ability to construct a (possibly infinite) family of open subsets that surrounds each point in a set, or to reverse that process by instantiating one or more of these neighborhoods around individual points. Through the encapsulation of the student's process conception, an object conception of the open set will be formed as the union or intersection of the family of neighborhoods that was imagined to be constructed above. This conception will shift toward a structural perspective, in which the union or intersection of the family of sets is primary, and the process of generating the family will become secondary. Evidence for an object conception would include the student's ability to perform actions on an open set, such as forming further unions or intersections with the newly constructed set, or to use the axiomatic open set to reason about mathematical properties such as the continuity of functions.

The genetic decomposition for the pointwise axiomatic conception is hypothetical because it was referenced in a limited way during the interviews. However, there was evidence to conclude that students may potentially build their *axiomatic* conception in a pointwise manner. Gavin's statement that "it could be the union of open balls around all of these points, you know, from 0 to 1" seemed to reference this conception, although he did not pursue this line of reasoning. Amy's attempts to build a union of half-open intervals also indicated that she was imagining the actions required for this conception. Yet, it seems doubtful that she had interiorized the union action, based on her inability to complete the task.

4.8.1.3 <u>A genetic decomposition for the setwise axiomatic open set</u>

As above, the construction of the *setwise axiomatic* open set conception also begins with actions on a set. However, in this case the neighborhood action can be bypassed if a student has a sufficiently developed schema for the limit of a sequence. Instead, in addition to interiorizing actions involved with the unions and intersections of set families (see Section 4.8.1.2-1), the *setwise axiomatic* conception involves the interiorization of the *nested intervals* action, in which a sub-family of basic open sets is specifically selected based on some criteria (e.g., in this case, sets with a convergent sequence of left-hand endpoints).

During a student's real analysis courses, she will have experiences reflecting on various uses of nested families of intervals of real numbers, as well as sequences of their endpoints (Lay, 2014). As the mental actions involved in constructing families are performed with different sets, the student reflects on them and begins to perceive them as a dynamic process for forming new sets, without having to imagine each interval or running through the sequences of endpoints to the limit. This is the interiorization of the *nested intervals* action, allowing the student to imagine building an open set as a union or intersection of such families. By interiorizing the neighborhood action into a mental process, the student is then capable of constructing a family of neighborhoods that may cover a set. The student may then perform the actions associated with the set operations on this family of neighborhoods, and interiorize her open set conception into a process.

By de-encapsulating and coordinating the *nested intervals* process with her set operations schema, the student will be able to construct a new axiomatic process for identifying open sets, distinct from the *pointwise axiomatic* approach. An individual with a process conception for the *setwise axiomatic* open set will be able to think of an open set in terms of structured families of neighborhoods, without the need to explicitly verify the containment of each point in a neighborhood. Evidence for a process conception for the *setwise axiomatic* approach would include the participant's ability to construct a nested family of open subsets $X_\alpha \subset Y$ that covers a given set Y, and to identify the set in terms of that family (whether through a union or finite intersection). The student should also be able to reverse their process conception by instantiating one or more of the subsets in the family. Finally, through the encapsulation of this process, an object conception of the open set will be formed as the union or intersection of the nested family of neighborhoods that was constructed above.

As with the *pointwise axiomatic* approach, the *setwise* conception will become more structural as it develops, so that the union or intersection of the nested family acquires greater salience for the student. The process involved in forming the family in this way may be recalled through the de-encapsulation of the open set object. However, the process the student recalls will be quite distinct from that obtained by de-encapsulating a *pointwise axiomatic* object. In the *setwise* case, evidence for an object conception would include the student's ability to perform actions on the open set, such as forming new nested families in such a way that the previously constructed open set is a member; or again, to use this version of the axiomatic open set to reason about mathematical properties such as the continuity of functions.

It is non-trivial for many students to recognize that an open boundary may be defined through the *nested intervals* action—via an infinite union of sets with closed boundaries. Successful participants seemed to rely on their past experiences with real analysis to enable this construction, resulting in the transfer of metric and Euclidean properties through their activities and abstractions. The use of metric definitions or properties of the real numbers was beneficial for the participants in some instances, but in other cases the previous knowledge interfered with their reasoning. For example, Saul recalled this conceptual tool immediately and used it to demonstrate that he had encapsulated the union (as a set operation) of a specifically selected family of open sets (via the interiorized *nested interval* action) into a *setwise axiomatic* object conception (see Figure 12). Gavin discovered that he could write the interval $(0,1)$ as the same union, although he approached it from a different direction (see Figure 13). In Section

4.8.3.2, I show that Amy and Wayne used similar approaches, although they reached that conception via the *interior* approach.

$$\bigcup\left[0+\frac{1}{\varepsilon},1\right) \quad \varepsilon=(2,\ldots,\infty)$$

Figure 13. Gavin's use of the *setwise axiomatic* approach.

4.8.2 Complement open set conception

The *complement* open set conception may work on either the first or second predicate levels, depending on a students' understanding of the sets and topologies being used. The participants in this study were observed to successfully use point-level reasoning by coordinating with their *limit point* schemas. However, set-level reasoning (i.e., identifying the complement of a set as closed based on the characteristics of the set, but without reasoning about individual points in the set) was not observed to be used effectively. For this reason, I will only outline a genetic decomposition for a *pointwise complement* open set conception below.

4.8.2.1 A mathematical analysis of the complement approach to open sets.

Here, I present a mathematical analysis of the formal concepts associated with the defining and structuring activities of one of my participants, Nolan. As with the axiomatic conceptions, Nolan's use of these concepts did not always match the formal theory, but it is provided as context for my interpretation of his work. The *complement* approach to the open set concept involves verification that the complement of a given set is a closed set. This requires two mental actions: 1) identifying the complement of the set in question, and 2) verifying that the complement, once identified, is a closed set.

The first action relies on the basic set operation of the complement, in which all elements of a subset A are excluded from the universal set X to form a new set $X \setminus A$. In the case of real numbers, finding the complement for an interval (a,b) involves identifying the union of two half-rays that exclude the interval $(-\infty,a]\cup[b,\infty)$.

For the second action listed above, there are multiple ways to show that a set is closed. I will discuss the only way I found conceptual evidence for in my data—multiple participants attempted to justify that a set was closed by showing that it contained all its limit points. This line of reasoning does not stem from the definition of a closed set, which was defined in Croom (1989, p. 100) as being the complement of an open set. Rather, the only students who reasoned with the limit point characterization for closed sets based their reasoning on Theorem 4.4(2) in Croom (1989. p. 104), which states that a subset *A* of a topological space *X* is "closed if and only if $A=\overline{A}$," where $\overline{A}$ is the derived set of *A*—the set of all the limit points of *A*. Thus, for a subset $A\subset X$, if its complement $X\setminus A$ contains all its own limit points $\overline{X\setminus A}$, then it must be closed. Then, by the definition of closed set, the subset *A* is open.

Identifying a limit point of a set requires a variation on the *neighborhood* action (see Section 4.8.1.1), in which *each* neighborhood of a given point *x* is instantiated to determine whether it intersects some set *Y* (rather than to determine if it is contained in *Y*). I will distinguish these two actions by referring to the former as the *neighborhood intersection* action (see Figure 14), and the latter as the *neighborhood containment* action (see Figure 10). The following genetic decomposition describes the logical pathway followed by Nathan, who was the only participant to use the *complement* approach in the task addressed in this section.

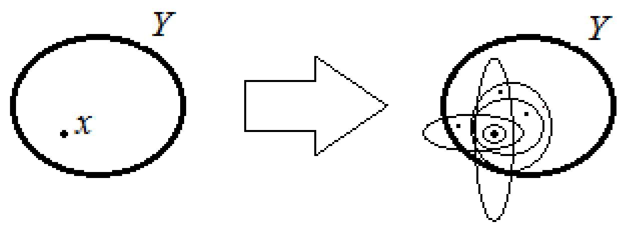


Figure 14. The *neighborhood intersection* action. This is a variation of the *neighborhood* action, but used to identify limit points of a set. The ellipses represent every neighborhood around *x*, each containing another element from *Y*. It is not possible to represent every neighborhood of *x* in a diagram, which caused problems for some of the participants in my study and is addressed in the next section.

4.8.2.2 A genetic decomposition for the complement open set.

The *complement* open set is formed from the coordination of two processes that stem from actions involving the student's *limit point* and *set operations* schemas: 1) identifying the complement of a set, and 2) identifying the limit points of a set (e.g. the set complement identified previously.

The set operation used here is the set *complement*. Typically, a student will have had previous experiences with this action in her analysis or set theory courses. By reflecting on these earlier experiences, the student interiorizes this action into a mental process, which however may be tied to a particular class of sets (e.g. real number intervals). Experiences finding the complements of different sets, as well as their topological implications, may provide the student with a process conception of greater depth for the complement operation.

The *limit point* process involves the interiorization of the *neighborhood intersection* action, in which the neighborhoods of x need not be contained in the superset Y, but are instead required to contain at least one other point from Y (see Figure 14). As this action is interiorized into a dynamic mental process, the student can imagine identifying all the limit points for a given set, even if it has infinitely many points. The student may or may not encapsulate this notion into the *derived set* object (i.e., the set of a set's limit points). If so, this object may need to be de-encapsulated for the student to use the limit point process in unfamiliar contexts. In either case, in unfamiliar topologies the *limit point* object will need to be de-encapsulated by the student to use it for the development of the *complement* open set conception. Thus, there may be differences among students who reason with the *complement* approach based on the development of their *limit point* schemas.

The *complement* open set process results from the coordination of the two processes just described, where the student identifies the complement of a given set and then uses the limit point process to identify its limit points. By coordinating these processes in many situations, the student establishes a unified mental process for checking whether the given set is open. Evidence for such a process conception would include the student's ability to verify that sets are open in a variety of contexts, through an analysis of limit points alone. Encapsulation of this process would then follow, by applying actions and processes to the coordinated *complement* process as a static entity. Evidence for a *complement* open set object conception may include identifying a set as open in the context of another task, such as determining continuity of a function (there was no evidence found in this data to support the belief that any participants had developed an object *complement* conception).

It is possible that a student's *complement* conception rests on her schemas for closed sets and set operations alone, without reflecting any direct relationship between her schemas for limit points and open sets. But it is also possible that the *open set* is encapsulated directly on the point-level—as a set which contains no limit points of its complement. Although limit points and interior points are mathematically connected ideas (i.e., a limit point for *X* cannot be an interior point of the complement of *X*), it is unclear from the interview data which specific connections Nolan had made between his conceptions for these ideas. He did not assert or indicate that he noticed any direct connections between the open set and the limit points of its complement.

Nolan demonstrated evidence for some of the processes outlined above. During his second attempt to use the *complement* approach, he used his *limit point* process conception, indicated by his action of testing several points both inside and outside of $(0,1)$ to verify whether or not they were limit points of the interval:

> "…so the question would be like, is 1 a limit point of it [of the interval $(-\infty,0]$]. And for the same reason, no, it's not. So, then is like, -1 a limit point of it? Yes. But everything [between] -1 and 0 would also be a limit point of it, which will all be contained in that set."

He seemed to use these actions as 'test cases', as he mentally collected the interval's limit points, to verify that the interval was closed. He showed an object conception for the *complement* set operation by identifying, and acting upon the complement of the interval $(0,1)$. He seemed to move easily between engaging the *complement* set operation as an object and as the process that formed it, as he selected points from both the interval and its complement on which to apply the *limit point* process. Although he initially struggled with it, Nolan's process conception of the *complement* open set was finally evidenced by his attempt to justify that $(0,1)$ was an open set through a logical consideration of its complement's limit points.

4.8.3 Interior open set conception

As with the *complement* approach, the *interior* approach may work on either the first or second predicate levels. A student may encapsulate the interior point as a point-level object, or encapsulate the same process into an *interior open set* (set-level) object. The participants in this study were observed to reason with this conception by using their *neighborhood* process conceptions. Specifically, they used the *neighborhood containment* process, rather than the *neighborhood intersection* process. Below, I outline a genetic decomposition for the *interior* approach to the open set concept.

4.8.3.1 A mathematical analysis of the interior approach to open sets

Here, I present a mathematical analysis of the formal theory involved in my interpretation of the participants' *interior* approaches to defining the open set. Unlike the other conceptions analyzed so far, the *interior open set* conception requires only one interiorized action, which was already discussed in Section 4.8.1.1: the *neighborhood containment* action. This was the action of identifying an open set X around individual points x in a set, which are themselves contained in the set $x \in X \subset Y$. A given set may be identified as an open set by verifying that this action can be performed on any one of its points. By selecting an arbitrary point in the set, this property can often be checked for an entire set with one calculation. The action involved in building this conception is the same as that used for constructing the *pointwise axiomatic* open set conception, but without the need to coordinate it with the union set operation. This leads to similarities between the two approaches.

4.8.3.2 A genetic decomposition for the interior open set

Construction of the *interior open set* begins with the application of the *neighborhood containment* action to points within various set contexts. Every point this action applies to is an interior point by its definition. This leads the way for the student to interiorize the *neighborhood containment* action into an *interior point* process; and subsequently to encapsulate this process into an object conception. These mechanisms take place on the first predicate level of points, and the constructions are point-level conceptions. Evidence for an object conception of the *interior point* may consist of the student's ability to identify and organize interior points in the context of a subset of points, such as the interior set.

Independently, by reflecting on the properties of the individual points for which this action is possible, a student interiorizes the action and forms a mental process allowing her to imagine constructing an open set around each point in a given set. By doing so, she forms an *interior set* process, which can then be encapsulated to form an object conception for the *interior set*. This is the conception required to form the *interior open set* conception, when the student can conceive of an open set as equal to its own interior. Evidence for an object conception of an *interior open set* may include the student's ability to identify and act on the interior of a given set, by comparing its membership to that of the whole set; performing set operations with it, such as unions, intersections or complements; and reasoning about more complex properties by referencing the open set via the *interior open set* conception.

Most of the participants had some level of success reasoning with the *interior* approach. Nolan demonstrated an object conception for both the *interior point* concept when he instantiated a basic open set for each interior point of the interval $(0,1)$ (see Figure 15(a)). His *interior open set* object conception was demonstrated when he established that $(0,1)$ was an open set based on the warrant that it was contained in its interior (see Figure 15(b)).

$$\forall x \in (0,1),\ [x,1) \subset (0,1) \qquad\qquad (0,1) \subset \operatorname{int}(0,1)$$

(a) (b)

Figure 15. Nolan's object conception for the *interior approach*. He formed an object conception for both: (a) *interior point* and the (b) *interior open set.*

4.8.4 Point-level affordances and the structure of topology

The results indicate that my participants were most successful in reasoning on the first predicate level involving points in the space; and, that certain approaches provided the participants with salient, pointwise means to encapsulate the *open set* object. Some participants were successful reasoning with the axiomatic approach at either the point- or set-levels, but the most successful attempts made use of the interior approach, which was shown to afford students with a straightforward, point-level proof strategy. This may account for the frequency of its use among my participants as a personal concept definition for open set.

Reasoning with and about set families seemed to present challenges for most of my participants, as the literature suggested it would (Narli, 2010; Zaskis & Gunn, 1997). While some participants, like Saul, were successful reasoning on higher predicate levels, the least difficulty seemed to arise when points were the focus of my participants' reasoning, as with Nolan and Wayne's attempts. It often helped my participants to translate the relevant theorem or definitions into the first-order language of points in the space as a means to de-encapsulate the underlying set-level objects into processes that could be more concretely reasoned with. This allowed participants to generate a clear proof strategy by attending to individual points or members of special classes of points, such as the interior or the derived set.

4.8.5 Conclusion

In this analysis, I have shown that my participants used three distinct conceptual approaches to the open set idea, each based on a mathematical characterization of open sets found in their textbook and class. These were the *axiomatic*, *complement*, and *interior* approaches. I examined the mathematics that formed the basis of the three approaches and wrote grounded genetic decompositions for the open set conceptions that each approach helped to construct. Evidence was provided to support the idea that my participants' conceptions for open sets were built on combinations of the three approaches, which determined the properties that were included in their personal concept definitions. Moreover, each of the conceptual approaches for defining the open set entailed pathways through the APOS structures that differed according to the predicate levels attended to by the participants during each conceptual approach.

The case of Gavin's defining activity was highlighted, in which he was observed to de-encapsulate a series of terms involved in the definition of a set's boundary. Despite accurate formal reasoning throughout this process, Gavin was unable to correctly answer the task. His secondary intuition (Fischbein, 1987), developed through his prior analysis course, led him to alter his definition of an open set (the final term in the chain of definitions he had de-encapsulated). Gavin fell back on a previous, metric-based definition, allowing him to align his formal understanding with the answer he believed was correct.

5. CONCEPTUAL ANALYSIS OF THE PROPERTY OF CONTINUITY

5.1 Introduction to the Continuity Analysis

This analysis is the second part of a larger study designed to answer the following research questions about students in an introductory topology class:

1) What distinctions and comparisons can be made between the ways that students manage their transition to an axiomatic understanding of continuous functions?

2) What obstacles do students face during this transition?

This chapter will detail the results and analysis from a sequence of clinical interviews (Ginsberg, 1981; Piaget, 1970), in which six undergraduate topology students responded to tasks involving continuous functions over the course of three one-on-one interviews. The tasks addressed increasingly generalized forms of the property of continuity[4] throughout the semester, culminating with a task in which the continuity of the given function varied according to the topology under consideration. The participants' responses to this series of tasks provided useful data for constructing hypothetical conceptual models of their mental operations, which were consistent with their observed mathematical activities.

I will answer the first of my research questions above through the continuity analysis by providing evidence to distinguish between the *generating* and *embodying* activities (see conceptual framework, Section 2.6.4) I observed in my participants' responses as they tried to make sense of the property of continuity in a novel topological context (the lower limit topology). The participants will be shown to generate exemplars

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[4] Although the term "continuity" has several meanings within various mathematical theories, I will use this term solely as shorthand for the property that differentiates continuous functions from non-continuous functions in topological spaces.

and prototypes as models for the properties they believed should belong to a continuous or non-continuous function. The participants embodying activity was interpreted as metaphorical mappings from perceptual and sensori-motor experiences related to their conceptions of continuity. Metonymical mappings (Johnson, 1987; Lakoff, 1987; Zandieh & Knapp, 2006) from previously understood mathematical contexts were also considered embodying activities.

As an answer to the second research question above, I will show how the participants' uses of exemplars, prototypes, metaphors, and metonymies interfered with their reasoning. This occurred when the participants mapped extraneous properties through their generating or embodying activities, or when they failed to map essential properties for continuous or non-continuous functions.

5.2 Purpose of the Continuity Analysis

The purpose of the continuity analysis was to model the mental activity of the participants as they tried to reconstruct their understanding of continuous functions to account for new stimuli; namely, reasoning within the novel context of the lower limit topology and the resulting disequilibria they faced. I assumed that reconstructing their understanding would involve two potential Piagetian mechanisms. They would first attempt to assimilate the new topology into their current schemes for classifying functions as continuous or non-continuous. When assimilation failed, the participants were expected to accommodate their continuity schemes in some way to make them fit with the new stimuli and remove the disequilibrium. My goals were to: 1) categorize the types of assimilation and accommodation I observed, 2) compare those types within and across episodes, and 3) examine the affordances for reasoning that these types of modifications provided for the participants.

5.3 Mathematical Background and Tasks for the Continuity Analysis

The continuity portion of the study consisted of three successive interviews, in which the six participants were presented with tasks involving continuous functions in increasingly abstract situations. The first set of tasks (see Figure 16) was used as a baseline to determine how the participants reasoned about the continuity of a standard, piecewise function on the real numbers. The second task (see

Figure 17) explored the participants' ways of thinking about a more general function, with a Cartesian product as the domain. This provided insight into how they reasoned about non-standard functions for which visual representations were less useful in detecting discontinuities. Both tasks provided a context in which to profile the participants' conceptualizations of sets, functions, and continuity. Thus, going into the third interview I had developed grounded models of each participant's ways of reasoning about those concepts. The third interview was approached as a test for these models within a set of mathematical situations that differed with respect to the property of continuity, yet were represented in perceptually similar ways.

The third set of tasks (see Figure 18) were used to elicit perturbations in the participants' awareness about representationally confusing applications of continuity in three distinct situations. The first of these tasks presented a standard, piecewise, real-number function on a restricted domain, which did not include a point of discontinuity in the domain, $x = 0$ (the function was therefore continuous on its domain). This setting was then modified such that the domain did include 0, assigning it to one of the connected components in the range, and therefore rendering a jump discontinuity (the modified function was therefore not continuous on its domain). Finally, the last situation modified the topological context of the previous one, by assigning the lower limit topology to both the domain and range (this modification caused the function to be continuous again). So, the participants were presented three perceptually similar situations which prompted them to repeatedly return to the same cognitive re-presentation and actively reflect on the essential differences that altered their meanings for the three tasks.

5.3.1 *Task 1.1 (A & B)*

In this task, the participants were asked to select intervals within the domain on which they believed the function to be A) continuous, and B) discontinuous, and to prove both assertions. A graphical representation of this function was not provided to the participants, but they were free to draw one, and many did. Other participants attempted to solve the problem syntactically, without representing the function in a visual or sensory manner. Most used some form of the limit definition for continuous functions on the real numbers, with varying degrees of formal accuracy.

5.3.2 *Task 2.3*

During the second interview, participants were asked to consider the function in this task. It sends each interior point of the unit disk to a point in the interval $(0,1)$ on the y-axis that corresponds to its distance from the origin in $\mathbb{R}\times\mathbb{R}$. It sends each boundary point and exterior point of the unit disk to the point $y=2$. As before, no visual representation was provided to the participants, though most chose to draw something like the image on the right in

Figure 17.

1.1 *(Interview 1, Task 1)*: Consider the following real-valued function:

$$f:\mathbb{R}\to\mathbb{R} \text{ where}$$

$$f(x)=\begin{cases} x, & x<2 \\ x+1, & x\geq 2 \end{cases}$$

A) Choose an open interval on which you believe the function *f* is continuous and prove it.

B) Choose an open interval on which you believe the function *f* is not continuous and prove it.

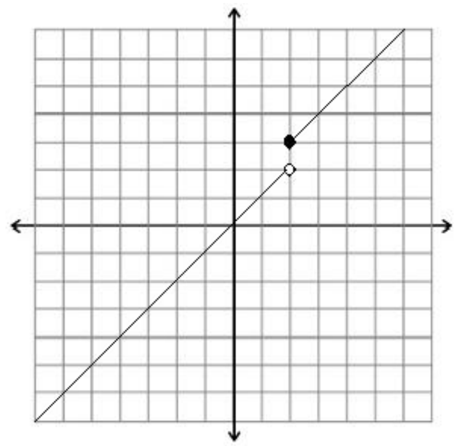


Figure 16. Task 1 from the first interview. This was used to gather baseline data about the participants' understanding of continuous functions in the standard, one-dimensional, real number context. The graph is provided as a visual aid for the reader. No graph or diagrammatic representation was provided to the participants.

2.3 *(Interview 2, Task 3)*: Determine whether the function *g* is continuous and prove your assertion.

$$g:\mathbb{R}\times\mathbb{R}\to\mathbb{R} \text{ where}$$

$$g(x,y)=\begin{cases} x^2+y^2, & x^2+y^2<1 \\ 2, & x^2+y^2\geq 1 \end{cases}$$

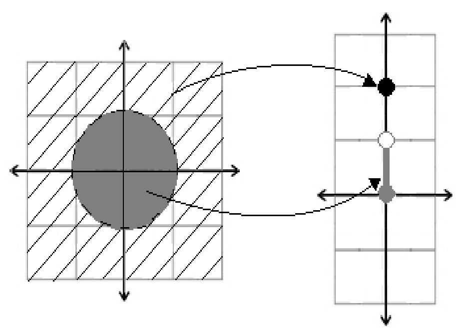


Figure 17. Task 3 from the second interview. This task required the participants to generalize their understanding of continuous functions to include a non-standard domain, and to consider two topologies at once (those generated by the 1-D and 2-D Euclidean metrics). Again, no graphical or diagrammatic representations of the task were provided to the participants.

Task 2.3 exposed the participants to a function that was less common to their experience, in that its domain was a Cartesian product and would typically be represented visually in two-dimensions. Moreover, the co-domain was one-dimensional, which led the participants to coordinate their reasoning about two topologies at once (i.e., two-dimensional Euclidean balls vs. one-dimensional real intervals). As mathematics majors, the participants had been exposed to physio-spatial examples of two-variable functions in their Calculus 3 and differential equations classes; and they may have explored largely syntactic proofs involving multi-variable functions in their analysis classes. However, based on participant feedback (see Discussion), this function seemed to be far from their typical experiences with functions in the past, especially with respect to the continuity of the function.

5.3.3 Task 3.2

In the final interview, participants were given a series of three related continuity tasks, in which each task entailed a perceptually minor adjustment to the previous one (see Figure 18 on the following page).

5.3.3.1 <u>Task 3.2(A)</u>

Participants were first asked to analyze the continuity of a piecewise-defined function, which was defined on the disconnected subset $(-1,0)\cup(0,1)$ of the real numbers. The component intervals of the domain were sent to 1 and -1, respectively, which is a similarly disconnected subset of the real numbers in the co-domain of the function. At this point in the semester, the participants had been exposed to two topological definitions for continuity, presented as the "definition," and "alternate definition" in the textbook (Croom, 1989, p. 115):

> **Definition:** Let $(X,\mathfrak{T})$ and $(Y,\mathfrak{T}')$ be topological spaces, $f: X\to Y$ a function, and *a* a point of *X*. Then *f* is **continuous at *a*** provided that for each open set *V* in *Y* containing $f(a)$ there is an open set *U* in *X* containing *a* such that $f(U)\subset V$. The function *f* is **continuous** if it is continuous at each point of its domain.

3.2(A): *Interview 3, Task 2(A):* Define a function $f:(-1,0)\cup(0,1)\to\mathbb{R}$ by the rule:

$$f(x)=\begin{cases}1, & x\in(0,1)\\ -1, & x\in(-1,0)\end{cases}.$$

Determine whether *f* is a continuous function or not using the standard topology on the real number line. Prove your assertion.

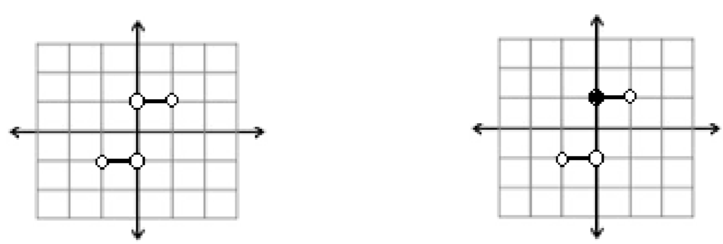
3.2(B): *Interview 3, Task 2(B):* Now define a function $f:(-1,0)\cup[0,1)\to\mathbb{R}$ by the rule:

$$f(x)=\begin{cases}1, & x\in[0,1)\\ -1, & x\in(-1,0)\end{cases}$$

Determine whether *f* is a continuous function or not using the standard topology on the real number line. Prove your assertion.

3.2(C): *Interview 3, Task 2(C):* Reconsider the function *f* from the task above.

Determine whether *f* is a continuous function or not using the *half-open interval topology** on the real number line. Prove your assertion.



(A) **(B & C)**

Figure 18. Task 2 from the third interview. This task was designed to elicit cognitive perturbations within the minds of the participants, and to promote their active attempts to re-equilibrate their conceptions. *The lower limit topology was referred to as the "half-open interval" topology in the class and textbook (Croom, 1989, p. 113). No graphical representations were provided to the participants during the task.

Alternate Definition: A function $f:(X,\mathfrak{I})\to(Y,\mathfrak{I}')$ is **continuous** means that for each open set *V* in *Y*, $f^{-1}(V)$ is an open set in *X*.

It is understandable that these non-intuitive and circuitous definitions might not be easy for students to conceptualize or relate to, compared to the more salient empirical observations that they might make about their visual representations of the task situation. Those participants that drew a graph were often led to surmise that the function was discontinuous without engaging the task in a reflective way, while those who reasoned syntactically encountered less difficulty accepting that the function was continuous, but still faced challenges when attempting to resolve their formal understanding with their empirical observations from the graph. The function is indeed continuous (despite having a "discontinuity" at 0), but most of the participants initially stated a belief that the function was non-continuous after inspecting the function's graph.

The discrepancy between the visually disconnected graph and the continuity of the function can be resolved: the function sends connected components of the domain to connected components of the range. This relationship, which had been presented and studied in the textbook and classroom, was not necessarily prevalent within all of the individual participants' conceptions. The relationship between the continuity of a function and the connectedness of the function's domain and range is expressed in Theorem 5.2 of the textbook and elaborated on the same page in a corollary to the theorem (Croom, 1989, p. 135):

> **Theorem 5.2:** Let X be a connected space and $f : X \to Y$ a continuous function from X onto a space Y. Then Y is connected.
>
> **Corollary:** If $f : X \to Y$ is a continuous function on the indicated spaces and X is connected, then the image $f(X)$ is a connected subspace of Y.

5.3.3.2 Task 3.2(B)

In the second part of the task, participants were asked to consider a similar function, but in this case with the *connected* domain $(-1,0)\cup[0,1)$. After deciding that the first function was continuous, many participants were unwilling to accept that this modification led to a discontinuous function, even when their definition was at odds with this interpretation. By re-engaging in the formal processes (see Section 6.7.2) they had used for Task 3.2(A), some of the participants found the key issue with the function's continuity more easily.

5.3.3.3 Task 3.2(C)

Finally, the participants were asked to modify the topology of the spaces on which the second function, 3.2(B), had been defined. They were asked to use the lower limit topology (for both the domain and co-domain) instead of the standard topology on the real numbers. In the lower limit topology, the previously defined function is once again continuous. Having previously established an action pattern for checking the continuity of this specific function, many of the participants were readily able to check and verify this property in this case.

5.4 Analytical Framework for the Continuity Analysis

For this section of the results, I report on a conceptual analysis (Glasersfeld, 1995) of the results from a sequence of clinical interviews (Garfield, 1981; Piaget, 1970) concerning my participants' conceptualization of the property of continuity for functions in various topological contexts. By choosing von Glasersfeld's (1995) framework, I set epistemological bounds on my study and specified the research goals that I could attain. For example, from the radical constructivist perspective researchers are inherently limited in what they can study about the mental activity of their participants. Von Glasersfeld (1995) maintains:

> Though we cannot watch how a language-user builds up his or her concepts we can investigate them by doing two things. First, examine what kind of situations the word is intended to describe; second, try to unravel, from a logical point of view, what elements the associated concept must incorporate in order adequately to reflect certain experiential situations.

Here, the "experiential situations" are encountered through the formal-symbolic world of axiomatic mathematics as it pertains to the field of topology. The "elements" to be incorporated are the mathematical structures presented to my participants; however, it is important to note that students may focus their attention on linguistic, perceptual, or other non-mathematical features as well. Von Glasersfeld (1995) explains that "individual students often make abstractions from the presented perceptual material that are quite different from those the teacher intends, to whom the material seems unambiguous" (p. 184). However, he goes on to remind the reader that it is *how* the students abstract these elements that govern their conceptual constructions:

> …both language and perceptual materials can provide experiential situations that may be conducive to reflections and abstractions a teacher wants to generate, but they are merely occasions, not causes. The students' concepts are determined by what they, as individual perceivers, come to abstract (empirical abstractions from their sensations, and reflective abstractions from the operations they themselves carry out in the process). (p. 185)

Because of the perceptual salience of the common properties presented in introductory topology classes, von Glasersfeld's (1995) distinction is significant. The continuity analysis will probe the mental actions the participants used to coordinate their sensori-motor knowledge and their formal understanding of continuous functions.

