

IMPROVED BLOWUP TIME ESTIMATES FOR FOURTH-ORDER DAMPED WAVE EQUATIONS WITH STRAIN TERM AND ARBITRARY POSITIVE INITIAL ENERGY

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ABSTRACT. We propose a new differential inequality that improve the upper bound of the blowup time estimate for nonlinear fourth-order damped wave equations with strain term and arbitrary positive initial energy. We also give two new initial conditions to expand the range of the initial data leading to the finite time blowup of solutions. We obtain a sharp result of finite time blowup for the special case of the new differential inequality. We illustrate our results with some simulations.

1. INTRODUCTION

We consider the initial boundary value problem (IBVP) for the nonlinear fourth-order damped wave equation with strain term

$$u_{tt} + \Delta^2 u + \sum_{i=1}^n \frac{\partial}{\partial x_i} \sigma_i(u_{x_i}) + mu_t = 0, \quad \text{in } (x, t) \in \Omega \times (0, T), \quad (1.1)$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad \text{in } x \in \Omega, \quad (1.2)$$

$$u = \frac{\partial u}{\partial \nu} = 0 \quad \text{or} \quad u = \Delta u = 0, \quad \text{on } (x, t) \in \partial\Omega \times (0, T), \quad (1.3)$$

where $m \geq 0$, $u_0(x)$ and $u_1(x)$ are the initial data, Ω is a bounded domain in \mathbb{R}^n ($n \geq 1$) with smooth boundary, T is the maximum existence time of the solution, and $\sigma_i(s)$ satisfies

- (H1) (i) $\sigma_i(s) \in C^1$ and $\sigma_i(0) = \sigma'_i(0) = 0$;
(ii) $\sigma_i(s)$ is monotone for $-\infty < s < \infty$ and is convex when for $s > 0$, and concave for $s < 0$;
(iii) $(p+1)G_i(s) \leq s\sigma_i(s)$ and $|\sigma_i(s)| \leq a|s|^p$ for some $a > 0$, with $1 < p < \infty$ if $n \leq 2$, and $1 < p < \frac{n}{n-2}$ if $n \geq 3$;
(iv) $G_i(s) = \int_0^s \sigma_i(v) dv$, $1 \leq i \leq n$.

Model (1.1) was introduced to describe the longitudinal motion of an elastoplastic bar [1], and then this model with the corresponding conditions attracted a lot of mathematicians' attention. Initial boundary-value problem (1.1)-(1.3) was considered in [3, 5, 14] by the potential well method proposed in [10, 11]. The

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finite time blowup of solutions with negative initial energy $E(0) < 0$ was treated by Yang in [14]. Liu and Xu obtained global solutions, and finite time blowup of solutions for positive initial energy $E(0) \leq d$ (d denotes the mountain pass level) in [5]. Han et al. [3] showed the finite time blowup of solutions for arbitrarily positive initial energy $E(0) > 0$. Relevant problems with damping and source terms were considered in [2, 6, 9, 15]. Some representative works [8, 12, 13] studied the control mechanism of the initial data on the dynamic behavior of solutions by the potential well method.

In studying the relationship between the dynamic behaviors of solutions and initial data in the framework of the potential well method, one of the central problems is investigating the finite time blowup of solutions. We often expect to utilize some differential inequalities as tools to prove the finite blowup of solutions and estimate blowup time. Korpusov proposed a differential inequality in [4] to estimate the upper bound of blowup time of solutions to the IBVP for a generalized dissipative high-order equation of Klein-Gordon type. This inequality was applied by Lin and Luo in [7] to do the same estimate for problem (1.1)-(1.3). As we know, the smaller the upper bound of blowup time we obtain, the better the estimate of blowup time is. In this paper, we propose a new differential inequality to improve that in [4] and obtain a better upper bound of blowup time. We also try to expand the range of the initial data leading to finite time blowup by giving two different sets of initial data. In a special case, we obtain a sharp result for finite time blowup to the new differential inequality. Some simulations are also conducted to verify the main results of the present paper.

The rest of this paper is organized as follows: Section 2 presents and proves theorems about the new differential inequality. Section 3 uses the theorems in Section 2 to prove the finite time blowup and improve the upper bound estimate of the blowup time of solutions.

