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NEHARI MANIFOLD APPROACH FOR FRACTIONAL $p(\cdot)$-LAPLACIAN SYSTEM INVOLVING CONCAVE-CONVEX NONLINEARITIES

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Abstract. In this article, using Nehari manifold method we study the multiplicity of solutions of the nonlocal elliptic system involving variable exponents and concave-convex nonlinearities,

$$\begin{gathered}
(-\Delta)^s_{p(\cdot)}u = \lambda a(x)|u|^{q(x)-2}u + \frac{\alpha(x)}{\alpha(x)+\beta(x)}c(x)|u|^{\alpha(x)-2}u|v|^{\beta(x)}, \quad x \in \Omega;\\
(-\Delta)^s_{p(\cdot)}v = \mu b(x)|v|^{q(x)-2}v + \frac{\alpha(x)}{\alpha(x)+\beta(x)}c(x)|v|^{\alpha(x)-2}v|u|^{\beta(x)}, \quad x \in \Omega;\\
u = v = 0, \quad x \in \Omega^c := \mathbb{R}^N \setminus \Omega,
\end{gathered}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$ is a smooth bounded domain, $\lambda, \mu > 0$ are parameters, and $s \in (0,1)$. We show that there exists $\Lambda > 0$ such that for all $\lambda + \mu < \Lambda$, this system admits at least two non-trivial and non-negative solutions under some assumptions on $q, \alpha, \beta, a, b, c$.

1. Introduction

In this article, we consider the nonlocal elliptic system with variable exponents,

$$\begin{gathered}
(-\Delta)^s_{p(\cdot)}u = \lambda a(x)|u|^{q(x)-2}u + \frac{\alpha(x)}{\alpha(x)+\beta(x)}c(x)|u|^{\alpha(x)-2}u|v|^{\beta(x)}, \quad x \in \Omega,\\
(-\Delta)^s_{p(\cdot)}v = \mu b(x)|v|^{q(x)-2}v + \frac{\alpha(x)}{\alpha(x)+\beta(x)}c(x)|v|^{\alpha(x)-2}v|u|^{\beta(x)}, \quad x \in \Omega,\\
u = v = 0, \quad x \in \Omega^c := \mathbb{R}^N \setminus \Omega,
\end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N, N \geq 2$ is a smooth bounded domain, $\lambda, \mu > 0$ are the parameters, $s \in (0,1)$, $p \in C(\mathbb{R}^N \times \mathbb{R}^N, (1,\infty))$ with $sp^+ < N$. Here $q, \alpha, \beta \in C(\overline{\Omega}, (1,\infty))$ are the variable exponents and $a, b, c : \overline{\Omega} \to [0,\infty)$ are the non-negative weight functions. The nonlocal operator $(-\Delta)^s_{p(\cdot)}$ is defined as

$$(-\Delta)^s_{p(\cdot)}u(x) := P.V. \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{N+s(x,y)p(x,y)}}dy, \quad x \in \mathbb{R}^N, \tag{1.2}$$

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where P.V. stands for Cauchy's principal value. Problems involving nonlocal operators have gained a lot of interest for research in recent years. Mathematical modeling of the problems in many areas like mechanics, population dynamics, thin obstacle problem, optimization and finance involve fractional Laplacian $(-\Delta)^s$ or fractional p-Laplacian $(-\Delta)_p^s$. We refer to [12, 24] for the basic results on problems involving nonlocal operators. Also, one can refer to [6, 11, 23, 26, 27] and the references therein for the existence, multiplicity, and regularity of the solutions of these problems.

In this work, our objective is to study the nonlocal elliptic problems with variable exponents. Operators involving variable growth are extensively studied due to the precision in the modeling of various phenomenon where the property of the subject under consideration depends on the point of the observation, for example, in image restoration, study of electrorheological fluid flow, non-Newtonian processes, etc. We refer to [2, 13, 15, 16, 28] and references therein for the study of the problems involving the local $p(x)$-Laplace operator, defined as $\Delta_{p(x)}u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$.

The fractional Sobolev spaces with variable exponents and the corresponding fractional $p(\cdot)$-Laplace operator $(-\Delta)_{p(\cdot)}^s$ were recently introduced by Kaufmann et al in [22]. Also, in [3, 4, 5, 21], the authors have established the basic properties of such spaces and studied the problems involving fractional $p(\cdot)$-Laplacian.

Using the Nehari manifold and the fibering map, in the case of local p-Laplacian, Brown and Wu [8] have obtained multiple solutions of an elliptic system with sign changing weight functions and concave-convex nonlinearities. In the nonlocal set-up, Sreenadh and Goyal [20] have studied the same for the single fractional p-Laplacian equation . Also, we cite [10] where the authors have studied the fractional p-Laplacian system involving concave-convex nonlinearities via Nehari manifold and fibering map. In [18], Pucci et al. have modified the definition of Nehari manifold and fibering map for the fractional (p,q)-Laplacian system and studied the corresponding Dirichlet problem. Recently Alves et al [1] have used this Nehari manifold method to prove the multiplicity of solutions for $p(x)$-Laplacian problems in the whole of $\mathbb{R}^N$.

Motivated by the above works, in this article, we address the multiplicity of the solutions of the nonlocal elliptic system with variable exponents involving concave and convex nonlinearities using the analysis of the fibering map and Nehari manifold. We note that the Nehari manifold approach through the fibering map analysis for the functional involving variable exponents is interesting due to the non-homogeneity that arises from the variable exponents. It is also worth mentioning that due to the presence of the variable exponents, most of the estimates do not hold immediately, unlike in the constant exponent set-up. Hence, in our present work, we need to carry out some extra careful analysis to overcome this issue. To the best of our knowledge, this is the first work dealing with fractional $p(\cdot)$-Laplacian system involving concave and convex nonlinearities using fibering-map approach.

Next, we set some notation. Let $\mathcal{D}$ be a domain. For any function $\Phi : \mathcal{D} \to \mathbb{R}$, we set

$$\Phi^- := \inf_{\mathcal{D}} \Phi(x), \quad \Phi^+ := \sup_{\mathcal{D}} \Phi(x). \tag{1.3}$$

We also define the function space

$$C_+(\mathcal{D}) := \{g \in C(\mathcal{D}, \mathbb{R}) : 1 < g^- \le g^+ < \infty\}.$$

To state our result, we assume that the variable exponents p, q, α, β and the weight functions a, b, c satisfy the following hypotheses:

(A1) The variable exponent $p \in C_+(\mathbb{R}^N \times \mathbb{R}^N)$.

(A2) The function p is symmetric, i.e., $p(x,y) = p(y,x)$ for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$.

(A3) The variable exponents $q, \alpha, \beta \in C_+(\overline{\Omega})$ and $p \in C_+(\mathbb{R}^N \times \mathbb{R}^N)$ satisfy
$$1 < q^- \le q^+ < p^- \le p^+ < \alpha^- + \beta^- \le \alpha^+ + \beta^+ < p_s^{*-},$$
where $p_s^*(x) = \frac{Np(x,x)}{N - sp(x,x)}$ is the critical exponent.

(A4) It holds that
$$\frac{p^-}{\alpha^+ + \beta^+} < \Big(\frac{p^- - q^+}{\alpha^+ + \beta^+ - q^+}\Big)\Big(\frac{\alpha^- + \beta^- - q^-}{p^+ - q^-}\Big).$$

(A5) The non-negative weight functions $a, b \in L^{q_*(x)}(\Omega)$, where
$$q_*(x) = \frac{\alpha(x) + \beta(x)}{\alpha(x) + \beta(x) - q(x)}.$$

(A6) The non-negative weight function $c \in L^\infty(\Omega)$.

Observe that, when all the exponents are constants, (A4) is equivalent to the condition $0 < p < \alpha+\beta$. Now we define the weak solution of (1.1) in the functional space E (defined in Section 2) as follows.

Definition 1.1. We say that $(u,v) \in E$ is a weak solution of (1.1), if we have
$$\begin{aligned}
&\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+sp(x,y)}}\,dx\,dy \\
&+ \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|v(x)-v(y)|^{p(x,y)-2}(v(x)-v(y))(\psi(x)-\psi(y))}{|x-y|^{N+sp(x,y)}}\,dx\,dy \\
&= \int_\Omega \Big(\lambda a(x)|u|^{q(x)-2}u\phi + \mu b(x)|v|^{q(x)-2}v\psi\Big)dx \\
&\quad + \int_\Omega \frac{\alpha(x)}{\alpha(x)+\beta(x)} c(x)|u|^{\alpha(x)-2}u|v|^{\beta(x)}\phi dx \\
&\quad + \int_\Omega \frac{\beta(x)}{\alpha(x)+\beta(x)} c(x)|v|^{\alpha(x)-2}v|u|^{\beta(x)}\psi dx \quad \text{for all } (\phi,\psi) \in E.
\end{aligned} \tag{1.4}$$

The main result in this article is stated as follows.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^N$, $N \ge 2$ be a smooth bounded domain, $s \in (0,1)$ and $p(\cdot,\cdot)$ satisfy (A1)–(A2) with $sp^+ < N$. Assume that the hypotheses (A3)–(A6) hold. Then there exists a positive constant $\Lambda = \Lambda(N,s,p,q,\alpha,\beta,a,b,c,\Omega)$ such that for any pair of positive parameters (λ,μ) with $\lambda + \mu < \Lambda$, (1.1) admits at least two non-trivial, non-negative weak solutions.*

2. Preliminary results

Here we recall the definition and some important properties of the Lebesgue spaces with variable exponents. For more details regarding these spaces, one can refer to [13, 16] and the references therein.

For $\gamma \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space
$$L^{\gamma(x)}(\Omega) = \big\{u : \Omega \to \mathbb{R} : u \text{ is measurable}, \int_\Omega |u|^{\gamma(x)} < +\infty\big\},$$

This space is a separable, reflexive Banach space equipped with the Luxemburg norm

$$\|u\|_{L^{\gamma(x)}(\Omega)} = \inf\Big\{\eta > 0 : \int_\Omega \Big|\frac{u}{\eta}\Big|^{\gamma(x)} \leq 1\Big\}.$$

We have the following Hölder-type inequality (see [13]) for variable exponents Lebesgue spaces.

Lemma 2.1. *Let $\gamma' \in C_+(\overline{\Omega})$ such that $\frac{1}{\gamma(x)} + \frac{1}{\gamma'(x)} = 1$. Then for any $u \in L^{\gamma(x)}(\Omega)$ and $v \in L^{\gamma'(x)}(\Omega)$ we have*

$$\Big|\int_\Omega uv dx\Big| \leq \Big(\frac{1}{\gamma^-} + \frac{1}{\gamma'^-}\Big)\|u\|_{L^{\gamma(x)}(\Omega)}\|v\|_{L^{\gamma'(x)}(\Omega)}.$$

Next, we recall [19, Lemma A.1].

Lemma 2.2. *Let $\nu_1(x) \in L^\infty(\Omega)$ such that $\nu_1 \geq 0$, $\nu_1 \not\equiv 0$. Let $\nu_2 : \Omega \to \mathbb{R}$ be a measurable function such that $\nu_1(x)\nu_2(x) \geq 1$ a.e. in Ω. Then for every $u \in L^{\nu_1(x)\nu_2(x)}(\Omega)$,*

$$\||u|^{\nu_1(\cdot)}\|_{L^{\nu_2(x)}(\Omega)} \leq \|u\|^{\nu_1^-}_{L^{\nu_1(x)\nu_2(x)}(\Omega)} + \|u\|^{\nu_1^+}_{L^{\nu_1(x)\nu_2(x)}(\Omega)}.$$

The modular function $\rho_\gamma : L^{\gamma(x)}(\Omega) \to \mathbb{R}$ is defined as

$$\rho_\gamma(u) = \int_\Omega |u|^{\gamma(x)} dx.$$

Now we state the following two lemmas from [16], which establish the relationship between the norm $\|\cdot\|_{L^{\gamma(x)}(\Omega)}$ and the corresponding modular function $\rho_\gamma(\cdot)$.

Lemma 2.3. *Let $u \in L^{\gamma(x)}(\Omega)$, then*

(i) $\|u\|_{L^{\gamma(x)}(\Omega)} < 1$ $(= 1; > 1)$ *if and only if* $\rho_\gamma(u) < 1$ $(= 1; > 1)$;
(ii) *If* $\|u\|_{L^{\gamma(x)}(\Omega)} > 1$, *then* $\|u\|^{\gamma^-}_{L^{\gamma(x)}(\Omega)} \leq \rho_\gamma(u) \leq \|u\|^{\gamma^+}_{L^{\gamma(x)}(\Omega)}$;
(iii) *If* $\|u\|_{L^{\gamma(x)}(\Omega)} < 1$, *then* $\|u\|^{\gamma^+}_{L^{\gamma(x)}(\Omega)} \leq \rho_\gamma(u) \leq \|u\|^{\gamma^-}_{L^{\gamma(x)}(\Omega)}$.

Lemma 2.4. *Let $u, u_m \in L^{\gamma(x)}(\Omega), m = 1, 2, 3, \cdots$. Then the following statements are equivalent.*

(i) $\lim_{m\to\infty} \|u_m - u\|_{L^{\gamma(x)}} = 0$;
(ii) $\lim_{m\to\infty} \rho_\gamma(u_m - u) = 0$;
(iii) u_m *converges to* u *in* Ω *in measure and* $\lim_{m\to\infty} \rho_\gamma(u_m) = \rho_\gamma(u)$.

2.1. **Fractional Sobolev spaces with variable exponents.** In this section, we discuss the properties of the fractional Sobolev spaces with variable exponents. These spaces have been introduced for the first time in [22]. Also, in [4, 5, 21], the authors have established some important properties of these spaces.