5.5 Overview of Results for the Continuity Analysis

In this section, I provide a summary and overview of the analytical themes for the continuity analysis, which were centered around the participants' coordination of meanings between: 1) the perceptual-spatial knowledge stored in their graphical, notational, and diagrammatic representations of the task situations; and 2) their formal and procedural understanding of mathematical situations. Rather than ranking or rating these two sources of potential meaning for the participants, I distinguished between the underlying bases for reasoning during their mathematical activities. I found that my participants used their intuitive, perceptual, and spatial knowledge of the task situation more often than their syntactic, formal-symbolic (reflective) understanding. More significantly, I noted that each of my participants managed the coordination of their empirical and reflective meanings in distinct ways.

The continuity analysis below is organized in a different manner from the open set analysis in Chapter 5. Rather than overviewing the analytical themes and then providing evidence to support them; the continuity analysis is organized by the participants' responses to each of the three parts of Task 3.2 in the chronological order of their activity during the task. Common themes from the responses will then be tied together and discussed as a retrospective conceptual analysis of the participants' mental actions associated with the property of continuity.

5.6 Continuity Analysis

I will present cases of reasoning observed during work on Task 3.2 (A, B, & C) by each of the participants. Selected excerpts from an interview with Maren will also be presented as a highlighted case to provide specific points of evidence targeting claims I will make in the analysis. Symbolic notation has been added to the transcript data to clarify the meanings of ambiguous verbal statements, such as the names of open, closed, or half-open intervals, etc. Numbers that are used in the context of the real number line are written as Arabic numerals; but outside of this context, they are written out in words. Short pauses are notated with ellipses…while longer pauses (>10 seconds) are notated with double ellipses……Deleted remarks are indicated with a bracketed ellipse […]. Pseudonyms were used throughout.

5.6.1 Saul's responses to the continuity tasks

Saul had a syntactic reasoning style and did not rely on visualizations for understanding. However, he did have strong mathematical intuitions from his prior experiences, and found himself challenged to reconcile his beliefs with the formal mathematical solutions he had found.

5.6.1.1 Saul's response to Task 3.2(A)

As he had done in previous tasks, Saul demonstrated a formal approach to understanding the continuity of the function for this task. He knew the alternative axiomatic definition (see Section 5.3.3), and after a few questions for clarification he was able to complete the task.

> *Interviewer:* I just want you to think about that function, and tell me if you think it's continuous or not…
>
> *Saul:* So, it's $(-1,1)$ minus 0 to the reals……and whether its continuous……
>
> *Interviewer:* Do you remember a definition for continuous function?
>
> *Saul:* Yeah the one-, I mean there's the epsilon-delta, but the one we lean on is the open set-, the pre-images of open sets are open sets.
>
> *Interviewer:* Ok.
>
> *Saul:* So, in that instance, for the image I would need to be concerned about open sets in the reals, um, I guess these would be intervals. I can say intervals in this sense?
>
> *Interviewer:* Well, I mean these [-1 and 1] are just points, right?
>
> *Saul:* Mhm, but they're in the…oh, okay.
>
> *Interviewer:* So, it really is just those two points in the reals as your image, or your range. But $\mathbb{R}$ is the whole co-domain…

Saul: Okay……is it okay to say that the points are open sets?

He asked whether he could use the term 'intervals' to refer to the open sets in this topology, likely recalling his previous issue in Task 3.1(B), during which he repeatedly confused the terms 'open interval' and 'open set'. This was related to his use of open intervals as a paradigmatic model (Fischbein, 1987) for open sets. However, further discussion showed that a another issue was involved here. Saul couldn't see how to use his definition of continuity, without having any open sets within the range of the function to check for the appropriate condition (i.e., that its pre-image under the function was an open set). I prompted him to apply the definition to the whole co-domain:

> *Interviewer:* I see what your issue is though, you're looking at these [points] and saying 'well how do I use these if I have to find an open set whose pre-image is open'? […] Now, keep in mind that your co-domain is all of $\mathbb{R}$. So, you just need to make sure that open sets in $\mathbb{R}$, that include these points…their pre-images are open.
>
> *Saul:* Yeah…Okay. That's what I was asking at first […] Well, then I would want to say…um, uh, no it's not continuous. Because I couldn't…wait……Well I guess it *is* continuous, um…this [indicates $(-1,1)$] is a union of two open sets…yeah I guess I would say continuous.

Saul seemed to have no problem adapting his use of the definition to the entire co-domain, and correctly surmised that the function was continuous on its domain. I asked him to describe his reasoning process, and he described being concerned about the function's left-hand continuity at the endpoint -1 of the domain:

> *Interviewer:* So, what were you thinking of before when you said it was discontinuous?
>
> *Saul:* I was worried about -1 going here, but -1 can't, so it's not a problem. I mean, -1 is not even in the domain […] so, as long as, if the domain is everything between -1 and 0, you can always have an open set arbitrarily close to -1 or 0, which produces this which can be open in the reals…reverse that and that would match the definition of continuous.

Interviewer: […] So, if you put an open interval around -1?

Saul: It's going to map back to this open interval, but that's just an open set.

His responses indicated that Saul had interiorized the formal process for checking topological continuity. However, it was unclear whether he had assimilated the elements that had empty pre-images into this scheme.

Interviewer: And everything that's not -1 in that interval would map…from where? What would its pre-image be?

Saul: The pre-image of 0, or any number arbitrarily close to -1, what would that pre-image be? I want to say it wouldn't exist…

Interviewer: So how do we talk about sets that don't exist in topology?

Saul: The empty set. Okay.

Interviewer: Is that an open set?

Saul: Yes.

Interviewer: So, if you had the pre-image of say, $(-2,0)$ here, it would be the union of this interval from that point and the empty set?

Saul: Oh, I see what you're saying. So, empty set union that would be the pre-image of that? I had never thought of it that way before, so yeah.

Interviewer: […] So that convinces you that its continuous?

Saul: Yeah, I mean it matches the definition.

Saul's approach had been based on the formal definition of continuous functions between topological spaces, but he had apparently not interiorized its use in situations involving a subspace of the co-domain. He indicated that he had not considered the pre-image of a point or set from outside a function's range as empty, although he had seemingly disregarded all the points besides -1 in his example above. Points outside the range seemed to create no perturbation for Saul, because he had not yet assimilated any examples of such points into his scheme for topological continuity. That is, he may have had experiences with subspaces before this, but he had apparently never had to explicitly justify that their pre-images are open sets.

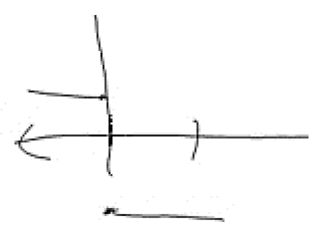
Until this point in the interview, Saul had not attempted to draw or represent the function in any sort of graphical or diagrammatic way. In fact, he had not written anything at all as he responded to the task. To find out how the informal aspects of his concept image might have played a role in his reasoning, I asked Saul about how he would represent this function in his mind. It surprised me to find that he did not conceptualize the task in a visual way at all:

> *Interviewer:* Does it feel continuous? I notice you haven't drawn it do you just, are you picturing it in your head? How are you picturing it?
>
> *Saul:* I mean, if I were going to draw it I guess, I would do it like, you know this…
>
> $$(-1,1)\setminus 0 \to \{1,-1\}$$
>
> *Interviewer:* So, that would be sort of, notationally describing it, right? Is that how you-, I mean, when you're thinking about this you don't have like a picture or a graph or anything in your head? You're just thinking about the symbols?
>
> *Saul:* I wouldn't know how to graph this […] Until you said the word graph, I didn't even think about it…
>
> *Interviewer:* So, this, so when you look at it you're just thinking along these notational lines, but it feels, intuitively it feels like a continuous function to you? I mean at first you said it was not continuous, but-
>
> *Saul:* Right, well I was just going with the definition and not worrying about what it looked like so much, um…yeah, I don't know. I guess one part of it is that if I see a function that splits to different things, I don't worry, I stop thinking about graphs…
>
> *Interviewer:* Okay, so like piecewise functions, you don't prefer to think in terms of visualizations?

Saul: I guess not. Yeah, I usually just, I mean they're elements…maybe I should try graphing them, but it's never occurred to me.

Saul indicated that he conceptualized piecewise functions in an entirely syntactic way. He made no connection between this function and his graphing scheme for real-number functions, and seemed to prefer to reason symbolically from the definition. I prompted Saul to draw the function out to see how he would represent it, and whether it would cause him to understand his solution in a different way:

Saul: For fun, I'll just try to draw it. I mean the domain is this, so you know, it's going to be like that but with a little hole here. And then, um…kind of like that I guess. Or, the other way around! Let's pretend…



Interviewer: So, when you look at that does that look continuous?

Saul: Well no, this is obviously not continuous. Yeah. That would be a faster way to do it.

Interviewer: So…it looks obviously *not* continuous?

Saul: Yeah.

Having drawn the function's graph, Saul noticed a discrepancy between the visual disconnectedness of the graph and his previous logical conclusion that the function was continuous. Saul pointed out that he had accidentally drawn the graph in the reverse of its conventional representation, his picture provided a clear perceptual signal that the graph must not be continuous. The disconnectedness of his graph generated a disequilibrium because he waivered in his conclusion, stating that it was "obviously not continuous." I reminded Saul that his new conclusion was contrary to the accurate reasoning he had just given for his claim that the function was indeed continuous and he then quickly resolved the issue:

> *Interviewer:* But without looking at it you just had a strong argument, I thought, that it was continuous?
>
> *Saul:* Right! so now I'm going to have to think why-, I mean I believe it's continuous because it matches the definition […] Um……oh…okay, I know why it's continuous. It's because we've restricted the domain…So, this isn't actually part of the graph, this is just…it would be like if you had, I don't know…is this, $f(x)=x^2$, is that one-to-one? You say no. Well, if the domain is $(0,\infty)$ then now it is, so, problem solved. And this is similar, we've just restricted the domain, and now it's continuous.
>
> *Interviewer:* Okay, so essentially 0 is not a part of the domain…
>
> *Saul:* So, we don't have to worry about it…right, this isn't actually part of the graph…

Once Saul had found an example of another property that varies according to restrictions in its domain, he was able to assimilate the idea of a function with a disconnected graph into his scheme for continuous functions.

5.6.1.2 <u>Saul's response to Task 3.2(B)</u>

Having reasoned through the visual discrepancy caused by his graph of the function in Task 3.2(A), Saul was now faced with a function that was representationally similar to the first function, but with a domain that did include the point of discontinuity, $x=0$. At first, Saul could not see any difference I the continuity between the two functions.

> *Saul:* Um, so now we're just doing that? We're including 0 and its going [to 1]. Pardon my notation…and now the question is, is this continuous?
>
> *Interviewer:* Right.

Saul: I don't see why it-…intuitively it seems like, yeah sure its continuous. Because I don't see any problems.

Interviewer: So, the empty hole is not up here, right, as part of that image. I mean it's not an empty hole any more, it's a closed hole!

Saul: Okay, yeah but I don't see why that would be a problem. As far as the definition, um, it doesn't seem like it really changes anything. It still, you know, I mean, that kicks back to the same…well not exactly the same, but still an open set.

Interviewer: Right, instead of two open sets.

It seemed that Saul was focusing on the pre-images of sets that included both points in the range and noting that the domain, presented as a union of two sets in the problem $(-1,0)\cup[0,1)$, was in fact a single open set $(-1,1)$. I asked him to consider sets around 1, as he had done in the previous problem.

Interviewer: So, what if I just focused around 1. If I put an open interval around 1?

Saul: Up here, in the image? An open interval around 1? I don't see why that would change anything. I mean, do you think it would?

Interviewer: Um, I don't know.

Saul: I don't see any...

Interviewer: Well what would be the pre-image of an open interval around 1. Let's say I put an open interval around 1 like $(0,2)$?

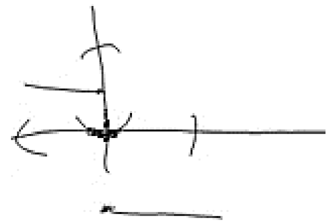
Saul: The pre-image would be 0…wait what do you want to do? Sorry.

Interviewer: Um, put an open interval from 0 to 2 in the co-domain and look at the pre-image of that.

Saul: $(0,2)$?

Interviewer: Yeah, I just want to include 1 in an open interval…

Saul: Like that [indicates an open interval he drew on the y-axis]?



Interviewer: Mhm. What's the pre-image of that?

Saul: Um…I mean, it would be this [indicates $[0,1)$] plus the empty set…union the empty set.

Interviewer: Okay.

Saul: I mean, because the 1 is going to go here, that's the pre-image of 1.

$$[0,1)$$

Interviewer: Mhm.

Saul: And then everything that's not 1 is that [the empty set]. So…

$$[0,1) \cup \emptyset$$

Interviewer: What about that set?

Saul: Is it open?......I'm thinking I want it to be open. Because I don't see why this isn't continuous.

Saul was convinced that the function was continuous, but his reasons for this belief were not clear. He said he couldn't find a "problem" that would alter his solution to the earlier task. He seemed willing to declare that $[0,1)$ was an open set based solely on his conviction that the function was continuous.

Interviewer: Now, we already established that the empty set is an open set.

Saul: Sure.

Interviewer: But in the standard topology is $[0,1)$ an open set?

Saul: Oh…well, no.

Interviewer: So, you're searching for ways to make it open?

Saul: Yeah.

Interviewer: But it's not, right? It's just not an open set in that topology.

Saul: No. And yet I am convinced that this is continuous!

Unlike in the previous task (see 5.6.1.1), Saul's intuition seemed to play a dominant role in his reasoning about the new situation. Despite his accurate formal reasoning in identifying the appropriate pre-images and identifying those pre-images as open or non-open sets, he was unwilling to abandon his previous claim that the new function was continuous. To resolve the two competing possibilities, I prompted Saul to consider the differences in properties between the functions in Task 3.2(A) and B, and their respective domains and co-domains:

> *Interviewer:* I think your definition is good: a function is continuous if every open set in the co-domain has an open pre-image in the domain……so you're convinced that its continuous in the same way that you were convinced it was continuous when it was still this function, right? That rule? Because when you said that you felt like this was continuous, you said it's because we restricted the domain. And so, it's okay that it [the graph] breaks, because we don't even acknowledge zero in the domain, right?
>
> *Saul:* Mhm.
>
> *Interviewer:* But now we're acknowledging zero, right? So, our domain is actually…what? It's the entire interval $(-1,1)$. Which is different from this domain [indicates $(-1,0)\cup(0,1)$] in what way?
>
> *Saul:* Um, it includes the point 0.
>
> *Interviewer:* So, that makes it one connected set, right?
>
> *Saul:* The fact that it splits over in steps in the output part doesn't change the input part, the domain, or the pre-image part. Is that correct?
>
> *Interviewer:* What do you mean by that?

> *Saul:* …I guess, I mean, I was thinking that this is not actually the pre-image of this function. This is just a rule, this [indicates $(-1,1)$] is the actual pre-image…
>
> *Interviewer:* Of the entire function?
>
> *Saul:* I guess I'm thinking that going by the definition, the pre-images of open sets have to be open sets. This is not the pre-image then.
>
> *Interviewer:* Well, that's the pre-image of-
>
> *Saul:* That particular value, sure…and this [indicates $[0,1)$] is part of this [indicates $(-1,1)$], which is an open set. I guess what I'm thinking is along the lines, if this [indicates $(-1,0)$] did not exist, then……then you would have a problem…

It was apparent that Saul was attempting to use the same reasoning as in the previous task to argue that restricting the domain to $[0,1)$ would render the function continuous. We discussed the fact that we were considering the function on its entire domain, and he decided that his reasoning was inaccurate. He pointed to his personal, unexplained intuition about the problem as the reason for his difficulty accepting that the function was discontinuous:

> *Saul:* It doesn't feel right […] Usually if I don't think something seems right, it's not right!

Finally, I asked Saul to consider the limit definition for continuity, rather than the topological definition. Considering the function through this less-abstract continuity checking process, he quickly recognized the function's discontinuity.

> *Interviewer:* So, if we think about epsilon and delta…
>
> *Saul:* They're going to zoom in here, and yeah…okay. Alright. I feel like I'm-, I feel like switching my frame of reference trips me up. Okay, okay.
>
> *Interviewer:* You found it to be very intuitive that it was continuous, the first one, despite the fact that it was broken…but then switching…

Saul: Switching reference, yeah, okay well…

Saul demonstrated a dynamic image for the limit concept that enabled him to recognize the function as non-continuous, whereas the topological definition of continuity had failed to provide him with a mechanism for appropriately checking the function. His image of 'zooming in' may have given him an experiential framework with which to reason, while the notion of an open set remained formal and abstract in his mind. Reasoning with those notions did not "feel right" despite his formal understanding of the definition.

5.6.1.3 Saul's response to Task 3.2(C)

After having reasoned through the previous two tasks, Saul had no problem recognizing that the function in Task 3.2(B) was continuous when given the lower limit (half-open interval) topology.

> *Interviewer:* Now what if this was, what if these were both in the half-open interval topology? […] I want the domain to stay exactly as it is, but now I want to say that basic open sets are half-open intervals.
>
> *Saul:* Okay. So, we're defining open sets on the domain as being half-open intervals. I got it, okay.
>
> *Interviewer:* But I want to stick to our new version of this function now [Task 3.2(B)], that you just decided you didn't want to be discontinuous, but you think the definition says it is…
>
> *Saul:* Okay, well then it is continuous.
>
> *Interviewer:* Because then…
>
> *Saul:* Well, it's just the 0 is included. But that would just go back here [indicates $[0,1)$] and be okay […] So yeah, it's continuous.

It was not surprising that Saul could reason effectively about this task, considering that he had already analyzed the function involved. He had established that the "problem" with its continuity would come from the pre-image of any set containing 1 but not -1; and he had already noted that this pre-image would be a union of the lower half-open interval $[0,1)$ and the empty set, both of which are open in the new topology.

5.6.2 Nolan's responses to the continuity tasks

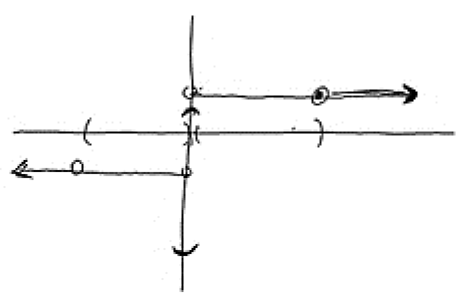
Nolan's interviews revealed a student with a strong formal understanding of concepts from analysis and calculus, as well as rigorous proof methods and syntactic procedures. He had strong intuitions from his experience with the standard topology on the real numbers, but he was willing to work hard to overcome his spatial intuitions and learn the content mathematically. As he said, "I knew my intuition wouldn't take me far in this class."

5.6.2.1 <u>Nolan's response to Task 3.2(A).</u>

At first, Nolan drew a graphical representation of the function and decided that it was not continuous. Although he referenced the formal definition, he still explicitly mentioned the need to find a warrant for this claim.

> *Nolan:* …Ok, is it continuous? So, it's going to be continuous if for all open sets in $\mathbb{R}$ …so let's just find an open set that doesn't work right?

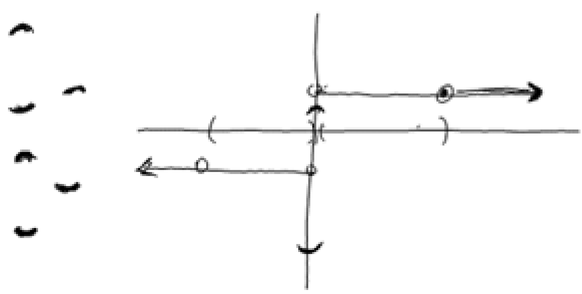
$$f: (-1,0)\cup(0,1) \to \mathbb{R} \quad ; \quad f(x) = \begin{cases} 1, & x \in (0,1) \\ -1, & x \in (-1,0) \end{cases}$$



By stating that he would attempt to "find an open set that doesn't work," Nolan indicated that he did not believe the function to be continuous. After drawing the graph of the function, Nolan wrote down the condition he would use as the criterion for deciding whether it was a continuous function or not. He then used it as a basis to look for a counterexample, having assumed that the function was discontinuous based on his graphical representation of it.

<u>cont:</u> $\forall\ O^{open} \in \mathbb{R}$, $f^{-1}(O)$ is open in $(-1,0) \cup (0,1)$.

> *Nolan:* I think it is a little tricky. Because I need to find an open set whose inverse image is not open. These [-1 and 1] are not good candidates to do so…
>
> *Interviewer:* So, you want to show that it's not continuous?
>
> *Nolan:* Right […] So then, I'm thinking…I think I can have an open set, there's probably more than three cases, I'm going to go with the three. Ok, so no matter how I split the open set over all this, they're all going to be open. I should choose something over here…um, so…yeah, let's try 0…because yeah, when *x* equals 0, the function is undefined.



In the figure above, the marks on the left-side of the graph represent open intervals in the co-domain. Nolan took these as three possible cases to check the pre-images were open. The representation is perceptually suggestive, being a disconnected graph, but Nolan attempted to provide a rigorous proof for his assertion.

Nolan: Right, so I feel like there's a hole here [indicating the origin], and a hole here [indicating -1 and 1 on the *y*-axis]. So, I have an inkling that this is going to be discontinuous, um…because of the jump.

Interviewer: Because there's a jump?

Nolan: Yeah, I mean that's probably the most obvious reason, but you have to prove it, right? So……

Interviewer: I guess the question is, what point is it discontinuous at?

Nolan: Which *x*-value? I think the origin…

Interviewer: And you just told me that's not in the domain…

Nolan: Oh, I hate this so much! It has to be discontinuous *in* the domain! […] So, let's take…

Although I did not examine his use of the term "discontinuous" at the time, it is important to note the difference between a *non-continuous* function (on some domain or subset of the domain) and a point of *discontinuity* of a function. A point of discontinuity "need not be in the domain" of the function (Thomas, 2005). This is a significant source of confusion for many students as early as calculus, because a function that is continuous on its domain may have one or more points of discontinuity (Shipman, 2012). Nolan's final response seemed to indicate that he understood that $x = 0$ is not a counterexample to the assertion that the function is continuous. In this context, it does not seem that he meant to assert that a point of discontinuity must be in the domain.

Without further reflection, we adopted the term "discontinuous" (rather than "non-continuous") for the remainder of the interview. I redirected Nolan back to the three cases he had drawn before.

Interviewer: So, out of these three cases here, these are the three kinds of open intervals you think will make a difference in the co-domain?

Nolan: Yeah. So, I mean…how about just…oh, this is breaking my brain! Because if I'm looking at $(0,1)$, I'm wondering, you know, what *x* gives me 0…um, none do right?

$$f^{-1}(0,1) = \emptyset$$

Interviewer: None of them give anything between 0 or 1…

Nolan: Right, and so, since I didn't include 1, my instinct tells me that this is empty, right? Which is open so that's okay. So then, um, let me try that one again […] Okay, so, I mean, I know where -1 goes, or where it comes from rather, I know where 1 comes from…um…

$$f^{-1}(-1,1) = (0,1) \cup (-1,0)$$

Interviewer: And everything in between?

Nolan: Is empty, so it's just a union…I would think it's just a union of those, since everything in between is empty.

Interviewer: Yep, that makes sense.

Nolan: That's still open, in the usual topology. So, I might need to go beyond 1…

Interviewer: So, you're convinced that it's discontinuous and you're just trying to find a good open set-

Nolan: Well, now a little bit I'm not so convinced that it's discontinuous, right? Because it's an important distinction, right? Like, discontinuous *in the domain*, right? Whereas, where I was like, you know, thinking about things, the point 0, which isn't in the domain, so it's inconsequential for what I'm trying to do. So, do you think it's a good idea to start trying to, like in this case, try to prove it true, and if you hit a wall maybe that's a good sign.

Interviewer: Try and prove that it *is* continuous?

Nolan: Yeah, and if you hit a wall, is that a sign maybe, or…?

Interviewer: That's one way to go about it…I mean, to prove it continuous, you would…

Nolan: Just let- [writing]…

$$f^{-1}(-1,1) = (0,1) \cup (-1,0) \quad \text{open}$$
$$f^{-1}(-3, \tfrac{1}{2}) =$$
$$f^{-1}(-1, \tfrac{1}{2}) = \emptyset$$
$$f^{-1}((-3,-1] \cup [-1, \tfrac{1}{2})) \quad \text{open}$$

Nolan worked out the pre-images of several intervals that were examples of the cases he had outlined previously. He was unable to find any interval that did not fit his criteria for continuity, and he began to reluctantly accept the new conclusion.

> *Interviewer:* You sort of had these three candidates, or three cases of the ways that you could take an image, and none of those are showing you a non-open pre-image, right?
>
> *Nolan:* Mhm, so just try to break it into cases and try to prove it?
>
> *Interviewer:* Do you think those cases exhaust what can happen here?
>
> *Nolan:* Yeah, and…and so, you know, everything below -1 on the image, um, is being mapped from nothing. As is, this $\frac{1}{2}$. And so, what if I do, like… -1, not including $\left(-1, \frac{1}{2}\right)$. That's just going to be empty…Mmm, I'm starting to think this is continuous then…Yeah, it's open too. I'm pretty sure that's open too, because this is going to be coming, this, the inverse image of this, because I know that *f* inverse of the union is the union of the *f* inverses. So, *f* inverse of this is going to come from here. This is going to be equal to that, and the *f* inverse of this is going to equal the same thing. So, I guess I could start proving it. I don't think there's……

Nolan accepted the conclusion of his logical train of thought, despite the perturbation it caused in his understanding.

> *Interviewer:* So, it bothers you that you can't find a way to prove it's not continuous?
>
> *Nolan:* It's just, I have the frickin' intuition, you know! Of like, it's a jump, it's going to be discontinuous, but the domain, right?! We're not going from $\mathbb{R}$ to $\mathbb{R}$ …

Interviewer: […] So, the only place it looks discontinuous is where it doesn't exist.

Nolan: Where it's not defined, yeah.

Interviewer: So, I mean…are you convinced that this is a continuous function?

Nolan: I think so, yeah. And I mean, I think, and this, like picture-wise, I drew these going on forever, but that's not what's happening.

Interviewer: Mhm, but even if it did, would that change the continuity of the function?

Nolan: No, because, a point from here could come over here, right? Real, it doesn't necessitate these values being in the domain…

Nolan: Yeah. So, if we were to just go from $(-1,1)$ instead, um, then it would be discontinuous at 0.

Summing up his understanding of the situation, Nolan anticipated the subject of the following task, and we moved on to it.

5.6.2.2 Nolan's response to Task 3.2(B).

Nolan seemed to have interiorized the operations involved in the formal definition of continuous function, based on his responses in the previous task. However, like Saul, when I presented Task 3.2(B) to Nolan, he felt that the function defined on the new domain should still be continuous. He seemed to base this response on the superficial similarities between his visual representations of this and the previous function:

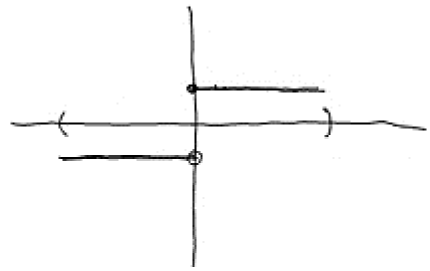
Interviewer: So now, it [the domain] is essentially $(-1,1)$ because I'm including 0 on the end of this interval.

Nolan: Yeah. I think it's wha-, it is what I wanted it to be!

Interviewer: Right, so would that make the function…not continuous?

Nolan: Yeah, so if I have…no, it would be continuous still, um because x equals 0, right, x equals 0 then $f(x)$ equals 1. It's not, it's defined at x

equals 0. I think it's still continuous, I mean, I think all we did was just continue the line…



Interviewer: Just drew that extra dot, right?

Nolan: Yeah.

Since he seemed not to have reflected on the three cases he had just examined in the previous task, I asked Nolan to focus his attention on those intervals, rather than the empirical properties of his representation. He examined them and eventually found the source of the issue.

Interviewer: So, well let's take a look at all the intervals you had-

Nolan: Oh, no! I have a hard time with continuity, let's see...we're going from $-1$to 1, that way, that way…alright, so […] Okay, an open set in the range…I mean, yeah, it's not going to be a union of open sets though, I think that's the moral of the story…right? […] Yeah, because, -2 coming to -1 is empty, but once you get to -1, that like, between -1 and 1, you know, we're empty, but then where this…

$$f^{-1}(-2,2) = (-1,0) \cup [0,1)$$

Interviewer: Yeah, at first glance, that looks like two non-open sets, but…

Nolan: Yeah, no, I should have known better, honestly.

$$= (-1,1)$$

Interviewer: Okay, so that one worked, so when you encompass everything [in the range] it's still not a problem…

Testing the next case, Nolan recognized the reason why the newly defined function was not continuous.

> *Nolan:* […] So I need to go 2, or -2 …um…we're not including 0, but we're including 1, so this is going to be…oh, this is where it fails! I like that. Oh, okay, now that's immediately clear, so I mean I know-, so okay, so I know this is not…okay, I know this part is not an open set…
>
> $$f^{-1}(0,2) = [0,1)$$
>
> *Interviewer:* Right…
>
> *Nolan:* in the range, right? But I-, yeah, and so I know- I just think-, right, I know the pre-image of that-, of an open set that gives me this pre-image, right? I know…I think that should be the way, that might be a helpful way for me to look at this, it's like, well here's a pre-image that's not open [indicating $[0,1)$], maybe I can find a corresponding image that is open.
>
> *Interviewer:* Right, or some open set that includes that [$[0,1)$] as a pre-image.
>
> *Nolan:* Yeah, like $(0,2)$.

After expending the mental effort to take into account the three cases he had previously investigated; Nolan was quickly able to recognize that the function was not continuous on its domain. I asked him what he had based his reasoning on and he pointed out gaps in his experience with functions.

> *Interviewer:* So, that's interesting…why is it you felt like, by changing this [adding 0 to the domain], it wouldn't change the continuity?
>
> *Nolan:* I figured…I just think-, I mean honestly I really think it's this whole misconception of…I mean the problem is a misconception that's derived from this idea, but it's just the whole thing—for my entire life I've been doing $\mathbb{R}\times\mathbb{R}$, you know, $\mathbb{R}$ to $\mathbb{R}$, and…it's just, I'm used to it being, like if it's $\mathbb{R}$ to $\mathbb{R}$, then if, um…I'll say it like this: I think it's trivial to see discontinuities from $\mathbb{R}$ to $\mathbb{R}$, for me, because I'm used to that, like I think this kind of plays into, I probably didn't practice enough of this stuff in this class, um…right, so my ideas of continuity are still doused in $\mathbb{R}$ to $\mathbb{R}$ …yeah, and I think that's just really clear…

Nolan had some difficulty articulating it, but I interpret his statement as a description of two specific mental challenges he confronts when reasoning about the continuity of topological functions. First, he emphasized that his past experiences with the property of continuity had largely been in the context of the real numbers in their entirety, rather than functions with subsets of the real numbers as the domain or range. Then, re-wording his thoughts, his last utterance was an implicit reference to a second challenge: that the continuity of functions is a salient, empirical property in the context of the standard topology on the real numbers. He claimed that his conceptions were "doused" in this experiential context, and he seemed to imply that his reasoning about more generalized topological functions was obstructed by this characteristic of his concept image for continuity.