2. IMPROVED DIFFERENTIAL INEQUALITY AND NEW INITIAL CONDITIONS

In this section, we present an improved differential inequality and some new initial conditions, which will be used in Section 3 to prove the finite time blowup and improve the upper bound estimate of the blowup time of solutions.

Theorem 2.1 (Improved differential inequality). *Suppose that $\Phi(t) \in \mathbb{C}^2$ satisfies*

$$\Phi\Phi'' - \alpha(\Phi')^2 + \gamma\Phi\Phi' + \beta\Phi \geq \mu\Phi^2, \quad \alpha > 1, \beta, \gamma, \mu \geq 0; \quad (2.1)$$

$$\Phi'(0) > \frac{\Phi(0)\theta}{\alpha - 1} > 0, \quad (2.2)$$

where

$$\theta = \frac{1}{2} \left(\gamma + \sqrt{\max\{\gamma^2 + 4(\alpha - 1)(\beta\Phi^{-1}(0) - \mu), 0\}} \right). \quad (2.3)$$

Then $\Phi(t)$ blows up in finite time T , where

$$T < \begin{cases} \frac{1}{2\theta - \gamma} \ln \left(\frac{(2\theta - \gamma)\Phi(0)}{(\alpha - 1)\Phi'(0) - \theta\Phi(0)} + 1 \right) & \text{if } 2\theta > \gamma, \\ \frac{\Phi(0)}{(\alpha - 1)\Phi'(0) - \gamma\Phi(0)/2} & \text{if } 2\theta \leq \gamma. \end{cases} \quad (2.4)$$

Proof. Dividing both sides of (2.1) by $\Phi^{1+\alpha}$ yields

$$\left(\frac{\Phi'}{\Phi^\alpha} \right)' + \gamma \frac{\Phi'}{\Phi^\alpha} + \frac{\beta}{\Phi^\alpha} \geq \frac{\mu}{\Phi^{\alpha-1}},$$

that is

$$\frac{1}{1-\alpha} (\Phi^{1-\alpha})'' + \frac{\gamma}{1-\alpha} (\Phi^{1-\alpha})' + \beta\Phi^{-\alpha} - \mu\Phi^{1-\alpha} \geq 0.$$

Let $z(t) = \Phi^{1-\alpha}(t)$. Then

$$z''(t) + \gamma z'(t) - \beta(\alpha-1)z^{\alpha_1}(t) + \mu(\alpha-1)z(t) \leq 0, \quad \alpha_1 = \frac{\alpha}{\alpha-1}. \quad (2.5)$$

We first assume $\gamma^2 + 4(\alpha-1)(\beta\Phi^{-1}(0) - \mu) > 0$. Let $y(t) = e^{\theta t}z(t)$, where θ is defined in (2.3). It is easy to see that $2\theta > \gamma$. Then

$$\begin{aligned} z'(t) &= (e^{-\theta t}y(t))' = -\theta e^{-\theta t}y(t) + e^{-\theta t}y'(t), \\ z''(t) &= e^{-\theta t}y''(t) - 2\theta e^{-\theta t}y'(t) + \theta^2 e^{-\theta t}y(t), \\ z''(t) + \gamma z'(t) - \beta(\alpha-1)z^{\alpha_1}(t) + \mu(\alpha-1)z(t) \\ &= e^{-\theta t} \left(y''(t) - 2\theta y'(t) + \theta^2 y(t) + \gamma y'(t) - \theta \gamma y(t) \right. \\ &\quad \left. - \beta(\alpha-1)e^{\theta(1-\alpha_1)t}y^{\alpha_1}(t) + \mu(\alpha-1)y(t) \right) \leq 0, \end{aligned}$$

i.e.,

$$y''(t) \leq (2\theta - \gamma)y'(t) - y(t) \left(\theta(\theta - \gamma) + \mu(\alpha-1) - \beta(\alpha-1)e^{-\frac{\theta t}{\alpha-1}}y^{\frac{1}{\alpha-1}}(t) \right). \quad (2.6)$$