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain and $p(\cdot, \cdot)$ satisfy (A1) and (A2). For any $x \in \mathbb{R}^N$, we denote

$$\overline{p}(x) := p(x, x).$$

Thus, $\overline{p} \in C_+(\overline{\Omega})$. Now we define the fractional Sobolev space with variable exponents as follows.

$$\begin{aligned} W &= W^{s,\overline{p}(x),p(x,y)}(\Omega) \\ &:= \Big\{u \in L^{\overline{p}(x)}(\Omega) : \int_{\Omega\times\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{\eta^{p(x,y)}|x - y|^{N+sp(x,y)}}\, dx\, dy < \infty,\ \text{for some } \eta > 0\Big\}. \end{aligned}$$

We set the seminorm as

$$[u]_{\Omega}^{s,p(x,y)} := \inf\Big\{\eta > 0 : \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{\eta^{p(x,y)}|x-y|^{N+sp(x,y)}}\,dx\,dy < 1\Big\}.$$

Then $(W, \|\cdot\|_W)$ is a separable reflexive Banach space (see [5]) equipped with the norm

$$\|u\|_W := \|u\|_{L^{\overline{p}(x)}(\Omega)} + [u]_{\Omega}^{s,p(x,y)}.$$

Now we state the following continuous and compact embedding theorem (see [21]).

Theorem 2.5. *Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $s \in (0,1)$ and $p(\cdot,\cdot)$ satisfy* (A1), (A2) *with $sp^+ < N$. Let $r \in C_+(\overline{\Omega})$ such that $1 < r^- \le r(x) < p_s^*(x) = \frac{N\overline{p}(x)}{N-s\overline{p}(x)}$ for $x \in \overline{\Omega}$. Then, there exits a constant $C = C(N,s,p,r,\Omega) > 0$ such that, for any $u \in W$,*

$$\|u\|_{L^{r(x)}(\Omega)} \le K\|u\|_W.$$

Moreover, this embedding is compact.

For studying nonlocal problems involving the operator $(-\Delta)^s_{p(\cdot)}$ with Dirichlet boundary datum via variational methods, we define another fractional type Sobolev spaces with variable exponents. One can refer to [24] and references therein for this type of spaces in fractional Laplacian framework. We set $\mathcal{Q} := \mathbb{R}^{2N} \setminus (\Omega^c \times \Omega^c)$ and define the new fractional Sobolev space with variable exponent as

$$\begin{aligned} X &= X^{s,\overline{p}(x),p(x,y)}(\Omega) \\ &:= \Big\{u : \mathbb{R}^N \to \mathbb{R} : u_{|\Omega} \in L^{\overline{p}(x)}(\Omega), \\ &\quad \int_{\mathcal{Q}} \frac{|u(x)-u(y)|^{p(x,y)}}{\eta^{p(x,y)}|x-y|^{N+sp(x,y)}}\,dx\,dy < \infty, \text{ for some } \eta > 0\Big\}. \end{aligned}$$

The space X is equipped with the norm

$$\|u\|_X := \|u\|_{L^{\overline{p}(x)}(\Omega)} + \inf\Big\{\eta > 0 : \int_{\mathcal{Q}} \frac{|u(x)-u(y)|^{p(x,y)}}{\eta^{p(x,y)}|x-y|^{N+sp(x,y)}}\,dx\,dy < 1\Big\},$$

where $[u]_X$ is the seminorm

$$[u]_X = \inf\Big\{\eta > 0 : \int_{\mathcal{Q}} \frac{|u(x)-u(y)|^{p(x,y)}}{\eta^{p(x,y)}|x-y|^{N+sp(x,y)}}\,dx\,dy < 1\Big\}.$$

Then $(X, \|\cdot\|_X)$ is a separable reflexive Banach space. Next, we define the subspace X_0 of X as

$$X_0 = X_0^{s,\overline{p}(x),p(x,y)}(\Omega) := \{u \in X : u = 0 \text{ a.e. in } \Omega^c\}.$$

We define the norm on X_0 as follows

$$\|u\|_{X_0} := \inf\Big\{\eta > 0 : \int_{\mathcal{Q}} \frac{|u(x)-u(y)|^{p(x,y)}}{\eta^{p(x,y)}|x-y|^{N+sp(x,y)}}\,dx\,dy < 1\Big\}.$$

Remark 2.6. *For $u \in X_0$, we obtain*

$$\int_{\mathcal{Q}} \frac{|u(x)-u(y)|^{p(x,y)}}{\eta^{p(x,y)}|x-y|^{N+sp(x,y)}}\,dx\,dy = \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^{p(x,y)}}{\eta^{p(x,y)}|x-y|^{N+sp(x,y)}}\,dx\,dy.$$

Thus, we have

$$\|u\|_{X_0} := \inf \Big\{ \eta > 0 : \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)-u(y)|^{p(x,y)}}{\eta^{p(x,y)}|x-y|^{N+sp(x,y)}}\,dx\,dy < 1 \Big\}.$$

Now we state the following continuous and compact embedding result for the space X_0. The proof follows from [3, Theorem 2.2, Remark 2.2] and [4, Lemma 2.1].

Theorem 2.7. *Let Ω be a smooth bounded domain in $\mathbb{R}^N$ and let $s \in (0,1)$. Let $p(\cdot,\cdot)$ satisfy* (A1) *and* (A2) *with $sp^+ < N$. Then for any $r \in C_+(\overline{\Omega})$ such that $1 < r(x) < p_s^*(x)$ for all $x \in \overline{\Omega}$, there exits a constant $C = C(N,s,p,r,\Omega) > 0$ such that for every $u \in X_0$,*

$$\|u\|_{L^{r(x)}(\Omega)} \le C\|u\|_{X_0}.$$

Moreover, this embedding is compact.

Definition 2.8. For $u \in X_0$, we define the modular $\rho_{X_0} : X_0 \to \mathbb{R}$ as

$$\rho_{X_0}(u) := \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy. \tag{2.1}$$

The interplay between the norm in X_0 and the modular function ρ_{X_0} can be studied in the following lemma.

Lemma 2.9. *Let $u \in X_0$ and ρ_{X_0} be defined as in* (2.1)*. Then we have the following results:*

(i) $\|u\|_{X_0} < 1$ $(= 1; > 1)$ *if and only if* $\rho_{X_0}(u) < 1 (= 1; > 1)$.
(ii) *If* $\|u\|_{X_0} > 1$, *then* $\|u\|_{X_0}^{p^-} \le \rho_{X_0}(u) \le \|u\|_{X_0}^{p^+}$.
(iii) *If* $\|u\|_{X_0} < 1$, *then* $\|u\|_{X_0}^{p^+} \le \rho_{X_0}(u) \le \|u\|_{X_0}^{p^-}$.

The next lemma can easily be obtained using the properties of the modular function ρ_{X_0} from Lemma 2.9.

Lemma 2.10. *Let $u, u_m \in X_0$, $m \in \mathbb{N}$. Then the following two statements are equivalent:*

(i) $\lim_{m\to\infty} \|u_m - u\|_{X_0} = 0$,
(ii) $\lim_{m\to\infty} \rho_{X_0}(u_m - u) = 0$.

Lemma 2.11 ([3, Lemma 2.3])**.** $(X_0, \|\cdot\|_{X_0})$ *is a separable, reflexive and uniformly convex Banach space.*

We define $E := X_0 \times X_0$ as the solution space corresponding to (1.1), equipped with the norm $\|(u,v)\| = \max\{\|u\|_{X_0}, \|v\|_{X_0}\}$. Clearly $(E, \|(\cdot,\cdot)\|)$ is a reflexive, separable Banach space.

3. Nehari manifold and fibering map analysis

Here first we discuss certain technical results regarding the Nehari manifold and the fibering map and the behavior of the energy functional corresponding to (1.1).

The energy functional $J_{\lambda,\mu}: E \to \mathbb{R}$ associated with (1.1) is defined as

$$
\begin{aligned}
J_{\lambda,\mu}(u,v) = & \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{1}{p(x,y)} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy \\
& + \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{1}{p(x,y)} \frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy \\
& - \int_\Omega \frac{1}{q(x)}\Big(\lambda a(x)|u|^{q(x)} + \mu b(x)|v|^{q(x)}\Big)dx \\
& - \int_\Omega \frac{1}{\alpha(x)+\beta(x)} c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx.
\end{aligned}
\tag{3.1}
$$

By a direct computation, it can be checked that $J_{\lambda,\mu} \in C^1(E,\mathbb{R})$ and for any $(\phi,\psi) \in E$, we have

$$
\begin{aligned}
& \langle J'_{\lambda,\mu}(u,v), (\phi,\psi)\rangle \\
& = \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+sp(x,y)}}\,dx\,dy \\
& + \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|v(x)-v(y)|^{p(x,y)-2}(v(x)-v(y))(\psi(x)-\psi(y))}{|x-y|^{N+sp(x,y)}}\,dx\,dy \\
& - \int_\Omega \Big(\lambda a(x)|u|^{q(x)-2}u\phi + \mu b(x)|v|^{q(x)-2}v\psi\Big)dx \\
& - \int_\Omega \frac{\alpha(x)}{\alpha(x)+\beta(x)} c(x)|u|^{\alpha(x)-2}u|v|^{\beta(x)}\phi dx \\
& - \int_\Omega \frac{\beta(x)}{\alpha(x)+\beta(x)} c(x)|v|^{\alpha(x)-2}v|u|^{\beta(x)}\psi dx.
\end{aligned}
$$

Therefore, the weak solutions of (1.1) are the critical points of the functional $J_{\lambda,\mu}$. One can note that $J_{\lambda,\mu}$ is not bounded below on E, but it is bounded below on the following subset of E. We define the Nehari manifold as

$$
\mathscr{N}_{\lambda,\mu} := \{(u,v) \in E \setminus \{(0,0)\} : \langle J'_{\lambda,\mu}(u,v), (u,v)\rangle = 0\}.
$$

Therefore, $(u,v) \in \mathscr{N}_{\lambda,\mu}$ if and only if

$$
\begin{aligned}
& \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy + \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy \\
& - \int_\Omega \Big(\lambda a(x)|u|^{q(x)} + \mu b(x)|v|^{q(x)}\Big)dx - \int_\Omega c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx = 0.
\end{aligned}
\tag{3.2}
$$

The Nehari manifold is closely associated with the behavior of the fibering maps $\varphi_{u,v}: \mathbb{R}^+ \to \mathbb{R}$, defined as $\varphi_{u,v}(t) = J_{\lambda,\mu}(tu,tv)$, where $(u,v) \in E$. These maps are first given by Drabek and Pohozaev in [14] and are discussed in detail in [9] and [20].

For $(u,v)\in E$, we have

$$
\begin{aligned}
\varphi_{u,v}(t) &= J_{\lambda,\mu}(tu,tv)\\
&= \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{t^{p(x,y)}}{p(x,y)}\Big\{\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}+\frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\Big\}\,dx\,dy\\
&\quad -\int_\Omega \frac{t^{q(x)}}{q(x)}\Big(\lambda a(x)|u|^{q(x)}+\mu b(x)|v|^{q(x)}\Big)dx\\
&\quad -\int_\Omega \frac{t^{\alpha(x)+\beta(x)}}{\alpha(x)+\beta(x)}c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx.
\end{aligned}
\tag{3.3}
$$

$$
\begin{aligned}
&\varphi'_{u,v}(t)\\
&= \langle J'_{\lambda,\mu}(tu,tv),(u,v)\rangle\\
&= \int_{\mathbb{R}^N\times\mathbb{R}^N} t^{p(x,y)-1}\Big\{\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}+\frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\Big\}\,dx\,dy\\
&\quad -\int_\Omega t^{q(x)-1}\Big(\lambda a(x)|u|^{q(x)}+\mu b(x)|v|^{q(x)}\Big)dx\\
&\quad -\int_\Omega t^{\alpha(x)+\beta(x)-1}c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx.
\end{aligned}
\tag{3.4}
$$

$$
\begin{aligned}
\varphi''_{u,v}(t) &= \int_{\mathbb{R}^N\times\mathbb{R}^N} (p(x,y)-1)t^{p(x,y)-2}\Big\{\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\\
&\quad +\frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\Big\}\,dx\,dy\\
&\quad -\int_\Omega (q(x)-1)t^{q(x)-2}\Big(\lambda a(x)|u|^{q(x)}+\mu b(x)|v|^{q(x)}\Big)dx\\
&\quad -\int_\Omega (\alpha(x)+\beta(x)-1)t^{\alpha(x)+\beta(x)-2}c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx.
\end{aligned}
\tag{3.5}
$$

Then using that $\varphi'_{u,v}(t)=\langle J'_{\lambda,\mu}(tu,tv),(u,v)\rangle$, we can see that $(tu,tv)\in\mathscr{N}_{\lambda,\mu}$ if and only if $\varphi'_{u,v}(t)=0$. In particular, $(u,v)\in\mathscr{N}_{\lambda,\mu}$ if and only if $\varphi'_{u,v}(1)=0$. Thus, it is natural to split $\mathscr{N}_{\lambda,\mu}$ into three parts corresponding to the points of local maxima, local minima and inflection of the function $\varphi_{u,v}$ as follows:

$$
\begin{aligned}
\mathscr{N}^+_{\lambda,\mu} &:= \{(u,v)\in\mathscr{N}_{\lambda,\mu}:\varphi''_{u,v}(1)>0\}\\
&= \{(tu,tv)\in E\setminus\{(0,0)\}:\varphi'_{u,v}(t)=0,\varphi''_{u,v}(1)>0\},\\
\mathscr{N}^-_{\lambda,\mu} &:= \{(u,v)\in\mathscr{N}_{\lambda,\mu}:\varphi''_{u,v}(1)<0\}\\
&= \{(tu,tv)\in E\setminus\{(0,0)\}:\varphi'_{u,v}(t)=0,\varphi''_{u,v}(1)<0\},\\
\mathscr{N}^0_{\lambda,\mu} &:= \{(u,v)\in\mathscr{N}_{\lambda,\mu}:\varphi''_{u,v}(1)=0\}\\
&= \{(tu,tv)\in E\setminus\{(0,0)\}:\varphi'_{u,v}(t)=0,\varphi''_{u,v}(1)=0\}.
\end{aligned}
$$

Hence, for any $(u,v) \in \mathscr{N}_{\lambda,\mu}$, from (3.2), (3.4) and (3.5), we deduce

$$\begin{aligned}\varphi''_{u,v}(1) &= \int_{\mathbb{R}^N\times\mathbb{R}^N} p(x,y)\Big\{\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} + \frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\Big\}\,dx\,dy \\ &\quad - \int_\Omega q(x)\Big(\lambda a(x)|u|^{q(x)} + \mu b(x)|v|^{q(x)}\Big)dx \\ &\quad - \int_\Omega (\alpha(x)+\beta(x))c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx.\end{aligned} \tag{3.6}$$

For a given pair of functions $(u,v) \in E$, we set

$$P(u,v) := \int_{\mathbb{R}^N\times\mathbb{R}^N} \Big\{\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} + \frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\Big\}\,dx\,dy,$$
$$Q(u,v) := \int_\Omega \Big(\lambda a(x)|u|^{q(x)} + \mu b(x)|v|^{q(x)}\Big)dx,$$
$$R(u,v) := \int_\Omega c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx.$$

In the next lemma, we obtain some estimations on P, Q and R.