> *Nolan:* But, I just think it would help a lot if I had been introduced to, really even like this kind of notation, right? Like, just the idea, in and of itself, because the whole paradigm of like, the Cartesian plane, it made me think that before this class, I wholeheartedly believed that, like you have a function from $\mathbb{R}$ to $\mathbb{R}$, you have functions from complex numbers to complex numbers, but I never really thought of having functions from intervals to intervals and stuff. So, I don't know…

5.6.2.3 Nolan's response to Task 3.3(C).

As with Saul, Nolan could reason through this task by returning to the cases he had initially examined. Altering the topology of the space did not seem to affect the required process significantly.

> *Interviewer:* So, what if this was in the half-open interval topology, would it be continuous or discontinuous?
>
> *Nolan:* So, the example I found was having a half-open interval which is not open in the usual topology. And so, I think, the question that comes to my mind is, is every counterexample I found from the usual topology, does that provide an inverse image that is half-open?

Interviewer: Half-open…

Nolan: And if so, then it will be continuous on the half-open interval topology. So, I think it is [continuous].

After this, Nolan and I continued to discuss his conceptions for a few more minutes. He re-emphasized his confusion over piecewise functions:

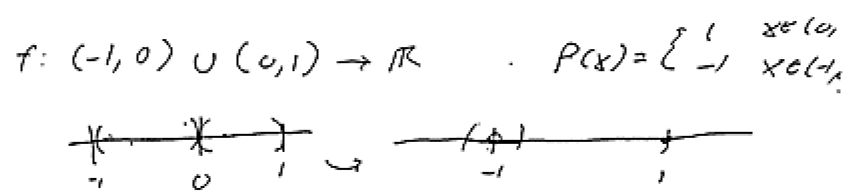
> I just don't think I've ever seen a piecewise function outside of this class that wasn't from $\mathbb{R}$ to $\mathbb{R}$, you know what I mean? […] I mean if this was from $\mathbb{R}$, if this piecewise function was from $\mathbb{R}$ to $\mathbb{R}$, then um, it would be discontinuous because of that jump, right?

5.6.3 ***Wayne's responses to the continuity tasks.***

In previous tasks, Wayne had reasoned in formal-syntactic ways, sometimes relying on visual representations to organize his understanding of formal definitions and theorems. At other times, he would reason without such representations, only writing his arguments in proof form after he had considered them in his mind. The visualizations Wayne used did not usually present obstacles to his reasoning.

5.6.3.1 <u>Wayne's response to Task 3.2(A).</u>

Wayne began by drawing a graphical representation of the function that was similar to Gavin's horizontal number line. However, Wayne included another copy of the number line to represent the domain. At first, he didn't believe the function to be continuous, but after applying his formal scheme for the topological definition of continuity, he changed his mind and began to prove that the function was continuous.



Interviewer: What's your hunch? Do you feel that it's continuous, or not?

Wayne: Um, well if I pick-, I don't think it is, because if I pick, you know, some point, um, -1, and put an open set around it…then in the pre-image, which would go somewhere in here [indicates $(-1,0)$]…is there an open set around it? Um……..ah, so…[writing]

PR:
Let $c \in (-1, 0)$. Then $f(c) = -1$, if we take an open set around $f(c)$ then the pre-image of the set should be open.

Interviewer: And what does that say?

Wayne: Uh, should be. Should be open.

Wayne was setting up an argument that the function should be continuous at any point in the interval $(-1,0)$, through a reversal of the function process, using the pre-image operator. He did not specify an open set in the range, so I was unsure how he was conceiving of this process, or how he warranted his claim that the pre-image of such a set would be open. I asked for elaboration.

Interviewer: So, what is the pre-image of -1?

Wayne: The pre-image of $-1$is, uh, anything from -1 to 0. It's in that set [indicates $(-1,0)$].

Interviewer: And that's an open set?

Wayne: Yeah, but…the open set around……

Wayne hesitated, recognizing that he had no scheme to account for any other points in his generic open interval besides $f(-1)$.

Interviewer: You're wondering what to do with all these other points out here [indicates points in the co-domain that aren't in the range of the function]?

Wayne: Mhm.

Interviewer: So, what's their pre-image? What's the pre-image of -0.9?

Wayne: Uh, the empty set. There isn't…yeah, so…

Interviewer: Is that open?

Wayne: No…

Interviewer: The empty set?

Wayne: Uh, I mean…it's open and closed isn't it?!

Interviewer: Hm…and we don't necessarily mind if it's closed, right?

Wayne: No.

Interviewer: You just want to make sure it's-

Wayne: Open, yeah so I guess…

Interviewer: You're just worried that you're capturing too much stuff out there?

Wayne: Yeah, yeah, that's what I was thinking. I was like, but I guess anything out here is the empty set, so…um…so I guess then you'd write pretty much the empty set should be open, which would be, uh, $(-1,0)$; and the empty set which is open, therefore, the pre-image is open, or, yeah, the pre-image is open. Thus, it's continuous.

Which would be $(-1,0)$ and $\emptyset$. Therefore the pre-image is open.

Wayne was easily able to assimilate empty pre-images into his formal scheme for the topological definition of continuity, and conclude that the function was continuous. Further discussion revealed that he understood the need to check this condition for other points in the domain as well.

5.6.3.2 Wayne's response to Task 3.2(B).

Unfortunately, when I gave Wayne the prompt for Task 3.2(B), I failed to draw his attention to its symbolic representation of the function carefully enough. Based on my verbal explanation of the function's differences from the function in Task 3.2(A), Wayne had the impression that the domain was $[-1,0)\cup[0,1]$ instead of $(-1,0)\cup[0,1)$. This affected his answer before I realized what had happened. Nevertheless, his reasoning with this version of the task's function was insightful as well.

> *Interviewer:* Mhm. So, let me change the problem slightly…So, I'm going to close this part of the interval and make it $[0,1)$, and whenever x is in $[0,1)$, we'll go to 1…I just want to see how that affects your answer.
>
> *Wayne:* Ok [writing]…Let c equal some point in…we'll just say it's equal to -1. Then the pre-image of c will be equal to some x in $[-1,0)$. If we take an open set around c…[writing]…

let $c = -1$. Then $f^{-1}(c) = x \in [-1,0)$. We take the open set around c. Then we have $\emptyset \cup [-1,0)$. This is not an open set Therefore, f is not contnous at point c.

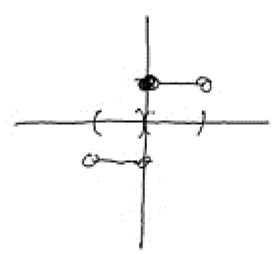
Based on his understanding of the task prompt, Wayne had found evidence to argue that the function was not continuous. It was at this point that I recognized our earlier miscommunication and corrected it.

However, other issues arose when analyzing Wayne's reasoning based on his interpretation of the task's function. While using his personal concept definition for *continuity at a point*, Wayne mistakenly selected the point c (at which he intended to check for continuity) from the range instead of the domain of the function. He then selected an "open set around c" and found its pre-image under the function. He did not indicate what that open set might be, but seemed to be imagining an open interval around $y=-1$ that would not contain $y=1$ (i.e., and interval (a,b) such that $a<-1$ and $b<1$).

Despite this inaccurate interpretation of the formal definition for continuity at a point, Wayne seemed to have little trouble reversing the function process to arrive at the correct pre-image of this generic open set. Based on his previous, more accurate interpretation of the continuity definition in Task 3.2(A), it seems likely that Wayne's mistake in this context was not the result of a significant misunderstanding. Further discussion clarified that Wayne recognized that he would need to define the function's point of discontinuity at a point in the domain, rather than the range.

At this point, I felt that Wayne's disconnected, one-dimensional representations of the domain and range may have been a kind of semiotic obstacle to appropriately interpreting the formal definition. I prompted him to consider a two-dimensional graph instead.

> *Interviewer:* Now, it's interesting to me that when you originally drew this, you drew it as just two single number lines. Do you ever think about drawing it, like, in $\mathbb{R}^2$?
>
> *Wayne:* On the plane? Yeah, I guess the reason I did it was just because it was on $\mathbb{R}$ […] Oh, okay. So, do you mind if I take a second here to draw it? [drawing] So, um…then you'd have $(0,1)$, and from there if it's in, uh, it'll be basically…open…open, and then the same thing from -1, open and open. So, yeah, I guess when you draw it like this, oh…yeah the discontinuity would make sense if it happened at 0.



Once he was encouraged to graph the function in a more conventional way, Wayne immediately perceived the discontinuity as a salient feature of his representation. He could pinpoint the location of the discontinuity, though he did not attempt to re-formulate his argument based on this new information.

5.6.3.3 <u>Wayne's response to Task 3.2(C).</u>

It was unfortunate that I did not prompt Wayne did not return to Task 3.2(A) to query any disparities he might have noticed between his new visual representation and his previous formal demonstration that the function was continuous. However, once we had corrected our earlier miscommunication, I hoped to learn if he felt any perturbation through the a similar discrepancy in Task 3.2(C).

> *Interviewer:* What would happen if it was in the half-open interval topology?
>
> *Wayne:* Okay. So, try to prove that it's continuous again?
>
> *Interviewer:* Or discontinuous, whatever you think it is. But you were thinking it was continuous this way [Task 3.2(A)], then discontinuous with the closed 0 [Task 3.2(B)], and now we're switching topologies on you.
>
> *Wayne:* Okay, so it's still closed here up at the 1. Alright, so, but now it's in the half-open interval topology. So, we'll just start the same way [writing], let's c equal 0 this time, and then f inverse of c is equal to x, which would be in the, um…wait a minute……okay, huh…I started without thinking…so then x is going to be in $[0,1)$, uh, and when we take the open set around c…yeah, we get the empty set and $[0,1)$, so this is an open set…therefore f is continuous at the point.

let $c = 0$. Then $f^{-1}(c) = x \in [0,1)$. We take the open set around c. We get $\emptyset \cup [0,1)$. This is an open set in $\mathcal{H}$ therefore, f is continuous at point c.

Although he continued to interpret the formal definition somewhat inaccurately, and again examined only one case before claiming to be convinced, Wayne did select the appropriate point from the domain, 0 (referring to its image as $c = 1$) to check for continuity. Again, the remainder of his formal reasoning was accurate and he correctly deduced the proper answer to the task.

I asked Wayne how he felt about his solution, and after a short discussion he explained that he had not used his graphical representation from Task 3.2(B) to reason about the new function. He had performed his analysis syntactically.

> *Wayne:* Mhm. I guess, you see that and it's just like, oh, that's a basis, a union of basis's so, that makes sense. Whereas, back there we were, it was

just, I was trying to think more visually, rather than just, oh yeah, that's open in this topology.

In order to verify the continuity of the function, he referenced the formal definition of *open set* as his criterion for checking whether the pre-image was open.

5.6.4 Maren's responses to the continuity tasks

In previous tasks, Maren had relied on her mathematical experiences and embodied metaphors to make sense of her idea of continuous functions (see Section 5.7). However, by the final interview, she demonstrated her capacity to reason axiomatically, by appropriately using aspects of the formal theory to justify her empirical observations.

5.6.4.1 <u>Maren's response to Task 3.2(A).</u>

Maren began by referencing what seemed to be a mathematically accurate personal concept definition for continuity. But then she faced the challenge of interpreting that definition in a case where the function's image contained no open sets.

> *Maren:* Okay, so I just want to show whether f is continuous or not? So, a continuous function would mean that an open set in $\mathbb{R}$ would have to be open in X, but we don't have open sets because there's just one member…and that's not an open set…so then our other rules of continuity are…if it's continuous at a point, but that has to do with sets too…

After some discussion, Maren realized that her definitions for continuity made reference to the co-domain of the function, and not its image, which offered a logical resolution for her disequilibrium. However, she still indicated some perturbation when attempting to use the definition in this way.

> *Maren:* I see what you're saying now. But if I-, the thing that's confusing is that we're using f inverse, yeah?
>
> *Interviewer:* Yeah, or the pre-image…
>
> *Maren:* But the only thing about using f inverse, I thought we were supposed to pick from the range, because those are the only things included. Like if f only maps to two points then how can we take a

different point and put it back to x, like f never mapped to it in the first place, how could we bring it back?

Interviewer: What would be the-

Maren: Its like if I drew you a parabola, x-squared. And then I tried to say, okay, take f inverse of -20.

Interviewer: What would that pre-image be?

Maren: It doesn't have one.

Interviewer: So, in mathematical terms, what do we say that is?

Maren: It's the empty set.

Interviewer: It's the empty set. That's part of the topology right?

Maren: Oh, you got me! And the empty set is an open set.

Interviewer: Right.

Maren: But then it gets awkward, because you could use that argument for other things.

Interviewer: Like?

Maren: I don't know!

Maren was initially confused by the similarities between the *pre-image* and *inverse function* concepts (and likely their identical notations as well). Since an inverse function has the properties of a function, each point in its domain must have an image under the function. However, the pre-image operator may act on any point in the co-domain, even if it returns nothing. Thus, Maren could not conceptualize the pre-image of a point that was not in the image of the function until she had disassociated pre-images from inverse functions in her mind. She was finally convinced of the logical reasonableness of the definition after connecting it with axiomatic descriptors, such as "open set" and "empty set," that permitted her to assimilate the new definition into her scheme for continuous functions.

Maren: So……if I were to do…the inverse of, I need an open set containing 1, so $(\frac{1}{2}, \frac{3}{2})$, $(0,2)$, whatever; and we did the inverse of that, only 1 would get mapped to the open set $(0,1)$, and since $(0,1)$ is open, does that mean it's continuous? […] So, it's continuous.

Maren did not consider the necessity of checking various cases, as Nolan had done. She seemed to anticipate that every case would yield an open pre-image, but I was unsure whether she had reflected on each possibility. I encouraged her to look at more cases and her response seemed to indicate that she had interiorized the process for checking continuity, and had applied it to several cases in her mind:

Interviewer: Now that has to be true for every open interval in the co-domain, right?

Maren: Well, anything that doesn't contain 1 or -1 will map to the empty set…so…

Interviewer: […] So, every open interval you can imagine over here [indicates the co-domain of the function] is…going to some open set [in the domain]?

Maren: Mhm.

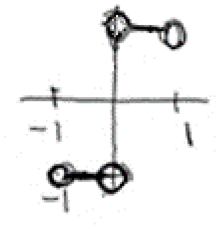
As Saul had done, Maren responded to the task only verbally, without writing any words or symbols, or drawing any graphs or diagrams. I asked her how she would represent the problem graphically:

Interviewer: Now, does that…does it feel like a continuous function to you?

Maren: Yes, yeah.

Interviewer: What does it look like, could you graph it?

Maren: Yeah, its-, I could graph it……It's a continuous function from a disconnected space onto another disconnected space. So, it's not going to be smooth, but the function itself is continuous, I think that's why I was so confused in our previous interview…[writing]…so we have, here's -1 and 0, and so we have 1 here, -1 here. So, then we have $(0,1)$, so we have open circles, and then we have -1 down here.



After considering what the function's graph would look like, Maren noted a key property that seemed to justify the graph's counterintuitive appearance in her mind:

> *Interviewer:* Mhm. So, that doesn't visually bother you for it to be continuous?
>
> *Maren:* No because it came from a disconnected space so we would expect it to map to a disconnected space. If it came from the real line then it wouldn't be continuous, because the real line is continuous.

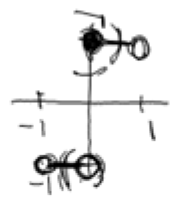
It seems clear that Maren had previously recognized and interiorized the relationship between continuous functions and connected graphs. Although she still demonstrated some confusion, in that she declared the real number line to be continuous, she did seem to have formed a distinction in her mind between the notions of *disconnected* and *continuous*.

5.6.4.2 Maren's response to Task 3.2(B)

At this point, I hypothesized that Maren was demonstrating a product of her reflective abstraction, which differentiated the property of continuity for functions; and the property of being disconnected, which is a property attributed to spaces. She seemed to be able to differentiate the two concepts and coherently justify her mathematical beliefs using the distinction. However, it was also possible that Maren was reasoning from an exemplar or prototype in her concept image; and that her use of these terms related to these analogies, rather than to the mathematical theorem in the textbook

(Croom, 1989), which I assumed she was referring to. To test this hypothesis, I modified the task to 3.2(B) and asked what she thought:

> *Maren:* So, then it would be…but if you do that…if you include the 0 then the union would be $(-1,1)$, because we're now including 0, and this is a connected space [indicating $(-1,1)$]. So, if we were to map it onto a disconnected space it wouldn't be continuous because I could take an open set in $\mathbb{R}$ from…I guess we would want to include 0 and part of it would be down there and part of it would be up here. Like if we did, I mean just this interval, in the range…if we did *f* inverse of $(-1,1)$, then from $(-1,0)$ would map down here [indicating $y=-1$] and $[0,1)$ would map up here [indicating $y=1$]. Right? Am I thinking about that wrong?



> *Maren:* […] So, if I take $(-1,1)$, -1 would map…it would be here [indicates lower segment on graph] right? It would be this whole line [*sic* segment].
>
> *Interviewer:* So, you get everything in here?
>
> *Maren:* It would be the whole line. And then 1 would be the closed line [*sic* segment], so it would be a union of this line and this line, right? Which is a disconnected space?
>
> *Interviewer:* So, if you took the pre-image of $(-1,1)$, you're going to get what? Isn't it $(-1,1)$? And, union with the empty set because of all the stuff in the middle?
>
> *Maren:* Mmm…
>
> *Interviewer:* So, I don't think that would prove it's discontinuous yet, because you're going to get an open set [in the pre-image]…

Maren was attempting to use her new understanding of the relationship between continuous functions and connected spaces, which she had coordinated as a result of an accommodation during or after our previous interview. The relationship is mathematically tied to Theorem 5.2 in Croom (1989) (see Section 5.3.3.1). However, she had initially attempted to apply it in a superficial way, without revisiting the processes involved in using the theorem. When she seemed to be stuck, I asked her to explicitly consider the pre-image of a set containing 1 from the co-domain:

> *Interviewer:* What if you took an open set around just one of those components, and it would be around the range, right, so you'd put an interval around there [indicates 1 on the *y*-axis]. And you're going to get what as a pre-image of that?
>
> *Maren:* A point?
>
> *Interviewer:* Which *x*'s go to 1?
>
> *Maren:* 0…to 1…closed…Oh! And it would be like this…[writing]...
>
> $$[0, 1)$$
>
> *Interviewer:* And that's not an open set?
>
> *Maren:* Mm-mm…
>
> *Interviewer:* So, you've got an open interval whose pre-image is not open, right?
>
> *Maren:* Yeah, that's better!

In her response to the first task, Maren seemed to have interiorized the process involved in checking the property of continuity. However, in the new, modified version of the task, she was unable to apply her new understanding in a systematic manner without significant prompting. She did indicate her understanding of the process, but a source of difficulty seemed to be her repeated confusion about the images and pre-images of sets. This confusion may have been detrimental to her ability to fully interiorize the process.

Before moving on to the third and final task, I asked Maren to elaborate on her understanding of the relationship between connected spaces and continuous functions. Her responses indicate significant self-reflection about the topic since our last interview.

Interviewer: So, you think in the last interview that you were a little bit confused about connectedness versus continuity?

Maren: I didn't realize that a function could have properties.

Interviewer: Oh, what do you mean by that?

Maren: Like a function can be contin-, like there's a difference between a continuous function and like…that's not describing the sets, it's describing the actual mapping. And I think I was getting confused, and looking at what the sets look like, and seeing if the sets were connected. And getting that confused with if the function is continuous.

Maren expressed a difficulty that is likely faced by many students in their attempts to reconcile and coordinate different conceptions that stem from different forms of abstraction. Maren used her reflective abstraction based on Theorem 5.2 to explicate her empirical abstractions concerning the connectedness of the function's domain and range. However, she encountered difficulty when attempting to abstract the processes involved in the definition of continuity.

5.6.4.3 <u>Maren's response to Task 3.2(C)</u>

Maren noted the similarities between the Task 3.2(C) and the previous problem. She used the same process in the context of the new topology and quickly decided that the new function was continuous.

Maren: So, if we're in the half-open interval topology, would still be written like this? Or would…okay, but would the -1 be included too?

Interviewer: No.

Maren: Okay, so this is…okay, so is -1, the question now is, is $(-1,0)$ considered an open set, right?

Interviewer: You told me that those were open sets, right? [Task 3.1(B)]

Maren: How do we write $(-1,0)$ as an open set?

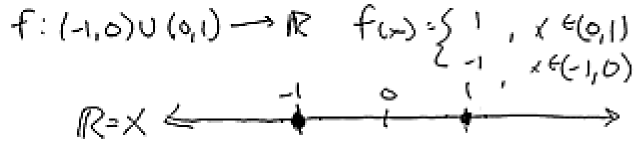
Interviewer: […] But what's the purpose of writing this as an open set? I mean, you know it's an open set now, but what is-

> *Maren:* Okay, because we want open sets to map to open sets, that's why I wanted to make sure that they were both open.
>
> *Interviewer:* Or, you want open sets to be mapped *from* open sets, right?
>
> *Maren:* Oh, sorry, the pre-image to be an open set.
>
> *Interviewer:* But the main thing is, before something failed that made it discontinuous when these were closed, right?
>
> *Maren:* Correct, but that's when we were under the usual topology, because this is not an open set in the usual topology. It is in the half-open interval topology. So…
>
> *Interviewer:* So, does that fix it?
>
> *Maren:* I think so, because if I were to take the pre-image of anything, I would get back one of these two sets essentially, right? The pre-image of anything containing -1, the pre-image of anything containing 1. And I would get one of these, and they're both open sets. And if I pick open sets, which I'm going to because that's what I'm trying to show, then yeah. I'm okay with that.

5.6.5 Gavin's response to Task 3.2(A)

In contrast to many of the other participants, Gavin had not learned or interiorized the process involved in the formal definition for continuity; therefore, I asked him to use his textbook to complete the task. As a result, Gavin's responses to the first task provide a window on an episode of highly syntactic reasoning with little semantic context. He began by writing out the symbolic representation of the function, and then drew the image of the function on a horizontal number line. He did not draw or indicate a graphical representation of the domain. Gavin stated that he didn't think the function was continuous, but could not recall the formal definition well enough to check his intuition:

> *Gavin:* […] So, I think intuitively it's not continuous because this, like, you know you can't, um…like you know the straight line test and stuff like that. I'm trying to remember the definition to use the inverse to test for continuity or not. I know there's a condition with the inverse.



Interviewer: The inverse images?

Gavin: Yeah, because $f(1)$, that would be, that would just go back to $(0,1)$, right? And then 1 and -1…

$$f^{-1}(1) = (0,1] \quad f^{-1}(-1) = (-1,0)$$

Interviewer: So, you're just not sure about what that inverse definition actually states?

Gavin: Yeah…

Since he was unable to recall the formal definition, I asked Gavin to use his textbook as an aid. He read the topological definition for continuous functions and then went through a process of relating the elements of the definition with the task situation:

Gavin: Yeah…[reads definition from textbook]…So, for some open set…*U* in, wait…so for each open set *V* in *Y*…so *V* is in *Y*, so it has to contain both of those, so it would be some open set around -1....the union of-, so some open set around those two [-1 and 1]…there's an open set in *X* such that the image, this is already an open set…so that the image of that is a subset of *V*, which is true, because the image of that is just these single points [indicates -1 and 1], and for any open set we pick around this that contains those two points, well it's going to be a union of those two, but it's always going to be… $f(U)$ is always going to be a subset of that…because it's only the singletons. And the domain is only, is defined from there to there [indicates -1 to 1]. So, it would be continuous at every point in there, so it's a continuous function I guess.

Gavin pieced together his solution through a detailed application of the definition to the task at hand. His activity demonstrated that he had not interiorized the processes involved in checking the definition, but he could reason through it syntactically and arrive at a mathematically accurate response.

Gavin: Because, using the definition, or at least I think I'm following it right, using the definition in the book…so the open set in *Y* is going to be the union of [writing], some open set around that:

V is an open set in Y
$$(-1) \cup (1)$$

Um…I have to include that [?] right? Yeah, 'containing the point *a*', so just some open set around those two. Um, if I let *U*, because *U* is the…yeah 'there is an open set', so I can just let *U* equal $(-1,0)\cup(0,1)$, so $f(U)$ is equal to -1 and 1, which is a subset of, well in this case a proper subset, but it's a subset of, um, our set *V*.

let $U = (-1,0) \cup (0,1)$
$f(U) = \{-1\}, \{1\} \subseteq V$

Interviewer: Where *V* was…

Gavin: Where *V* was just some open set around -1 and 1.

Interviewer: So, you didn't think it should be continuous because it seemed…

Gavin: Yeah, just because with all of the…before this class, in all of my classes if something doesn't look connected, then it's not continuous.

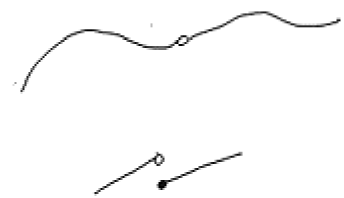
Gavin weighed his syntactic use of the formal definition against his empirical observations about the graph's connectedness. I prompted him to consider similar cases in his prior experiences with continuous functions, to explore whether he might have begun to interiorize the mental operations involved in the formal definition and perceive this solution as a generalization of his previous knowledge.

Interviewer: Can you think of an example in the [standard] real number to real number functions where you're sending something similar to this, where you have a disconnected domain being sent to a disconnected range. Can you think of any examples of standard functions that are like that?

Gavin: That wouldn't be, or that would be continuous?

Interviewer: Yes, but maybe seem discontinuous?

Gavin: No because when I think of it I think of stuff like [drawing], you know with a hole, or I even think of stuff [drawing]…something like that, but…



Interviewer: So, like for this one here [top curve]; what's the domain of this? I mean if we were just to think of a generic example.

Gavin: Um, *a* to *b*. I guess we could include *a*. And then *b* to *c*.

Interviewer: Okay, and then you're excluding *b*.

Gavin: Yeah, and then *b* is excluded, yeah.

Gavin did not seem to notice the similarities between his prototypical representation of a function with a discontinuity (though continuous on its domain), and the more abstract function he had just declared to be continuous through a syntactic application of the textbook definition to a function he had not been able to visualize. Thus, Gavin's syntactic approach to the task helped him to accurately answer the task prompt, but was not helpful in interiorizing the processes involved, or in relating those processes to his previously-formed, prototypical representations. Due to his lack of an interiorized scheme for the continuity process, Gavin's attempt to answer Task 3.2(B) is not reported. He made no attempt to answer Task 3.2(C).

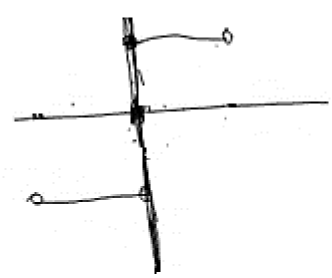
5.6.6 *Amy's responses to the continuity tasks.*

Amy's scheme for checking continuity was connected to many examples, visual representations, and superficial aspects of her concept image. The formal limit definition for *continuous function* seemed disconnected from the axiomatic definition, and there was no clear organization to the array of concepts she put forward to help her understand the property. Her responses were insightful to describe how she attempted to find such an organization during the tasks.

5.6.6.1 Amy's response to Task 3.2(A).

Amy began by immediately drawing a two-dimensional Cartesian graph for the function. The salient feature of the graph's disconnectedness prompted her to declare the function to be not continuous.

> *Amy:* Hmm......so, if x is between 0 and 1, it would go to 1……so everything between here would just be like this……and then, from -1 to 0……but those aren't included.



> *Interviewer:* And those go to -1 right?
>
> *Amy:* Oh, jeez…inside of it is continuous but its…the domain is $(-1,0)$ union $(0,1)$ ……Hmm, now I'm confusing myself with, like intervals, but-
>
> *Interviewer:* Mhm, now we're in the usual topology here, so open intervals are open. They are the basic open sets…Do you have a hunch?
>
> *Amy:* Because of 0?
>
> *Interviewer:* 0 makes it…

Amy: Like it's not included…like um…you know it would be open. And same thing for this [indicates 0 in both component intervals of the domain]…

Interviewer: So, does that make you feel like its continuous or not?

Amy: …So, when x is 0, it can either be up here or down here.

Interviewer: Well, or neither actually, because we're not actually including 0 in the domain, right?

Amy: Right.

Interviewer: So, 0 is not assigned to anything.

Amy: Right, and a continuous function can't have breaks.

Interviewer: So when you look at this break, you're thinking that that makes it not continuous?

Amy: That it's not continuous.

Amy stated that her criteria for evaluating the continuity of this function involved the empirical attribute of the connectedness of the function's graph. Next, I attempted to elicit a more reflective response from her, by asking her for a definition.

Interviewer: So, what's your-, do you have a definition for continuous function?

Amy: Um……I'm trying to think, like……looking at different pictures of functions in my head, um……

Interviewer: Mhm, like what functions?

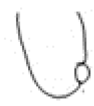
Amy: Well I'm just looking at parabolas, like this.

Interviewer: Mhm, so those feel continuous?

Amy: Mhm, and I'm only looking at these because um, you know like the one-to-one stuff. But I'm thinking about, the only thing that doesn't make it continuous is that, or every uh, point in the range, there's not-, if there's one case where like, uh, one point in the range does not have a point in the domain that points to it.

Interviewer: Oh.

Amy: Like right here…if there's one point where the image does not have a pre-image, or…oh man, now I'm thinking about this for like, the step function or something where it's open…I don't know I'm just thinking!



Amy was unable to recall a definition for a continuous function, so she was imagining various exemplars for functions that she considered continuous, and one that she considered not continuous. It is unclear which properties she was attributing to these prototypes, but she mentioned the injective property. Significantly, she seemed to describe the surjective property as a candidate for the definition of continuity. In other words, if there is a point in the "range" (or "image") of the function which has no pre-image, she apparently conceptualized it as a hole in the graph. However, the non-continuous function's graph she drew did not seem to meet this condition, since the function's non-injectivity ensured that each point in the image did indeed have a pre-image.

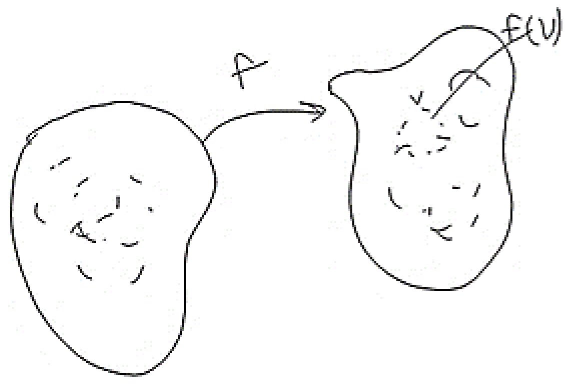
After she seemed unable to continue with her line of reasoning, I asked Amy to look up the definition, to see how she made sense of her examples through a more syntactic, formal lens. After only a moment of looking at the two formal definitions the textbook provides, Amy was easily able to express her understanding in those terms, drawing an accurate diagrammatic representation of the definition as an aid.

Interviewer: Maybe you want to look up the definition of continuous function? […] Do you feel pretty comfortable with those definitions?

Amy: Um…yeah but, it's hard for me when I go from, like I don't know, I'm always thinking about these pictures, um…

Interviewer: They don't necessarily translate well into-

Amy: Yeah, because I know there's a bunch of, I guess you'd call them loopholes or whatever…so I get myself confused. I mean I know, for *every* open set in here [indicates co-domain in diagram], you know, it gets…that means there's an open set around here [indicates domain in her diagram]. Like, *if* there's an open set over here, then that means there's an open set over here that's like pointing to it, or…Like the alternate [definition], for each open set, these are V, you know…oh wait, this is $f(V)$, this whole thing…Um, the pre-image is open in here. I know that, but like…



Given her initial difficulty, it was surprising that Amy was able to provide an accurate diagram for the topological definition for continuity, although she had been unable to recall the definition directly in formal terms. I attempted to have her relate her description to the task at hand:

Interviewer: So, what are open sets on-, so this [indicates $\mathbb{R}$ in the symbolic description of the function] is the co-domain now right? So, what are the open sets there? Instead of those blobs.