Note that

$$\begin{aligned} y'(t) &= \theta e^{\theta t}z(t) + e^{\theta t}z'(t) \\ &= e^{\theta t} (\theta\Phi^{1-\alpha}(t) + (1-\alpha)\Phi^{-\alpha}(t)\Phi'(t)) \\ &= e^{\theta t}\Phi^{-\alpha}(t) (\theta\Phi(t) - (\alpha-1)\Phi'(t)). \end{aligned}$$

By (2.2), we have

$$y'(0) = \Phi^{-\alpha}(0) (\theta\Phi(0) - (\alpha-1)\Phi'(0)) < 0. \quad (2.7)$$

We claim that $y'(t) < 0$ for all $t < T$. Arguing by contradiction, we suppose that there exists a $t_0 \in (0, T)$, such that

$$y'(t_0) = 0, \quad y'(t) < 0, \quad 0 < y(t) < y(0) \quad (2.8)$$

for $t \in [0, t_0)$. In fact, $y(0) = \Phi^{1-\alpha}(0) > 0$. By the definition of the maximum existence time T and the continuity of $y(t)$ in t , the sign of $y(t)$ cannot change over $[0, T)$, because $y(t^*) = 0$ means that the solution $\Phi(t)$ blows up at the finite time t^* . Hence we have $y(t) > 0$ over $[0, T)$. Integrating (2.6) from 0 to t , we find

$$\begin{aligned} y'(t) &\leq y'(0) + \int_0^t \left((2\theta - \gamma)y'(\tau) \right. \\ &\quad \left. - y(\tau) \left(\theta(\theta - \gamma) + \mu(\alpha-1) - \beta(\alpha-1)e^{-\frac{\theta\tau}{\alpha-1}}y^{\frac{1}{\alpha-1}}(\tau) \right) \right) d\tau. \end{aligned} \quad (2.9)$$

Now

$$\begin{aligned} &\theta(\theta - \gamma) + \mu(\alpha-1) - \beta(\alpha-1)e^{-\frac{\theta\tau}{\alpha-1}}y^{\frac{1}{\alpha-1}}(\tau) \\ &\geq \theta(\theta - \gamma) + \mu(\alpha-1) - \beta(\alpha-1)y^{\frac{1}{\alpha-1}}(0) \\ &= \theta(\theta - \gamma) + \mu(\alpha-1) - \beta(\alpha-1)\Phi^{-1}(0). \end{aligned} \quad (2.10)$$

It follows from (2.3) that

$$\begin{aligned}\theta(\theta - \gamma) &= \left(\frac{1}{2}\sqrt{\gamma^2 + 4(\alpha - 1)(\beta\Phi^{-1}(0) - \mu)} + \frac{\gamma}{2}\right) \\ &\quad \times \left(\frac{1}{2}\sqrt{\gamma^2 + 4(\alpha - 1)(\beta\Phi^{-1}(0) - \mu)} - \frac{\gamma}{2}\right) \\ &= (\alpha - 1)(\beta\Phi^{-1}(0) - \mu).\end{aligned}\quad (2.11)$$

Combining (2.10) and (2.11), we obtain

$$\theta(\theta - \gamma) + \mu(\alpha - 1) - \beta(\alpha - 1)e^{-\frac{\theta t}{\alpha-1}}y^{\frac{1}{\alpha-1}}(t) \geq 0 \quad \text{for } t \leq t_0. \quad (2.12)$$

From (2.9), we have

$$y'(t) \leq y'(0) < 0, \quad t \in [0, t_0],$$

which is a contradiction with (2.8). Hence $y'(t) \leq y'(0)$ for all $t < T$ and

$$y(t) \leq y(0) + y'(0)t, \quad (2.13)$$

which implies that $\Phi(t)$ blows up in finite time.

To estimate the blowup time T , we use (2.9) and (2.12) to obtain

$$y'(t) \leq y'(0) + \int_0^t (2\theta - \gamma)y'(\tau) d\tau = y'(0) + (2\theta - \gamma)(y(t) - y(0)). \quad (2.14)$$

For convenience, let $\sigma = -y'(0) > 0$. Rewrite (2.14) as

$$\frac{d}{dt} \left(e^{-(2\theta-\gamma)t} y(t) \right) \leq -(\sigma + (2\theta - \gamma)y(0)) e^{-(2\theta-\gamma)t}. \quad (2.15)$$