Lemma 3.1. *Let $(u,v) \in E$. Then we have the following:* Is this what you had in mind

(i)

$$P(u,v) \ge \begin{cases} \|(u,v)\|^{p^+}, & \text{if } \|(u,v)\| < 1 \\ \|(u,v)\|^{p^-}, & \text{if } \|(u,v)\| > 1, \end{cases}$$
$$P(u,v) \le \begin{cases} 2\|(u,v)\|^{p^-}, & \text{if } \|(u,v)\| < 1 \\ 2\|(u,v)\|^{p^+}, & \text{if } \|(u,v)\| > 1. \end{cases}$$

(ii) *There exists a constant $C_1 = C_1(N,s,p,q,\alpha,\beta,a,b,\Omega) > 0$ such that*

$$Q(u,v) \le C_1(\lambda+\mu)\max\{\|(u,v)\|^{q^-}, \|(u,v)\|^{q^+}\}.$$

(iii) *There exists a constant $C_2 = C_2(N,s,p,\alpha,\beta,c,\Omega) > 1$ such that*

$$R(u,v) \le C_2 \max\{\|(u,v)\|^{r^-}, \|(u,v)\|^{r^+}\}.$$

Proof. (i) Clearly $P(u,v) = \rho_{X_0}(u) + \rho_{X_0}(v)$. Hence, we have

$$\max\{\rho_{X_0}(u), \rho_{X_0}(v)\} \le P(u,v) \le 2\max\{\rho_{X_0}(u), \rho_{X_0}(v)\} \tag{3.7}$$

For $\|(u,v)\| > 1$, there are two cases.

Case I: $\|u\|_{X_0} > 1$ and $\|v\|_{X_0} > 1$. Then from Lemma 2.9 , we obtain

$$\|u\|_{X_0}^{p^-} < \rho_{X_0}(u) < \|u\|_{X_0}^{p^+} \text{ and } \|v\|_{X_0}^{p^-} < \rho_{X_0}(v) < \|v\|_{X_0}^{p^+}. \tag{3.8}$$

Thus, from (3.7) and (3.8), we obtain

$$P(u,v) \le 2\max\{\|u\|_{X_0}^{p^+}, \|v\|_{X_0}^{p^+}\} = 2\|(u,v)\|^{p^+};$$
$$P(u,v) \ge \max\{\|u\|_{X_0}^{p^-}, \|v\|_{X_0}^{p^-}\} = \|(u,v)\|^{p^-}.$$

Case II: $\|v\|_{X_0} < 1 < \|u\|_{X_0}$: Then $\|(u,v)\| = \|u\|_{X_0}$. Now Lemma 2.9 implies that

$$\|u\|_{X_0}^{p^-} < \rho_{X_0}(u) < \|u\|_{X_0}^{p^+} \quad \text{and} \quad \|v\|_{X_0}^{p^+} < \rho_{X_0}(v) < \|v\|_{X_0}^{p^-}. \tag{3.9}$$

Combining (3.7) and (3.9), we deduce that

$$P(u,v) \le 2\max\{\|u\|_{X_0}^{p^+}, \|v\|_{X_0}^{p^+}\} = 2\|(u,v)\|^{p^+},$$
$$P(u,v) \ge \max\{\|u\|_{X_0}^{p^-}, \|v\|_{X_0}^{p^-}\} = \|(u,v)\|^{p^-}.$$

Next, for $\|(u,v)\| < 1$, we have $\|u\|_{X_0} < 1$ and $\|v\|_{X_0} < 1$. By Lemma 2.9, we obtain

$$\|u\|_{X_0}^{p^+} < \rho_{X_0}(u) < \|u\|_{X_0}^{p^-} \quad \text{and} \quad \|v\|_{X_0}^{p^+} < \rho_{X_0}(v) < \|v\|_{X_0}^{p^-}. \tag{3.10}$$

Hence, from (3.7) and (3.10), it follows that

$$P(u,v) \le 2\max\{\|u\|_{X_0}^{p^-}, \|v\|_{X_0}^{p^-}\} = 2\|(u,v)\|^{p^-},$$
$$P(u,v) \ge \max\{\|u\|_{X_0}^{p^+}, \|v\|_{X_0}^{p^+}\} = \|(u,v)\|^{p^+}.$$

Thus, we obtain (i).

(ii) Using Hölder's inequality (Lemma 2.1), Sobolev-type embedding (Lemma 2.7) and Lemma 2.2, we obtain

$$\begin{aligned}
Q(u,v) &= \int_\Omega \Big(\lambda a(x)|u|^{q(x)} + \mu b(x)|v|^{q(x)}\Big)dx \\
&\le 2\lambda\|a\|_{L^{q_*(x)}(\Omega)}\||u|^{q(\cdot)}\|_{L^{\frac{\alpha(x)+\beta(x)}{q(x)}}(\Omega)} + 2\mu\|b\|_{L^{q_*(x)}(\Omega)}\||v|^{q(\cdot)}\|_{L^{\frac{\alpha(x)+\beta(x)}{q(x)}}(\Omega)} \\
&\le 2\lambda\|a\|_{L^{q_*(x)}(\Omega)}\Big\{\|u\|^{q^-}_{L^{\alpha(x)+\beta(x)}(\Omega)} + \|u\|^{q^+}_{L^{\alpha(x)+\beta(x)}(\Omega)}\Big\} \\
&\quad + 2\mu\|b\|_{L^{q_*(x)}(\Omega)}\Big\{\|v\|^{q^-}_{L^{\alpha(x)+\beta(x)}(\Omega)} + \|v\|^{q^+}_{L^{\alpha(x)+\beta(x)}(\Omega)}\Big\} \\
&\le K_1\Big[\lambda\Big\{\|u\|_{X_0}^{q^-} + \|u\|_{X_0}^{q^+}\Big\} + \mu\Big\{\|v\|_{X_0}^{q^-} + \|v\|_{X_0}^{q^+}\Big\}\Big] \\
&\le C_1(\lambda+\mu)\max\Big\{\|u\|_{X_0}^{q^-}, \|u\|_{X_0}^{q^+}, \|v\|_{X_0}^{q^-}, \|v\|_{X_0}^{q^+}\Big\} \\
&= C_1(\lambda+\mu)\max\Big\{\max\Big\{\|u\|_{X_0}^{q^-}, \|v\|_{X_0}^{q^-}\Big\}, \max\Big\{\|u\|_{X_0}^{q^+}, \|v\|_{X_0}^{q^+}\Big\}\Big\} \\
&= C_1(\lambda+\mu)\max\Big\{\|(u,v)\|^{q^-}, \|(u,v)\|^{q^+}\Big\},
\end{aligned}$$

where

$$\begin{aligned}
K_1 &= 2\big(\|a\|_{L^{q_*(x)}(\Omega)} + \|b\|_{L^{q_*(x)}(\Omega)}\big) \\
&\quad \times \max\big\{\big(C(N,s,p,\alpha,\beta,\Omega)\big)^{q^-}, \big(C(N,s,p,\alpha,\beta,\Omega)\big)^{q^+}\big\}
\end{aligned}$$

and $C_1 = 4K_1$.

(iii) Using Young's inequality, Lemma 2.2 and Lemma 2.7, we deduce

$$\begin{aligned}
R(u,v) &= \int_\Omega c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx \\
&\le \|c\|_{L^\infty(\Omega)}\int_\Omega |u|^{\alpha(x)}|v|^{\beta(x)}dx \\
&\le \|c\|_{L^\infty(\Omega)}\int_\Omega \Big\{\frac{\alpha(x)}{\alpha(x)+\beta(x)}|u|^{\alpha(x)+\beta(x)} + \frac{\beta(x)}{\alpha(x)+\beta(x)}|v|^{\alpha(x)+\beta(x)}\Big\}dx \\
&\le \|c\|_{L^\infty(\Omega)}\Big[\Big\{\|u\|^{\alpha^++\beta^+}_{L^{\alpha(x)+\beta(x)}(\Omega)} + \|u\|^{\alpha^-+\beta^-}_{L^{\alpha(x)+\beta(x)}(\Omega)}\Big\} \\
&\quad + \Big\{\|v\|^{\alpha^++\beta^+}_{L^{\alpha(x)+\beta(x)}(\Omega)} + \|v\|^{\alpha^-+\beta^-}_{L^{\alpha(x)+\beta(x)}(\Omega)}\Big\}\Big]
\end{aligned}$$

$$
\begin{aligned}
&\le K_2\Big[\Big\{\|u\|_{X_0}^{\alpha^+ +\beta^+}+\|u\|_{X_0}^{\alpha^- +\beta^-}\Big\}+\Big\{\|v\|_{X_0}^{\alpha^+ +\beta^+}+\|v\|_{X_0}^{\alpha^- +\beta^-}\Big\}\Big]\\
&\le C_2\max\Big\{\|u\|_{X_0}^{\alpha^- +\beta^-},\|u\|_{X_0}^{\alpha^+ +\beta^+},\|v\|_{X_0}^{\alpha^- +\beta^-},\|v\|_{X_0}^{\alpha^+ +\beta^+}\Big\}\\
&= C_2\max\Big\{\max\Big\{\|u\|_{X_0}^{\alpha^- +\beta^-},\|v\|_{X_0}^{\alpha^- +\beta^-}\Big\},\max\Big\{\|u\|_{X_0}^{\alpha^+ +\beta^+},\|v\|_{X_0}^{\alpha^+ +\beta^+}\Big\}\Big\}\\
&= C_2\max\Big\{\|(u,v)\|^{\alpha^- +\beta^-},\|(u,v)\|^{\alpha^+ +\beta^+}\Big\},
\end{aligned}
$$

where

$$
K_2=\|c\|_{L^\infty(\Omega)}\max\big\{\big(C(N,s,p,\alpha,\beta,\Omega)\big)^{\alpha^- +\beta^-},\big(C(N,s,p,\alpha,\beta,\Omega)\big)^{\alpha^+ +\beta^+}\big\}
$$

and $C_2=4K_2+1$. □

In the following lemma, we characterize the critical points of $J_{\lambda,\mu}$ as the local minimizers of $J_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}^+$ (or $\mathcal{N}_{\lambda,\mu}^-$).

Lemma 3.2. *Let $(u^*,v^*)\in\mathcal{N}_{\lambda,\mu}^+$ (or $\mathcal{N}_{\lambda,\mu}^-$) be a local minimizer for $J_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}^+$ (or $\mathcal{N}_{\lambda,\mu}^-$). Then (u^*,v^*) is a critical point of $J_{\lambda,\mu}$.*

Proof. First assume that $(u^*,v^*)\in\mathcal{N}_{\lambda,\mu}^+$ is a local minimizer for $J_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}^+$. Let $I_{\lambda,\mu}(u,v)=\langle J'_{\lambda,\mu}(u,v),(u,v)\rangle$. Note that for $(u,v)\in E\setminus\{0\}$ with $I_{\lambda,\mu}(u,v)=0$, we have $\varphi''_{u,v}(1)>0$ if and only if $\langle I'_{\lambda,\mu}(u,v),(u,v)\rangle>0$. Since (u^*,v^*) is a local minimizer for $J_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}^+$, using Lagrange's multiplier theorem we obtain a real number τ such that

$$
J'_{\lambda,\mu}(u^*,v^*)=\tau I'_{\lambda,\mu}(u^*,v^*).
$$

Therefore,

$$
0=\langle J'_{\lambda,\mu}(u^*,v^*),(u^*,v^*)\rangle=\tau\langle I'_{\lambda,\mu}(u^*,v^*),(u^*,v^*)\rangle=\tau\phi''_{(u^*,v^*)}(1).
$$

Since $(u^*,v^*)\in\mathcal{N}_{\lambda,\mu}^+$, we obtain that $\phi''_{(u^*,v^*)}(1)>0$ and hence $\tau=0$. This completes the proof. Similarly we can prove the result when $(u^*,v^*)\in\mathcal{N}_{\lambda,\mu}^-$ is a local minimizer for $J_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}^-$. □

Next we show that the set of points of inflection of the function $\varphi_{u,v}$ is empty for certain values of the parameters λ and μ.

Lemma 3.3. *There exists $\delta>0$, given by*

$$
\delta=\frac{1}{C_1}\Big(\frac{\alpha^-+\beta^--p^+}{\alpha^-+\beta^--q^-}\Big)\Big(\frac{p^--q^+}{C_2(\alpha^++\beta^+-q^+)}\Big)^{\frac{p^+-q^-}{\alpha^-+\beta^--p^+}}
$$

such that for any pair of $(\lambda,\mu)\in\mathbb{R}^+\times\mathbb{R}^+$ with $\lambda+\mu<\delta$, we have $\mathcal{N}_{\lambda,\mu}^0=\emptyset$, where the positive constants C_1,C_2 are given as in Lemma 3.1.