Amy: Open sets in the co-domain……

Interviewer: So, like, if we're in the usual topology…

Amy: Open intervals.

Interviewer: Mhm. So, if we wanted to show that it's not continuous, what you're saying is that we need to find an open interval on the *y*-axis?

Amy: Mhm…that's not open in the domain.

Interviewer: Okay, whose pre-image is not an open interval in the domain…

Amy: Okay, so……yeah it's just the 0 part, the union part……

Interviewer: That bothers you?

Amy had a limited conception of the main axiomatic definition for continuity and it was apparent that she had not interiorized the complex process it involves. When it was clear that she could not continue, I prompted her by relating parts of her diagram to the task at hand. However, she continued to apply the continuity checking process in the forward direction rather than the reverse.

Amy: Yeah, like if I were to pick $\left(-\frac{1}{2},\frac{1}{2}\right)$….

Interviewer: Now, you mean $\left(-\frac{1}{2},\frac{1}{2}\right)$ on the co-domain, on the *y*, or do you mean on the *x*?

Amy: Oh yeah…

Interviewer: Because these are all on *x* right? So we want to be on $\mathbb{R}$, which is the *y*-axis in this case.

Amy: Oh yeah, oh yeah……So if I find…any um…if I want to show it's continuous then I have to show that for every open set along here, then its open in $\mathbb{R}$, well it's going to be open in $\mathbb{R}$, but I have to see if I can find an open interval along here.

Interviewer: But it would be down here, right? Because this is the domain, $(-1,0)\cup(0,1)$. So, any open interval you find has to come from an open interval down here, right? It's just like, this picture, except this this the *x*-axis and this is the *y*-axis. And of course, they're intervals instead of circles.

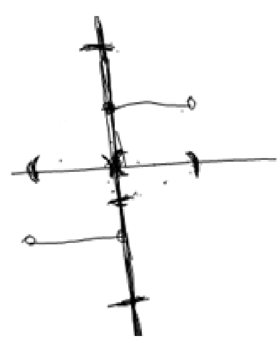
Amy: Mhm…that's confusing me a lot.

Amy continued to require guidance to apply her limited scheme for the formal definition, and I continued to prompt her to see how she would make sense of her conceptions. It became apparent that Amy's one issue was related to Amy's understanding of the pre-image of a set.

> *Amy:* I have trouble, yeah, like I haven't had a lot of examples that I've gone over or studied.
>
> *Interviewer:* Mhm. So, what would be the-, so if I picked an interval here, like from $(0,2)$, on the *y*-axis, what would be its pre-image? Or its inverse image?
>
> *Amy:* That's what I have problems with. Like when you gave me that packet and you asked for the, I think it was the inverse image, and like the pre-image I think, to draw it. I wasn't sure how to do that, I mean I thought about it but whatever I put I was like, I really don't know.

The "packet" she was referring to was the pre-assessment worksheet (see Appendix C), on which the participants were asked to draw the pre-images of a given set under a real-valued function. I prompted her to think about the pre-image of a single point at a time.

> *Interviewer:* So, let's just start with a point then. If I just start with 1, the pre-image of the point 1 on the *y*-axis, is everything in the *x*-axis that points to it right?
>
> *Amy:* Everything from 0 to 1.
>
> *Interviewer:* So, the pre-image of 1 is just $(0,1)$…So you get that whole set, right? And the pre-image of -1 would be…[Amy drawing on graph] mhm, up to 0, right.



I guided Amy through the process of finding pre-images of intervals in the co-domain as well, using the interval $(0,2)$ as an example. When she seemed to understand, I asked her to apply the process I had gone through with her to a different interval.

> *Interviewer:* […] So, the pre-image of that if I just pick $(0,2)$, the pre-image of that is really just $(0,1)$, I mean we can say the empty set, but $\varnothing \cup (0,1)$ is just $(0,1)$, right? And the empty set is open too right?
>
> *Amy:* Yes…wow! So, no matter what, if we're above the *x*-axis, however big the interval is always going to be back to here [indicates $(0,1)$], and everything down here [below the *x*-axis], since we can use everything, is going to go back to here [indicates $(-1,0)$]. And these are open, so then it *is* continuous.
>
> *Interviewer:* Okay, what if I picked, like $(-2,2)$? What would I get then?
>
> *Amy:* Oh, wait, okay…so if you went from -2 to 2, then it would be the union of these two…and that is open.
>
> *Interviewer:* That's still open. So, you can't find any sets in the image that-
>
> *Amy:* That make a problem.
>
> *Interviewer:* That would make it discontinuous?
>
> *Amy:* Nope. Wow! See this is what I'm talking about. All those math classes where you have to just look at pictures like, ugh…yeah, it's not good to teach like that.

Amy was frustrated at her habitual need to reason visually, given the inaccuracies it caused in her reasoning.

> *Interviewer:* So, when you look at this function you want to say it is discontinuous.
>
> *Amy:* Just because from looking at it!
>
> *Interviewer:* Mhm, but you just can't prove it…
>
> *Amy:* Yeah.

5.6.6.2 Amy's response to Task 3.2(B)

As I had observed with Saul and Nolan, Amy believed that the function in Task 3.2(B) would remain continuous despite the change to its rule.

> *Amy:* Now we have everything from -1 to 1. Okay, so then…now we're including 0……I want to say no, but I'm trying to think why? Like, what changes…it goes to 1, and it's equal……But now this 1 is included…when *x* is 0…So before when we went from 0 to 2, we came back over here [indicates $(0,1)$].
>
> *Interviewer:* Mhm. You got $(0,1)$…
>
> *Amy:* I'm trying to think like an interval say around, that goes around here [indicates just below 0 on her graph] and upwards.
>
> *Interviewer:* What would be the pre-image of that?
>
> *Amy:* Because now we're below 0. If we're just below 0 we go back here [indicates $(-1,0)$], and then if we're above we go back here [indicates $[0,1)$].

I pointed out to Amy, and she agreed, that the pre-image of points "just below" 0 should be empty; but I was more interested to see what she thought of the other pre-image she had mentioned, $[0,1)$.

> *Interviewer:* Is that an open pre-image?
>
> *Amy:* Now I'm like, well I can't find a problem because…
>
> *Interviewer:* Do you feel like it's continuous, or not continuous anymore?
>
> *Amy:* ……I'm thinking it *is,* but just because, I'm still trying to figure what would make it a problem. So, I'm trying to think about all the cases. I mean, what would make it…
>
> *Interviewer:* What is the pre-image of the set you just came up with? Like, if you picked, it looks like you're picking $(-\frac{1}{2},2)$ or something?
>
> *Amy:* Mhm.

Interviewer: So, what do you get when you go backwards through the function? Like where does $-\frac{1}{2}$ come from in the function?

Amy: Like, this is what is confusing because going back to $(0,2)$, $(0,5)$, or whatever [on the *y*-axis], we're going to get this interval [indicates $[0,1)$ on the *x*-axis]. And then if we went from $-\frac{1}{2}$ to above 1 [on the *y*-axis]…But now that 0 is included, if we went from $-\frac{1}{2}$ to 2…like, I don't see a problem with that. I don't see...

Interviewer: So, you're taking the pre-image of this open set…

Amy: So, it would be…so everything outside of 0, right? That's an empty set. That would be an empty set union with this interval, *half-open* interval I guess.

Interviewer: Mhm, so you took a pre-image of an open set…an open interval…

Amy: Mhm.

Interviewer: Did you get an open pre-image?

Amy: If we consider that [indicates $[0,1)$] open!

Amy did not seem to reflect on her conception of open sets when she found a pre-image that was not an open set in the standard topology on the real numbers. She seemed to be looking for justification that the half-open interval should be open. At this point, we moved to Task 3.2(C).

5.6.6.3 Amy's response to Task 3.2(C)

As with the other participants, Amy was able to modify her reasoning from the previous Tasks to quickly resolve Task 3.2(C).

Amy: Well…I would say, okay this is now…I'm going back to that it doesn't have to look like a half-open interval to be open…so then, yes?

Interviewer: Yeah, so I mean if it does look like a half-open interval then it's open.

Amy: But it doesn't have to be, because I was thinking about the other side.

Interviewer: Right, because open intervals are also open.

Amy: So, if this was changed to the half-open interval topology, then it would be [continuous].

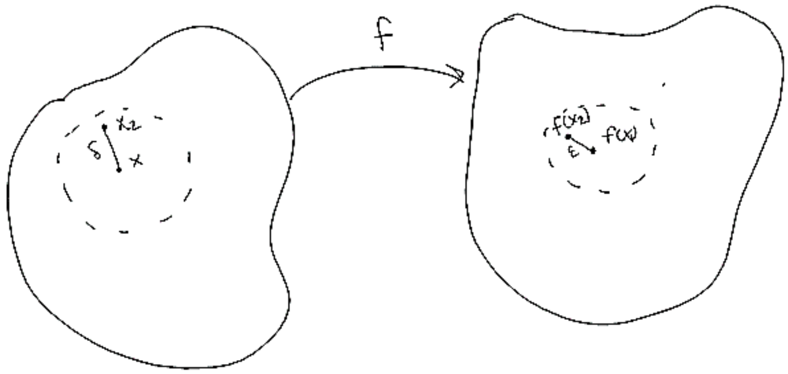
Amy asked about open intervals under the lower limit topology, likely because of her experiences during Task 3.1(B), which she had completed earlier in the interview. It seemed that she wanted to be sure that both interval forms would qualify as open pre-images, in order to cover every possible case.

5.6.6.4 Amy's coordination of her formal and axiomatic understanding

After concluding this discussion, Amy seemed to have a more coherent and interiorized process for the topological definition of continuous function. However, during the discussion, she pointed out that she knew the epsilon-delta definition of continuity well. I asked if she could relate this to her solutions for Task 3.2.

Interviewer: So, can you relate this picture to the epsilon-delta definition of continuity?

Amy: Yeah, I could try and draw it…So, um, like we have x right? And then um, $f(x)$…so, what was it, if…it was like the distance from two points is less than delta, then I know there's something else though…That has to be less than epsilon. So if X is a set of points over here, and this distance is delta, then, um…these distances have to be less than epsilon.



$$|x_1 - x_2| < \delta$$

$$|f(x_1) - f(x_2)| < \varepsilon$$

Interviewer: […] Yeah, but you feel more comfortable with this picture then with that definition? Or not necessarily?

Amy: Um, the picture helps a lot. Um…when I first saw it being drawn, understanding this…because when I first heard this definition…I mean they don't tell you, distance, I mean you should know that, but all you see is, like x minus y and then $f(x)$ minus $f(y)$…and then you're like…what do I care about that?!

Interviewer: So, when did you come to the understanding that that was like a distance?

Amy: When a professor had said it. That was a few years ago…Yeah, I know it's like this minus this, but I don't see it as distance, I see it as-, it's so weird, I see it as measurement, but I'm not seeing it is as a distance. Like when you draw an open ball and you're like, radius, the distance from here to here, yeah; but when I see it just like, like uh…or it could be like this, you know?

$$|3 - 8|$$

I'm just thinking um…the value I guess. I'm not thinking distance. Even if you're saying it like, on a number line, like 3 minus 8. Like, okay, you're going to have five units…I don't think…I never thought of it as distance.

5

Amy seemed intrigued by the way she had conceptualized the absolute value operators within the epsilon-delta definition of the limit. Since she learned to overlay her diagrammatic representation for continuity with the formal notation she had learned in previous mathematics courses, she was able to form a new connection in her mind between this symbolic notation and the perceptual idea of one-and two-dimensional Euclidean distance.

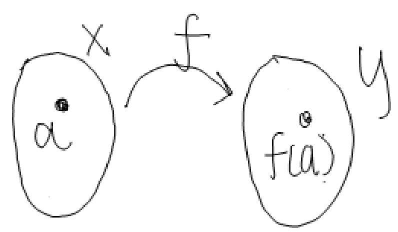
5.7 Highlighted Case: Maren's Use of Metaphor and Prototype

Maren's discussion about her conception of functions and continuity provides an illustrative example of several common themes I observed as my participants used metaphorical and metonymical structures in their reasoning. During the semi-structured discussion portion of one interview, I asked Maren to explain her understanding of a continuous function in her common experience of real-valued functions as well as through her new experiences of functions between abstract topological spaces. Her responses indicate that she reasoned analogically about the properties she considered intrinsic to a continuous function in both contexts. Her efforts resulted in a conceptual blending originating from two sources: 1) her visual metaphors for topological spaces and continuous functions; and, 2) properties she gathered in her analysis of two real-valued exemplars. Evidence for Maren's conceptual blending will be presented in the following excerpts; where, despite her use of real-valued exemplars to generalize her conceptions, there was an apparent shift in the way that Maren conceptualized functions between topological spaces as opposed to her conception of functions on the real numbers.

5.7.1 *Maren's formal and metaphorical conceptions of functions*

Maren stated that her understanding of functions was largely based on her experiences in her modern algebra class. Her initial conception seemed to rely on the assumption of two properties, some of which were irrelevant to the formal mathematical concept. These included her expectation that every function should be injective ("one-to-one") and surjective ("onto"). Later in the interview she would revise these expectations as they became untenable with respect to other conclusions she was less willing to abandon.

> *Interviewer:* So, what do you think a function is?
>
> *Maren:* So, when I see function, I actually think of the definition from modern [algebra]; that's the one that stuck with me I guess. And it's like, a function goes from the domain *X* to the co-domain, which we usually call *Y*. Um, and I'm pretty sure it's one-to-one and onto, to be a function. It has to be one-to-one. Um, and then I can draw that for you if you'd like?
>
> *Interviewer:* Sure, yeah. And, what would one-to-one be in that drawing…



> *Maren:* Oh okay, yeah. So here's a set *X*, and here is a set *Y*, so a function *f* goes from *X*, which is the domain to *Y*, which is the co-domain, or you put *f*, where *f* is some, like, operation, so you put these numbers in the domain into *f*, and your output is what's in the co-domain. And it has to be one-to-one, which means that for every *a* that's in *X*, $f(a)$ is in *Y*, and if, um, the definition of one-to-one is that if $f(a_1)$ equals $f(a_2)$, um, then $a_1 = a_2$, I think that's an if and only if…

$$f(a_1) = f(a_2) \Leftarrow\Rightarrow \\ a_1 = a_2$$

It was initially unclear what Maren meant by statements like "a function *f goes* from *X* to…*Y*; but she did spell out a few defining properties that she considered necessary, such as injectivity and surjectivity. Throughout the study, I found that many of my participants mistook the injective property, or "one-to-one," for what I will refer to as the *function property*. Namely, that for each point *x* in the domain, there is one and only one point *y* in the co-domain such that $f(x) = y$. Although this property is not usually introduced independently as a property of functions, it is the defining criteria for a relation to be classified as a function. It is equivalent to the reverse implication of Maren's "if and only if" statement in the definition above, which she seemed to include as an afterthought.

The forward implication of Maren's statement is the injective property, which is not a property that is necessary for all functions. It appears that, in addition to accurately considering the function property, her personal concept definition for functions also included at least the injective property, if not the surjective property, "onto," as necessary characteristics of a function. To verify that she was not accidentally using the term "one-to-one" to describe what she intended to mean the function property, I inquired further about her definition. Her answer revealed that she indeed believed injectivity to be a necessary property of any function:

> *Maren:* So, a contradiction would be like $y = x^2$. And you would have a 4 and -4, and they would both go to sixteen. So the *a*'s, a_1 and a_2 don't equal, but $f(a_1)$ and $f(a_2)$ are equal, therefore it's not one-to-one.

$y = x^2$

4 $\quad$ -4

16

Interviewer: Okay, so is this a function?

Maren: No.

Interviewer: Okay, so only-, so functions have to be one-to-one, and you said onto? Or just one-to-one?

Maren: I can't remember if they have to be both or not.

Interviewer: Not sure. What is onto?

Maren: Um, for every y in the co-domain, there exists an x in the domain, such that $f(x) = y$. So for every element in the co-domain, there is one in the domain that we put into f, that we got that. So there's no number in here that's just like random, yeah.

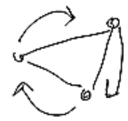
Her definitions for the injective and surjective properties were accurate, although she was initially unsure whether she believed that surjectivity was also required of any function. Eventually, she went on to explain why she considered both to be necessary properties for any function:

Maren: Well, because we can't really take any inverse of it. Because that's kind of what's nice about functions like these, or functions in general, is that for us to be able to take an inverse, it has to be one-to-one and onto which is why we're interested in them. Because, stuff like, sine, you can't take the inverse of it because there's infinite x's that correspond to the same three y's, generally that we plot, 0, 1, and -1. So it's like, that's why we have to restrict the domain.

My interest in this exchange stemmed not only from the fact that Maren had attached extraneous properties to her definition of a function, but also that she was able to state accurate personal concept definitions for each of those added properties, providing internally consistent justifications for why those properties (or at least one of them) should be included in the definition. My goal was to investigate how Maren might use this previous conception of real-valued functions to re-conceptualize functions in the context of abstract topological spaces. I asked her what she imagined the points a and $f(a)$ to be in her drawing:

> *Maren:* I mean, it could be a number; it could just be anything, like when, for example, um, in modern [algebra] we did the cycles, like one, two, three, and that was like a set, a set of the cycles, and then we would operate them like, on themselves or something, and that would give us the separate set. So to me it's kind of like, whatever we're working with. Generally, we compare them with numbers, because you know, when you learn something new you apply it something you know. So that's the easiest way to learn something, but um…that's the easiest way to explain it is with numbers.

$$\underline{(123)} \cdot \underline{(123)} = (132)$$



Although she repeatedly referred to "numbers," her example had to do with the composition of permutations, which she remembered learning as a group operation in her modern algebra class. She referred to the permutations as "cycles," and claimed that cycles did not represent functions themselves. She only considered the group operation as a function, with the set of cycles as her domain:

> *Maren:* Well the actual…I consider the function, like mainly…I mean I want to use the word "operation" that's being done. So, like this dot would be the function.

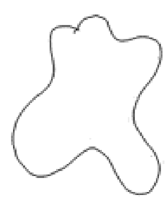
$$\underline{(123)} \cdot \underline{(123)}$$

It is unclear why Maren did not consider the permutations themselves to be functions, or how the binary group operation was assimilated into her originally stated conception for functions, which was not binary in nature. Her repeated use of the term "operation," and her action-oriented descriptions, indicated that she may have been working with an un-encapsulated process conception (Dubinsky, 1994) for functions, or what Sfard (1994) termed an operational conception. The lack of an object conception for functions is further supported by her unwillingness to classify the permutation cycles as functions. She seemed to have reified these cycles as conceptual objects in their own right, and she did not de-encapsulate those objects into their functional processes to note their equivalence with her personal concept definition for functions.

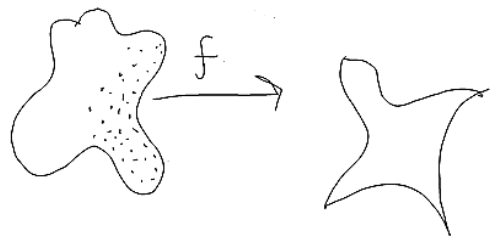
5.7.2 Maren's metaphor for general topological functions

Having learned about Maren's conception for functions from her previous experiences, I steered the conversation toward her understanding of functions between more general topological spaces. She expressed her conception of general topological spaces through a commonly-used prototypical visual representation—a two-dimensional, closed, Euclidean figure. She then mentally overlaid her conception of functions onto this representation, constructing the conceptual blend of a function as a process that changes the shape of her visual prototype:

> *Interviewer:* Now […] you're dealing with functions on points in these abstract spaces. How do you think about those?
>
> *Maren:* […] So, I imagine just like, some blob…that's what I imagine.



> *Maren:* So you want me to demonstrate like, a function? So if we just have some blob, and we want to operate on a function onto it, I think of like, this blob is made of a bunch of little, like its constructed of a bunch of little things, like little elements; and so, when we have our function, *f*, we put all these little elements into the function, whatever it does, and it brings us, like, maybe some different shape, or something. Um, so, we'll like, operate on them and it will bring us a different shape. So, for the discrete metric it makes everything exactly one distance apart. So we're kind of like, just morphing the shape, the space, that's kind of like what, it's kind of confusing to me still…



Interviewer: Right, so when you think about this [indicates previously written real-numbered function], you're thinking of an operation that changes a number to a number; but when you think about this, the operation more like, changes a shape? That's sort of how you conceptualize it?

Maren: Mhm. That's what I was thinking. Could be wrong!

While Maren considered a real-valued function in a pointwise and abstract manner (as a mapping process between any two sets); she seemed to think of functions between topological spaces in a different way. Despite recognizing that the concept's definition doesn't change in these different contexts, her conceptualization of the idea was more holistic and spatially-oriented in the latter context of general topological spaces. She seemed to use the visual prototype of two-dimensional Euclidean "blobs" to represent general topological spaces in her mind. The instantiation of this paradigmatic example (Zandieh & Knapp, 2006) allowed Maren to conceive of a function as a transformation of the entire *shape* in the plane, albeit through an "operation" on its individual elements. Despite the fact that the domain and co-domain from her original example were both topological spaces in their own right (as subsets of the real numbers), she appeared not to consider these as typical examples of the concept. Her need to represent functions in a spatial way only occurred when she attempted to consider topological spaces as abstract entities, which were not the real numbers with their standard topology.

5.7.3 Maren's metaphor for continuous topological functions

With an understanding of how she conceptualized functions in real number and general topological contexts, I asked Maren to discuss her conception of the property of continuity. I was interested in observing the extent to which her response would be influenced by those prior conceptions, and in which ways she might modify the structure of her scheme to accommodate the unfamiliar context.

> *Interviewer:* Now when you get to continuous functions on, sort of, general metric or topological spaces, have you gotten sort of stumped by not having visualizations there?
>
> *Maren:* So when we're talking about continuous topological spaces, I think that, like, essentially, what it's saying…okay those are just as confusing to me as visualizing metric spaces.
>
> *Interviewer:* Right, because you just can't-
>
> *Maren:* I can't see it!
>
> *Interviewer:* So you can't tell whether you have to pick your pencil up or not?
>
> *Maren:* Well, okay so…well, and also because a function is defined as from set *A* to…the codomain, or from the domain to the codomain. So if you're just saying, like there's a space and it has a function, well there has to be a codomain, or that would fail. So then, if you have just some space, you would have to have a function applied to it to where it would go upon some other space. So then if every point in the first space is mapped to another point on the second space, um, so then, what that would mean would be, for every point that we pick, or every element in that space that we pick, it would be transported somehow to another space; and if every point is transformed smoothly, then I could see how it would kind of flow, like a bridge almost. Like, so, if you have like a city that's separated by a river, right? That would be like an asymptote. But if you build a bridge, then they could all smoothly-

Maren's struggle to re-conceptualize continuous functions was evident as she talked her way through these mental images. She finally resorted to the metaphor of a bridge connecting two parts of a city separated by a river. Later in the interview, this metaphor seemed to evolve in such a way that the river became the target concept for continuous functions. She added:

> So then, um, I just think of continuous as just like a smooth, um, flowing like a river. Like it's smooth, and it doesn't have any breaks or jumps, and it's not separated by any asymptotes, so you could almost think of this original space that we had as separated by some kind of asymptote, where both functions, or both sides of the space are getting really close to each other, but they never actually go like that, right? So they get really close to each other, and then when we apply the function that I had given an example for, where they-, this one goes here and this one goes here, well now that function that we applied…if we were try to take, oh my goodness! If we were to try to take…I don't know…

Still faced with the difficulties of translating her conception of continuous real-valued functions into an understanding of continuous topological functions, Maren seemed to require more conceptual structure on which to base her translation. She next turned to specific examples of discontinuous real-valued functions to discern which properties she expected to fail when a topological function is not continuous.

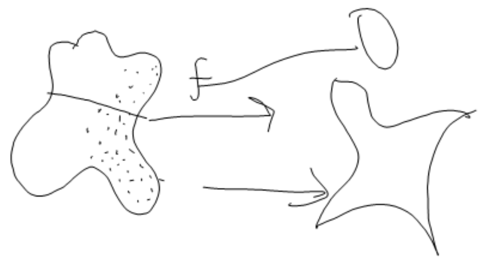
While her criteria that every function should be one-to-one and onto were not immediately apparent in her visual prototype for a function, the properties that did persist in providing structure to her concept image had their source in the metaphors of a bridge and flowing river. She made use of these images by extracting two properties from them and interpreting those visual properties in a mathematical way. These mathematized properties were: 1) the connectedness of the function's graph, and 2) the differentiability of the function itself. In relation to the former property, she indicated that "it doesn't have any breaks or jumps, and it's not separated by any asymptotes;" while, with respect to the latter she noted that "the derivative at every point exists, which means there's no cusp." While the connectedness of a graph is related to the continuity of the function that creates it, differentiability is only conversely related to continuity. Associating these properties with continuous functions is a common occurrence for many students (Tall & Vinner, 1981; Dreyfus, ??); but, of interest in Maren's case, she connected them to a strong metaphorical image, which she then attempted to transfer to the more abstract context of functions between topological spaces.

5.7.4 Maren's exemplars for non-continuous functions

Maren did recall one version of the formal definition for a function to be continuous in the real number context—that the left- and right-hand limits of the function must exist and agree with one another, as well as with the function's value at the point in question. However, she encountered difficulty when attempting to translate this definition into the more abstract setting of topological functions. Her attempt to assimilate abstract topological spaces into her scheme for continuous functions then turned toward her examination of two examples of real-valued, functions, both of which she considered to be discontinuous. These were the reciprocal function $y = 1/x$, and a self-generated piecewise function. She used these examples as foundations on which to build a definition in the context of abstract topological spaces. Here, I show how the properties that were apparent to her in these examples framed this process. I began by asking Maren to describe how she would interpret the notion of continuity within her conceptualization of functions on abstract topological spaces:

> *Interviewer:* Right, so, in this situation then, we talked about a function that sort of morphs shapes in these topological spaces…
>
> *Maren:* Which is okay with me.
>
> *Interviewer:* So what would be true about how this function brings these points to those points if we were going to say this was continuous? Versus if it was discontinuous, what would fail to be true?
>
> *Maren:* Right…um, so then every point, okay so maybe, maybe if it's a disconnected space, then the points would map to another disconnected space. Because I'm pretty sure that that's a property of disconnected

spaces. So let's say that the disconnected space in here, like maybe it's just that they're pretty much touching each other. But let's say it maps to, like a space, like this half maps to this space and this half maps, like another circle [[drawing]].



Maren: Then those would be like, completely separate. It would be one-to-one and onto, but…when we take the limit…we put the domain to get a value from the co-domain…So if the co-domain is split up into two different pieces, then there wouldn't be…it would be like this, like if it's split up, then…we would take the limit and it would just stop. And then we would try to continue it…

5.7.4.1 Maren's use of the reciprocal function as an exemplar

Based on her earlier representation of a function on topological spaces, Maren had created a preliminary representation of the property of *dis*continuity in this context. In order to clarify her discussion of limits in this context, and the separation she drew in the domain space, I asked her whether the distance between points in the image was an important element in her definition, which she attempted to clarify with a more detailed diagram and explanation:

Interviewer: So the essential thing you're saying is that, because at first you said that these two spaces were really close together, although disconnected. And now they're going to two places, disconnected still, but now farther apart?

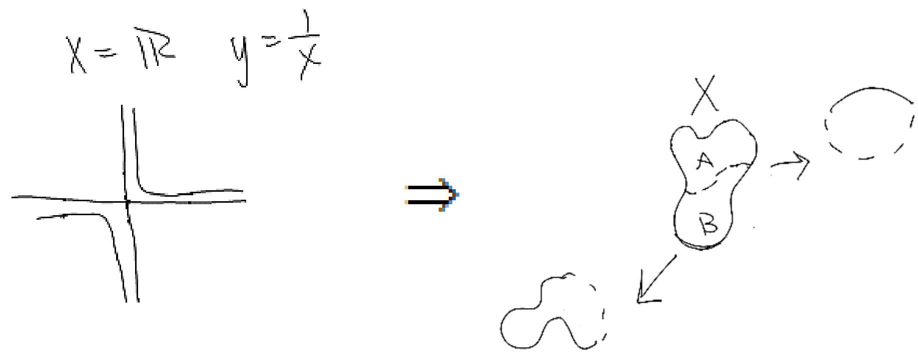
Maren: It's a huge distance, yeah.

Interviewer: So…it sounds like there's this underlying theme of distance in your idea of continuous [functions] and limits.

Maren: And we're not supposed to apply to that to topological spaces I'm pretty sure.

Interviewer: Right, but we don't need to [discuss] topological spaces *per se*. But, […] just this idea of, somehow your conception of these—both limits and continuous function— has to do with how far apart the image is compared to how far apart it was in the domain?

Maren: Right because when we had, okay so let's compare it to this one. So here, the set of X is the real numbers, so then when we do $y = 1/x$, we get our x's are now split into two completely different fields, and they're not at all connected. So if we were to have some space that's disconnected here, wherever, right [draws figures below]…



Maren: And we have an *a* and a *b* and we have all those fun properties, and we'll say *A*, like the function gets mapped, or we apply a function onto this space, then like what if *a* goes to some half-open set and *b* goes like, over to some weirdness as well, then there's no…like even though there's a function, we don't get like a nice continuous co-domain. The co-domain is not continuous. I think that's, that's what it is right? Because that's what we're really looking at, we're looking at, because when we're looking at a limit, we're looking at, if the *y*-value and the *x*-value…well…

I interpreted Maren's drawing as an attempt to use her concept image for discontinuous, real-number functions to help her assimilate her continuity scheme to include functions on topological spaces. She instantiated the reciprocal function as an exemplar from which to draw properties that she could use to conceptualize discontinuous functions in more abstract situations. It is noteworthy that she disconnects both the domain and co-domain in her diagram, just as the graph of the reciprocal function is disconnected by both horizontal and vertical asymptotes. She also denotes the images of the boundary of the disconnected regions in her drawing with dashed curves, which I interpret as an attempt to preserve the 'open-ness' of the boundary in the pre-image. This is apparently a reflection of the connected components' open boundaries (at the asymptotes) in the domain and co-domain of the real-valued function she is modelling.

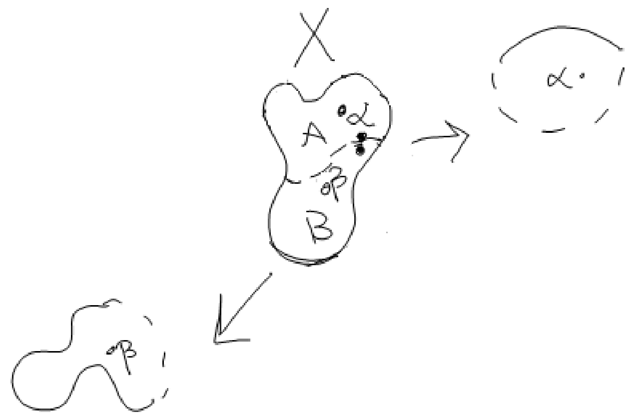
Mathematically, her representation is problematic for a number of reasons. First, it is not at all necessary that the co-domain of a discontinuous function be disconnected, although that is the case for the example she emulated. Similarly, but perhaps of greater concern, there is no reason for the domain of a discontinuous function to be disconnected. In fact, this hypothetical function, as she has drawn it, might be interpreted to represent a *continuous* function that happens to send a disconnected space to another disconnected space. So long as the image of each connected component is also connected, there is nothing obvious in the diagram to prevent this function from being continuous.