Integrating (2.15) from 0 to t , we obtain

$$\begin{aligned}e^{-(2\theta-\gamma)t} y(t) &\leq y(0) - \frac{\sigma + (2\theta - \gamma)y(0)}{2\theta - \gamma} \left(1 - e^{-(2\theta-\gamma)t}\right) \\ &= y(0)e^{-(2\theta-\gamma)t} - \frac{\sigma}{2\theta - \gamma} \left(1 - e^{-(2\theta-\gamma)t}\right),\end{aligned}$$

i.e.,

$$0 < y(t) \leq y(0) - \frac{\sigma}{2\theta - \gamma} \left(e^{(2\theta-\gamma)t} - 1 \right), \quad t \in [0, T].$$

So we obtain

$$\frac{(2\theta - \gamma)}{\sigma} y(0) + 1 > e^{(2\theta-\gamma)t}, \quad t \in [0, T].$$

Hence

$$T < \frac{1}{2\theta - \gamma} \ln \left(\frac{2\theta - \gamma}{\sigma\Phi^{\alpha-1}(0)} + 1 \right) = \frac{1}{2\theta - \gamma} \ln \left(\frac{(2\theta - \gamma)\Phi(0)}{(\alpha - 1)\Phi'(0) - \theta\Phi(0)} + 1 \right).$$

If $\gamma^2 + 4(\alpha - 1)(\beta\Phi^{-1}(0) - \mu) \leq 0$, then, setting $\theta = \gamma/2$ in (2.6), we have

$$\begin{aligned}y''(t) &\leq -y(t) \left(\frac{-\gamma}{4} + \mu(\alpha - 1) - \beta(\alpha - 1)e^{-\frac{\theta t}{\alpha-1}}y^{\frac{1}{\alpha-1}}(t) \right) \\ &< -y(t) \left(-\frac{\gamma}{4} + (\alpha - 1)(\mu - \beta y^{\frac{1}{\alpha-1}}(0)) \right) \leq 0,\end{aligned}$$

which implies that $y(t) < y(0) + y'(0)t$. Hence, $\Phi(t)$ blows up in finite time and $T < y(0)/|y'(0)|$. \square

Remark 2.2. It is easy to verify that for $\mu = 0$,

$$\Phi'(0) > \frac{\gamma}{\alpha - 1}\Phi(0) + \sqrt{\frac{2\beta}{2\alpha - 1}\Phi(0)} > \frac{\Phi(0)}{2(\alpha - 1)} \left(\gamma + \sqrt{\gamma^2 + 4\beta(\alpha - 1)\Phi^{-1}(0)} \right),$$

while $\Phi'(0) > \frac{\gamma}{\alpha - 1}\Phi(0) + \sqrt{\frac{2\beta}{2\alpha - 1}\Phi(0)}$ is required in [4]. Hence in Theorem 2.1, this condition is relaxed to include a large range of initial data.

Remark 2.3. We consider some specific examples to show that by applying the improved differential inequality (Theorem 2.1), we can obtain some better estimates of the blowup time in some cases than the results derived by the method proposed in [4]. We also show some numerical simulations in Figure 1.

Let $\alpha = 2, \beta = \gamma = 1, \Phi(0) = 2$, and denote T^* as the upper bound of blowup time and consider the following three cases:

- (i) If $\Phi'(0) = 2.74$ and $\mu = 0$, then (2.2) is satisfied, but the conditions of [4] are not satisfied. So we can get $T^* \approx 3.51$ by the improved differential inequality, and $T^* \approx 1.71$ by numerical computations when (2.1) is equality (Figure 1, (a), the dotted line).
- (ii) If $\Phi'(0) = 3.2$ and $\mu = 0$, then both (2.2) and the conditions of [4] are satisfied. We can obtain $T^* \approx 6.12$ based on [4], $T^* \approx 1.23$ by the improved differential inequality, and $T^* \approx 1.11$ by numerical computations when (2.1) is equality (Figure 1, (a), the solid line).
- (iii) If we choose $\Phi'(0) = 2.4$ and $\mu = 0.3$, then (2.2) is satisfied and the solution blows up with blowup time $T^* < 2.37$ based on (2.4). The numerical blowup time $T^* \approx 1.8$ when (2.1) is equality (Figure 1, (b), the left-hand solid line). Also, numerical computations show that the solution will blow up when $\Phi'(0) = 1.1$ (Figure 1, (b), the dotted line), but the solution won't blow up when $\Phi'(0) = 1.0$ (2.4) (Figure 1, (b), the right-hand solid line).