Proof. We prove this lemma by contradiction. Let us assume that there exist $\lambda,\mu>0$ with $\lambda+\mu<\delta$ such that $\mathcal{N}_{\lambda,\mu}^0\neq\emptyset$. Hence, there is $(u,v)\in\mathcal{N}_{\lambda,\mu}^0$. Now, if $\|(u,v)\|<1$, then using (3.2),(3.6) and Lemma 3.1 $(i),(ii)$, we obtain

$$
\begin{aligned}
0&=\varphi''_{(u,v)}(1)\\
&\le p^+P(u,v)-q^-Q(u,v)-(\alpha^-+\beta^-)R(u,v)\\
&=(p^+-(\alpha^-+\beta^-))P(u,v)+(\alpha^-+\beta^--q^-)Q(u,v)\\
&\le (p^+-(\alpha^-+\beta^-))\|(u,v)\|^{p^+}+(\alpha^-+\beta^--q^-)C_1(\lambda+\mu)\|(u,v)\|^{q^-}.
\end{aligned}
$$

This implies

$$\|(u,v)\|^{p^+-q^-} \le \frac{(\alpha^- + \beta^- - q^-)}{(\alpha^- + \beta^- - p^+)} C_1(\lambda+\mu). \tag{3.11}$$

Again using (3.2), (3.6) and Lemma 3.1 (i), (iii), we deduce

$$\begin{aligned}
0 &= \varphi''_{(u,v)}(1)\\
&\ge p^- P(u,v) - q^+ Q(u,v) - (\alpha^+ + \beta^+) R(u,v)\\
&= (p^- - q^+) P(u,v) - (\alpha^+ + \beta^+ - q^+) R(u,v)\\
&\ge (p^- - q^+)\|(u,v)\|^{p^+} - (\alpha^+ + \beta^+ - q^+) C_2 \|(u,v)\|^{\alpha^- + \beta^-}.
\end{aligned}$$

This yields

$$1 \ge \|(u,v)\|^{\alpha^- + \beta^- - p^+} \ge \frac{(p^- - q^+)}{C_2(\alpha^+ + \beta^+ - q^+)}. \tag{3.12}$$

Combining (3.11) and (3.12), we obtain

$$\lambda + \mu \ge \frac{1}{C_1}\Big(\frac{\alpha^- + \beta^- - p^+}{\alpha^- + \beta^- - q^-}\Big)\Big(\frac{p^- - q^+}{C_2(\alpha^+ + \beta^+ - q^+)}\Big)^{\frac{p^+ - q^-}{\alpha^- + \beta^- - p^+}},$$

which is a contradiction.

Next, if $\|(u,v)\| > 1$, again using (3.2), (3.6) and Lemma 3.1 $(i), (ii)$, we find that

$$0 = \varphi''_{u,v}(1) \le (p^+ - (\alpha^- + \beta^-))\|(u,v)\|^{p^-} + (\alpha^- + \beta^- - q^-) C_1(\lambda+\mu)\|(u,v)\|^{q^+},$$

that is,

$$\|(u,v)\|^{p^- - q^+} \le \frac{(\alpha^- + \beta^- - q^-)}{(\alpha^- + \beta^- - p^+)} C_1(\lambda+\mu). \tag{3.13}$$

On the other hand, by taking into account (3.2), (3.6) and Lemma 3.1 (i), (iii), it follows that

$$0 = \varphi''_{u,v}(1) \ge (p^- - q^+)\|(u,v)\|^{p^-} - (\alpha^+ + \beta^+ - q^+) C_2 \|(u,v)\|^{\alpha^+ + \beta^+},$$

that is,

$$\|(u,v)\|^{\alpha^+ + \beta^+ - p^-} \ge \frac{(p^- - q^+)}{C_2(\alpha^+ + \beta^+ - q^+)}. \tag{3.14}$$

Thus, combining (3.13) and (3.14), we obtain

$$\lambda + \mu \ge \frac{1}{C_1}\Big(\frac{\alpha^- + \beta^- - p^+}{\alpha^- + \beta^- - q^-}\Big)\Big(\frac{p^- - q^+}{C_2(\alpha^+ + \beta^+ - q^+)}\Big)^{\frac{p^- - q^+}{\alpha^+ + \beta^+ - p^-}}. \tag{3.15}$$

Since $0 < \big(\frac{p^- - q^+}{C_2(\alpha^+ + \beta^+ - q^+)}\big) < 1$ and $\frac{p^- - q^+}{\alpha^+ + \beta^+ - p^-} < \frac{p^+ - q^-}{\alpha^- + \beta^- - p^+}$, from (3.15) we infer that

$$\lambda + \mu \ge \frac{1}{C_1}\Big(\frac{\alpha^- + \beta^- - p^+}{\alpha^- + \beta^- - q^-}\Big)\Big(\frac{p^- - q^+}{C_2(\alpha^+ + \beta^+ - q^+)}\Big)^{\frac{p^+ - q^-}{\alpha^- + \beta^- - p^+}},$$

which is a contradiction. The proof is complete. □

In the next result, we discuss the behavior of the functional $J_{\lambda,\mu}$ on $\mathscr{N}_{\lambda,\mu}$.

Lemma 3.4. *For $\lambda + \mu < \delta$, $J_{\lambda,\mu}$ is coercive and bounded below on $\mathscr{N}_{\lambda,\mu}$.*

Proof. Let $(u,v) \in \mathscr{N}_{\lambda,\mu}$. Then for $\|(u,v)\| > 1$, from (3.1) and (3.2) and Lemma 3.1 (ii), we deduce

$$
\begin{aligned}
&J_{\lambda,\mu}(u,v)\\
&\geq \frac{1}{p^+}P(u,v) - \frac{1}{q^-}Q(u,v) - \frac{1}{\alpha^- + \beta^-}R(u,v)\\
&= \Big(\frac{1}{p^+} - \frac{1}{\alpha^- + \beta^-}\Big)P(u,v) - \Big(\frac{1}{q^-} - \frac{1}{\alpha^- + \beta^-}\Big)Q(u,v)\\
&\geq \Big(\frac{1}{p^+} - \frac{1}{\alpha^- + \beta^-}\Big)\|(u,v)\|^{p^-} - C_1(\lambda+\mu)\Big(\frac{1}{q^-} - \frac{1}{\alpha^- + \beta^-}\Big)\|(u,v)\|^{q^+}.
\end{aligned}
\tag{3.16}
$$

Since from (A3), we have $1 < q^- \leq q^+ < p^- \leq p^+ < \alpha^- + \beta^-$, (3.16) yields that $J_{\lambda,\mu}(u,v) \to +\infty$ as $\|(u,v)\| \to +\infty$. Therefore, $J_{\lambda,\mu}$ is coercive and bounded below on $\mathscr{N}_{\lambda,\mu}$. $\square$

Lemma 3.5. *We have the following results:*

(i) *If $(u,v) \in \mathscr{N}^+_{\lambda,\mu}$, then $Q(u,v) > 0$.*
(ii) *If $(u,v) \in \mathscr{N}^-_{\lambda,\mu}$, then $R(u,v) > 0$.*

Proof. (i). Since $(u,v) \in \mathscr{N}^+_{\lambda,\mu}$, we have $\phi''_{(u,v)}(1) > 0$. Thus, using (3.2) and (3.6), we obtain

$$
\begin{aligned}
0 < \varphi''_{(u,v)}(1) &\leq p^+P(u,v) - q^-Q(u,v) - (\alpha^- + \beta^-)R(u,v)\\
&= \{p^+ - (\alpha^- + \beta^-)\}P(u,v) + (\alpha^- + \beta^- - q^-)Q(u,v).
\end{aligned}
$$

This implies that

$$
Q(u,v) \geq \frac{(\alpha^- + \beta^- - p^+)}{(\alpha^- + \beta^- - q^-)}P(u,v) > 0.
$$

(ii). Since $(u,v) \in \mathscr{N}^-_{\lambda,\mu}$, we have $\phi''_{(u,v)}(1) < 0$. Thus, taking into account (3.2) and (3.6), we obtain

$$
\begin{aligned}
0 > \varphi''_{(u,v)}(1) &\geq p^-P(u,v) - q^+Q(u,v) - (\alpha^+ + \beta^+)R(u,v)\\
&= (p^- - q^+)P(u,v) - (\alpha^+ + \beta^+ - q^+)R(u,v),
\end{aligned}
$$

that is,

$$
R(u,v) \geq \frac{(\alpha^+ + \beta^+ - p^-)}{(\alpha^+ + \beta^+ - q^+)}P(u,v) > 0.
$$

$\square$

From Lemma 3.3 and Lemma 3.4, we conclude that for any pair of parameters $(\lambda,\mu) \in \mathbb{R}^+ \times \mathbb{R}^+$ with $\lambda + \mu < \delta$, $\mathscr{N}_{\lambda,\mu} = \mathscr{N}^-_{\lambda,\mu} \cup \mathscr{N}^+_{\lambda,\mu}$ and $J_{\lambda,\mu}$ is coercive and bounded below on $\mathscr{N}^-_{\lambda,\mu}$ and $\mathscr{N}^+_{\lambda,\mu}$. Therefore, we can define

$$
\theta_{\lambda,\mu} = \inf_{(u,v)\in\mathscr{N}_{\lambda,\mu}} J_{\lambda,\mu}(u,v), \quad \theta^+_{\lambda,\mu} = \inf_{(u,v)\in\mathscr{N}^+_{\lambda,\mu}} J_{\lambda,\mu}(u,v),
$$
$$
\theta^-_{\lambda,\mu} = \inf_{(u,v)\in\mathscr{N}^-_{\lambda,\mu}} J_{\lambda,\mu}(u,v).
$$

The following two lemmas give the signs of $\theta^+_{\lambda,\mu}$ and $\theta^-_{\lambda,\mu}$, respectively.

Lemma 3.6. *If $\lambda + \mu < \delta$, then $\theta_{\lambda,\mu} \leq \theta^+_{\lambda,\mu} < 0$.*

Proof. Let $(u,v) \in \mathcal{N}^+_{\lambda,\mu}$. Then $\varphi''_{u,v}(1) > 0$. Now combining (3.2) and (3.6), we obtain

$$\begin{aligned} 0 < \varphi''_{u,v}(1) &< p^+ P(u,v) - q^- Q(u,v) - (\alpha^- + \beta^-) R(u,v) \\ &= (p^+ - q^-) P(u,v) - (\alpha^- + \beta^- - q^-) R(u,v), \end{aligned}$$

that is,

$$R(u,v) < \frac{(p^+ - q^-)}{(\alpha^- + \beta^- - q^-)} P(u,v). \tag{3.17}$$

Using (3.1), (3.2) and (3.17), we deduce

$$\begin{aligned} &J_{\lambda,\mu}(u,v) \\ &\le \frac{1}{p^-} P(u,v) - \frac{1}{q^+} Q(u,v) - \frac{1}{\alpha^+ + \beta^+} R(u,v) \\ &= \Big(\frac{1}{p^-} - \frac{1}{q^+}\Big) P(u,v) + \Big(\frac{1}{q^+} - \frac{1}{\alpha^+ + \beta^+}\Big) R(u,v) \\ &\le \Big\{\Big(\frac{1}{p^-} - \frac{1}{q^+}\Big) + \Big(\frac{1}{q^+} - \frac{1}{\alpha^+ + \beta^+}\Big) \frac{(p^+ - q^-)}{(\alpha^- + \beta^- - q^-)}\Big\} P(u,v) \\ &= \Big\{\frac{(q^+ - p^-)(\alpha^+ + \beta^+) + p^-(\alpha^+ + \beta^+ - q^+)\frac{(p^+ - q^-)}{(\alpha^- + \beta^- - q^-)}}{p^- q^+ (\alpha^+ + \beta^+)}\Big\} P(u,v). \end{aligned} \tag{3.18}$$

From (A4), we have $(q^+ - p^-)(\alpha^+ + \beta^+) + p^-(\alpha^+ + \beta^+ - q^+)\frac{(p^+ - q^-)}{(\alpha^- + \beta^- - q^-)} < 0$. Hence, (3.18) implies that $J_{\lambda,\mu}(u,v) < 0$. Therefore, from the definition of $\theta_{\lambda,\mu}$ and $\theta^+_{\lambda,\mu}$, it follows that $\theta_{\lambda,\mu} \le \theta^+_{\lambda,\mu} < 0$. □

Lemma 3.7. *If $\lambda + \mu < \Big(\frac{q^-}{p^+}\Big)\delta$, then $\theta^-_{\lambda,\mu} > K$, where K is some positive constant depending on $N, s, p, q, \alpha, \beta, a, b, \lambda, \mu, \Omega$.*

Proof. Let $(u,v) \in \mathcal{N}^-_{\lambda,\mu}$. Then $\varphi''_{u,v}(1) < 0$. Therefore, from (3.12) and (3.13), we obtain

$$\|(u,v)\| \ge \begin{cases} \Big\{\frac{(p^- - q^+)}{C_2(\alpha^+ + \beta^+ - q^+)}\Big\}^{1/(\alpha^- + \beta^- - p^+)}, & \text{if } \|(u,v)\| < 1 \\ \Big\{\frac{(p^- - q^+)}{C_2(\alpha^+ + \beta^+ - q^+)}\Big\}^{1/(\alpha^+ + \beta^+ - p^-)}, & \text{if } \|(u,v)\| > 1. \end{cases} \tag{3.19}$$

Now for $\|(u,v)\| < 1$, plugging (3.2) into (3.1) and using Lemma 3.1 (i), (ii) and (3.19), we deduce that

$$\begin{aligned} &J_{\lambda,\mu}(u,v) \\ &\ge \frac{1}{p^+} P(u,v) - \frac{1}{q^-} Q(u,v) - \frac{1}{\alpha^- + \beta^-} R(u,v) \\ &= \Big(\frac{1}{p^+} - \frac{1}{\alpha^- + \beta^-}\Big) P(u,v) - \Big(\frac{1}{q^-} - \frac{1}{\alpha^- + \beta^-}\Big) Q(u,v) \\ &\ge \Big(\frac{1}{p^+} - \frac{1}{\alpha^- + \beta^-}\Big) \|(u,v)\|^{p^+} - \Big(\frac{1}{q^-} - \frac{1}{\alpha^- + \beta^-}\Big) C_1(\lambda + \mu) \|(u,v)\|^{q^-} \\ &= \|(u,v)\|^{q^-} \Big[\Big(\frac{1}{p^+} - \frac{1}{\alpha^- + \beta^-}\Big) \|(u,v)\|^{p^+ - q^-} \\ &\quad - \Big(\frac{1}{q^-} - \frac{1}{\alpha^- + \beta^-}\Big) C_1(\lambda + \mu)\Big] \end{aligned}$$

$$
\begin{aligned}
&\geq \Big\{\frac{(p^- - q^+)}{C_2(\alpha^+ + \beta^+ - q^+)}\Big\}^{\frac{q^-}{(\alpha^- + \beta^- - p^+)}} \Big[\Big(\frac{1}{p^+} - \frac{1}{\alpha^- + \beta^-}\Big) \\
&\quad \times \Big\{\frac{(p^- - q^+)}{C_2(\alpha^+ + \beta^+ - q^+)}\Big\}^{\frac{(p^+ - q^-)}{(\alpha^- + \beta^- - p^+)}} \\
&\quad - \Big(\frac{1}{q^-} - \frac{1}{\alpha^- + \beta^-}\Big) C_1(\lambda + \mu)\Big] := d_1
\end{aligned} \tag{3.20}
$$

If

$$
\begin{aligned}
\lambda + \mu &< \Big(\frac{q^-}{p^+}\Big)\delta \\
&= \Big(\frac{q^-}{p^+}\Big)\frac{1}{C_1}\Big(\frac{\alpha^- + \beta^- - p^+}{\alpha^- + \beta^- - q^-}\Big)\Big\{\frac{(p^- - q^+)}{C_2(\alpha^+ + \beta^+ - q^+)}\Big\}^{\frac{(p^+ - q^-)}{(\alpha^- + \beta^- - p^+)}},
\end{aligned}
$$

then

$$
\lambda + \mu < \frac{\alpha^- + \beta^- - p^+}{p^+(\alpha^- + \beta^-)}\Big\{\frac{(p^- - q^+)}{C_2(\alpha^+ + \beta^+ - q^+)}\Big\}^{\frac{(p^+ - q^-)}{(\alpha^- + \beta^- - p^+)}} \frac{(\alpha^- + \beta^-)q^-}{\alpha^- + \beta^- - q^-}.\frac{1}{C_1},
$$

that is,

$$
\begin{aligned}
&\Big(\frac{1}{p^+} - \frac{1}{\alpha^- + \beta^-}\Big)\Big\{\frac{(p^- - q^+)}{C_2(\alpha^+ + \beta^+ - q^+)}\Big\}^{\frac{(p^+ - q^-)}{(\alpha^- + \beta^- - p^+)}} \\
&- \Big(\frac{1}{q^-} - \frac{1}{\alpha^- + \beta^-}\Big) C_1(\lambda + \mu) > 0,
\end{aligned}
$$

and thus, from (3.20), we obtain that $d_1 > 0$.