This issue could have stemmed from Maren's choice of the reciprocal function as an exemplar in the first place. This function, considered on its proper domain of $\mathbb{R}-\{0\}$ is actually a continuous function, despite the great confusion it causes students due to the appearance of its graph and its treatment in calculus textbooks. Maren did specify $X=\mathbb{R}$ to be the domain on which she was considering the function, allowing her to accurately claim it as an example of a discontinuous function. Yet, technical accuracy was apparently not sufficient for her to construct a formal conception of discontinuity out of this exemplar.

As I was still unclear how the notion of distance played a role in Maren's concept image for continuous functions, I asked her to elaborate on the representation she had provided, which provoked her to adopt a new exemplar and reformulate her definition again:

> *Interviewer:* What if, instead of whole spaces, what if we just talked about, within the space, just take two points, little *a* and little *b*, and

there, you know, what would…[[Maren drawing]]…so in your perspective, a continuous function would not send these things far apart from each other?



Maren: Well they could be far apart but the points in between them should be connected to each other.

Interviewer: Oh, so continuity and connectedness are-

Maren: This is getting worse! So like, okay, so…So for it to be continuous…so here, well because all these points are different, that's why this is so strange. It's because all of these points are different. But I guess that like, like here, these really tiny points, like really close to the boundary line…they get completely thrown separately.

Interviewer: Right, so points that were close together are far apart.

Maren: It's just awkward.

Interviewer: What about the point-, what about that point next to the boundary that's in A?

Maren: This one right here?

Interviewer: Yeah, what if I-, what if it was to be sent over here?

Maren: No, it can't because this is a disconnected space…

Interviewer: Mhm, so if it's a continuous function, it needs to go with the other points in this region.

Maren: Yeah.

Interviewer: If it's a discontinuous function, could it then go over there?

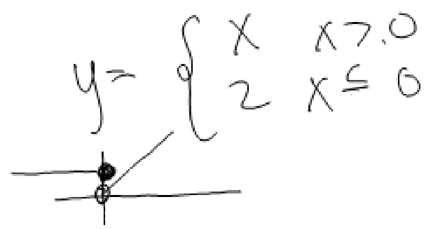
Maren: I guess.

Interviewer: I mean, is that a function thing that it's got to stick to-, with its neighbors? Or is it a continuous function thing?

5.7.4.2 Maren's use of a piecewise function as an exemplar

Maren had extracted as much information (properties) as she could from the reciprocal function, but was still unsatisfied with her answer. She searched for another exemplar that would provide more information about the properties she believed were necessary for a function to be discontinuous.

> *Maren:* Well, okay, so if we think about this like, like what if we thought about, okay think of it like a piecewise. So if we had like $y = x$ for $x > 0$, $y = 2$ for $x \le 0$, then…what you would be saying is that, even though we're picking from the same, from one *X*, they're getting sent to two completely different places and these are obviously not continuous, because I had to pick my pencil up and re-draw a different line, and the limit as we're coming from here does not equal the limit as we're coming from this way. But it's just so much easier to see when it's numbers. It's not, like when it's spaces, it's…meh…



> *Maren:* Mhm, yeah, so if we're picking points from this one, single set, you would want it to go onto another smooth, single co-domain. Here we have two co-domains and I think that's why it's not continuous.
>
> *Interviewer:* Okay, so if the domain is not disconnected? But then it got sent to two different [subsets of the co-domain]?
>
> *Maren:* Then that would definitely not be continuous.

Here, Maren seemed to have switched to examine another known exemplar of a discontinuous function with a different property from the last one—in this case, the function was given a connected domain. Altering this property allowed her to narrow her focus on which specific properties should be necessary for a discontinuous function. She noted that her domain was "one, single set," and attempted to distinguish it from the "two co-domains." While I adopted her terminology within the interview, she was more accurately referring to the function's images of the sets of positive and non-positive real numbers. She apparently thought of these images as two disconnected subsets of a single co-domain $(0,\infty)\subset\mathbb{R}$. However, since these images were actually $(0,\infty)$ and $\{2\}$, they are not disjoint and the co-domain for her exemplar was in fact connected.

Maren was unable to attend further to the defining property for a discontinuous function, which is stated through limits in the context of real numbers. She had no analogous tool at her disposal for abstract topological functions, and so relied on her interpretation of the empirical properties of the function's graph (her perception of the connectedness of the domain and co-domain) as her criteria for structuring the notion of a continuous function between topological spaces. Therefore, both of the exemplars Maren used as bases for her reasoning led her away from the defining property for continuous functions, and prevented her from seeing past the visual representation she had used.

5.8 Results and Conclusions from the Continuity Analysis

In this section, I will present a theoretical interpretation of the analyses from this chapter, based on my observations and in consideration of research literature from multiple fields. The participants' informal bases of reasoning will be discussed, such as perceptual and non-perceptual intuitions. I interpret such underlying intuitions as belonging to the "non-verbal representation system" of Paivio (2010). They are representative of the "perceptual space" of Piaget & Inhelder (1948/1967) and the "pragmatic paradigm" of Job & Schneider (2014). Accommodations designed to coordinate these dual worlds of mathematical understanding are discussed next, followed by an examination of the role of the participants' use and representation of properties in learning about axiomatic systems.

5.8.1 Reasoning foundations

Throughout the interviews, all the participants seemed to attempt to assimilate the task stimuli into their current mental schemes. They pieced together their different ways of understanding the property of continuity and reacted when those pieces failed to align with the formal theory as they understood it.

5.8.1.1 Perceptual intuition in the participants' reasoning

Fischbein (1987) classified intuitions by their roles and origins. Although he distinguished perceptual knowledge from intuition, he established that perceptual or sensory-based information can play a large part in generating human intuitions. He referred to these as *semantic, affirmational* intuitions (p. 52), which occur when "a non-intuitive axiomatic meaning" has one or more "related intuitive meanings." His examples included the "intuitive correspondent" meanings that accompany the mathematical concepts of points, straight lines and forces. Similarly, my participants seemed to understand the meaning of continuous functions in multiple ways, including: graphically (connectedness of the graph), behaviorally (trace the graph without lifting the pencil), temporally (uninterrupted duration), and axiomatically (according to the formal definition).

For this reason, perceptual cues played a significant role in the mental operations of all the participants during the continuity tasks. However, the participants varied in the extent to which these cues were integrated with their formal mathematical understanding. In Task 3.2(A), participants reacted to a function that was continuous on its domain, but which is conventionally represented as a perceptibly disconnected graph. The disparity between the function's continuity and its graph's connectedness was not initially a source of perturbation for most of the students. Only Nolan and Amy drew a conventional, two-dimensional graph without being prompted (see Figure 19).

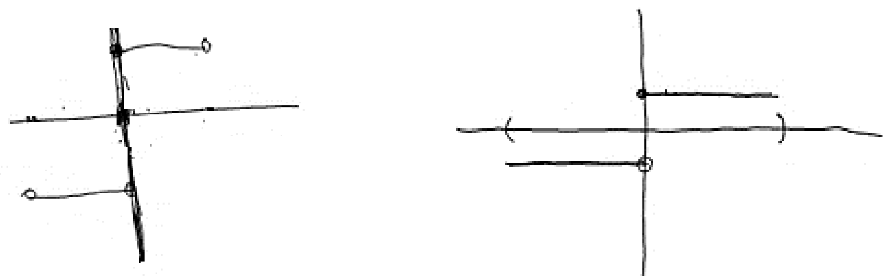


Figure 19. Two-dimensional representations of the function in Task 3.2(A). Amy (left) and Nolan's (right) representations.

Gavin and Wayne both drew one-dimensional representations of the function. Gavin's graph made no indication of the domain of the function, while Wayne's employed two copies of the horizontal real number line to represent it (see Figure 20). On the other hand, Saul and Maren did not draw graphs. Until they were prompted, neither made any attempt to draw or represent the function in a visual way at all.

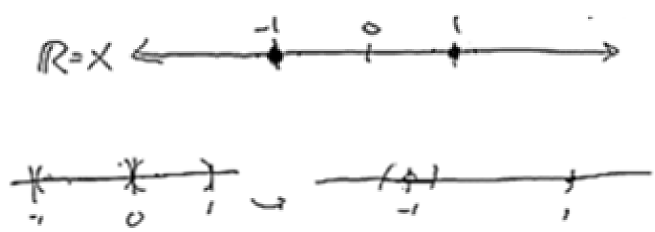


Figure 20. One dimensional representations of the function in Task 3.2(A). Gavin (top) and Wayne's (bottom) representations.

Each participant encountered the task in a unique way—both perceptually and mathematically. But, there were many commonalities among the responses. Table 6 summarizes the forms of visual representations drawn by the participants, both initially and after they were prompted. Each of the participants eventually drew a conventional two-dimensional representation, either after my prompting or their own reflection.

Table 6: Visual representations used by the participants

	Initial	**When Prompted**
Saul	None	2-D graph
Nolan	2-D graph	n/a
Maren	None	2-D graph
Gavin	1-D graph (excluding domain)	2-D graph (non-Cartesian)
Wayne	1-D graph (including domain)	2-D graph
Amy	2-D graph	Non-Cartesian diagram

5.8.1.2 Non-perceptual intuition in the participants' reasoning

Table 7 catalogs the reactions of five of the participants when they saw a graph of the function in Task 3.2(A). Each of the participants expressed an apparently intuitive and belief that the function was discontinuous. Their use of certain phrases, such as "inkling," "intuitively," "obviously," and "would make sense if…" all point to non-mathematical bases for their reasoning, while expressions such as "the jump," "can't have breaks," and "if it happened at 0" indicate that the basis for reasoning was in fact the disconnected graph. This was expected, and my goal as teacher-researcher was to elicit a more detailed explanation for their belief. I challenged each participant to formalize and make coherent their explanations as much as possible.

Some of the participants also demonstrated the use of non-perceptual intuitions as well. Rather than intuitions grounded in sensori-motor experience, these were "secondary intuitions" (Fischbein, 1987) built up from the participants' past experiences. The participants' responses in Table 7 were interpreted as seeking warrants for their intuition based responses. However, these intuitions were not based on salient visual features of a graph or diagram. These seemed to be non-perceptual intuitions, although no attempt was made to distinguish specific intuitional bases in each case. This is a potential area for future research.

Table 7: Participants' responses to visualizations for Task 3.2(A)

Nolan	So, I have an inkling that this is going to be discontinuous, um…because of the jump.
Gavin	So, I think intuitively it's not continuous…
Saul	Well no, this is obviously not continuous.
Amy	Right, and a continuous function can't have breaks.
Wayne	So, yeah, I guess when you draw it like this, oh…yeah the discontinuity would make sense if it happened at 0.

Intuition-Based Warrants

Saul: Is it open?......I'm thinking I want it to be open. Because I don't see why this isn't continuous […] Intuitively it seems like, yeah sure its continuous. Because I don't see any problems.

Nolan: I think all we did was just continue the line…

Amy: If we consider that [indicates $[0,1)$] open!

Interviewer: But it's not, right? It's just not an open set in that topology.

Saul: No. And yet I am convinced that this is continuous!

Nolan: I think it's trivial to see discontinuities from $\mathbb{R}$ to $\mathbb{R}$ …my ideas of continuity are still doused in $\mathbb{R}$ to $\mathbb{R}$.

Figure 21. Participants' intuition-based warrants.

5.8.2 Accommodations observed in the participants' reasoning

The participants demonstrated evidence that they had modified their schemes for continuous functions in two distinct ways. First, those participants who were able to complete each part of Task 3 on their own, appeared to have accommodated their schemes to incorporate the axiomatic definition of continuity. I called these *formalizing* *accommodations*. Second, when the participants were attempting to explain the discrepancy between their visual observations and the formal solutions to the tasks, many of them actively accommodated their schemes for continuous functions to overcome the perturbation. This is what I referred to as an *explicating accommodation*. It differs from a *formalizing accommodation* that would serve to rigorize a student's mathematical conceptions. Instead, an explicating accommodation serves to link the student's formal-symbolic and embodied perspectives of a perceptually salient mathematical situation. It is a way to link the participants' "pragmatic" and "deductive" models of the situation (Job & Schneider, 2014). Both formalizing and explicating accommodations may help to build conviction for the student, but an explicating accommodation is a reaction to perceptual sources of information and need not be sufficient to formally complete the task on its own.

Figure 22 outlines my model for the types of cognitive activity I interpreted from the participants' responses to Tasks 3.2, parts A and B. Below, I describe each of the shaded accommodation types in the lower half of the figure.

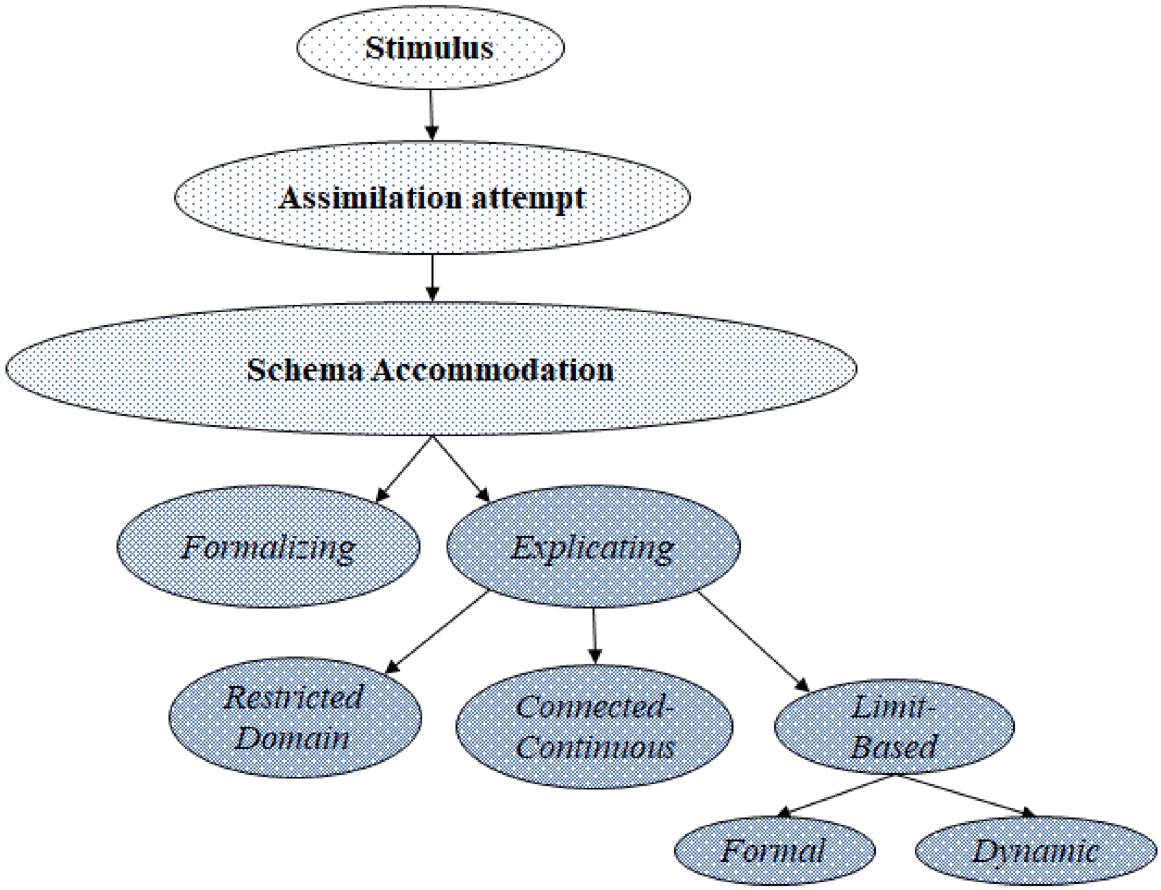


Figure 22. Analytical model of cognitive responses types. Accommodation types are shaded and categorized in the lower half of the figure.

5.8.2.1 Accommodations that formalize mathematical knowledge

In one way or another, all the participants attempted to use some version of the formal definition of continuity to complete the task. Although some of the participants, such as Gavin and Amy, did not have formal schemes that were developed enough to do so without guidance, their responses to the guidance they received (whether from the textbook or teacher-researcher) were still theoretically insightful. For the other participants, the uses of their personal concept definitions for continuity indicated they possessed schemes that were accommodated to the axiomatic mathematical structure of the task (to varying degrees). Had this not been the case, their strategies would have involved metric or real number versions of the terms and definitions they used, and may not have been successful in some topological settings.

5.8.2.2 Accommodations that explicate perceptual knowledge and intuition

As they worked through the task, the participants began to incorporate increasingly formal meanings into their explanations, and compared their present conceptions with exemplars and prototypes from their concept images. Most indicated feeling a perturbation associated with the discrepancy between the visual representation of the function and its formal mathematical classification as continuous. Some of the participants were emphatic about maintaining their perceptual basis of understanding, despite mounting evidence from their formal reasoning about the definition. Amy believed the function was continuous "just because from looking at it!" Nolan was explicit about his perceptual association, exclaiming that "it's just, I have the frickin' intuition, you know! Of like, it's a jump, it's going to be discontinuous…" while Saul expressed his past success in relying on his intuition: "It doesn't feel right […] Usually if I don't think something seems right, it's not right!"

It was during such moments of cognitive disequilibrium that many of the participants felt the need to search for a formal mathematical explanation for the discrepancy between their conceptual understanding (after correctly 'proving' the task) and their perceptual awareness of the visual characteristics of the graph. The explanation they sought was an explicating accommodation that would adjust their mental scheme for continuity in such a way that their perceptual information made sense. In some cases, the explicating accommodation had occurred at some time before the interviews. For example, after considering his existing scheme for the limit definition of continuity, Saul immediately recognized the discontinuity in the function for Task 3.2(B).

> *Interviewer:* So, if we think about epsilon and delta…
>
> *Saul:* They're going to zoom in here, and yeah…okay. Alright. I feel like I'm-, I feel like switching my frame of reference trips me up. Okay, okay.

But in other cases, I interpreted the participants' actions to mean that they were actively modifying their schemes during the interview, rather than recalling a prior scheme. Both Saul and Maren seemed to actively search for and find a formal mechanism that would link their dual systems of representation.

Table 8: Explicating Accommodations Used by the Participants

	Task 3.2(A)	**Task 3.2(B)**
Saul	Restricted Domain / Axiomatic	Limit-Dynamic
Nolan	Restricted Domain	Same
Maren	Connected-Continuous	None (researcher guided)
Wayne	Axiomatic	Same
Amy	None (researcher guided)	Limit-Formal
Gavin	None (syntactic-textbook)	n/a

5.8.2.2.1 Restricted domain accommodation

However, Saul was also quick to look for a mathematical explanation for the discrepancy: "Right! So now I'm going to have to think why-, I mean I believe it's continuous because it matches the definition." He quickly noticed that the function was not defined at the point of discontinuity. In explaining this argument, he described an accommodation he had made to his scheme for checking continuity based on his previous experience of noticing how the properties of functions may change due to restrictions of the domain. The fact that Saul had apparently modified his scheme to reflect this information is not sufficient to claim an accommodation had been made. However, his use of this modification during his work on subsequent tasks seems to provide verification.

Nolan appeared to have made a similar accommodation when he asked: "I mean if this was from $\mathbb{R}$ …it would be discontinuous because of that jump, right?" He went on to express being encumbered by "the whole paradigm of like, the Cartesian plane," claiming that he had never "seen a piecewise function outside of this class that wasn't from $\mathbb{R}$ to $\mathbb{R}$." Although their reasoning differed in many respects, both Saul and Nolan recognized that the continuity of the function depended on the domain on which it is defined. I will refer to this as the *restricted domain accommodation.*

One important aspect of the restricted domain accommodation is that it was not made for the purpose of accomplishing the task or formalizing any prior mathematical knowledge. For example, Saul had already correctly reasoned that the function should be discontinuous using the formal, topological definition for continuity, and he stated his conviction that this answer was correct. But, he made his new accommodation as a way to reconcile his two competing conceptions—based respectively on his formal-symbolic knowledge and his empirical perceptions. He prioritized the former, and sought a formal explanation for why the latter failed to agree.

5.8.2.2.2 Connected-continuous accommodation

In contrast to the other participants, Maren never reported any perturbation about the graph's disconnectedness. Before drawing the graph, she reflected on its attributes and declared that "it's a continuous function from a disconnected space onto another disconnected space. So, it's not going to be smooth, but the function itself is continuous."

Her meaning for the term “smooth” was not clear, but she had apparently formed a link between her schemes for continuous functions and connected spaces, allowing her to develop a coordinated scheme to address such a visual discrepancy with formal reasoning. For her, a continuous function has the property that connected components of the domain are sent to connected components of the co-domain. The function in Task 3.2(A) fits that criterion (see Figure 18), resolving any potential perturbation in her thinking. This is another example of an explicating accommodation, which I will refer to as the *continuous-connected accommodation*.

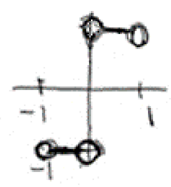


Figure 23. Maren’s graph for the function in Task 3.2(A)

Again, it is important to note that Maren only used this accommodation to explicate her “pragmatic model” (Job & Schneider, 2014), not to determine the continuity of the function itself. For that, she had used her personal concept definition, stating that “an open set in $\mathbb{R}$ would have to be open in X.” Her use of this definition, though not entirely precise, must have been brought about by a functional accommodation prior to the interview, because she expressed it spontaneously and without prompting. On the other hand, the new warrant Maren gave seems to indicate that for her, a function that preserves connected subsets is permitted to be continuous.

This explicating accommodation that Maren used would not be appropriate for determining the continuity of many functions, such as modified versions of the Topologist’s Sine Curve (Munkres, 2000/1975) or Conway’s Base-13 Function (Oman, 2010). Such functions are not continuous, but do send connected subsets of the domain to connected subsets of the co-domain. Therefore, Maren could not have used such an accommodation as a way to solve the task; any attempt to do so would have potentially been logically problematic. However, in this context, her use of the accommodation was not for the purpose of answering the task directly. Rather, she was trying to justify the continuity of the function in spite of its graph's apparent disconnectedness. That is, the purpose of her reasoning was to avoid a contradiction between her formal-symbolic and embodied schemes for continuous functions, which this theorem allowed her to do in a logically permissible manner.

5.8.2.2.3 Axiomatic accommodation

At times, participants also used their understanding of the axioms and formal definitions to explicate their perceptions, which I will refer to as an *axiomatic accommodation*. As with the other forms of accommodation this involves an acceptance of the outcomes from formal reasoning, despite inconsistent perceptual signals. However, unlike the restricted domain or continuous-connected accommodations, an axiomatic accommodation entails a suppression of the participant's embodied schemes. In these cases, participants indicated that they simply disregarded the perceptual information and accepted the results 'on faith.' Nolan demonstrated this when he said "I try not to let the truth, what appear to be the truths of nature, bother me, because I can't do anything about it…I knew my intuition wasn't going to take me far in this [class]."

For instance, based on his initial response to the task it appeared that Wayne had previously accommodated his continuous function scheme to the topological definition. He realized that "a union of basis's" formed an open set, and that was sufficient for his conviction that the function was continuous. He indicated that he was comfortable basing his reasoning on the formal definition when he said: “back there we were, it was just, I was trying to think more visually, rather than just, oh yeah, that’s open in this topology.” He coordinated his knowledge of the formal definition for topological continuity with the symbolic notation that indicated which sets were included in the basis of the topology. These mathematical properties of the situation seemed to be more useful for his reasoning than the visual representations he had been trying to use before. Both were salient features of the function, but the visual cues apparently failed to provide him a coherent strategy for continuing the proof.

Similarly, Saul indicated the potential for an axiomatic accommodation when he said he “was just going with the definition and not worrying about what it looked like so much.” His basis for reasoning about the situation was to ensure that it “matches the definition, which might indicate that Saul had accommodated his understanding in such a way that his perceptual experiences could be completely explicated by the formal-axiomatic definition. However, as Steffe & Thompson (2000) noted, an accommodation cannot be inferred from a single instance of a student using it. It is important to look at Saul’s responses to modified versions of the task.

At first during Task 3.2(B) Saul seemed to disregard the formal definition in favor of his “feeling” that the function was still continuous. The slight modification he had made to his visual representation did not seem to register to him as significant in terms of the function’s continuity. Rather than checking cases as he had seemed to do in Task 3.2(A), Saul fixed his attention on superficial differences in the graph. This may indicate that Saul’s apparently axiomatic scheme was not fully accommodated for the explication of his perceptual knowledge. It is important to note that students' schemes are likely to be in flux on any occasion. It was not surprising to find that any of the participants' ways of understanding were still developing.

5.8.2.2.4 Limit-Dynamic and Limit-Formal Accommodations

When Saul finally provided an explicating accommodation for Task 3.2(B), it was by referencing his spatial and kinetic (Sierpinska, 1987) conceptions of the limit process, which allowed him to think about the function in terms of his previous knowledge from real analysis. Explaining why he did not feel perturbed by the discontinuity of the function's graph, he referenced a dynamic metaphor in which the function "zooms in here" on the point of discontinuity. I called this the *limit-dynamic* accommodation, and it was only observed on this occasion. I considered the limit-dynamic accommodation to be an explicating accommodation based on a previously achieved formalizing accommodation.

Unlike Saul's limit point conception, Amy referenced a syntactic conception of the limit of a function by referring to the formal, epsilon-delta definition of limit at a point. It is unclear whether her conception was static (Sierpinska, 1987) or in some sense dynamic, because her language remained entirely formal. As a result, I named this the *limit-formal* accommodation, which was observed in only this episode.

5.8.3 Coordinating the conceptual and perceptual systems

It may be the case that the successful learner in this context must use her sensori-motor and representational worlds in coordination with each other to construct a meaningful, multi-modal conceptual system. In the reverse sense, a lack of coordination of these two worlds may manifest as obstacles to the student's understanding. Table 9 describes the parallel representational systems observed in the participants explicating accommodations during the tasks. For most of the accommodations, the participants' formal basis for reasoning was tied to a specific perceptual basis. However, in the axiomatic accommodation, any perceptual basis was suppressed in favor of the conceptual representation system for continuous functions. In these cases, the explicating accommodation did not serve to connect the formal and perceptual systems, but rather allowed the participants' formal understanding to play a more dominant role than their perceptual knowledge.

Table 9: Bases for Explicating Accommodations

Accommodation	Formal Basis	Perceptual Basis
Connected-Continuous	Theorem 5.2	Continuity preserves connectedness
Restricted Domain	Definition of function	Undefined subsets of the domain are occluded
Limit-Formal	Limit definition of continuity	N/A
Limit-Dynamic	Limit definition of continuity	Dynamic movement toward discrepant point
Axiomatic	Axiomatic definition of continuity	Concepts prioritized and perceptual knowledge suppressed

5.8.4 Participants' attention to properties and attributes

Reflecting on the completed task, many of the participants traced their difficulties to a focus on the incorrect mathematical properties; or, looking at those properties in an incorrect way. Some of the participants stated that they should have been focusing on different properties, or the properties of different mathematical objects. For example, Maren explained that she had been attending to the properties of the function's underlying space, the domain, rather than concentrating on the properties of the function as a transformation of that space.

> *Maren:* I didn't realize that a function could have properties […] I think I was getting confused, and looking at what the sets look like, and seeing if the sets were connected. And getting that confused with if the function is continuous.

Nolan expressed a similar thought, when he said the following:

> *Nolan:* …it's more than what the function is defined on that makes it discontinuous or not. Like, it's more than just what you're defining it on, which I think is a misconception I was carrying into this class…is everything in the domain being mapped? That's a function, it doesn't have anything to do with continuity. So, I mean, that's a really big difference.

Other students had inaccurately interpreted properties and their interactions, or confused two separate properties that functions could have. Maren conflated the injective property with the function property, as did Amy. But Amy also seemed to relate the continuity property to the surjective property in an interesting way.

> *Interviewer:* So, what's your-, do you have a definition for continuous function?

> *Amy:* Um……I'm trying to think, like……looking at different pictures of functions in my head, um……

Figure 24. Amy's representation of functions from her example space. She used these to formulate her conception of continuity.

> *Amy:* …and I'm only looking at these because um, you know like the one-to-one stuff. But I'm thinking about, the only thing that doesn't make it continuous is that…every point in the range…if there's one case where like one point in the range does not have a point in the domain that points to it.

Figure 25. Amy’s visualization of a non-continuous function. She used this as a basis for her description of the property of continuity.

Her representation was apparently not injective or surjective. However, her example and accompanying statement closely matched one element of the formal concept definition for *a point of discontinuity*; i.e., a point at which the function is undefined. When she stated that a discontinuity means that “one point in the range does not have a point in the domain that points to it,” she likely meant the co-domain, according to standard terminology, since by most definitions, everything in the range has a pre-image by definition. Her choice of a function which was neither surjective *nor* injective did not seem to challenge her reasoning in this regard.

It was clear from her discussion during the interview that Amy preferred to reason with visual re-presentations from her concept image. She had discussed how her understanding for continuous functions was based on the picture in Figure 26.

> *Amy:* If you were to talk to me any day about open, I mean continuous, I’m going to draw this picture, and then stuff like this actually helped me a lot in my analysis class for proofs, where we weren’t even using numbers…

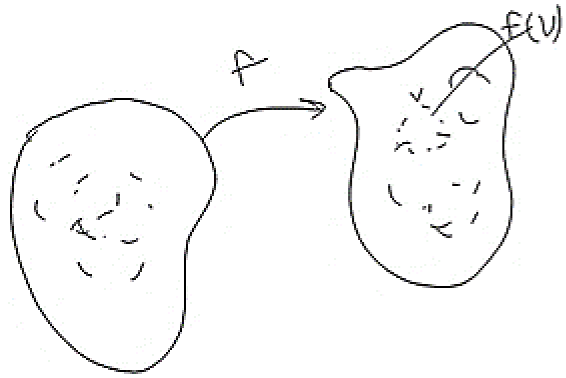


Figure 26. Amy's visual representation of a continuous function.

On the other hand, Saul felt more comfortable reasoning procedurally for certain functions, for which he seemed to disregard any sort of graphical or visual re-presentation, other than formal-symbolic notation identified in the Task 3.2(A) prompt.

> *Saul:* I guess one part of it is that if I see a function that splits to different things, I don't worry, I stop thinking about graphs…

$$(-1,1)\setminus 0 \longrightarrow \{1,-1\}$$

Figure 27. Saul's inscription when reasoning about the continuity of the function in Task 3.2(A).

> *Saul:* I wouldn't know how to graph this […] Until you said the word graph, I didn't even think about it…

5.8.5 Axiomatic Reasoning

Wayne's reasoning about the interior of a set (see 4.6.5.3), and the explicating accommodations made by Saul and Maren (see 5.8.2.2) demonstrate a distinction between formal-symbolic reasoning from what I refer to as *axiomatic reasoning*. This refers to semantic forms of reasoning that would seem to benefit axiomatic understanding, but which were not performed in a formal context. Some students with limited formal understanding were still able to recognize the essential axiomatic structure and reason with it.

I identified four axiomatic traits (or axiomatic 'habits of mind') that allowed students to reason effectively about the tasks, even with relatively little formal knowledge. First, I identified cases of the "anontologization" (Freudenthal, 2002) of a student's conceptions. That is, the severing of the student's mathematical world from her knowledge of physical reality. This is indicated when a student is able to perceive that a logical train of thought is correct, despite a perceptual or intuitive discrepancy. Wayne's axiomatic explicating accommodation demonstrated this characteristic (see Section 5.6.3.2).

Second, I found cases of my participants' awareness of the global relationships between the elements of topological theory. This was indicated when a participant connects concepts from different predicate levels or in more generalized ways. For example, when Amy translated the epsilon-delta definition for continuous, real-valued functions into her understanding of the alternative axiomatic definition for continuous functions, she demonstrated an understanding of one such global relationship (see 5.6.6.4). Other examples included Nolan's shift from the set-level to the point-level in his complement approach to the open set (see 4.6.3), and many of the participants' explicating accommodations as well (see 5.8.2.2).

The third characteristic of axiomatic reasoning is the *thematization* (Arnon, et al., 2014; Baker, et al., 2000) of the *axiom scheme*: Specific topologies can be seen as placeholders for the abstract axiom system of topology. This is indicated when a student can pass between topologies and see inter-relations between the underlying elements in each context. Each of the participants showed signs of a thematized axiom scheme when they correctly answered Task 3.2(C) by shifting their previous reasoning from the standard topology to the lower limit topology on the real numbers (see 5.8.2.2).

Finally, the expectation of consistent outcomes when using alternative formulations of concepts within the bounds of topological theory. Similar to “concept consistency” (Alcock & Simpson, 2011), this trait is indicated when a student recognizes that there are no “exceptions” or “tricks,” and that there should be a single logical result from any line of axiomatic reasoning. This characteristic was demonstrated on many occasions, when participants stated that a contradiction would need to be resolved within their reasoning. For example, when Saul realized that he would have to find an explanation why his representation of a function showed a discrepancy with his formal reasoning (see 5.6.1.2).

The relative independence of formal and axiomatic reasoning indicates that axiomatization (as a psychological process) operates with different obstacles than formalization, leading to the possibility that axiomatization could be taught prior to introducing formal theory. This would allow students to form axiomatic conceptions earlier in their learning development. Given the exploratory nature of this research, many such questions must be addressed in future studies with more specific research goals.

5.8.6 Conclusion

In this analysis, I have attempted to highlight the multi-modal aspect of my participants’ understanding of continuous functions in topological contexts. Evidence was provided to support the idea that my participants’ understanding of topology was grounded in multiple modes of meaning-making and understanding. Each in their own way, the participants worked to make sense of the interplay between their perceptual knowledge and the conceptual framework they built around it.

Wayne and Maren reasoned syntactically, and both could use formal reasoning to explicate their semantic understanding about the changing continuity of the functions in the tasks. Saul and Nolan both struggled to overcome their intuitions, despite their accurate understanding of the formal definitions and processes involved in the tasks. Gavin and Amy had not interiorized the processes involved with checking continuity enough to compare directly with the other participants. However, their responses were each insightful in different ways. Gavin provided the perspective of a student reading the definition and building his conception in the moment; while Amy brought a collage of perceptual and conceptual meanings together, and tried to make sense of them during the interview.

Finally, the case of Maren was highlighted, in which she was observed (during an earlier interview) to actively construct her understanding of continuous functions in terms of her prior conceptions and physio-spatial metaphors. Each of these non-formal *processes of understanding* (Sierpinska, 1994) played an integral role in the participants' attempts to reconstruct meaning around their conceptions of continuity in light of the new axiomatic structure imposed on it.

6. CONCLUSIONS

6.1 Summary and Overview

To advance the research on students' reasoning in advanced mathematics, I conducted a semester-long qualitative study to illuminate how six undergraduate mathematics majors approached the transition to axiom-based reasoning in their introductory topology course. The purpose of this research was to build theoretical interpretations of the conceptual activities of individuals as they worked to accommodate their mental schemas for the open set and continuous function concepts. Through a series of clinical interviews, I observed and interpreted my participants' mathematical efforts as they completed proof tasks in the context of topologies with which they were unfamiliar. I found that they employed diverse strategies and reasoned with multiple formulations and conceptions of the open set and continuous function constructs as they embedded their less formal and more formal ways of understanding into schemas that would reflect the axiomatic system of topology.

The research questions I addressed in this study are as follows. For undergraduate students in an introductory topology class:

1) What distinctions and comparisons can be made between the various ways that students manage their transition to an axiomatic understanding of continuous functions?

2) What obstacles do students face during this transition?

Some answers to these questions were examined in the analysis and results reported in Chapters 4 and 5; which involved respectively: an APOS analysis of the open set concept, and a radical constructivist conceptual analysis of the continuous function concept.

6.2 Research Question #1

What distinctions and comparisons can be made between the various ways that students manage their transition to an axiomatic understanding of continuous functions?

Chapters 4 and 5 addressed distinctions and comparisons in the participants' responses to the tasks, by delineating categories for the range of mathematical activities observed during the interviews.

		Axiomatic Approaches		
		Pointwise Axiomatic	*Setwise Axiomatic*	*Familywise Axiomatic*
Predicate Level	*Family*	*Object:* open set as the union of a family of sets constructed pointwise $\bigcup_{x\in O}[x,1)$	*Object:* open set as the union of a family of sets constructed setwise $\bigcup_{n\in\mathbb{Z}^+}[1/n,1)$	*Object:* open set as a recognized member of the family of the topology $[0,1)\in\mathfrak{T}_{\mathrm{L}}$
	Set	*Process:* Construct a family of basic open subsets around each point in the set $\{[x,1)\}_{x\in O}$	*Process:* Construct a family of basic open sets by defining criteria, e.g., a convergent sequence of left-hand endpoints $\{[1/n,1)\}_{n\in\mathbb{Z}^+}$	N/A
	Point	*Action:* Identify a basic open subset around individual points $[x,1)\subset(0,1)$	N/A	N/A

Figure 28. Three axiomatic approaches to the open set construct, differentiated by predicate levels. Arrows indicate interiorizations and encapsulations required to achieve each construction.

In Chapter 4, I compared my participants' mental constructions for the open set concept through an APOS analysis (Arnon, et al., 2014) that resulted in multiple genetic decompositions for a diverse range of conceptual approaches to the open set concept. These decompositions were distinguished along two dimensions: 1) the distinct mathematical approaches to the open set concept that could be interiorized by the participants as mental processes, and then encapsulated into object conceptions; and 2) the range of predicate levels, or tiers in the categorical hierarchy of points, sets, and set families that participants could choose to reason about. These conceptions, arranged according to those two dimensions of abstraction, are listed in Figure 28 and Figure 29.

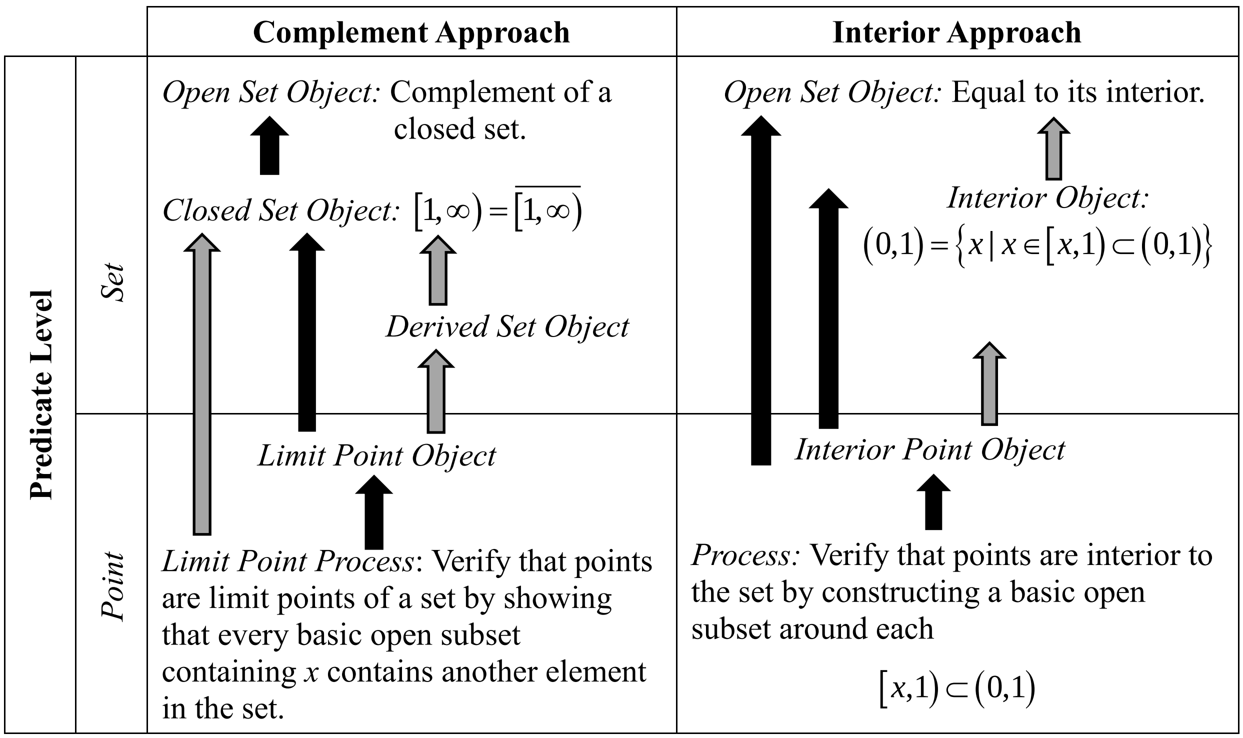


Figure 29. The complement and interior approaches to the open set concept. Arrows indicate the potential encapsulation of processes, with variations on the different objects that may be formed along the way. Black arrows indicate paths that were observed in this study, while grey arrows indicate hypothetical paths to encapsulating the open set object through the respective approaches.

Through the conceptual analysis reported in Chapter 5, I explored the participants' changing conceptions for continuous functions as they progressed through the semester in their topology class. It was reported how some of the participants had "extracted meaning" from the formal axiomatic structure of topology to make sense of the tasks; while others "gave meaning" to the definitions by building up their understanding from other aspects of their concept images (Pinto & Tall, 1999). This distinction was operationalized as a class of scheme accommodations that served to explicate (Dawkins,

2015) the participants' perceptual knowledge (rather than as an accommodation to complete the task or formalize their prior mathematical understanding). These accommodations are listed in Table 10. Through these distinctions, different forms of reasoning about the property of continuity were shown to be overlaid onto a common, abstract re-presentation of the idea of a function.

Table 10: Explicating Accommodations

Accommodation	Formal Basis	Perceptual Basis
Connected-Continuous	Theorem 5.2	Continuity preserves connectedness
Restricted Domain	Definition of function	Undefined subsets of the domain are occluded
Formal-Limit	Limit definition of continuity	Dynamic movement toward discrepant point
Axiomatic	Axiomatic definition of continuity	Concepts prioritized over precepts

6.3 Research Question #2

What obstacles do students face during the transition to an axiomatic understanding of continuous functions?

Several obstacles were evident in the participants' responses to the tasks in both Chapters 5 and 6. An epistemological obstacle (Sierpinska, 1994) involving the coordination of *perceptual* and *conceptual* space (Job & Schneider, 2014; Piaget & Garcia, 1983/1989) was found to create challenges for the participants' axiomatic reconstruction of their understanding. Discussing the role of epistemological obstacles in the history of scientific thought, Sierpinska (1994) reminds us that

> …these obstacles are, contrary to the connotations that the word 'obstacle' can bring to mind, positive. They are positive in the sense that they constituted the ground of the 'epistemological space' that determined, in a way, the kind of scientific questions and ways of approaching them, characteristic of a given epoch. (p. 134)

Similarly, the epistemological obstacles involved in understanding the self-referencing notion of an open set in the axiomatic system of topology can serve as a map of the "epistemological space" of the participants, delineating the types of concepts they may construct. On the other hand, topic-specific (non-epistemological) obstacles were also apparent throughout the participants' attempts to reconcile the large number of alternative approaches to defining concepts within the axiomatic structure of topology. These included: participants' use of inappropriate definitions (metric or real number topology instead of the lower limit topology); an excessive number of potential defining approaches learned without an understanding of the global relationships between them; and reliance on paradigmatic examples and visual prototypes for property generation rather than deduction from the formal definitions. Non-epistemological obstacles seemed to offer little benefit to the participants' attempts to make sense of the tasks.

Although each of the conceptual approaches found in Chapter 4 had mathematical bases in the textbook and shared knowledge of the class, some of the approaches were more closely tied to salient perceptual features of the participants' re-presentations (e.g. the interior and complement approach). On the other hand, the axiomatic approaches required the participants to detach their reasoning from the perceptual domain. Different predicate levels for each of the conceptual approaches compounded the difficulties that some students faced, when they failed to de-encapsulate the mental objects associated with particular concepts based on the axiomatic definition of open set (e.g. interior, boundary, closure). For example, many participants were quick to declare a set of the form (a,b) open, yet reluctant to declare sets like $[a,b)$ to be open, even in the lower limit topology. Their previous mathematical experiences with the term 'open' had become reified to reflect the symbolic form of the set's description, rather than the axiomatic theory that defines the concept. Rather than de-encapsulating their mental objects to examine their properties, they unconsciously and automatically reasoned from superficial appearances.

Moreover, as Gavin's case demonstrated (see Chapter 4), the concepts of *boundary point*, *closure*, *derived set* and *limit point* all relied on his personal concept definition of an *open set*, which he modified in such a way as to exclude a particular set from consideration. Since this modification essentially re-established a metric definition of *open set*, he was willing to accept it without further perturbation. Thus, Gavin adapted to the epistemological gap in his understanding by modifying his 'theory' and embracing a form of the concept from his earlier experiences.

In Chapter 5, the explicating accommodations made by the participants were for the purpose of overcoming the epistemological obstacle that was illuminated by the contributions of Piaget & Garcia (1983/1989), Pavio (2010), and Job & Schneider (2014). The participants' varying degrees of reliance on visual representations for the tasks provided an opportunity to contrast the affordances and obstacles of cognitive representational systems that are heavily weighted toward the perceptual or the conceptual forms of understanding. I classified both extremes as epistemological obstacles. An overreliance on spatial intuition and sensori-motor perceptions was shown to impede some participants' abilities to formalize or axiomatize their understanding. But, an overreliance on formalization was also shown to be detrimental to some participants' ability to coordinate their dual representation systems.

6.4 Theoretical Contributions of the Research

The research questions explored in this study were inspired by current trends in undergraduate mathematics education research. In the past several decades, an emerging movement has been underway to expand theories of cognitive development (both cognitive and historical) to elaborate research on learning in advanced mathematics (cf., Alcock & Simpson, 2002; Arnon, et al., 2014; Dawkins, 2015; Lakoff & Nuñez, 2000; Piaget & Garcia, 1983/1989; Sierpinska, 1994; Tall, 2013). This growing body of theory contributes to our understanding of the connections between cognition at varying levels of mathematical abstraction. The research presented here builds on these theories to interpret and explain the conceptions of an under-researched population—undergraduate topology students.

One reason that this population is important to study is the subject itself. The field of topology stands at the intersection of human sensori-motor knowledge and the axiomatic conceptual structures we embed within our perceptions (Moore, 2008). The field provides us with an opportunity to study the cognitive process of axiomatization as a natural human endeavor—as an evolution of our basic psychological operations. In the words of Piaget and Garcia (1983/1989), the axiomatization of topology stands at the highest level of the human "psychogenesis of space" (p. 112).

Through my analysis of the participants' mathematical activities during the tasks, I have presented evidence that property-based concepts like *open set* and *continuity* may be experienced in more 'primitive' and perceptual ways prior to the formation of formal-symbolic conceptions of either idea. As the participants' ways of understanding evolved over the course of the semester, their non-verbal "coding systems" (Paivio, 2010) were not simply replaced by verbal and conceptual systems; but seemed to play a role in the development and reconstruction of the formal concept itself (Piaget & Garcia, 1983/1989; Piaget & Inhelder, 1948/1967).

For example, consider the highlighted case of Maren's use of metaphor, exemplar, and prototype to understand the continuity of topological functions (see 5.7). The influences of her earlier experiences (her example space) and her perceptual experiences of continuity in physical space (her metaphors and physical instantiations) were clearly present in her descriptions and visual representations of discontinuous topological functions (see 5.7.3). There were many similar episodes in which a participant mentioned physio-spatial attributes as warrants for logical claims, or "borrowed properties" (Hazzan, 1994) from well-known exemplars to make sense of previously known concepts in new, more abstract contexts.

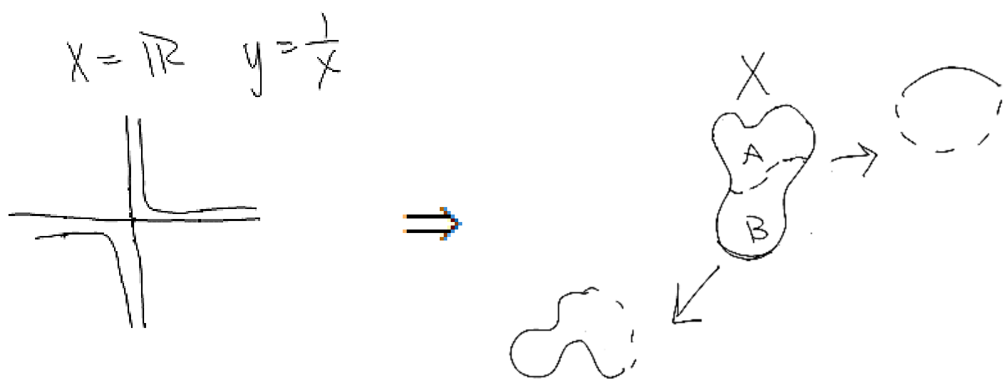


Figure 30. The influence of visual attributes of a graph. Salient visual characteristics led to Maren's reconstruction of the continuity property in topological spaces.

Further examples of the participants' dual conceptions of space can be found in the explicating accommodations they made to reconcile their formal solutions to the continuity tasks and the perceptual feature of their graphs' disconnectedness. For example, Saul (see 5.8.1) and Maren (see 5.8.2.2.2) both invoked specific elements of the formal theory to justify the discrepancy they noticed. Saul focused on the role of the domain in the formal definition of *function*, while Maren expressed her interpretation of Theorem 5.2 (Croom, 1989, p. 75). On their own, these theoretical nuances would not be sufficient to complete the task. However, in this case the participants had already accomplished the task. Instead, they accommodated their schemas to overcome a perturbation caused by the interaction of their two representation systems.

In other words, Saul and Maren used these accommodations to bring their "dual coding systems" (Paivio, 2010) into coherence. On the other hand, Wayne and Nolan seemed to use a different approach to the duality of their understanding. Rather than coordinating the mathematical world with the sensori-motor world, they seemed to suppress the latter and give priority to the former. Neither Wayne or Nolan seemed to note a necessity that their conceptual and perceptual coding systems should provide equivalent outcomes in every situation, and they deferred to their formal conceptual systems regardless of any perceptual feedback. I refer to this as *anontologization,* Freudenthal's (2002) term for the cutting of mathematical theory from its perceptual links and origins. Further research is needed to determine whether such "anontologization" (Freudenthal, 2002) is beneficial too students' long term axiomatic thinking.

6.4.1 Informal axiomatic reasoning

This study also contributes to the study of axiomatization of mathematical content by presenting evidence for the independence of axiomatic reasoning from formal reasoning. Cases such as Wayne's reasoning about the interior of a set (see 5.6.3), and the explicating accommodations made by Saul and Maren (see 5.6.2) serve to differentiate formal reasoning from what I have called *axiomatic reasoning* (see 5.8.5). I observed reasoning traits that would seem to benefit axiomatic understanding, but which were not performed in a formal context. Some students with limited formal understanding were still able to recognize the essential axiomatic structure and reason with it. I outlined four traits that were apparently beneficial for axiomatic reasoning.

6.4.2 Properties and attributes as a research focus

By exploring the participants' transformative uses of properties during the accommodation of their schemas to axiomatic contexts, my study contributes to an emerging perspective on the construction of axiomatic mathematical understanding in general. I have demonstrated the utility of a fine-grained analysis of student conceptions of both mathematical and non-mathematical properties when it comes to the transition to axiomatic levels of understanding.

Mathematically, many of the participants indicated their realization that they had been focusing on different properties than they should have; or, they realized that they had inaccurately interpreted those properties and their interactions (see 5.7.3). Also, in a perceptual sense, attributes of their re-presentations for the properties of functions played a significant role in the way the participants interpreted the mathematical structures they called upon, which was indicated by the multiple approaches to defining the open set that were reported in Chapter 4, as well as the explicating accommodations described in Chapter 5. Moreover, the highlighted cases of Gavin and Maren illuminated connections between their perceptual and intuitive attributes they noticed on the one hand, and the mathematical properties they used on the other. Considering the duality referred to in Chapter 2, the dialectic interaction between these participants' verbal and non-verbal systems of representation played itself out in both cases.

6.5 De-Limitations for the Study

I created analytical bounds and de-limitations for the study through my participant sampling and content selection choices, my research questions and coding paradigm, my adherence to a radical constructivist epistemology, and the overall analytical objectives I pursued.

I sought out candidates to participate in the interviews who demonstrated: moderate to high levels of understanding of the course content, symbolic fluency, and high degrees of motivation and interest in the subject. These choices influenced the types of interactions and responses I observed, and avoided the inadvertent examination of superficial factors, such as inadequate study skills of a lack of motivation, as potential obstacles to reasoning. Moreover, this study was de-limited by the specified objectives of the analysis, which was to interpret the mathematical activities of my participants from their own perspectives; and to guide them to elaborate coherent, deductive records of their conceptions at the time of the interviews. I made no prior assumptions about the reasoning activities of the participants.

An undergraduate and introductory course was chosen to specify the research domain, while the *open set* and *continuous functions* topics were chosen for their theoretical importance in the field. By specifying the course setting and content topics, I was able to focus my research within a narrow band of the advanced mathematical curriculum and explore my participants' conceptual reasoning in depth. My conceptual and analytical frameworks also de-limited my research, in the sense that the research questions and coding paradigm were the result of three semesters of preliminary analysis and theory-building, ultimately resulting in the frameworks for this study. The utility of those frameworks resulted from being applied in specific contexts that allowed for detailed comparisons of the participants' reasoning about closely related concepts.

Finally, the radical constructivist epistemology set bounds on the nature of my study, and de-limited the claims it was possible to make. I explicitly stated that my analysis was about indirectly interpreting my participants' experiences, as they interpreted a mathematical reality that we all equally had no direct way of experiencing. For this reason, my research goals involved guiding the students to build a coherent and consistent 'record' of their own ideas; rather than judging the accuracy of that record.

6.6 Limitations of the Study

Several limitations also apply to this research, and should be considered carefully along with the findings presented. These limitations relate to the research population and sampling, some procedural issues, and the inherent reporting limitations for a large collection of data from a small sample.

Although the participant sampling was systematically conducted based on guidelines that had been designed over three semesters of preliminary research, due to timing and location constraints there was little choice in the initial classroom population. Nevertheless, the class that was selected for participation was taught by a professor familiar with my preliminary research, who organized his curriculum in a clear manner (evidenced by my classroom observations and field notes), based on a common approach to the content (see Textbook Analysis), and who provided extensive feedback and support for his students. Therefore, the choice is theoretically justified, and not simply a matter of convenience. There was also little choice in the level of the course selected. The original research design involved data collection from two classes, an undergraduate and a graduate course; however, the graduate course was unexpectedly not offered during the semester of the main study.

Another significant limitation was the large amount of data I collected, and the relatively small amount data I was able to analyze and report on. By de-limiting my analysis (see Section 6.5), I bounded the study and chose discrete analytical objectives based on my conceptual framework and chosen content topics. To the extent that this report relays a complete and coherent narrative about relevant themes in the research literature, it may overcome this limitation.

My inexperience as a researcher and the fluidity of interactions during a clinical interview or teaching experiment led to some miscommunications with the participants, as well as several missed opportunities to elicit further elaboration where it would have been useful. With only six interview participants such omissions can be damaging to the coherency of the analysis. I have attempted to clarify participant meanings (as I interpreted them) as carefully as possible.

As with many qualitative research studies, my interpretations as a researcher, as a teacher, and as one who constructs his own mathematical meanings had an impact on how I collected, interpreted, analyzed and reported my data. Thus, in numerous ways, the legitimacy of the findings reported here rely on my knowledge and professional reputation. Having expressed this clearly, it is for the reader to determine whether I have used this knowledge to contribute usefulness to the findings I have presented.

6.7 Recommendations for Future Research

These findings represent only the beginnings of an exploration into abstract student understanding in topology and other axiomatically-structured mathematical content. I intend to further this research program by: 1) gathering further evidence to support the grounded genetic decompositions for students' conceptions of the open set construct (see Chapter 4); 2) establishing other examples of students' coordination of their sensori-motor and representational schemes (see Chapter 5); and 3) elaborating on the theoretical categories presented throughout both analyses.

6.7.1 Student approaches to defining axiomatic objects

Chapter 4 highlighted the need to investigate the interactions between the different forms of abstraction that are involved in the construction of the mental structures students create during their courses in advanced mathematics. It would be useful to explore factors that enable students to develop such formal and/or axiomatic conceptions, and whether those conceptions provide any benefits for subsequent learning in topology. More research could also be conducted on the connections between various conceptual approaches to the content, and whether other factors might mediate the affordances and obstacles offered by each approach.

6.7.2 Focusing on students' representations of properties

The central unit of analysis for this research was the participants' uses, representations, and instantiations of mathematical properties and descriptive attributes of the mental objects they conceptualized. This novel focus offered a more nuanced view of the mental constructions that inhabited their mathematical and perceptual worlds. This study demonstrates the fruitfulness of such an approach when investigating students' conceptual transitions to axiomatic systems. Similar research could prove useful for the analysis of students' understanding in other axiomatic fields, such as advanced courses in algebra, geometry, or analysis.

6.7.3 Instructional interventions

The topology-specific aspect of this research would also benefit from a wider survey of teaching styles, levels of content abstraction, and commonly-used textbooks. This would help to understand the didactical implications of students learning through the various conceptual approaches; and to gauge different instructional approaches to the topology curriculum. If the axiomatic formalization of students' understanding in topology is an educational goal for most professors, it would be useful to study whether the centrality of the axiomatic definition is conveyed appropriately (by professors and textbooks) to students. It would also be beneficial to investigate whether it is possible for students to succeed in topology with alternative, and less explicitly axiomatic, conceptions of its fundamental notions. Answers to these questions may help to determine which pedagogical choices influence which properties and processes that students attend to as they reflect on their conceptions of topological ideas.

6.8 Final Remarks

Through an emphasis on the ways my participants' represented mathematical properties and perceptual attributes, I have provided a narrative mapping of their mental conceptions over the course of the semester. By selecting topology as the content matter, I could observe the participants during a key time of transition in their trajectory towards formalization of their mathematical understanding. Moreover, I could observe the participants engage in the re-conceptualization of properties with which they had a great deal of experience.

In this spirit, I close with quotations from two of the participants, that I believe capture the essence of their challenges after a semester in the class.

Maren: So, essentially, when we're talking about a topology, putting a topology on a space…we're kind of just taking a space that already exists and like, organizing it in a specific way, right? That's kind of what a topology is…but we're just kind of picking the way that we're going to order the numbers. That's what we're doing with topologies.

Nolan: Right, and I think there's a price you pay, but there's also a convenience for using more abstract, right? I mean you don't have to take and keep track of all these epsilons and deltas and all the images of all the epsilons and deltas. And you know, you just have to keep track of one set, and, but you know you pay for that in the level of abstraction. *You have less knobs to turn but you have a bigger knob and it's harder to turn…*

APPENDIX SECTION

Appendix A: First Interview Protocol

The first interview was the most structured of the three and involved two parts. First, the participants were asked as series of questions designed to elicit their conceptualization of the main content topics involved in a metric understanding of topological ideas. This was the extent to which they had been introduced to the curriculum at that point in the semester. Moreover, the questions were designed to determine the alignment of the participants' personal concept definitions and example spaces with the formal concept definition for each content topic. The second part of the interview tasked the participants with proving several statements about each of the following content areas: 1) continuous functions, 2) the intermediate value theorem for real-valued functions, and 3) open balls in three-dimensional Euclidean space. Responses to Task 1.1 from this interview are reported in the results.

Section I: Definitions, Examples, and Categories

<u>Elicitation</u>: *Discussion of participant's definitions and example spaces for the concepts of:*

- Function (on survey)
- Limit of a function
- Continuous function on the real numbers (on survey)
- Open and closed balls in a metric space
- Boundary and interior points in a metric space

<u>Alignment</u>: *Discussion of discrepancies in participants' statements from above.*

- For each concept above, explain how any one of the examples you chose agrees with the definition you gave (i.e., justify why your choice is an example of the definition).
- Which, if any, metaphors do you use when thinking about these concepts?
- What, if any, visual imagery comes to your mind when you think about these concepts?

  Possible questions:
  - *Can you think of an example of a continuous function that has a disconnected graph?*

- *Does every subset of a metric space have an interior/boundary? Can you think of a subset of any metric space that has no interior/boundary?*
- *Can a ball (set) in a metric space be both/neither open and/nor closed? Provide an example.*
- *Can you think of a continuous function from one metric space to another space with a different metric? How would you prove that the function you chose is continuous?*

Section II: Proof and Problem-Solving

Consider the following real-valued function:

$$f:\mathbb{R}\to\mathbb{R} \text{ where } f(x)=\begin{cases} x, & x<2 \\ x+1, & x\geq 2 \end{cases}$$

1.1 a) Choose an open interval on which you believe the function *f* is continuous and prove it.

Possible responses:

- *Incorrect interval chosen including x=2 (unlikely). Why continuous?*
- *Correct interval chosen on one or other side of x=2. Why continuous?*
    - *No gaps in the interval*
    - *It is a linear function and they are known to be continuous.*
    - *Limit criteria hold for every value in the interval.*
    - *Epsilon-delta proof.*

b) Choose an open interval on which you believe the function *f* is not continuous and prove it.

Possible responses:

- *Incorrect interval chosen not including x=2 (unlikely). Why not continuous?*
- *Correct interval chosen including x=2. Why not continuous?*
    - *Gap in the interval*
    - *Requires two formulas*
    - *Limit doesn't exist*
    - *Function doesn't exist*
    - *Epsilon-delta proof*

1.2 a) Do you recall the intermediate value theorem from calculus? If so, specifically state the theorem and its meaning to you.

Prompt as needed for a precise formal statement, or provide the following:

Intermediate Value Theorem for Real-Valued Functions
If $f:\mathbb{R}\to\mathbb{R}$ *is continuous on the closed interval* $[a,b]$ *and k is any number between* $f(a)$ *and* $f(b)$, *then there is at least one number c in* $[a,b]$ *such that* $f(c)=k$.

b) What is it about the property of continuity that makes this theorem true? Use any words, pictures or diagrams you need to explain your answer.

Possible responses:

- *Unsure*
- *Correct response*
  - *Related to connectedness of graph*
  - *Related to connectedness of real number line*
  - *Related to limits of the function/epsilon-delta argument*

c) Can you think of a non-continuous function for which this theorem would not be true? Represent the function in any way you can to explain your answer.

Possible responses:

- *Unsure/incorrect example*
- *Correct example*
  - *Represented graphically*
  - *Represented algebraically*
  - *Represented verbally*

d) Can you restate this theorem in a way that would apply to a continuous function in any general metric space?

Possible responses:

- *Unsure/incorrect generalization*
- *Correct generalization referencing arbitrary metric*
- *Specific example provided for an alternative metric*

1.3 a) Determine the shape of an open ball in the max and taxicab metrics in three-dimensional, Euclidean space—the points $(x,y,z)\in\mathbb{R}^3$. Represent your solution in any way you can.