Similarly for $\|(u,v)\| > 1$, again plugging (3.2) in (3.1) and using Lemma 3.1 $(i), (ii)$ and (3.19), we obtain

$$
\begin{aligned}
&J_{\lambda,\mu}(u,v) \\
&\geq \frac{1}{p^+}P(u,v) - \frac{1}{q^-}Q(u,v) - \frac{1}{\alpha^- + \beta^-}R(u,v) \\
&= \Big(\frac{1}{p^+} - \frac{1}{\alpha^- + \beta^-}\Big)P(u,v) - \Big(\frac{1}{q^-} - \frac{1}{\alpha^- + \beta^-}\Big)Q(u,v) \\
&\geq \Big(\frac{1}{p^+} - \frac{1}{\alpha^- + \beta^-}\Big)\|(u,v)\|^{p^-} - \Big(\frac{1}{q^-} - \frac{1}{\alpha^- + \beta^-}\Big)C_1(\lambda + \mu)\|(u,v)\|^{q^+} \\
&= \|(u,v)\|^{q^+}\Big\{\Big(\frac{1}{p^+} - \frac{1}{\alpha^- + \beta^-}\Big)\|(u,v)\|^{p^- - q^+} \\
&\quad - \Big(\frac{1}{q^-} - \frac{1}{\alpha^- + \beta^-}\Big)C_1(\lambda + \mu)\Big\} \\
&\geq \Big\{\frac{(p^- - q^+)}{C_2(\alpha^+ + \beta^+ - q^+)}\Big\}^{\frac{q^+}{(\alpha^+ + \beta^+ - p^-)}} \Big[\Big(\frac{1}{p^+} - \frac{1}{\alpha^- + \beta^-}\Big) \\
&\Big\{\frac{(p^- - q^+)}{C_2(\alpha^+ + \beta^+ - q^+)}\Big\}^{\frac{(p^- - q^+)}{(\alpha^+ + \beta^+ - p^-)}} - \Big(\frac{1}{q^-} - \frac{1}{\alpha^- + \beta^-}\Big)C_1(\lambda + \mu)\Big]
\end{aligned} \tag{3.21}
$$

Combining that $\frac{(p^- - q^+)}{(\alpha^+ + \beta^+ - p^-)} < \frac{(p^+ - q^-)}{(\alpha^- + \beta^- - p^+)}$ and $\frac{(p^- - q^+)}{C_2(\alpha^+ + \beta^+ - q^+)} < 1$, and taking into account (3.20) and (3.21), we deduce that

$$
J_{\lambda,\mu}(u,v)
$$

$$
\begin{aligned}
&\geq \Big\{\frac{(p^- - q^+)}{C_2(\alpha^+ + \beta^+ - q^+)}\Big\}^{\frac{q^+}{(\alpha^- + \beta^- - p^+)}} \Big[\Big(\frac{1}{p^+} - \frac{1}{\alpha^- + \beta^-}\Big) \\
&\quad \times \Big\{\frac{(p^- - q^+)}{C_2(\alpha^+ + \beta^+ - q^+)}\Big\}^{\frac{(p^+ - q^-)}{(\alpha^- + \beta^- - p^+)}} - \Big(\frac{1}{q^-} - \frac{1}{\alpha^- + \beta^-}\Big) C_1(\lambda + \mu)\Big] \\
&\geq \Big\{\frac{(p^- - q^+)}{C_2(\alpha^+ + \beta^+ - q^+)}\Big\}^{\frac{(q^+ - q^-)}{(\alpha^- + \beta^- - p^+)}} d_1 := d_2 > 0.
\end{aligned}
$$

Finally by choosing $K = \min\{d_1, d_2\} > 0$, the proof is complete. □

The next lemma describes the nature of the map $\varphi_{u,v}$. We refer to [8] and [10] for the similar results in the case of local p-Laplacian and nonlocal p-Laplacian, respectively, and [1, 15] for variable exponent Laplacian.

Lemma 3.8. *For $(u, v) \in E\backslash\{(0, 0)\}$, there exists $\delta' > 0$ such that for all $\lambda+\mu < \delta'$, we have the following:*

- (i) *If $Q(u, v) = 0$, then there exists a unique $t^- = t^-(u, v)$ such that $(t^-u, t^-v) \in \mathscr{N}^-_{\lambda,\mu}$ and $J_{\lambda,\mu}(t^-u, t^-v) = \sup_{t\geq 0} J_{\lambda,\mu}(tu, tv)$.*
- (ii) *If $Q(u, v) > 0$, then there exist $t^* > 0$ and unique positive numbers $t^+ = t^+(u, v) < t^- = t^-(u, v)$ such that $(t^-u, t^-v) \in \mathscr{N}^-_{\lambda,\mu}$, $(t^+u, t^+v) \in \mathscr{N}^+_{\lambda,\mu}$ and*

$$
J_{\lambda,\mu}(t^+u, t^+v) = \inf_{0\leq t\leq t^*} J_{\lambda,\mu}(tu, tv), \quad J_{\lambda,\mu}(t^-u, t^-v) = \sup_{t\geq 0} J_{\lambda,\mu}(tu, tv).
$$

Proof. (i) Using the given assumption, for $0 < t < 1$ sufficiently small, we obtain

$$
\varphi_{u,v}(t) > \frac{t^{p^+}}{p^+} P(u, v) - \frac{t^{\alpha^+ + \beta^+}}{\alpha^+ + \beta^+} R(u, v) > 0
$$

and for $t > 1$ sufficiently large, we obtain

$$
\varphi_{u,v}(t) < \frac{t^{p^+}}{p^-} P(u, v) - \frac{t^{\alpha^+ + \beta^+}}{\alpha^- + \beta^-} R(u, v) < 0.
$$

Hence, $\varphi_{u,v}$ achieves its maximum at some point $t^-(u, v)$ on $[0, \infty)$. Thus, $\varphi'_{u,v}(t^-) = \langle J'_{\lambda,\mu}(t^-u, t^-v), (u, v)\rangle = 0$. Set $(t^-u, t^-v) := (\overline{u}, \overline{v})$. Then $\langle J'_{\lambda,\mu}(\overline{u}, \overline{v}), (\overline{u}, \overline{v})\rangle = 0$, which implies $(\overline{u}, \overline{v}) \in \mathscr{N}_{\lambda,\mu}$. Thus, from (3.2), we obtain

$$
P(\overline{u}, \overline{v}) = R(\overline{u}, \overline{v}). \tag{3.22}
$$

Now we define the function $\Theta_{\overline{u},\overline{v}} : [0, \infty) \to \mathbb{R}$ as $\Theta_{\overline{u},\overline{v}}(t) = J_{\lambda,\mu}(t\overline{u}, t\overline{v})$. We know that $\Theta_{\overline{u},\overline{v}}(1) = J_{\lambda,\mu}(\overline{u}, \overline{v}) = \max_{t\in[0,\infty)} \Theta_{\overline{u},\overline{v}}(t)$ and $\Theta'_{\overline{u},\overline{v}}(1) = \langle J'_{\lambda,\mu}(\overline{u}, \overline{v}), (\overline{u}, \overline{v})\rangle = 0$. For $t > 1$, by (3.22), we deduce that

$$
\begin{aligned}
\Theta'_{\overline{u},\overline{v}}(t) &= \langle J'_{\lambda,\mu}(t\overline{u}, t\overline{v}), (\overline{u}, \overline{v})\rangle \\
&\leq t^{p^+-1} P(\overline{u}, \overline{v}) - t^{\alpha^- + \beta^- - 1} R(\overline{u}, \overline{v}) < 0,
\end{aligned}
$$

and on the other hand, for $t \in (0, 1)$, again using (3.22), we obtain

$$
\begin{aligned}
\Theta'_{\overline{u},\overline{v}}(t) &= \langle J'_{\lambda,\mu}(t\overline{u}, t\overline{v}), (\overline{u}, \overline{v})\rangle \\
&\geq t^{p^--1} P(\overline{u}, \overline{v}) - t^{\alpha^+ + \beta^+ - 1} R(\overline{u}, \overline{v}) > 0.
\end{aligned}
$$

This shows that the point t^- is unique. Hence, the result follows.

(ii) To prove this lemma, first we set

$$f_1(t) := \int_{\mathbb{R}^N\times\mathbb{R}^N} t^{p(x,y)}\Big\{\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} + \frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\Big\}\,dx\,dy,$$
$$f_2(t) := \int_\Omega t^{q(x)}\Big(\lambda a(x)|u|^{q(x)} + \mu b(x)|v|^{q(x)}\Big)dx,$$
$$f_3(t) := \int_\Omega t^{\alpha(x)+\beta(x)} c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx.$$

The f_i's are continuous and strictly increasing functions with $f_i(0) = 0$ for $i = 1,2,3$. Also, we have the following observations:

(I) $\lim_{t\to 0^+} f_3(t)/f_1(t) = 0$.
(II) $\lim_{t\to+\infty} f_2(t) = +\infty$.
(III) $\lim_{t\to+\infty}(f_1 - f_3)(t))/f_2(t) = 0$.
(IV) $f_1 - f_3$ has unique point of maximum, say $t_{\max}$ and $(f_1 - f_3)(t) \to -\infty$ as $t \to +\infty$.
(V) There exists $\tilde{t} \in (0, t_{\max})$ such that $\frac{f_1-f_3}{f_2}$ is strictly increasing on $(0,\tilde{t})$.

From (I), we note that $(f_1 - f_3)(t) > 0$ for $t \to 0^+$ sufficiently small. Hence, by (V) and intermediate value theorem, for each choice of the pair $(\lambda,\mu) \in \mathbb{R}^+ \times \mathbb{R}^+$ with $f_2(\tilde{t}) < (f_1 - f_3)(\tilde{t})$, there exists a unique $t^+ = t^+(\lambda,\mu) \in (0,\tilde{t})$ such that

$$\frac{(f_1 - f_3)(t^+)}{f_2(t^+)} = 1. \tag{3.23}$$

Since $\frac{(f_1-f_3)}{f_2}$ is strictly monotone increasing in $(t^+,\tilde{t})$, from (3.23), we obtain

$$1 = \frac{(f_1 - f_3)(t^+)}{f_2(t^+)} < \frac{(f_1 - f_3)(t)}{f_2(t)} \quad \text{for all } t \in (t^+,\tilde{t}),$$

that is,

$$f_2(t) < (f_1 - f_3)(t) \quad \text{for all } t \in (t^+,\tilde{t}). \tag{3.24}$$

Now we can fix $(\lambda^*,\mu^*) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that for all $\lambda \in (0,\lambda^*), \mu \in (0,\mu^*)$, taking into account (3.24), we have

$$f_2(t) < (f_1 - f_3)(t) \text{ for all } t \in (t^+, t_{\max}). \tag{3.25}$$

Since $f_1 - f_3$ is strictly decreasing in $(t_{\max},\infty)$ and f_2 is monotonically increasing in $(0,\infty)$, by (II) and (3.25), there exists a unique positive real number $t^- > t_{\max}$ such that

$$f_2(t^-) = (f_1 - f_3)(t^-) \text{ for all } (\lambda,\mu) \in (0,\lambda^*) \times (0,\mu^*). \tag{3.26}$$

Hence, combining (3.23) and (3.26), we yield that the function $\varphi'_{u,v}(t) = f_1 - f_2 - f_3$ has exactly two nontrivial zeroes, $t^+ < t^-$, that is, t^+ and t^- are critical points of $\varphi_{u,v}(t)$. For $\delta' := \lambda^* + \mu^*$, we can choose $\lambda^*,\mu^* > 0$ sufficiently small such that $\delta' < \delta$, where δ is given as in Lemma 3.3. Since $\varphi_{u,v}(0) = 0$ and $\varphi_{u,v}(t) < 0$ for $t \to 0^+$ sufficiently small, we obtain that $\varphi'_{u,v}(t) < 0$ for all $t \in (0,t^+)$ and $\varphi'_{u,v}(t) > 0$ for all $t \in (t^+,t_{\max})$ and $\varphi'_{u,v}(t^+) = 0$. Again, by the Lemma 3.3, we have $\mathcal{N}^0_{\lambda,\mu} = \emptyset$ and thus, we infer that $\varphi_{u,v}$ attains a local minimum at t^+ and consequently $\varphi''_{u,v}(t^+) > 0$. Hence, $(t^+u, t^+v) \in \mathcal{N}^+$.

Similarly, since $\varphi'_{u,v}(t) > 0$ for all $t \in [t_{\max},t^-)$, $\varphi'_{u,v}(t) < 0$ for all $t > t^-$, and $\varphi'_{u,v}(t^-) = 0$, using the fact that $\mathcal{N}^0_{\lambda,\mu} = \emptyset$ from Lemma 3.3, it follows that t^- is the point of global maximum for $\varphi_{u,v}$ and consequently $\varphi''_{u,v}(t^-) < 0$. Hence,

$(t^-u, t^-v) \in \mathscr{N}^-$. Now by appealing to Lemma 3.6 and Lemma 3.7, we obtain $\varphi_{u,v}(t^+) < 0$ and $\varphi_{u,v}(t^-) > 0$. Also, by the above discussions, $\varphi_{u,v}$ is strictly increasing on $[t^+, t^-]$ and strictly decreasing for all $t > t^-$ with $\varphi_{u,v}(t) \to -\infty$ as $t \to +\infty$. Thus, there exists a unique $t^* \in (t^+, t^-)$ such that $\varphi_{u,v}(t^*) = 0$. Therefore,

$$J_{\lambda,\mu}(t^+u, t^+v) = \varphi_{u,v}(t^+) = \inf_{0\le t\le t^*} \phi_{u,v}(t) = \inf_{0\le t\le t^*} J_{\lambda,\mu}(tu, tv),$$
$$J_{\lambda,\mu}(t^-u, t^-v) = \varphi_{u,v}(t^-) = \sup_{t\ge 0} \phi_{u,v}(t) = \sup_{t\ge 0} J_{\lambda,\mu}(tu, tv).$$

This completes the proof. □

4. Existence of multiple solutions

In this section, we will prove the existence of at least two distinct non-trivial and non-negative weak solutions of (1.1). The next two propositions ensure the existence of minimizers for the functional $J_{\lambda,\mu}$ in $\mathscr{N}^+_{\lambda,\mu}$ and $\mathscr{N}^-_{\lambda,\mu}$, respectively, which serve as the weak solutions of (1.1). We set $\delta_0 := \min\left\{\left(\frac{q^-}{p^+}\right)\delta, \delta'\right\}$, where δ and δ' are given as in Lemma 3.7 and Lemma 3.8, respectively.

Proposition 4.1. *For $\lambda + \mu < \delta_0$, the functional $J_{\lambda,\mu}$ has a minimizer (u_0, v_0) in $\mathscr{N}^+_{\lambda,\mu}$, which satisfies the following assertions:*

(i) $J_{\lambda,\mu}(u_0, v_0) = \theta^+_{\lambda,\mu} < 0$,
(ii) (u_0, v_0) *is a solution of* (1.1)

Proof. (i) Since $J_{\lambda,\mu}$ is bounded below on $\mathscr{N}_{\lambda,\mu}$ and hence on $\mathscr{N}^+_{\lambda,\mu}$, there exists a minimizing sequence $\{(u_m, v_m)\} \subset \mathscr{N}^+_{\lambda,\mu}$, such that

$$\lim_{m\to\infty} J_{\lambda,\mu}(u_m, v_m) = \inf_{(u,v)\in\mathscr{N}^+_{\lambda,\mu}} J_{\lambda,\mu}(u, v).$$

By Lemma 3.4, we have $J_{\lambda,\mu}$ is coercive on $\mathscr{N}^+_{\lambda,\mu}$, which implies that the sequence $\{(u_m, v_m)\}$ is bounded on E. Therefore, there exists $(u_0, v_0) \in E$ such that, passing to a subsequence,

$$u_m \rightharpoonup u_0, \quad v_m \rightharpoonup v_0 \quad \text{in } X_0 \text{ as } m \to \infty$$

and hence, using Sobolev-type embedding result (Lemma 2.7), we have

$$u_m \to u_0 \quad \text{strongly in } L^{q(x)}(\Omega), \text{ and } L^{\alpha(x)+\beta(x)}(\Omega),$$
$$v_m \to v_0 \quad \text{strongly in } L^{q(x)}(\Omega) \text{ and } L^{\alpha(x)+\beta(x)}(\Omega),$$
$$u_m(x) \to u_0(x) \text{ and } v_m(x) \to v_0(x) \quad \text{a.e. in } \Omega$$

as $m \to \infty$. Now by applying Lemma 2.4 and Lebesgue dominated convergence theorem, one can check that

$$\begin{gathered}\lim_{m\to\infty} \int_\Omega a(x)|u_m|^{q(x)}dx = \int_\Omega a(x)|u_0|^{q(x)}dx,\\ \lim_{m\to\infty} \int_\Omega b(x)|v_m|^{q(x)}dx = \int_\Omega b(x)|v_0|^{q(x)}dx;\end{gathered} \tag{4.1}$$

and

$$
\begin{aligned}
\lim_{m\to\infty}\int_\Omega a(x)\frac{|u_m|^{q(x)}}{q(x)}dx &= \int_\Omega a(x)\frac{|u_0|^{q(x)}}{q(x)}dx,\\
\lim_{m\to\infty}\int_\Omega b(x)\frac{|v_m|^{q(x)}}{q(x)}dx &= \int_\Omega b(x)\frac{|v_0|^{q(x)}}{q(x)}dx.
\end{aligned} \tag{4.2}
$$

Also, by Lemma 5.1 and Lemma 5.2 (see Appendix), we have

$$
\begin{gathered}
\lim_{m\to\infty} R(u_m,v_m) = R(u_0,v_0),\\
\lim_{m\to\infty}\int_\Omega \frac{c(x)|u_m|^{\alpha(x)}|v_m|^{\beta(x)}}{\alpha(x)+\beta(x)}dx = \int_\Omega \frac{c(x)|u_0|^{\alpha(x)}|v_0|^{\beta(x)}}{\alpha(x)+\beta(x)}dx,
\end{gathered} \tag{4.3}
$$

respectively. We claim that $(u_0,v_0)\not\equiv(0,0)$. Note that $Q(u_0,v_0)>0$. Indeed, if not then from (4.1),

$$
Q(u_m,v_m)\to Q(u_0,v_0)=0 \quad \text{as } m\to\infty. \tag{4.4}
$$

Since $(u_m,v_m)\in\mathscr{N}^+_{\lambda,\mu}$, using (3.1) and (3.2), we obtain

$$
J_{\lambda,\mu}(u_m,v_m)\geq\Big(\frac{1}{p^+}-\frac{1}{\alpha^-+\beta^-}\Big)P(u_m,v_m)-\Big(\frac{1}{q^-}-\frac{1}{\alpha^-+\beta^-}\Big)Q(u_m,v_m).
$$

Now letting $m\to\infty$ in the both side of the last expression and using (4.4), we obtain

$$
\lim_{m\to\infty} J_{\lambda,\mu}(u_m,v_m)\geq 0. \tag{4.5}
$$

But Lemma 3.6 gives $\lim_{m\to\infty}J_{\lambda,\mu}(u_m,v_m)=\inf_{(u,v)\in\mathscr{N}^+_{\lambda,\mu}}J_{\lambda,\mu}(u,v)<0$, which contradicts (4.5). Thus, the claim is proved and we obtain that $(u_0,v_0)\in E\setminus\{(0,0)\}$.

Next, we claim that $u_m\to u_0$ and $v_m\to v_0$ strongly in X_0 as $m\to\infty$. If not, then $u_m\not\to u_0$ or $v_m\not\to v_0$ in X_0 as $m\to\infty$. Therefore, using Lemma 2.10 and Brezis-Lieb lemma (see [7]), it follows that either

$$
\begin{aligned}
&\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{1}{p(x,y)}\frac{|u_0(x)-u_0(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy\\
&<\liminf_{m\to\infty}\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{1}{p(x,y)}\frac{|u_m(x)-u_m(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy\\
&\text{or}\\
&\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{1}{p(x,y)}\frac{|v_0(x)-v_0(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy\\
&<\liminf_{m\to\infty}\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{1}{p(x,y)}\frac{|v_m(x)-v_m(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy.
\end{aligned} \tag{4.6}
$$

Thus, combining (3.1), (4.2), (4.3) and (4.6), we obtain

$$
\begin{aligned}
&\lim_{m\to\infty} J_{\lambda,\mu}(u_m,v_m)\\
&= \liminf_{m\to\infty} \Big[\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{1}{p(x,y)}\Big\{\frac{|u_m(x)-u_m(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\\
&\quad + \frac{|v_m(x)-v_m(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\Big\}\,dx\,dy\\
&\quad - \int_\Omega \frac{1}{q(x)}\Big(\lambda a(x)|u_m|^{q(x)} + \mu b(x)|v_m|^{q(x)}\Big)dx\\
&\quad - \int_\Omega \frac{1}{\alpha(x)+\beta(x)} c(x)|u_m|^{\alpha(x)}|v_m|^{\beta(x)}dx\Big]\\
&\geq \liminf_{m\to\infty} \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{1}{p(x,y)}\frac{|u_m(x)-u_m(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy\\
&\quad + \liminf_{m\to\infty} \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{1}{p(x,y)}\frac{|v_m(x)-v_m(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy\\
&\quad - \lim_{m\to\infty}\int_\Omega \frac{1}{q(x)}\Big(\lambda a(x)|u_m|^{q(x)} + \mu b(x)|v_m|^{q(x)}\Big)dx\\
&\quad - \lim_{m\to\infty}\int_\Omega \frac{1}{\alpha(x)+\beta(x)} c(x)|u_m|^{\alpha(x)}|v_m|^{\beta(x)}dx\\
&> \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{1}{p(x,y)}\frac{|u_0(x)-u_0(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy\\
&\quad + \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{1}{p(x,y)}\frac{|v_0(x)-v_0(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy\\
&\quad - \int_\Omega \frac{1}{q(x)}\Big(\lambda a(x)|u_0|^{q(x)} + \mu b(x)|v_0|^{q(x)}\Big)dx\\
&\quad - \int_\Omega \frac{1}{\alpha(x)+\beta(x)} c(x)|u_0|^{\alpha(x)}|v_0|^{\beta(x)}dx\\
&= J_{\lambda,\mu}(u_0,v_0)
\end{aligned}
\tag{4.7}
$$

By Lemma 3.8 (ii), for $(u_0,v_0)\in E\setminus\{(0,0)\}$, there exists a positive real number $t_0^+(u_0,v_0)$ such that $(t_0^+u_0,t_0^+v_0)\in\mathcal{N}^+_{\lambda,\mu}$. Again, considering the assumption $u_m\not\to u_0$ or $v_m\not\to v_0$ in X_0, we have

$$
\rho_{X_0}(t_0^+u_0) < \liminf_{m\to\infty}\rho_{X_0}(t_0^+u_m) \quad\text{or}\quad \rho_{X_0}(t_0^+v_0) < \liminf_{m\to\infty}\rho_{X_0}(t_0^+v_m). \tag{4.8}
$$

Furthermore, appealing Lemma 2.4 and Lebesgue dominated convergence theorem, we obtain

$$
Q(t_0^+u_0,t_0^+v_0) = \lim_{m\to\infty} Q(t_0^+u_m,t_0^+v_m) \tag{4.9}
$$

and by Lemma 5.1 (see Appendix), we obtain

$$
R(t_0^+u_0,t_0^+v_0) = \lim_{m\to\infty} R(t_0^+u_m,t_0^+v_m). \tag{4.10}
$$

Taking into account (3.4), (4.9), (4.10) and (4.8), we deduce that

$$
\begin{aligned}
&\lim_{m\to\infty} \varphi'_{u_m,v_m}(t_0^+)\\
&= \liminf_{m\to\infty} \Big[\int_{\mathbb{R}^N\times\mathbb{R}^N} (t_0^+)^{p(x,y)-1} \frac{|u_m(x)-u_m(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy\\
&\quad + \int_{\mathbb{R}^N\times\mathbb{R}^N} (t_0^+)^{p(x,y)-1} \frac{|v_m(x)-v_m(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy\\
&\quad - \int_\Omega (t_0^+)^{q(x)-1}\Big(\lambda a(x)|u_m|^{q(x)} + \mu b(x)|v_m|^{q(x)}\Big)dx\\
&\quad - \int_\Omega (t_0^+)^{\alpha(x)+\beta(x)-1} c(x)|u_m|^{\alpha(x)}|v_m|^{\beta(x)}dx\Big] \qquad (4.11)\\
&\geq \frac{1}{t_0^+}\Big[\liminf_{m\to\infty} \rho_{X_0}(t_0^+u_m) + \liminf_{m\to\infty} \rho_{X_0}(t_0^+v_m) - \lim_{m\to\infty} Q(t_0^+u_m, t_0^+v_m)\\
&\quad - \lim_{m\to\infty} R(t_0^+u_m, t_0^+v_m)\Big]\\
&> \frac{1}{t_0^+}\Big[\rho_{X_0}(t_0^+u_0) + \rho_{X_0}(t_0^+v_m) - Q(t_0^+u_0, t_0^+v_0) - R(t_0^+u_0, t_0^+v_0)\Big]\\
&= \varphi'_{u_0,v_0}(t_0^+) = 0.
\end{aligned}
$$

Thus, for m large enough, $\varphi'_{u_m,v_m}(t_0^+) > 0$. Since $(u_m, v_m) \in \mathscr{N}^+_{\lambda,\mu}$ for all $m \in \mathbb{N}$, we have $\varphi'_{u_m,v_m}(1) = 0$ and $\varphi''_{u_m,v_m}(1) > 0$. Then using Lemma 3.8 *(ii)*, we obtain $\varphi'_{u_m,v_m}(t) < 0$ for all $t \in (0,1)$ and therefore, from (4.11), we must have $t_0^+ > 1$. Since $(t_0^+u_0, t_0^+v_0) \in \mathscr{N}^+_{\lambda,\mu}$, again by Lemma 3.8 *(ii)*, we obtain that $\varphi_{u_0,v_0}(t)$ is monotone decreasing on $(0, t_0^+)$. Hence using (4.7), we obtain

$$
J_{\lambda,\mu}(t_0^+u_0, t_0^+v_0) \leq J_{\lambda,\mu}(u_0,v_0) < \lim_{m\to\infty} J_{\lambda,\mu}(u_m,v_m) = \inf_{(u,v)\in\mathscr{N}^+_{\lambda,\mu}} J_{\lambda,\mu}(u,v).
$$

This is a contradiction to the fact that $(t_0^+u_0, t_0^+v_0) \in \mathscr{N}^+_{\lambda,\mu}$. So, $(u_m, v_m) \to (u_0, v_0)$ strongly in E as $m \to \infty$ and thus, $(u_0, v_0) \in \mathscr{N}_{\lambda,\mu}$. Since Lemma 3.3 gives that $\mathscr{N}^0_{\lambda,\mu} = \emptyset$ and from Lemma 3.6, we have $J_{\lambda,\mu}(u_0,v_0) = \lim_{m\to\infty} J_{\lambda,\mu}(u_m,v_m) < 0$, we infer that $(u_0, v_0) \in \mathscr{N}^+_{\lambda,\mu}$.