Provide the following if requested or needed:

Max metric: $d_{\max} = \max\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\}$

Taxicab metric: $d_{\text{taxi}} = |x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|$

Possible responses:

- *Unsure*
- *Confuses the two metrics*
- *Uses two-dimensional case to find correct/incorrect solution*
- *Correct description*
  - *Verbal*
  - *Graphical*
  - *Algebraic*

b) Prove that the open ball in either of the above metrics is an open set.

If this is not possible, ask if participant can prove this for an open ball in the standard metric of two-dimensional Euclidean space or an open interval in $\mathbb{R}$.

Possible responses:

- *Unsure/incorrect proof*
- *An open set is a union of open balls*
- *Its complement is closed*
- *Equal to its interior/every point is interior*
- *Contains no boundary points/not limit points of exterior*

c) Prove that the closed ball in either of the above metrics is a closed set.

If this is not possible, ask if participant can prove this for a closed ball in the standard metric of two-dimensional Euclidean space, or a closed interval in $\mathbb{R}$.

Possible responses:

- *Unsure/incorrect proof*
- *A closed set is a union of closed balls*
- *Its complement is open*
- *Equal to its closure*
- *Contains its boundary*

Appendix B: Second and Third Interview Protocols

The second and third interviews were less structured than the first, and were focused on following the participants' lines of reasoning through proof and justification tasks, in whichever direction they might take. These interviews took place later in the semester, as the participants had begun to make the transition from real number and metric space conceptions towards more abstract conceptions and personal concept definitions.

Due to the less structured nature of these interviews, anticipated responses and follow-up questions for the tasks were not established. However, impromptu questions were asked when necessary to clarify participants' statements and resolve contradictions or inconsistencies with their previous activities and explanations. In some cases, these tasks were presented in a different order to accommodate the flow of activities a participant chose to engage in. The tasks in the second interview involved the following content areas: 1) open and closed sets in two-dimensional Euclidean space, 2) continuous functions between topologically distinguishable Euclidean spaces, and 3) interior and boundary points in the lower limit ("half-open interval") topology. The tasks in the third interview involved: 1) continuous functions in a sequence of representationally similar, but mathematically different contexts; and 2) a topological theorem involving the preservation of connectedness via continuous functions. Responses to Tasks 2.3 and 3.1 are reported in the results.

Second Interview Tasks

2.1 a) Let $A=\{(x,y)\in\mathbb{R}^2 \mid x>0\}$. Determine whether A is an open set or not, and prove it.

b) Let $B=\{(x,y)\in\mathbb{R}^2 \mid y=0\}$. Determine whether B is a closed set or not, and prove it.

2.2 Determine whether the function g is continuous and prove your assertion.

$$g:\mathbb{R}\times\mathbb{R}\to\mathbb{R} \text{ where } g(x,y)=\begin{cases} x^2+y^2, & x^2+y^2<1 \\ 2, & x^2+y^2\geq 1 \end{cases}$$

2.3 a) Find the interior and boundary of [0,1) in the half-open interval topology.

b) Prove that (0,1) is an open set in the half-open interval topology.

c) Prove that the real number line is disconnected in this topology.

Third Interview Tasks

3.1 a) Define a function $f:(-1,0)\cup(0,1)\to\mathbb{R}$ by the rule $f(x)=\begin{cases} 1, & x\in(0,1) \\ -1, & x\in(-1,0) \end{cases}$.

Determine whether f is a continuous function or not using the standard topology on the real number line. Prove your assertion.

b) Define a function $f:(-1,0)\cup[0,1)\to\mathbb{R}$ by the rule $f(x)=\begin{cases} 1, & x\in[0,1) \\ -1, & x\in(-1,0) \end{cases}$.

Determine whether f is a continuous function or not using the standard topology on the real number line. Prove your assertion.

c) Reconsider the function f from the task above. Determine whether f is a continuous function or not using the half-open interval topology on the real number line. Prove your assertion.

3.2 Let X and Y be topological spaces, and $A\subseteq X$ a connected subset of X. Let $f:X\to Y$ be a continuous function from X to Y. Prove that $f(A)$ is a connected subset of Y.

Appendix C: Preliminary Assessment

The content of the preliminary assessment of the participants' conceptions is presented here. The form has been changed to accommodate a smaller space within the appendix section.

The following is an assessment of your understanding of the definitions of certain concepts related to the idea of continuous functions. Please attempt to answer each of the questions as completely and accurately as possible from your own recollection. If you cannot recall a precise formal definition for any of the concepts, write a description of your intuitive idea for what the term means. There are no right or wrong answers; the survey is meant to discover what your understanding consists of and how you construct that understanding. Thank you for your participation!

Functions

Define the following terms in the context of mathematical functions and give an example of each:

- Function
- Injective ("one-to-one") and Surjective ("onto")
- Pre-image and Image
- Inverse Function

Sequences and Limits

Define these terms in the context of sequences of real numbers and give an example of each:

- Sequence
- Limit of a Sequence

Which of the following represents a sequence? Circle those that are and be prepared to justify your response:

$\{1, 2, 3, 4, 5, 6, \ldots\}$	$\{x^n \mid n \in \mathbb{N}\}$	$\left\{1, 17, -8, \frac{2}{3}, 0, -2, 11, \frac{1}{5} \ldots\right\}$
$\{0.9, 0.99, 0.999, 0.9999\ldots\}$	$\{0, 0, 0, 0, 0, 0, \ldots\}$	$a_n = \frac{n^2}{2n+1}$
$\{\ldots -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots\}$		$\{2.7, 2.71, 2.718, 2.7182, 2.71828, \ldots\}$

Metric Spaces

Define these terms in the context of <u>metric spaces</u> and give an example of each:

- Metric
- Limit Point
- Boundary Point
- Interior Point
- Open Set

Topological Spaces

Define these terms in the context of <u>topological spaces</u> and give an example of each:

- Topology
- Open set
- Closed set
- Limit point
- Connected

Set Theory

Define these terms in the context of set theory and provide an example of each:

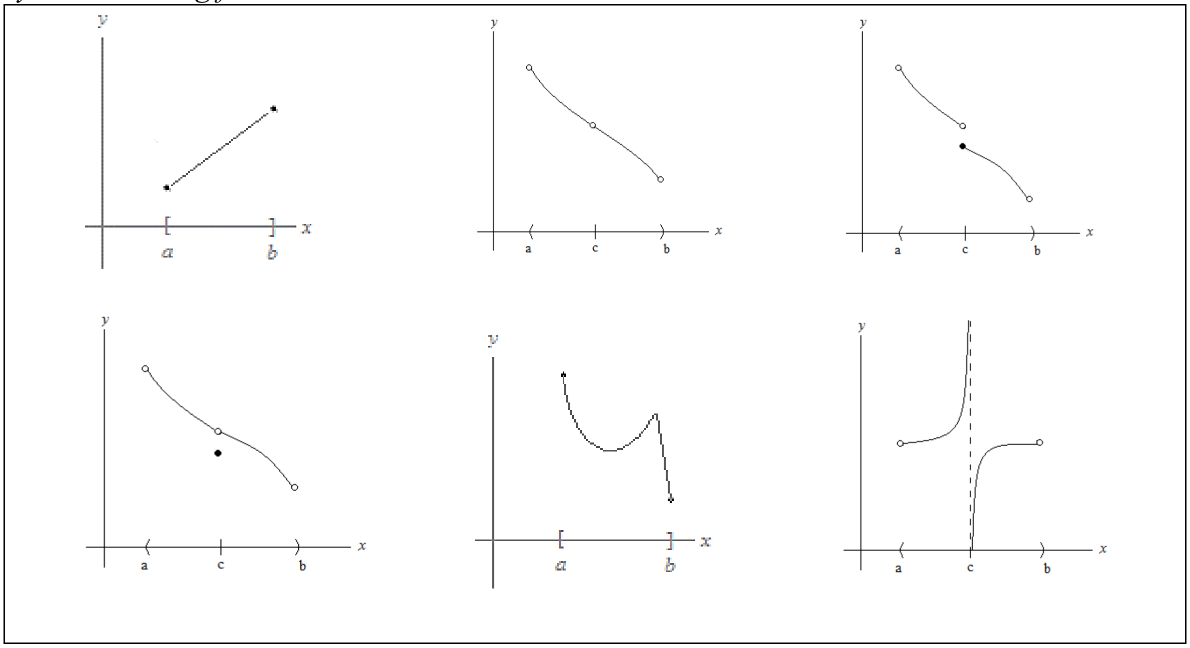
- Set and Subset
- Cardinality
- Finite and Infinite
- Countable and Uncountable
- Infinitesimal

Continuity

*Define the property of **continuity** in the following settings:*

- Continuity of a real-valued function on an open interval
- Continuity of a function in a general metric space
- Continuity of a function in a general topological space

Which (if any) of the following real-valued functions do you think is continuous on the indicated domain? Circle those you believe to be continuous and be prepared to explain your reasoning for each.



Symbolic Notation

Write a phrase or sentence that describes the precise meaning of each of the following symbolic expressions:

- $\{x \in \mathbb{R} | x \notin \mathbb{Q}\}$
- For $\{b_n\}_{n=1}^{\infty}$, let $B = \{n \in \mathbb{Z}^+ \mid b_n < 1\}$
- $\{\{a_n\}_{n=1}^{\infty} | \exists n, a_n = 0\} \neq \emptyset$
- $V \subset \{U_n\}$ and $U_1 \in \{U_n\}_{n=1}^{\infty}$
- $\exists N \in \mathbb{N}: \forall n \geq N, d(a_n, L) < \varepsilon$

Appendix D: Demographic Survey

The demographic survey was designed to provide information that would permit me to build a basic profile for each participant, including their academic preferences and affective reactions to different forms of mathematics they had experienced. They were also asked some fundamental questions about their understanding of set theory, functions, and continuity. This form has been adapted to accommodate a smaller size within these appendices.

Name: ________________________________ Major: ________________________

Are you an: □ Undergraduate student □ Graduate student

1. Please indicate the mathematics courses you have taken prior to this semester and the grades you received in the classes you've taken:

TSU Course (or equivalent)	Final Course Grade
□ 3315 Modern Geometry	
□ 3323 Differential Equations	
□ 3330 Introduction to Advanced Mathematics	
□ 2471 Calculus I	
□ 2472 Calculus II	
□ 3373 Calculus III	
□ 3377 Linear Algebra	
□ 3380 Analysis I	
□ 4315 Analysis II	
□ 4330 General Topology	

2. Among the classes listed above, which did you enjoy a) the most, and b) the least? Explain.

3. How do you feel about this class so far?
    - ☐ Enjoying it
    - ☐ Neutral/unsure
    - ☐ Not enjoying it

4. How well are you understanding the course material so far?
    - ☐ I understand everything pretty well.
    - ☐ I understand most of it.
    - ☐ I understand somethings, confused about others.
    - ☐ I don't understand very much.
    - ☐ I don't understand anything so far.

5. What would you change about Dr. Snyder's lectures to help you understand the material better? (You may choose multiple responses).
    - ☐ Nothing, he explains the material very well.
    - ☐ Illustrating more examples of important concepts.
    - ☐ More pictures and visualizations.
    - ☐ Explaining the applications of concepts.
    - ☐ Relating ideas to the mathematics I have learned previously.

6. What else would you change about this class to help you understand the material better?

7. What is the likelihood that you will drop this class if not satisfied with your first few grades:
    - ☐ I will probably drop the class.
    - ☐ I could possibly drop the class
    - ☐ I'm unsure
    - ☐ It's unlikely I will drop the class.
    - ☐ I definitely won't drop the class

8. How much or little does it help when your mathematics professors draw pictures on the board during lecture?
    - ☐ Always helps a lot
    - ☐ It usually helps
    - ☐ Helps a little
    - ☐ Never helps

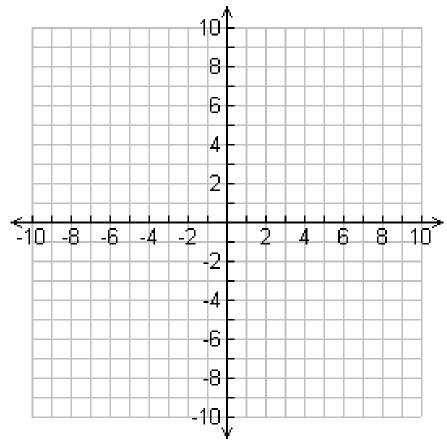
9. How frequently do you visualize or draw pictures of a concept to aid in understanding or proving mathematical statements?
   - ☐ Always
   - ☐ Usually
   - ☐ Sometimes
   - ☐ Occasionally
   - ☐ Never

10. How frequently do you use specific examples of a concept to aid in understanding or proving mathematical statements?
    - ☐ Always
    - ☐ Usually
    - ☐ Sometimes
    - ☐ Occasionally
    - ☐ Never

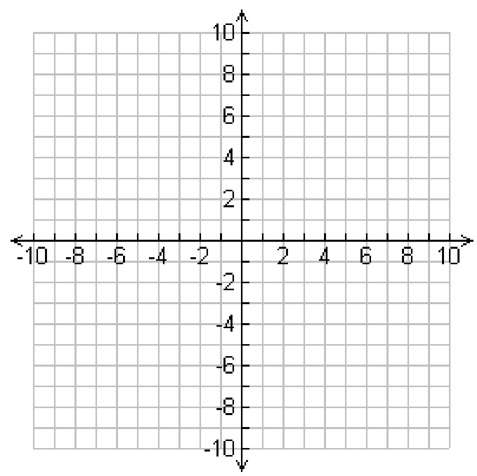
11. How frequently do you rely only on the definition of a concept to aid in understanding or proving mathematical statements?
    - ☐ Always
    - ☐ Usually
    - ☐ Sometimes
    - ☐ Occasionally
    - ☐ Never

12. How difficult is it for you to memorize mathematical definitions?
    - ☐ Very difficult
    - ☐ Fairly difficult
    - ☐ Neutral/unsure
    - ☐ Fairly easy
    - ☐ Very easy

13. On the grid below, draw the **image** of the interval (-3, 2) under the function $f(x) = x^2$



12. On the grid below, draw the **pre-image** of the interval (-1, 8) under $g(x) = x^3$.



14. Describe the meaning of the following symbols in plain English (as if you were explaining it to a non-mathematical friend):

- $\sqrt{2} \in \{x \in \mathbb{R} \mid x \notin \mathbb{Q}\}$
- $\exists N \in \mathbb{N}: \forall n \geq N, d(a_n, L) < \varepsilon$
- $\forall \varepsilon > 0,\ \exists \delta > 0$ such that $|x-3| < \delta \Rightarrow |f(x) - f(3)| < \varepsilon$

15. Compute the following:

- $\bigcap_{n \in \mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right)$
- $\bigcup_{n \in \mathbb{N}} \left(0, \frac{2}{n}\right)$

16. Write a sentence negating each of the following statements:

- For all $x \in [0, 2]$, $f(x) = 0$.
- No horizontal line intersects the graph of *f* twice or more.
- For each *k* there exists *c* such that $x > c \Rightarrow f(x) > k$
- If $a > 0$, then $ax^2 + bx + c = 0$ has no real solutions whenever $c > 0$.

17. Why is this function continuous on the indicated interval?

18. Let *X* be a set with subsets *A* and *B*. Prove: if $A \subset B$ then $X \setminus B \subset X \setminus A$.

19. Let $f : X \to Y$ be a function between sets *X* and *Y*. Let $C \subset Y$. Use set notation to define the preimage of *C* under *f*:

$$f^{-1}(C) = \{ \quad | \qquad\qquad\qquad\qquad \}$$

20. Let *X* and *Y* be sets, and $f : X \to Y$ a function. Suppose *C* and *D* are subsets of *Y*, and $C \subset D$. Prove: $f^{-1}(C) \subset f^{-1}(D)$.

Appendix E: Preliminary Research Methods

Here I will introduce and describe the three semesters of preliminary research that were conducted to establish the design, protocols, and coding paradigm for the main study, the methodology of which is described in Chapter 3. I include a detailed account of the procedures used in each of the three semesters leading up to the main study, running from the Spring semester of 2014 through the Spring semester of 2015. This includes the preliminary textbook review and analysis, and three semesters of task-based interview data with thirty-seven participants from topology and real analysis courses.

Overview and Timeline

I initiated my research with broad questions about how students might learn in general topology and similar advanced mathematical subjects. I began by reviewing and analyzing twelve commonly-used introductory topology textbooks, which informed my understanding of the expectations that might be placed on students in an introductory topology class. The textbook analysis continued throughout the preliminary research studies, during which I specified my research goals through an ongoing analysis of interview data collected from three semesters of interviews with volunteers from: 1) an undergraduate real analysis class, 2) an undergraduate topology course, and 3) a graduate topology course. The interview tasks, questions, and cues for each subsequent interview were informed by an analysis of the previous semester's data, as I developed a coding paradigm (Strauss & Corbin, 1998) around the mental actions and processes students used when learning about real analysis and topology. The fourth semester was the culmination of the study, bringing together all the previous analyses to generate questions and tasks that would span three interviews each for seven students in an undergraduate topology class (see Chapter 3). The responses of six of these students were then used to create case analyses that will be reported in the results (Chapters 4 & 5). Table 11 provides an overview of the four semesters.

Table 11: Class Information by Semester

Semester	Course		Under/Grad	Participants	Research Activity	Textbook
Spring 2014	3380	Real Analysis	UG	11	1 Interview each	Lay (2014)
Fall 2014	4330	Gen. Topology	UG	14	Class Observations 1 Interview each	Croom (1989)
Spring 2015	5330	Topology I	G	12	Class Observations 1 Interview each	Instructor Notes
Fall 2015	4330	Gen. Topology	UG	6 (9)*	Class Observations Pre-Assessment Quiz and Test Review 3 Interviews each	Croom (1989)

Research Goals

My initial research questions were broadly concerned with how undergraduate students make sense of their first exposure to general topology. To understand the cognitive processes that are involved in students' transitions to the axiomatic definitions presented in topology, I chose an analytical framework with explanatory power, that stayed within the context of my participants' dynamic conceptions and their attempts to build meaningful cognitive structures from the content. This framework incorporated various methods from the grounded theory tradition with analytical flexibility and theory-building power, including: the constant comparative method (Glaser & Strauss, 1967), an evolving coding paradigm (Strauss & Corbin, 1998), the development of reflexivity as a researcher (Charmaz, 2007/2010), and the maintenance of theoretical sensitivity (Kelle, 2010).

I hoped to use these techniques to build a conceptual framework that could explicate the structural components of the participants' concept images, as well as the relationships between them (see Chapter 2). This framework eventually manifested within the research as a coding paradigm (Strauss & Corbin, 1998) consisting of fifteen secondary research questions that helped to guide my analysis during the main study (see Chapter 3)

Research Design

Grounded theory methods were appropriate for analyzing my preliminary data for several reasons. Grounded theory is known as a powerful means to analyze qualitative data, especially in studies of an explorative nature (Bryant & Charmaz, 2007/2010; Creswell, 2013; Strauss & Corbin, 1998). It provides a framework for the researcher to inductively build theory out of empirical data without overstepping the deductive bounds of scientific inquiry. Moreover, given the extended time frame of my full study (four consecutive semesters) and concomitant changes in participants and instructional settings, a flexible research method was needed with the power to build theory through repeated comparisons across varied cases. I chose an approach to grounded theory that would provide such power and flexibility, while continuing to enable the integration of the extant literature from diverse fields of study (e.g., pure mathematics, mathematics education, cognitive psychology, epistemology, etc.). The approach I selected was a synthesis of the methods of Strauss and Corbin (1998) and Charmaz (2014) and several others (e.g., Kelle, 2007/2010; Strubing, 2007/2010).

Building theory through grounded research. Broadly speaking, the method of grounded theory is a "systematic, inductive, and comparative approach for conducting inquiry for the purpose of constructing theory" (Charmaz, 2009). As a research method, it provides a comprehensive way to categorize and connect qualitative data, allowing researchers to remain in contact with their data, simultaneously collecting and analyzing until a theory emerges (Bryant & Charmaz, 2007/2010). Some versions of grounded theory also provide avenues to reconnect existing theory with new empirical data (Kelle, 2010). Considering the broad and varied definitions for grounded theory methodology, and its origination in institutional settings that differ greatly from the settings of this research, my use of grounded theory methods was highly selective.

The goal of grounded theory is to provide a methodological structure for building new theory in the social sciences. Theory building goes hand-in-hand with the collection and interpretation of empirical data, but is not itself an entirely data-driven process. New knowledge must be created through a synthesis of empirical observations and cognitive sense-making, which relies on the researcher's theoretical sensitivity and knowledge of her field (Strauss & Corbin, 1998). These methods allow this interplay to take place in a guided way, consistently checked by new comparisons among the data and emerging concepts.

According to Strauss & Corbin (1998, p. 22), "theory denotes a set of well-developed categories (e.g., themes, concepts) that are systematically interrelated through statements of relationship to form a theoretical framework that explains some relevant social, psychological, educational, nursing, or other phenomenon." These "statements of relationship" are the key step in building an explanatory, rather than a descriptive, theory. This is because "however much we can describe [a] social phenomenon with a theoretical concept, we cannot use it to explain or predict. To explain or predict, we need a theoretical statement, a connection between two or more concepts" (Hage, 1972, p. 34, as cited in Strauss & Corbin, 1998, p. 22).

The development of concepts with which to form relationships is essential to scientific endeavor, as Strauss and Corbin (1994) explain:

> Theory consists of *plausible* relationships proposed among *concepts* and *sets of concepts*. (Though only plausible, its plausibility is to be strengthened through continued research). Without concepts, there can be no propositions, and thus no cumulative scientific (systematically theoretical) knowledge based on these plausible but testable propositions. (p. 278, as cited in Strübing, 2007/2010, p. 586)

Reflecting these theory-building goals, my research centered around constructing initial concepts related to how students think about continuous functions in axiomatic contexts.

Pre-existing categories and theoretical codes in grounded theory. An ongoing debate among proponents of grounded theory centers on the researcher's use of previous knowledge and established research to aid in the formation of theoretical categories in the analysis. Authors from one prominent school of thought have found the use of pre-existing categories problematic (Glaser, 1992, 2002), and wished to see all "codes and categories emerge directly from the data," without looking to previous research theories to interpret the new findings (Reichertz, 2007/2010, p. 215).

However, other researchers have maintained that "observation and the development of theory are necessarily always already theory guided" (Reichertz, 2007/2010, p.215); and it is therefore necessary to establish a clear and explicit relationship with previous research as any new theory is built up from empirical data. Creswell (2013) pointed out that some qualitative researchers have maintained the use of "prefigured codes" in many areas of research to build on existing theory, rather than starting from scratch. More adamantly, Kelle (2007/2010) argued that theoretical insight is *essential* to building grounded theory, and that "theoretical categories, whether empirically grounded or not, cannot start *ab ovo*, but have to draw on already existing stocks of knowledge" (p. 197).

The value of a researcher's theoretical insights was noted by the originators of grounded theory. While warning against attempts to force the data into categories, Glaser and Strauss (1967) insisted that to uncover grounded theoretical models a researcher would need "theoretical sensitivity." This was defined as the "ability to have theoretical insight into [one's] area of research, combined with an ability to make something of [one's] insights" (p. 46). For Glaser and Strauss, the meaning of this theoretical sensitivity evolved in divergent ways over time; however, both agreed that previous theoretical knowledge can, in some ways, "support the emergence of new categories" (Kelle, 2007/2010, p. 192).

My research efforts began as an exploration of student activity, with few expectations of what I might observe during my interviews. Nevertheless, I did bring theoretical insights to bear on the analysis as I sought to make sense of my findings. In the interim between each of the preliminary studies, a review of the literature among several academic fields of research took place, aligned with questions informed by my most recent findings. This provided a robust connection between the results of my grounded theory methods and previously established knowledge; and, provided a framework for an inductive analysis of my participants' responses.

Grounded theory approaches applied in the preliminary research. The analytical method used for the initial period of research was primarily a synthesis of two approaches to grounded theory, those of Strauss and Corbin (1998) and Charmaz (2014). While Strauss and Corbin (1998) laid the foundation for my understanding and use of grounded theory methods, Charmaz (2014) reinforced a constructivist perspective on my procedures for collecting and analyzing data.

Strauss and Corbin's (1998) approach to grounded theory. Theoretical constructs, adapted from the existing research in a variety of fields, were needed to situate my research and to suggest new paths of inquiry. Since the research was exploratory, I hoped to be guided by previous findings, but also free to form new connections and explanations of the phenomena I would observe. To achieve this, I followed the analytical approach proposed by Strauss and Corbin (1998). Through their emphasis on the researcher's theoretical sensitivity to other findings in the field, the grounded theory methods elaborated by these authors allowed me to develop a theoretical perspective out of my data while incorporating insights from diverse fields of study, yet without compromising the primacy of my observations.

Constructivist Grounded Theory. To further understand my epistemological stance as a researcher in this project, I also looked to constructivist developments in grounded theory. Charmaz (2009, 2014) encouraged researchers to position themselves reflexively with respect to their participants. My interpretation of this perspective has guided and tempered my evolving analysis by emphasizing three mediating factors: my role as a researcher, the contextual role of the classroom and professor, and the recognition of the entire research process as a social construction.

Setting and Participants

Over the course of three semesters, I interviewed thirty-seven students with task-based protocols as preliminary research for the main study. These students came from three different courses that varied in content, organization, and level of advanced reasoning expected. Below, I describe the context of each participating class, and the criteria I used to select the participants.

Sampling procedures. Although I was limited in my selection of classroom contexts for my research, I used the longevity of the preliminary study phase as a way to ensure a diverse pool of participants for the interviews, sampling students from a variety of content areas, academic levels, and pedagogical contexts. During the main study in the fourth semester, I used criteria developed during the preliminary studies to select theoretically interesting participants for a series of interviews.

Many grounded theorists (Charmaz, 2014; Glaser & Strauss, 1967; Strauss & Corbin, 1998) have stressed the method of “theoretical sampling,” in which a researcher samples new participants for study “on the basis of emerging concepts, with the aim being to explore the dimensional range or varied conditions along which the properties of concepts vary” (Strauss & Corbin, 1998, p.73). Although the populations I sampled for the preliminary studies were in some ways convenience samples—due to limitations on the number of students enrolled in relevant mathematics courses at the time—multiple preliminary studies were conducted to examine a broader subset of the overall population of topology students. Repeated preliminary studies allowed me to interview students with more broadly varying characteristics; including three levels of content presentation (pre-requisite, undergraduate, and graduate), two distinct pedagogical approaches, and students with and without prior experience in a topology class. This provided a range of conceptual levels and participants' mathematical maturities to study, despite limitations on the pool of potential participants. Similarly, during the textbook analysis, the texts were chosen to encompass a broad range of approaches, both conceptually and didactically.

Research participation by semester. As described above, the nature of the participating classes and students evolved each semester through a deliberate process of extended theoretical sampling. In this section, I will describe the classes, professors, and general student make-up for all the semesters, with a more thorough and detailed description of the main study participants and the criteria used to select them. Outlined in Table 12 are contexts and sampling aims for the textbook analysis and each of the four semesters of the study.

First semester class and participants. All the students enrolled in the Spring-2014 offering of Math 3380 were invited to participate in my initial research. Out of these, eleven students took part in a one-time interview, of approximately ninety minutes, during the final weeks of the semester. The professor of the course, Dr. S., was a full professor at Texas State University who, at that time, had taught Math 3380 eight times over the course of his twenty-seven-year long career. Dr. S. chose Lay (2014) as the textbook for the class, organizing the curriculum around its first six chapters.

Table 12: Theoretical Selection Criteria and Sampling Aims

Sampling Type	Activity/Course Semester	Number of Participants	Selection Criteria for Theoretically-Derived Samples
Theoretical	Textbook Analysis --------------- Summer/ Spring 2014	12 Textbooks	• *Criteria/Context:* introductory topology textbooks that are popular and/or commonly used in the U. S. • *Sampling Aim:* to explore common mathematical and didactic approaches used by introductory topology textbooks
	Undergraduate Real Analysis --------------- Spring 2014	11	• *Criteria/Context:* voluntary to all students in an undergraduate real analysis course • *Sampling Aim:* to explore pre-requisite understanding of students prior to their introduction to topology
	Undergraduate Topology --------------- Fall 2014	14	• *Criteria/Context:* voluntary to all students in an undergraduate topology course • *Sampling Aim:* to explore the target research population
	Graduate Topology --------------- Spring 2015	12	• *Criteria/Context:* voluntary to all students in a graduate topology course • *Sampling Aim:* to explore an alternative didactic approach in a research population similar to the target population
Case	Undergraduate Topology --------------- Fall 2015	7(6)*	• *Context:* six students selected from an undergraduate topology course • *Selection Criteria:* ▪ definition alignment ▪ concept consistency ▪ example space structure ▪ formalization stage ▪ central metaphor/basis for reasoning • *Sampling Aim:* to build cases within the target research population

The interview tasks were designed to investigate the participants' understanding of functions, sequences (convergence, boundedness, limits), sets (open, closed, boundedness), and continuity. A semi-structured portion of the interview protocol was also used to probe the participants' history with, and affective responses to, these concepts. In total, approximately eighteen hours of audio and digital pen recordings of interviews were made in this first semester.

The students interviewed during the first semester of the study encompassed a wide spectrum of formal mathematical sophistication and previous understanding. Although the interviews were semi-structured by a series of prepared questions and tasks, the conversations varied greatly between participants, depending on their level of understanding and confidence with different topics. The student's conceptualizations of functions, sequences, limits, and continuity were the primary focus of the tasks, but the participants were encouraged to discuss related components of their knowledge as well, which altered the direction of the interview from participant to participant.

Second semester class and participants. Fourteen students chose to participate during the Fall-2014 semester of the study. These students were enrolled in Math 4330, an undergraduate course in introductory, point-set topology. This class was taught by Dr. T., a full professor at Texas State University who, at that time, had taught Math 4330 fifteen times over the course of his thirty-eight year-long career. Dr. T. chose Croom (1989) as the textbook for the class, organizing the curriculum around the first six chapters.

During this semester of the study, only one participant had previously taken a topology class. Two distinct clusters of students emerged in the interview process—a smaller group of four students who felt confident in their understanding of the new material, and the remainder of the students who expressed frustration about their difficulties in understanding throughout the semester. Thus, while the interviews were designed to be more structured than in the first semester, many participants were encouraged to speak to whatever their level of understanding was, rather than working through tasks they could not do.

Third semester class and participants. In the third semester, Spring-2015, I regularly observed a graduate, introductory topology course, Math 5330, taught by Dr. B., a forty-five-year member of the faculty in the department of mathematics. Although this was the first time Dr. B. had taught the graduate course, he had taught undergraduate topology (Math 4330) five times previously and had supervised two master's theses in topology. All the students enrolled in this class were invited to be interviewed for the study; and twelve chose to participate.

Dr. B. chose not to employ a central textbook for the course, although he did recommend several possible resources students could turn to on their own. He preferred to generate his own class notes, making use of various textbooks and web-based resources, depending on the topic, over the course of the semester. He took an abstract approach to teaching the subject, as described in the textbook analysis section of this chapter.

The students interviewed during the third semester of the study were graduate students, seven of whom had never taken a topology class before. The other five of the students had been in Dr. T's topology class the previous semester and two had passed that class. Four of these students had also taken part in the interviews during that semester as well. This round of interviews focused on tasks related more closely to students' conceptualization of functions, limits, open and closed sets, and continuous functions within topological contexts. In general, the participants had less difficulty working through the tasks this semester, allowing the interviews to be more structured than in the previous two semesters. On occasions when a student was unable to move forward on a task, they were permitted to use their class textbook or notes as an aid.