(ii) Using Lemma 3.2, we conclude that (u_0, v_0) is a solution of (1.1). $\square$

Proposition 4.2. *If $\lambda + \mu < \delta_0$, then $J_{\lambda,\mu}$ has a minimizer (w_0, z_0) in $\mathscr{N}^-_{\lambda,\mu}$ such that the following assertions hold:*

(i) $J_{\lambda,\mu}(w_0, z_0) = \theta^-_{\lambda,\mu} > 0$.
(ii) (w_0, z_0) *is a non-semi trivial solution of* (1.1).

Proof. (i) From Lemma 3.4, $J_{\lambda,\mu}$ is bounded below on $\mathscr{N}^-_{\lambda,\mu}$. Hence, there exists a minimizing sequence $\{(w_m, z_m)\} \subset \mathscr{N}^-_{\lambda,\mu}$ such that

$$
\lim_{m\to\infty} J_{\lambda,\mu}(w_m, z_m) = \inf_{(u,v)\in\mathscr{N}^-_{\lambda,\mu}} J_{\lambda,\mu}(u,v).
$$

Again from Lemma 3.4, we have $J_{\lambda,\mu}$ is coercive, which implies that the sequence $\{(w_m, z_m)\}$ is bounded on E and thus, there exists $(w_0, z_0) \in E$ such that up to a subsequence, $(w_m, z_m) \rightharpoonup (w_0, z_0)$ weakly and by Sobolev-type embedding result

(Theorem 2.7), we obtain

$$\begin{gathered}w_m \to w_0, z_m \to z_0 \quad \text{strongly in } L^{q(x)}(\Omega) \text{ and } L^{\alpha(x)+\beta(x)}(\Omega),\\ w_m(x) \to w_0(x) \text{ and } z_m(x) \to z_0(x) \quad \text{a.e. in } \Omega\end{gathered}$$

as $m \to \infty$. Using Lemma 2.4 and Dominated convergence theorem, we have

$$\begin{gathered}\lim_{m\to\infty} \int_\Omega a(x)|w_m|^{q(x)}dx = \int_\Omega a(x)|w_0|^{q(x)}dx \ \text{ and}\\ \lim_{m\to\infty} \int_\Omega b(x)|z_m|^{q(x)}dx = \int_\Omega b(x)|z_0|^{q(x)}dx.\end{gathered} \tag{4.12}$$

Also, Lemma 5.1 (see Appendix) gives

$$R(w_m, z_m) \to R(w_0, z_0) \quad \text{as } m \to \infty. \tag{4.13}$$

Next, we have $(w_0, z_0) \not\equiv (0, 0)$. Indeed, if $(w_0, z_0) = (0, 0)$, from (4.13), we obtain

$$R(w_m, z_m) \to R(w_0, z_0) = 0 \quad \text{as } m \to \infty. \tag{4.14}$$

Since $(w_m, z_m) \in \mathcal{N}^-_{\lambda,\mu}$, using (3.1), (3.2) and Lemma 3.6, we have

$$\begin{aligned}0 &< K < J_{\lambda,\mu}(w_m, z_m)\\ &\leq \Big(\frac{1}{p^+} - \frac{1}{q^-}\Big)P(w_m, z_m) + \Big(\frac{1}{q^-} - \frac{1}{\alpha^- + \beta^-}\Big)R(w_m, z_m) + o_m(1).\end{aligned}$$

Now letting $m \to \infty$, from the last expression and (4.14), we obtain

$$0 < K < \lim_{m\to\infty} J_{\lambda,\mu}(w_m, z_m) \leq 0,$$

which is a contradiction. Thus, $(w_0, z_0) \in E\setminus\{(0, 0)\}$. If $Q(w_0, z_0) = 0$, then we use Lemma 3.8 (i) and if $Q(w_0, z_0) > 0$, then we use Lemma 3.8 (ii). In both the cases, there exists a positive real number $t_0^- = t_0^-(w_0, z_0)$ such that $(t_0^- w_0, t_0^- z_0) \in \mathcal{N}^-_{\lambda,\mu}$. Next, we claim that $w_m \to w_0$ strongly in X_0 and $z_m \to z_0$ strongly in X_0 as $m \to \infty$.

Suppose the claim does not hold. Then $t_0^- w_m \not\to t_0^- w_0$ or $t_0^- z_m \not\to t_0^- z_0$ in X_0 as $m \to \infty$. This implies that either

$$\rho_{X_0}(t_0^- w_0) < \liminf_{m\to\infty} \rho_{X_0}(t_0^- w_m) \tag{4.15}$$

or

$$\rho_{X_0}(t_0^- z_0) < \liminf_{m\to\infty} \rho_{X_0}(t_0^- z_m). \tag{4.16}$$

Furthermore, using the same assumption, we can have the following as in Proposition 4.1: either

$$\begin{aligned}&\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{1}{p(x,y)} \frac{|t_0^- w_0(x) - t_0^- w_0(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy\\ &< \liminf_{m\to\infty} \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{1}{p(x,y)} \frac{|t_0^- w_m(x) - t_0^- w_m(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy\\ &\text{or}\\ &\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{1}{p(x,y)} \frac{|t_0^- z_0(x) - t_0^- z_0(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy\\ &< \liminf_{m\to\infty} \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{1}{p(x,y)} \frac{|t_0^- z_m(x) - t_0^- z_m(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy.\end{aligned} \tag{4.17}$$

Note that, using Lemma 2.4 and Lebesgue dominated converges theorem, we deduce that

$$\begin{aligned} &\lim_{m\to\infty}\int_\Omega \frac{1}{q(x)}\Big(\lambda a(x)|t_0^- w_m|^{q(x)}+\mu b(x)|t_0^- z_m|^{q(x)}\Big)dx \\ &=\int_\Omega \frac{1}{q(x)}\Big(\lambda a(x)|t_0^- w_0|^{q(x)}+\mu b(x)|t_0^- z_0|^{q(x)}\Big)dx. \end{aligned} \tag{4.18}$$

Also, by Lemma 5.2 (see Appendix), we have

$$\begin{aligned} &\lim_{m\to\infty}\int_\Omega \frac{1}{\alpha(x)+\beta(x)}c(x)|t_0^- w_m|^{\alpha(x)}|t_0^- z_m|^{\beta(x)}dx \\ &=\int_\Omega \frac{1}{\alpha(x)+\beta(x)}c(x)|t_0^- w_0|^{\alpha(x)}|t_0^- z_0|^{\beta(x)}dx. \end{aligned} \tag{4.19}$$

Thus, combining (3.1), (4.17), (4.18) and (4.19), we obtain

$$\begin{aligned} &\lim_{m\to\infty} J_{\lambda,\mu}(t_0^- w_m, t_0^- z_m) \\ &=\liminf_{m\to\infty}\Big[\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{1}{p(x,y)}\frac{|t_0^- w_m(x)-t_0^- w_m(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy \\ &\quad+\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{1}{p(x,y)}\frac{|t_0^- z_m(x)-t_0^- z_m(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy \\ &\quad-\int_\Omega \frac{1}{q(x)}\Big(\lambda a(x)|t_0^- w_m|^{q(x)}+\mu b(x)|t_0^- z_m|^{q(x)}\Big)dx \\ &\quad-\int_\Omega \frac{1}{\alpha(x)+\beta(x)}c(x)|t_0^- w_m|^{\alpha(x)}|t_0^- z_m|^{\beta(x)}dx\Big] \\ &\geq \liminf_{m\to\infty}\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{1}{p(x,y)}\frac{|t_0^- w_m(x)-t_0^- w_m(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy \\ &\quad+\liminf_{m\to\infty}\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{1}{p(x,y)}\frac{|t_0^- z_m(x)-t_0^- z_m(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy \\ &\quad-\lim_{m\to\infty}\int_\Omega \frac{1}{q(x)}\Big(\lambda a(x)|t_0^- w_m|^{q(x)}+\mu b(x)|t_0^- z_m|^{q(x)}\Big)dx \\ &\quad-\lim_{m\to\infty}\int_\Omega \frac{1}{\alpha(x)+\beta(x)}c(x)|t_0^- w_m|^{\alpha(x)}|t_0^- z_m|^{\beta(x)}dx \\ &> \int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{1}{p(x,y)}\frac{|t_0^- w_0(x)-t_0^- w_0(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy \\ &\quad+\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{1}{p(x,y)}\frac{|t_0^- z_0(x)-t_0^- z_0(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy \\ &\quad-\int_\Omega \frac{1}{q(x)}\Big(\lambda a(x)|t_0^- w_0|^{q(x)}+\mu b(x)|t_0^- z_0|^{q(x)}\Big)dx \\ &\quad-\int_\Omega \frac{1}{\alpha(x)+\beta(x)}c(x)|t_0^- w_0|^{\alpha(x)}|t_0^- z_0|^{\beta(x)}dx \\ &= J_{\lambda,\mu}(t_0^- w_0, t_0^- z_0). \end{aligned} \tag{4.20}$$

Again, using that $w_m \to w_0$ and $z_m \to z_0$ in $L^{q(x)}(\Omega)$, we obtain

$$\lim_{m\to\infty} Q(t_0^- w_m, t_0^- z_m) = Q(t_0^- w_0, t_0^- z_0) \tag{4.21}$$

and Lemma 5.1 (see Appendix) gives us

$$\lim_{m\to\infty} R(t_0^- w_m, t_0^- z_m) = R(t_0^- w_0, t_0^- z_0). \tag{4.22}$$

Considering (3.4), (4.15), (4.21) and (4.22), we obtain

$$\begin{aligned}
&\lim_{m\to\infty} \varphi'_{w_m,z_m}(t_0^-)\\
&= \liminf_{m\to\infty} \Big[\int_{\mathbb{R}^N\times\mathbb{R}^N} (t_0^-)^{p(x,y)-1} \frac{|w_m(x)-w_m(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy\\
&\quad + \int_{\mathbb{R}^N\times\mathbb{R}^N} (t_0^-)^{p(x,y)-1} \frac{|z_m(x)-z_m(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy\\
&\quad - \int_\Omega (t_0^-)^{q(x)-1}\Big(\lambda a(x)|w_m|^{q(x)} + \mu b(x)|z_m|^{q(x)}\Big)dx\\
&\quad - \int_\Omega (t_0^-)^{\alpha(x)+\beta(x)-1} c(x)|w_m|^{\alpha(x)}|z_m|^{\beta(x)}dx\Big]\\
&\ge \frac{1}{t_0^-}\Big[\liminf_{m\to\infty} \rho_{X_0}(t_0^- w_m) + \liminf_{m\to\infty} \rho_{X_0}(t_0^- z_m) - \lim_{m\to\infty} Q(t_0^- w_m, t_0^- z_m)\\
&\quad - \lim_{m\to\infty} R(t_0^- w_m, t_0^- z_m)\Big]\\
&> \frac{1}{t_0^-}\Big[\rho_{X_0}(t_0^- w_0) + \rho_{X_0}(t_0^- z_m) - Q(t_0^- w_0, t_0^- z_0) - R(t_0^- w_0, t_0^- z_0)\Big]\\
&= \varphi'_{u_0,v_0}(t_0^-) = 0.
\end{aligned} \tag{4.23}$$

For m large enough, $\varphi'_{w_m,z_m}(t_0^-) > 0$. Since $(w_m, z_m) \in \mathscr{N}^-_{\lambda,\mu}$ for all $m \in \mathbb{N}$, we have $\varphi'_{w_m,z_m}(1) = 0$ and $\varphi''_{w_m,z_m}(1) < 0$ for all $m \in \mathbb{N}$. Now using the Lemma 3.8, we obtain $\varphi'_{w_m,z_m}(t) < 0$ for all $t > 1$. Then from (4.23), we must have $t_0^- < 1$. Since $(t_0^- w_0, t_0^- z_0) \in \mathscr{N}^-_{\lambda,\mu}$, again using Lemma 3.8, we obtain that 1 is the global maximum point for $\varphi_{w_m,z_m}(t)$. Therefore, from (4.20), it follows that

$$\begin{aligned}
J_{\lambda,\mu}(t_0^- w_0, t_0^- z_0) &< \lim_{m\to\infty} J_{\lambda,\mu}(t_0^- w_m, t_0^- z_m) \le \lim_{m\to\infty} J_{\lambda,\mu}(w_m, z_m)\\
&= \inf_{(u,v)\in\mathscr{N}^-_{\lambda,\mu}} J_{\lambda,\mu}(u,v).
\end{aligned}$$

This is a contradiction to the fact that $(t_0^- w_0, t_0^- z_0) \in \mathscr{N}^-_{\lambda,\mu}$. Hence, $(w_m, z_m) \to (w_0, z_0)$ strongly in E as $m \to \infty$ and $(w_0, z_0) \in \mathscr{N}$. Now using the fact that $\mathscr{N}^0_{\lambda,\mu} = \emptyset$ from Lemma 3.3 and noticing that $J_{\lambda,\mu}(w_0, z_0) = \inf_{(u,v)\in\mathscr{N}^-_{\lambda,\mu}} J_{\lambda,\mu}(u,v) > 0$, we conclude that $(w_0, z_0) \in \mathscr{N}^-_{\lambda,\mu}$.