Data Collection for the Preliminary Studies

Interview data were collected during three semesters, in which I investigated students' content understanding in three distinct but related courses. These included an undergraduate real analysis class, an undergraduate topology class and a graduate level topology class. Discussions with professors of the participating classes were used to clarify the expected learning goals of their students, providing a source of appropriate content for each semester's interviews.

Overview and timeline for data collection. Due to the extended timeframe of the preliminary research period, which included three semester-long preliminary studies, data were collected from students during four different classes offered during consecutive semesters at Texas State University-San Marcos (TSU). Including the textbook analysis and preliminary phase of the study, data were analyzed from the spring semester of 2014 through the fall semester of 2015.

Based on the textbook analysis and a review of content-specific literature, initial interview questions and tasks were established to be used for the first semester of the study, Spring 2014. During this semester, I interviewed students in an undergraduate real analysis class, Math 3380. The content for this course is pre-requisite to the topics that are commonly introduced in an introductory topology course, allowing me to gauge the state of understanding of eleven students in the lead-up to their first taste of topology and the axiomatization of the real analysis content.

During the second semester of the study, Fall 2014, I observed an undergraduate introductory topology course, Math 4330, and interviewed fourteen students in the final weeks of the semester. Based on the first semester's analysis, I revised my previous interview protocol and added tasks about specific topological notions, such as: open/closed sets, continuous functions, and the topological axioms. I retained informative questions about pre-topological concepts as well.

The third semester, Spring 2015, provided the opportunity to observe a graduate topology class, Math 5330, which was taught with a more abstract approach than in the previous semester's undergraduate topology class. In the final weeks of the semester I interviewed twelve of the students in this class, using a newly modified protocol based on information gathered in the second semester. The highly abstract pedagogical approach chosen by the professor for this class offered a new perspective on the content. Although the investigation of alternative pedagogical approaches is beyond the scope of this research, it is mentioned here as documentation of the influence this experience had on the direction of my research, and my subsequent design of the interview tasks for the main study.

Role and limitations on the researcher during data collection. The founders of grounded theory (Glaser and Strauss, 1967) originally made the case for their method from a positivist standpoint, "in the sense that representation is ultimately unproblematic once a neutral point of reference can be assured for the researcher" (Bryant, 2007, p. 107). This was reasonable at that time, considering the dominance of positivism in fields of scientific inquiry during the 1960's, but such a perspective has now been "severely discredited" (p. 108) by "the extensive critiques of positivism that have emerged in the last 40 years. Any 'guarantees' of neutrality these days can only be given once objectivist GTM [grounded theory methodology] can be seen to have engaged with constructivist arguments" (Bryant, 2007, p. 107). From the constructivist perspective, the researcher is an active participant in the activity of the participants, and is far from neutral. Strauss & Corbin (1998) elaborate:

> …[Glaser & Strauss (1967)] emphasized the interplay of data and researcher, that is, of data themselves and the researcher's interpretation of meaning. Because no researcher enters into the process with a completely blank and empty mind, interpretations are the researcher's abstractions of what is in the data. These interpretations, which take the form of concepts and relationships, are continuously validated through comparisons with incoming data.

Reflecting this perspective in the current study, the goal of the researcher during the interview process was that of a facilitator, to elicit clear and precise communication from the participants. Expectations were expressed directly, and prompts were used to provoke the participants to act in ways that could be interpreted by the coding paradigm. Thus, as the researcher I played an active role in eliciting and interpreting the data. Rather than attempting to limit my influence and participation, I attempted to use my position as a "knowledgeable other" (Vygotsky, 1978) as an asset within the processes of data collection and analysis.

Data Analysis for the Preliminary Studies

Analysis of data during the preliminary research phase was ongoing and cyclical, following the guidelines of Strauss & Corbin's (1998) coding scheme. Through a textbook analysis and three iterations of their prescribed coding cycle, a theoretical perspective began to emerge, and would eventually become the coding paradigm described in Chapter 5.2, as well as the conceptual framework illustrated in Chapter 3.2. The following is an account of the analyses that led to the development of these analytical tools for the main study.

Overview and timeline of analysis. Prior to my preliminary research, I began a review of introductory topology textbooks, to orient my research toward the possible instructional sequences and learning expectations that might be involved in such a course. This ongoing textbook analysis provided insight into the concepts and issues I would address during the preliminary interviews and the main study. During the preliminary phase of the research (see Table 13), my analysis of the data underwent three semester-long iterations of the coding cycle described by Strauss & Corbin (1998). During the semester of the main study, case-oriented data were collected and analyzed based on the results of those preliminary analyses. In addition, throughout the analysis research articles from several fields were continuously reviewed to help make sense of the data. Comparisons, constructs and salient insights from this ongoing consultation with the literature are reflected in the emergent theory using *theoretical codes* (Strauss & Corbin, 1998).

Table 13: Timeline of the Analysis and Coding Cycles

	Activity/ Course -------------- Semester	Open Coding (O)	Axial Coding (A)	Selective Coding (S)
TA	Textbook Analysis -------------- Spring 2014	• *Focus:* Didactic Approach • *Concepts:* Geometric-Intuitive, Metric-Analytic, Abstract-Axiomatic	N/A	N/A
I	Math 3380 Undergrad Real Analysis -------------- Spring 2014	• *Focus:* Basis of Student Reasoning • *Concepts:* Twenty-six action-oriented codes related to student reasoning	• *Primary Axial Category:* Mathematical Activities • *Sub-Categories:* Mathematical Approach, Conceptual Structures	N/A
II	Math 4330 Undergrad Topology -------------- Fall 2014	• *Focus:* Properties • *Concepts:* Definitions, Examples, Categories, Metaphors	• *Primary Axial Category:* Use and Recognition of Properties • *Sub-Categories:* Definitions, Examples, Categories, Metaphors, Embodiment, Language	• *Core Relationships:* Concept image sub-structures built from properties: PCD, Example Space, Content Taxonomy, Central Metaphor.
III	Math 5330 Graduate Topology -------------- Spring 2015	• *Focus:* Levels of Abstraction • *Concepts:* Prototype, Exemplar, Rule Abstraction, Instantiation, (De)-Encapsulation	• *Primary Axial Category:* Use of Abstraction and Instantiation • *Sub-Categories:* Exemplars-Prototype Spectrum, Abstraction, and Instantiation	• *Core Relationships:* Dimensions of increasing/decreasing abstraction of properties within the student's concept image.

Textbook analysis. The purpose of the textbook analysis was to analyze the various didactical approaches that students might face during the introduction to topology. In total, twelve topology-related textbooks, used widely in introductory topology courses across the U.S., were analyzed in the preliminary data collection process. One of these, Croom (1989), was chosen by the participating professor as the course textbook for the final semester of the study. Three overarching content categories emerged, expressed as the *geometric-intuitive*, *metric-analytical*, and *abstract-axiomatic* approaches to topology. Additionally, I described a series of local sub-categories concerning the authors' approaches to specific topics, including limits, open and closed sets, and continuous functions.

The goal of the textbook analysis was to discern the intended learning paths that authors expect students to follow while transitioning to an axiomatic understanding of continuity. These instructional sequences represent classical categorization schemes for the central notion of continuous functions and several pre-requisite and co-requisite concepts. These logical structures represent the authors' learning goal for the concept images of their readers', but are not likely to be representative of the natural categorization schemes that most students adopt upon their first introduction to topology.

The textbooks' mathematical approaches to topology. I found three main categories to explain the general approach of the textbooks in presenting the overarching themes of the course. I named these categories the *metric-analytic*, *geometric-intuitive*, and *abstract-axiomatic* approaches.

Metric-Analytic. The metric-analytic trajectory was most common among the textbooks I surveyed. In this approach, concrete examples and definitions from real analysis are gradually generalized to metric spaces, and then abstracted further to topological spaces. Textbooks devoted specifically to metric spaces were often found to follow this analytic route, leading students from known examples of continuous real-valued functions through a generalization process that culminates in an abstract definition of continuous functions based on open sets (Conover, 1975/2003; Croom, 1989; Searcóid, 2007).

For instance, Searcóid (2007) began his textbook by introducing "the prototype" for the study to come, which would be the Euclidean distance function. From this prototype, the notions of distance, boundary, closure and interior were specified and then defined in more abstract ways within the context of general metric spaces. Similarly, in Croom's (1989) development of these concepts, the open sets, closed sets, and continuous functions were each defined three separate times—once for the context of real numbers, once for metric spaces, and once for abstract topological spaces.

Geometric-Intuitive. From a different perspective, a few textbooks took advantage of the subject's geometric origins, approaching continuous functions from the perspective of continuous deformations of common surfaces—the subject matter of algebraic topology (Armstrong, 1979; Flegg, 1974).

Flegg (1974) placed continuity behind twelve chapters of geometrically-flavored, physically intuitive topics, such as the Euler characteristic, surfaces, connectivity, and non-Euclidean geometries. Continuity was only presented later as a property of transformations of surfaces—presented in a brief, non-rigorous manner. For this reason, this text is only described as an example of one possible, alternative learning trajectory. It appears to be less commonly used and was not the approach chosen by the professor for this study.

Abstract-Axiomatic. Finally, some textbooks began with an abstract definition of open sets, based on a given topology, and proceeded to define continuous functions through them (Crossley, 2005; Munkres, 1975/2000; Willard, 1970/2004). Commonly known, concrete examples from real and complex analysis were then shown to be special cases of these abstract concepts. For instance, Munkres (1975/2000) is a popular textbook for introductory topology courses, and follows an abstract trajectory. In this approach, topologies were defined early through open sets and topological bases, along with an open set definition for continuous functions. Several abstract examples were provided for each definition, and topics from real analysis were included as special cases.

Each of the approaches outlined above are certainly valid hypothetical avenues for learning this material, given their classical categorization schemes; but it seems reasonable to assume that students will face different challenges and obstacles as they reconstruct their understanding of continuous functions, depending on which of these approaches is used in the classroom.

The textbooks' approaches to specific content areas. After considering the textbooks' broad approach to the subject of topology, they were then classified based on their specific instructional approaches to the following topics: open and closed sets, sequences and limits, and continuous functions.

The approaches varied widely with respect to these topics, affected in some cases by the author's need to construct the concepts from prior knowledge, and in others by their willingness to present an abstract definition without explicit motivation. Textbooks that were representative of particular approaches are highlighted below, and a table outlining the whole sample is provided in Table 14.

Approach to open/closed sets. Taking the abstract-axiomatic approach, Munkres (1975/2000) defined an open set strictly in terms of the axioms of topology, with no reference to metrics or the real numbers. He follows up on his definition of open sets by defining the dual concept of a closed set as the complement of an open set. All of the properties for open sets are then shown to emerge in this new form, without referencing limit points in the way some other texts do. Elements such as limit points, boundary points, interior points, and so on were derived secondarily from the open set definition.

On the other hand, Searcóid (2007), Buskes & van Rooij (1997), and Flegg (1974) each defined open and closed sets in terms of boundary inclusion, and only later deduce the complementarity property. Croom (1989) and Gamelin & Greene (1983) introduced open sets via the generalization of open real intervals and open balls in a metric space, respectively. Croom (1989) goes on to define closed sets in terms of complements, as Munkres (1975/2000) did; but Gamelin & Greene (1983) use closure of a set for this purpose instead.

*Approach to sequences and limits***.** Due to the nature and definition of sequences in topological spaces, the two central notions of a limit, related 1) to continuity and 2) to sequences, are fundamentally interwoven. Theorems ,such as the following from Munkres (1975/2000) illuminate this relationship:

> *Theorem:* Let $f: X \to Y$. If the function *f* is *continuous*, then for every convergent sequence $x_n \to x$ in *X*, the *sequence* $f(x_n)$ *converges* to $f(x)$. The converse holds if *X* is metrizable.

Thus, sequences are deeply significant in the study of continuous functions; useful as approximations to functions in larger domains, and in the process of checking continuity. They are, in fact, functions in their own right with a discrete domain, usually given as the natural numbers, and with some subset of the real numbers as the range:

$f: \mathbb{N} \to \mathbb{R}$. The images of these functions appear as lists of values that many students are led to associate with the concept of a sequence.

However, it is equally possible to introduce the axiomatic concept of a continuous function without ever mentioning sequences or their limits. The limit point concept is usually defined independently of sequences, in terms of open sets (Moore, 1995), and all of the other pre-requisite concepts for continuous functions can be derived from this. In the analysis of the texts, the treatment of sequences was more varied than any of the other category. Some textbooks focused on the metric introduction of sequences, introducing the Cauchy property that many students conflate with the definition of convergence (Buskes & van Rooj, 1997; Croom, 1989; Searcóid, 2007). Others focused on the topological notion of sequences, illustrating sequential versions of convergence, and topological properties such as compactness (Munkres, 1975/2000; Willard, 1970/2004). Meanwhile, some textbooks ignored sequences altogether (Crossley, 2005; Flegg, 1974; Mendelson, 1962/1990)

Approach to continuous functions. In terms of abstraction and generalization, there are essentially two pathways leading to an axiomatic understanding of continuous functions:

1) Students may abstract from their previous experiences with the concept to construct the notion in increasingly general settings; honing the real number prototype into a more abstract set of properties. This is the approach taken by many authors of textbooks on real analysis and metric spaces, as well as several authors of topology textbooks (Croom, 1989; Mendelson, 1962/1990; Searcóid, 2007).

2) Students may also construct, *a priori,* an axiomatic concept image for continuous functions via the notion of topological spaces, instantiating the real numbers and other familiar contexts as examples of this abstract form of understanding. This alternative approach is favored by many authors as well (Crossley, 2005; Munkres, 1975/2000; Willard, 1970/2004)

Variations to these two approaches are notable as well. Buskes and van Rooj (2007) focused heavily on sequences, using sequential continuity interchangeably with other continuity conditions throughout the text, while generalizing the metric space prototype and students' previous experiences with real number sequences. Munkres (1975/2000) also emphasized sequences, but approached the topic abstractly, from the topological axioms. Metric spaces like the real number lined were presented as early examples, but not offered as a prototype for the new concepts.

Table legend for the textbook analysis. Table 14 contains five topical categories that illustrate different choices in presenting material. These include each author's general approach, definition of open and closed sets, approach to sequences and limits, and approach to continuous functions. In the "General Approach" column, which describes the textbooks' overall perspectives on topological content, entries include: "Geometric," "Metric" and "Abstract." These are not exclusive categories, as some authors have blends of the three approaches.

For the category "Definition of Open Set," authors used a variety of alternative approaches. "Neighborhood" indicates an abstract definition based on the primitive notion of that name. It is distinct from the "Open Ball" definition, which emphasizes a metric. Both definitions are compatible with a learning trajectory that generalizes students' prior understanding, where examples from the real numbers serve as the prototype for open sets in more abstract spaces. The use of the term "boundary" describes a more detailed approach to open sets, defining them and their closed set counterparts in relation to the inclusion or exclusion of their boundaries. On the other hand, many authors choose to introduce open sets as members of a topology without any reference to a metric; this is indicated by "Topological."

The category "Definition of Closed Set" is closely related to the definition of open sets. As mentioned, authors who defined open sets in relation to their boundaries, also defined closed sets in this manner. "Closure," identifies a similar definition that invokes the closure of a set, the collection of its limit points. Most authors defined a closed set as the complement of an open set; a popular, but conceptually weak definition. This is indicated with "Complement."

The authors' uses of sequences and limits were varied. Many textbooks avoid the mention of sequences altogether, "N/A," or provide only a cursory treatment of the topic, "Light". However, some textbooks cover the topic in detail, "Heavy." Treatment of sequences can be further distinguished by the settings in which they are introduced. A "Metric" treatment of sequences will usually introduce concepts from the real numbers such as Cauchy sequences and the property of completeness. These will then be generalized to metric sequences. However, sequences also exist in arbitrary topological spaces with no metrics. Nevertheless, they can still play an important role in these settings. Such a treatment is indicated by "Topological."

The textbooks' treatments of continuous functions ranged from purely abstract, topological introductions, to the generalizing strategy described above (e.g. Searcóid, 2007). Several authors offered a definition of sequential continuity and showed it's equivalence to the standard definition. The learning trajectories are indicated in the order of presentation.

Table 14: Categories for the textbook analysis

Author	General Approach	Definition of Open Set	Definition of Closed Set	Sequences and Limits	Continuous Functions
Armstrong (1979)	Geometric Topological	Neighborhood	Complement	N/A	Topological
Buskes & van Rooij (1997)	Metric	Boundary	Boundary	Metric Heavy	Real-Metric-Top/Seq
Conover (1975/2003)	Metric Topological	Topological	Complement	Metric/Top Heavy	Real-Metric-Topological
Croom* (1989)	Metric	Open Ball	Complement	Metric Light	Real-Metric-Topological
Crossley (2005)	Metric Topological	Topological	Complement	Topological Light	Topological
Flegg (1974)	Geometric	Neighborhood	Boundary	N/A	Real
Gamelin & Greene (1983)	Metric Topological	Open Ball	Closure	Metric Light	Metric-Topological
Mendelson (1962/1990)	Metric	Neighborhood	Complement	N/A	Real-Metric-Topological
Morris (2015)	Topological	Topological	Complement	Metric Heavy	Real-Topological
Munkres (1975/2000)	Topological	Topological	Complement	Topological Heavy	Topological Sequential
Searcóid (2007)	Metric	Boundary	Boundary	Metric Heavy	Real-Metric-Topological
Willard (1970/2004)	Metric Topological	Topological	Complement	Topological Heavy	Topological

Conclusions from textbook analysis. The survey of popular topology-related textbooks reached theoretical saturation (Strauss & Corbin, 1998) in relation to my research goals. Codes for each of the five categories demonstrate a large amount of redundancy, with only minor variations among the twelve sampled texts. This is an indication that the textbook analysis had come close to exhausting the discovery of distinct, popular approaches to the material represented by the categories. Nevertheless, these codes represented a large variety of potential didactical approaches to the wider subject of continuous functions and topological spaces. Any one (or a blend) of the above approaches might be chosen by topology professors, with a different emphasis on examples, prototypical abstractions, categorization rules, or metaphors. Table 14 was used as a template for the initial range of instructional possibilities for the content, although the *metric-analytic* approach best fits the final semester's approach to the content.

Analysis procedures for the preliminary studies. Data from the three semesters of preliminary interviews was analyzed in the spirit of the grounded theory methodology described by Strauss and Corbin (1998). I carried out three iterations of the coding cycle described by these authors, as my tentative coding paradigm developed through the introduction of data from new classroom contexts. I do not consider here whether classroom context or pedagogical approach to the material had any influence on the ways that students reasoned. However, I note that these two aspects did play a role in the early development of my coding paradigm and the overarching theory that I will present in this report. Each semester of the preliminary studies offered a unique perspective on my research questions, due to their differences in content, academic level, and/or the professor's pedagogical approach, as described in the section on data collection.

Coding procedures for the preliminary studies. Grounded theory methods are designed to: 1) generate abstract concepts (categories) from empirical data, 2) organize these concepts into a categorical structure, and 3) relate categories and sub-categories through a coherent, useful, and potentially testable theory (Strauss & Corbin, 1998).

These three goals are reflected in the three forms of coding suggested by Strauss and Corbin (1998), respectively: *open*, *axial*, and *selective* coding.

The first stage, open coding, is used to identify and develop concepts, or themes, that emerge from the data. This is a free-form coding process, in which ideas are allowed to emerge from the data without reference to pre-conceived categories or theoretical constructs. Next, through axial coding, central and subordinate categories are established and their interactions are examined, leading to a preliminary conceptual ordering of the open codes. Finally, during selective coding, relationships and interconnections between the ordered categories are described to build the emerging theory, often with one of the categories chosen as the core focus for the study.

Generating concepts with open coding. Strauss and Corbin (1998, p. 101) defined open coding as "the analytic process through which concepts are identified and their properties and dimensions are discovered in data." Through this process, a researcher develops abstract concepts to represent central ideas and repeated patterns in the data; that is, they "conceptualize" the observed phenomena. The mechanism for this process is the grouping of similar observations according to their shared properties, thereby building "categories" that can be compared and related to each other. Such concepts, or categories, are "the building blocks of theory," and when placed into an appropriate conceptual ordering they greatly reduce large amounts of data to manageable quantities. Once these initial categories have been created, their properties are specified through the development of subcategories. These can then be analyzed to determine how the "concepts (categories) vary dimensionally along those properties" (Strauss & Corbin, 1998, p. 121).

From the constructivist perspective, Charmaz (2014) similarly emphasized that "initial coding" requires exploring data for analytic ideas while remaining "open to all possible theoretical directions indicated by your readings of the data" (p. 114). Codes are constructed by problematizing the participants' responses and interpreting their underlying meanings, and then naming and categorizing theoretically meaningful exchanges. These codes are created through the interactions of the researcher and her observations; they should be "provisional, comparative, and grounded in the data" (Charmaz, 2014, p. 117).

Organizing concepts with axial coding and the coding paradigm. Once open coding has provided categorical labels for the phenomena observed in the data, a second round of coding highlights the important elements in the nascent analysis. These tend to be more conceptual, abstract codes than the primarily descriptive initial, or open, codes. Strauss and Corbin (1998) defined axial coding as "the act of relating categories to subcategories along the lines of their properties and dimensions" (p.124). In this phase of the analysis, the researcher explores how the various phenomena (categories) "crosscut and link" with each other on a conceptual, rather than descriptive, level. This is called axial coding because the analysis revolves around the 'axis' of an individual category, as the researcher makes connections between the category being focused on and other categories or their subcategories. This permits the researcher to "reassembl[e] data that were fractured during open coding" (p. 124).

Axial coding begins to construct a narrative that tentatively describes how the observed phenomena fit together. Strauss and Corbin (1998) refer to this narrative as the coding "*paradigm"* (p. 127-128), which they describe as "nothing more than a perspective taken toward the data, another analytic stance that helps to systematically gather and order data in such a way that structure and process are integrated” (p. 128). In this study, my coding paradigm evolved over three semesters of preliminary studies, to form the collection of primary and secondary research questions presented at the beginning of this chapter. The five themes used to organize the secondary research questions can be regarded as the axial codes from the preliminary phase of the research.

Integrating concepts into theory with selective coding. According to Strauss and Corbin (1998), *theory* does not fully emerge from an analysis “until the major categories are integrated into a larger theoretical scheme” (p. 142). This is the purpose of selective coding, which is the “process of integrating and refining categories” (p. 143). Strauss and Corbin (1998) point out that this phase of the coding cycle is more involved with the researcher’s interpretations and interaction with the data:

> As with all phases of analysis, integration is an interaction between the analyst and the data. Brought into that interaction is the analytic gestalt, which includes not only who the analyst is but also the evolution of thinking that occurs over time through immersion in the data and the cumulative body of findings that have been recorded in memos and diagrams. (p. 144)

In selective coding, a *central category* is chosen to represent the main theme of the research. Strauss and Corbin (1998) offer criteria for choosing this category. It must have analytic power, appear frequently in the data, and provide a logical and consistent explanation of the data that holds up to variation and contradictory cases. This central category explicates the *core relationship* between all the phenomena observed in the study.

In the second and third semester preliminary studies, selective coding was used to develop two successive explanatory statements, which reflected the overall connections I perceived in the data at that point. These statements acted as guides for the research that followed as I attempted to collect and analyze data through those broad questions. These, together with their axial categories, constituted the state of my developing coding paradigm in the semesters leading up to the main study.

Coding cycles in the preliminary studies. There were three iterations of the coding cycle (Strauss & Corbin, 1998) in the preliminary analysis phase of this research, some of which progressed further through the cycle than others. Through the constant comparative method and a reflexive theoretical stance, I returned to the earlier stages of the cycle whenever I felt that the present study had reached theoretical saturation (Kelle, 2009; Strauss & Corbin, 1998). I then collected more data and began the cycle again.

First iteration of coding cycle (I). In the first iteration of coding for the preliminary studies, I reviewed LiveScribe recordings and transcripts from task-based interviews with students in an introductory Real Analysis class for interview protocols). It consisted of one round each of open (I-O) and axial (I-A) coding of the transcripts.

First semester of open coding (I-O). Open coding of the first semester's interview transcripts was performed episodically, focusing on each of the individual tasks and topics as singular units for each participant. Coding revealed a broad set of categories related to the large variety of participants' responses to the questions and tasks. In total, I identified twenty-six specific concepts related to the structure of students' concept images and associated forms of reasoning. Many of these categories were eventually abandoned or subsumed by related categories during subsequent analyses, so their details will be excluded here.

First semester axial coding (I-A). However, it is important to note the categories that were used to organize the concepts above during axial coding. During I-A, criteria were constructed to describe and classify: participants' mathematical approaches to the tasks, the mathematical activities they engaged in, and the conceptual structures with which they reasoned. The following are examples of some of these concepts (italics) that were grouped within each of the three categories (bold) above:

- A participant's use of *formal-syntactic logic*, *procedures*, or *representations* would be categorized as a **mathematical approach**;
- A participant's engagement in *warranting*, *mathematizing*, *modeling*, *analogizing*, or *abstracting* would constitute **mathematical activity**; and,
- A participant's use of *definitions*, *examples*, *mental categories*, *symbols*, or *metaphors* would be considered a **conceptual structure**.

I re-organized my research aims around these constructs, influencing the design and analysis of the second round of interviews. Therefore, these categories (from I-A) were the building blocks for the interview questions and tasks used for the second open coding cycle (II-O).

Second iteration of coding cycle (II). The next iteration of coding for the preliminary studies encompassed a second semester, in which I observed an undergraduate topology class and analyzed task-based interviews with fourteen of its students. Open (II-O), axial (II-A), and selective (II-S) coding were used to develop the emerging theory during this phase of the analysis.

Second semester open coding (II-O). Following up on the theme of the previous semester's analysis, I began the process of coding within a narrower scope, both in terms of content and my evolving coding paradigm. The second semester's participants were given tasks and asked to justify their reasoning in relation to the concepts of: limits, continuous functions, open/closed sets, and limit/interior/boundary points. The resulting LiveScribe recordings and transcriptions were coded for ways in which the participants engaged in mathematical activities and used various mathematical approaches and conceptual structures (from I-A); as well as the affordances and obstacles they encountered in each situation. My findings during this coding cycle came to be bounded by specific instances of the participants' use of four structural elements of the concept image, used as their bases for reasoning on the tasks: *definitions, examples, metaphors*, and *categorization criteria*.

Second semester axial coding (II-A). I noticed that the *properties* of the mathematical objects referenced by my participants during their mathematical activity took on greater significance during the second round of axial coding. These properties often seemed to influence which mathematical approach might be taken by a participant, or how a conceptual structure might be used. Although the collection of properties attended to by the participants were often mathematical, they were also found to be linguistic, metaphorical, sensory, perceptual, kinesthetic, spatial, or otherwise embodied properties associated with important concepts within a participant's mind. Therefore, both the participants' mathematical and non-mathematical **uses and recognition of properties**, emerged as the central category for the second round of axial coding. *Definitions*, *examples*, and *categorization criteria* were coded as mathematical sub-categories; while *metaphor, embodiment,* and *language* were coded as non-mathematical sub-categories.

Second semester selective coding (II-S). Through selective coding, the core relationship between "concept image structure and the building blocks of properties" was established. This emerged through my reflection on the axial categories, prompting a realization that each of the axial categories from II-A are related to my participants' understanding and uses of either mathematical properties or the nonmathematical attributes they had focused on.

For example, I noted that mathematical *definitions* are nothing more than lists of delimiting properties that specify a mathematical object; and that *examples* are specific instantiations of one or more of the properties listed in the definition. I found that my participants' *criteria for categorizing* mathematical phenomena was typically based on shared properties among the members of the category or taxonomy. Further, participants' *metaphors* relied on the abstraction of specific properties to form mappings across distinct domains of experience. This perspective guided my analysis of third semester's interviews and observations—that students' uses and recognition of both mathematical and non-mathematical properties undergirds the conceptual structures that they build and reason with.

Third iteration of the coding cycle (III). The final iteration of the preliminary coding cycle spanned a third semester of classroom observations and twelve student interviews, this time in a graduate level topology course. Once again, open (III-O), axial (III-A) and selective (III-S) coding schemes were used to explore new avenues in the theory through the establishment of categories and relationships between them.

Third semester open coding (III-O). Differences began to emerge between the ways that participants used the sub-structures of their concept images for open/closed sets, functions, and continuous functions. At times, highly abstract *prototype*s were used to construct visual or linguistic representations that captured a generalized concept; while in other cases, specific *exemplars* were instantiated to study individual properties. I also noticed connections between my participants' *abstractions* of properties to form general categories of mathematical objects, and their *instantiation* of those properties into exemplars for those categories.

Third semester axial coding (III-A). Theoretical coding provided a way to see these behaviors as linked in an *exemplar-prototype spectrum*, which became an important sub-category within the axial category of participants' *uses of abstraction and instantiation.* Another important sub-category that emerged at this point was the notion of *contextual abstraction and instantiation*. This refers to the way a participant made use of their concept image's structure, which often seemed to depend on the context of the task. For example, more abstract mathematical settings could lead a student to seek out extra-mathematical metaphors (Pimm, 1987), while familiar, analytical settings prompted the use of procedural calculations with less analogizing.

Third semester selective coding (III-S). Finally, a second process of selective coding established the core relationship between properties, the concept image structures built from those properties, and the "dimension of abstraction and instantiation involved in the use of concept image structures." These relationships guided the further evolution of the coding paradigm that would be used in the theoretical sampling for, and analyses of, the multiple case study that took place in the final semester of the study.

The three semesters of analysis of preliminary study data became integrated into a coherent theoretical narrative that would ultimately provide a basis for theoretical sampling and task formation in the final semester of the study. In this way, I could choose participants for a semester-long study of students with characteristics suitable to continued theorizing along more specific lines of inquiry.

Theoretical codes used in the preliminary analyses and main study. As described above, I consistently looked to theoretical literature to inform my analysis, and to provide descriptive terminology as explanatory codes for my observations. In Table 15, I list examples of such codes, with a timeline of their introduction to the analysis.

Table 15: Relevant Authors and Concepts from the Literature

Activity/ Semester	Theoretical Codes
Textbook Analysis --------------- Summer/ Spring 2014	• Moore (1994)—History of topological ideas • Tall (1990), Thurston (1990)—Advanced mathematical thinking and learning • Piaget & Garcia (1983/1989), von Glasersfeld (1995, 2007), Vygotsky (1978, 1986)—Constructivism, radical constructivism, social constructivism • Tall & Vinner (1981)—Concept image and concept definition
Undergraduate Real Analysis --------------- Spring 2014	• Sfard (1991, 1994)—Reification, process/object duality • Dubinsky (1991), Arnon, et al. (2014)—APOS theory • Watson & Mason (2002, 2005), Goldenberg & Mason, 2008—Example spaces • Dawkins (2012), Gravemeijer (1999), Parameswaran (2010)--Definitions
Undergraduate Topology --------------- Fall 2014	• Sierpinska (1990)—Acts and processes of understanding • Tall (2013)—Three worlds of mathematics
Graduate Topology --------------- Spring 2015	• Slavit (1994), Confrey & Costa (1996), Dörfler (1995)—Property-based view of reification and understanding • Lakoff, et al. (1987, 2000)—Cognitive psychology, metaphor, embodiment • Dawkins (2015), de Villiers (1986), Freudenthal (1973)—Axiomatization
Undergraduate Topology --------------- Fall 2015	• Rosch, et al. (1974, 1976, 1981)—Prototype theory, natural categorization • Barsalou, et al. (1999), Hampton (2003), Ross (1996)—Taxonomic theory, prototype-exemplar spectrum, context effects

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