(ii) Using Lemma 3.2, we infer that (w_0, z_0) is a solution of (1.1). Now we will show that (w_0, z_0) is not semi-trivial, that is, (w_0, z_0) is not of the form $(u, 0)$ (or $(0, v)$). The proof follows adapting the similar approach as in [10]. If $(u, 0)$ (or $(0, v)$) is a semi-trivial solution of (1.1), then from (1.4) we obtain

$$\rho_{X_0}(u) = \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy = \lambda \int_\Omega a(x)|u|^{q(x)}dx.$$

Therefore,

$$J_{\lambda,\mu}(u,0) = \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{1}{p(x,y)} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy - \lambda \int_\Omega \frac{1}{q(x)} a(x)|u|^{q(x)}dx$$

$$\begin{aligned}
&\le \frac{1}{p^-}\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy-\frac{\lambda}{q^+}\int_\Omega a(x)|u|^{q(x)}dx\\
&=\Big(\frac{1}{p^-}-\frac{1}{q^+}\Big)\rho_{X_0}(u)<0,
\end{aligned}$$

since by Lemma 3.7, $J_{\lambda,\mu}(w_0,z_0)>0$, it follows that (w_0,z_0) is not semi-trivial. $\square$

Proof of Theorem 1.2. Define $\Lambda=\delta_0$ (given as in Section 4). Let (u_0,v_0) be obtained as in Proposition 4.1. Now using Lemma 3.5 and the fact that $(u_0,v_0)\in\mathscr{N}^+_{\lambda,\mu}$, for $(|u_0|,|v_0|)\in E\setminus\{(0,0)\}$, we have $Q(|u_0|,|v_0|)=Q(u_0,v_0)>0$, and thus by Lemma 3.8 (ii), there exists $t_1>0$ such that $(t_1|u_0|,t_1|v_0|)\in\mathscr{N}^+_{\lambda,\mu}$. This implies

$$0=\varphi'_{|u_0|,|v_0|}(t_1)\le\varphi'_{u_0,v_0}(t_1). \tag{4.24}$$

Combining (4.24) with the facts that $(u_0,v_0)\in\mathscr{N}^+_{\lambda,\mu}$, $\varphi'_{u_0,v_0}(1)=0$, and again using Lemma 3.8 (ii), we obtain $t_1\ge 1$. This yields

$$J_{\lambda,\mu}(t_1|u_0|,t_1|v_0|)\le J_{\lambda,\mu}(|u_0|,|v_0|)\le J_{\lambda,\mu}(u_0,v_0)=\inf_{(u,v)\in\mathscr{N}^+_{\lambda,\mu}}J_{\lambda,\mu}(u,v).$$

Therefore, there exists a non-negative minimizer for $J_{\lambda,\mu}$ on $\mathscr{N}^+_{\lambda,\mu}$, which is a solution of (1.1) by Lemma 3.2.

Next, we assert that there exists a non-negative minimizer for $J_{\lambda,\mu}(w,z)$ on $\mathscr{N}^-_{\lambda,\mu}$. Indeed, for $(|w_0|,|z_0|)\in E\setminus\{(0,0)\}$, by Lemma 3.8, there exists $t_2>0$ such that $(t_2|w_0|,t_2|z_0|)\in\mathscr{N}^-_{\lambda,\mu}$, where (w_0,z_0) is as in Proposition 4.2. Since $(w_0,z_0)\in\mathscr{N}^-_{\lambda,\mu}$, again by Lemma 3.8, we obtain

$$J_{\lambda,\mu}(t_2|w_0|,t_2|z_0|)\le J_{\lambda,\mu}(t_2w_0,t_2z_0)\le J_{\lambda,\mu}(w_0,z_0)=\inf_{(u,v)\in\mathscr{N}^-_{\lambda,\mu}}J_{\lambda,\mu}(u,v).$$

Thus, we obtain a non-negative minimizer for $J_{\lambda,\mu}$ on $\mathscr{N}^-_{\lambda,\mu}$, which is a solution of (1.1), thanks to Lemma 3.2.

Hence for all $0<\lambda+\mu<\Lambda$, (1.1) admits two non-trivial and non-negative solutions in $\mathscr{N}^+_{\lambda,\mu}$ and $\mathscr{N}^-_{\lambda,\mu}$, respectively. Since $\mathscr{N}^+_{\lambda,\mu}\cap\mathscr{N}^-_{\lambda,\mu}=\emptyset$, these solutions are distinct. This completes the proof. $\square$

5. Appendix

Lemma 5.1. *Let $\{u_m\},\{v_m\}$ be any two bounded sequences in X_0 and c,α,β be as in Theorem 1.2. Then*

$$\lim_{m\to\infty}\int_\Omega c(x)|u_m|^{\alpha(x)}|v_m|^{\beta(x)}dx=\int_\Omega c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx.$$

Proof. Since $\{u_m\}$ and $\{v_m\}$ are bounded sequences in X_0 and X_0 is reflexive, up to subsequences, $u_m\rightharpoonup u$ and $v_m\rightharpoonup v$ weakly in X_0 as $m\to\infty$. First we claim that

$$\begin{aligned}
&\lim_{m\to\infty}\int_\Omega|u_m-u|^{\alpha(x)}|v_m-v|^{\beta(x)}dx\\
&=\lim_{m\to\infty}\int_\Omega|u_m|^{\alpha(x)}|v_m|^{\beta(x)}dx-\int_\Omega|u|^{\alpha(x)}|v|^{\beta(x)}dx
\end{aligned}\tag{5.1}$$

For $t \in (0,1)$, we note that

$$\begin{aligned} &\int_\Omega \int_0^1 \alpha(x)|u_m - tu|^{\alpha(x)-2}(u_m - tu)u|v_m|^{\beta(x)}\,dx\,dt \\ &- \int_\Omega \int_0^1 \beta(x)|u_m - u|^{\alpha(x)}|v_m - tv|^{\beta(x)-2}v(v_m - tv)\,dx\,dt \\ &= \int_\Omega |u_m|^{\alpha(x)}|v_m|^{\beta(x)}dx - \int_\Omega |u_m - u|^{\alpha(x)}|v_m - v|^{\beta(x)}dx. \end{aligned} \tag{5.2}$$

Set

$$\begin{gathered} f_m(x,t) := |u_m - tu|^{\alpha(x)-2}(u_m - tu)|v_m|^{\beta(x)} \\ g_m(x,t) := |u_m - u|^{\alpha(x)}|v_m - tv|^{\beta(x)-2}(v_m - tv). \end{gathered}$$

Now from the given assumptions, we have

$$\begin{gathered} f_m(x,t) \to (1-t)^{\alpha(x)-1}|u|^{\alpha(x)-2}u|v|^{\beta(x)} \quad \text{a.e. in } \mathbb{R}^N \times (0,1) \text{ as } m \to \infty, \\ g_m(x,t) \to 0 \quad \text{a.e. in } \mathbb{R}^N \times (0,1) \text{ as } m \to \infty. \end{gathered} \tag{5.3}$$

Next, using Hölder's inequality and Sobolev-type embedding result (Theorem 2.7), we obtain

$$\begin{aligned} &\int_\Omega \int_0^1 |f_m|^{\frac{\alpha(x)+\beta(x)}{\alpha(x)+\beta(x)-1}}\,dx\,dt \\ &\leq \| \, |u_m - tu|^{\{(\alpha-1)(\frac{\alpha+\beta}{\alpha+\beta-1})\}(\cdot)} \|_{L^{\frac{\alpha(x)+\beta(x)-1}{\alpha(x)-1}}(\Omega\times(0,1))} \\ &\quad \times \| \, |v_m|^{\beta(\cdot)} \|_{L^{\frac{\alpha(x)+\beta(x)-1}{\beta(x)}}(\Omega\times(0,1))} < M_1, \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} &\int_\Omega \int_0^1 |g_m|^{\frac{\alpha(x)+\beta(x)}{\alpha(x)+\beta(x)-1}}\,dx\,dt \\ &\leq \| \, |u_m - u|^{\alpha(\frac{\alpha+\beta}{\alpha+\beta-1})(\cdot)} \|_{L^{\frac{\alpha(x)+\beta(x)-1}{\alpha(x)}}(\Omega\times(0,1))} \\ &\quad \times \| \, |v_m|^{(\beta-1)\frac{\alpha+\beta}{\alpha+\beta-1}(\cdot)} \|_{L^{\frac{\alpha(x)+\beta(x)-1}{\beta(x)-1}}(\Omega\times(0,1))} < M_2, \end{aligned} \tag{5.5}$$

where M_1, M_2 are positive constants independent of m. Hence, the sequences $\{f_m\}$ and $\{g_m\}$ are uniformly bounded in $L^{\frac{\alpha(x)+\beta(x)}{\alpha(x)+\beta(x)-1}}(\Omega \times (0,1))$ and thus we have, up to subsequences,

$$\begin{gathered} f_m \rightharpoonup (1-t)^{\alpha(x)-1}|u|^{\alpha(x)-2}u|v|^{\beta(x)} \quad \text{weakly in } L^{\frac{\alpha(x)+\beta(x)}{\alpha(x)+\beta(x)-1}}(\Omega \times (0,1)), \\ g_m \rightharpoonup 0 \quad \text{weakly in } L^{\frac{\alpha(x)+\beta(x)}{\alpha(x)+\beta(x)-1}}(\Omega \times (0,1)). \end{gathered} \tag{5.6}$$

as $m \to \infty$. Using (5.6), we deduce that

$$\begin{aligned} \lim_{m\to\infty} \int_\Omega \int_0^1 \alpha(x) f_m u dx dt &= \lim_{m\to\infty} \int_\Omega \int_0^1 \alpha(x) f u\,dx\,dt \\ &= \lim_{m\to\infty} \int_\Omega |u|^{\alpha(x)}|v|^{\beta(x)}dx, \end{aligned} \tag{5.7}$$

$$\lim_{m\to\infty} \int_\Omega \int_0^1 \beta(x) g_m v\,dx\,dt = 0. \tag{5.8}$$

Thus, plugging (5.7) and (5.8) into (5.2), we obtain (5.1). Note that, from Theorem 2.7 and Lemma 2.4, we have

$$\int_\Omega |u_m-u|^{\alpha(x)+\beta(x)}dx \to 0 \quad \text{and} \quad \int_\Omega |v_m-v|^{\alpha(x)+\beta(x)}dx \to 0 \quad \text{as } m\to\infty.$$

Now using the above and Young's inequality, we have

$$\begin{aligned} &\int_\Omega |u_m-u|^{\alpha(x)}|v_m-v|^{\beta(x)}dx \\ &\le \int_\Omega \Big\{\frac{\alpha(x)}{\alpha(x)+\beta(x)}|u_m-u|^{\alpha(x)+\beta(x)} + \frac{\alpha(x)}{\alpha(x)+\beta(x)}|v_m-v|^{\alpha(x)+\beta(x)}\Big\}dx \\ &\le \frac{\alpha^+}{\alpha^-+\beta^-}\int_\Omega |u_m-u|^{\alpha(x)+\beta(x)}dx + \frac{\beta^+}{\alpha^-+\beta^-}\int_\Omega |v_m-v|^{\alpha(x)+\beta(x)}dx \\ &\to 0 \quad \text{as } m\to\infty. \end{aligned} \tag{5.9}$$

Thus, inserting (5.9) into (5.1), we obtain

$$\lim_{m\to\infty}\int_\Omega |u_m|^{\alpha(x)}|v_m|^{\beta(x)}dx = \int_\Omega |u|^{\alpha(x)}|v|^{\beta(x)}dx. \tag{5.10}$$

Now

$$\begin{aligned} &\Big|\int_\Omega c(x)|u_m|^{\alpha(x)}|v_m|^{\beta(x)}dx - \int_\Omega c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx\Big| \\ &\le \|c\|_{L^\infty(\Omega)}\int_\Omega \big||u_m|^{\alpha(x)}|v_m|^{\beta(x)} - |u|^{\alpha(x)}|v|^{\beta(x)}\big|dx. \end{aligned} \tag{5.11}$$

We define

$$w_m := |u_m|^{\alpha(x)}|v_m|^{\beta(x)} + |u|^{\alpha(x)}|v|^{\beta(x)} - \big||u_m|^{\alpha(x)}|v_m|^{\beta(x)} - |u|^{\alpha(x)}|v|^{\beta(x)}\big| \ge 0.$$

Since $u_m(x)\to u(x)$ and $v_m(x)\to v(x)$ a.e. in $\mathbb{R}^N$ as $m\to\infty$, we have

$$w_m(x) \to 2|u(x)|^{\alpha(x)}|v(x)|^{\beta(x)} \quad \text{a.e. in } \mathbb{R}^N \text{ as } m\to\infty.$$

Thus, by Fatou's Lemma,

$$\liminf_{m\to\infty}\int_\Omega w_m(x)dx \ge 2\int_\Omega |u|^{\alpha(x)}|v|^{\beta(x)}dx. \tag{5.12}$$

Again from (5.10), we find that

$$\begin{aligned} &\limsup_{m\to\infty}\int_\Omega w_m(x)dx \\ &\le \lim_{m\to\infty}\int_\Omega |u_m|^{\alpha(x)}|v_m|^{\beta(x)}dx + \lim_{m\to\infty}\int_\Omega |u|^{\alpha(x)}|v|^{\beta(x)}dx \\ &\quad - \limsup_{m\to\infty}\int_\Omega \big||u_m|^{\alpha(x)}|v_m|^{\beta(x)}dx - |u|^{\alpha(x)}|v|^{\beta(x)}\big|dx \\ &= 2\int_\Omega |u|^{\alpha(x)}|v|^{\beta(x)}dx - \limsup_{m\to\infty}\int_\Omega \big||u_m|^{\alpha(x)}|v_m|^{\beta(x)} - |u|^{\alpha(x)}|v|^{\beta(x)}\big|dx \end{aligned} \tag{5.13}$$

Combining (5.12) and (5.13), we have

$$\limsup_{m\to\infty}\int_\Omega \big||u_m|^{\alpha(x)}|v_m|^{\beta(x)} - |u|^{\alpha(x)}|v|^{\beta(x)}\big|dx \le 0,$$

that is,

$$\lim_{m\to\infty}\int_\Omega \big||u_m|^{\alpha(x)}|v_m|^{\beta(x)} - |u|^{\alpha(x)}|v|^{\beta(x)}\big|dx = 0.$$

Thus, combining the above together with (5.11), we obtain our final result. □

The proof of the next lemma is similarly to the one for Lemma 5.1, using that $\alpha, \beta \in C_+(\overline{\Omega})$.

Lemma 5.2. *Let $\{u_m\}, \{v_m\}$ be any two bounded sequences in X_0 and c, α, β be as in Theorem 1.2. Then*

$$\lim_{m\to\infty} \int_\Omega \frac{1}{\alpha(x)+\beta(x)} c(x)|u_m|^{\alpha(x)}|v_m|^{\beta(x)}dx = \int_\Omega \frac{1}{\alpha(x)+\beta(x)} c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx.$$

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