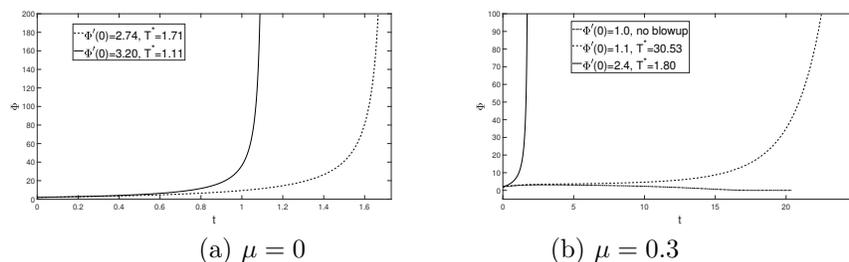


FIGURE 1. Blowup solutions for $\Phi\Phi'' - \alpha(\Phi')^2 + \gamma\Phi\Phi' + \beta\Phi = \mu\Phi^2$

Next, we present a theorem that allows us to estimate the upper bound of blowup time for some new sets of initial data not included in Theorem 2.1.

Theorem 2.4 (New initial condition I). *Suppose that $\Phi(t) \in \mathbb{C}^2$ satisfies (2.1) with $\mu = 0$ and*

$$\Phi'(0) > \frac{\gamma\Phi(0)}{\alpha - 1} + \frac{\beta(\alpha - 1)}{\gamma\alpha} > 0. \tag{2.16}$$

Then $\Phi(t)$ blows up in finite time $T \leq \Gamma$, where Γ is the unique solution of

$$\Phi(0) = \frac{\alpha - 1}{\alpha\gamma^2} \left((\alpha - 1)^2 \beta \left(1 - e^{-\frac{\gamma\Gamma}{\alpha-1}} \right) + \omega(e^{\gamma\Gamma} - 1) \right), \quad (2.17)$$

and

$$\omega = \alpha\gamma \left(\Phi'(0) - \frac{\gamma\Phi(0)}{\alpha - 1} - \frac{\beta(\alpha - 1)}{\alpha\gamma} \right). \quad (2.18)$$

Proof. Let $\theta = \gamma$ and $\mu = 0$ in (2.6). Then (2.6) becomes

$$y''(t) \leq \gamma y'(t) + \beta(\alpha - 1)e^{-\frac{\gamma t}{\alpha-1}} y^{\alpha_1}(t), \quad \alpha_1 = \frac{\alpha}{\alpha - 1}. \quad (2.19)$$

Multiplying (2.19) on both sides by $e^{\frac{\gamma t}{\alpha-1}}$ yields

$$e^{\frac{\gamma t}{\alpha-1}} y''(t) \leq \gamma y'(t) e^{\frac{\gamma t}{\alpha-1}} + \beta(\alpha - 1) y^{\alpha_1}(t),$$

i.e.,

$$\frac{d}{dt} \left(e^{\frac{\gamma t}{\alpha-1}} y'(t) \right) \leq \left(\gamma + \frac{\gamma}{\alpha - 1} \right) y'(t) e^{\frac{\gamma t}{\alpha-1}} + (\alpha - 1) \beta y^{\alpha_1}(t). \quad (2.20)$$

Integrating (2.20) from 0 to t , we obtain

$$e^{\frac{\gamma t}{\alpha-1}} y'(t) \leq y'(0) + \int_0^t \left(\frac{\alpha\gamma}{\alpha - 1} y'(\tau) e^{\frac{\gamma \tau}{\alpha-1}} + (\alpha - 1) \beta y^{\alpha_1}(\tau) \right) d\tau. \quad (2.21)$$

Note that, from (2.16) and (2.18),

$$\begin{aligned} & \frac{\alpha\gamma}{\alpha - 1} y'(0) + (\alpha - 1) \beta y^{\alpha_1}(0) \\ &= \frac{\alpha\gamma}{\alpha - 1} \Phi^{-\alpha}(0) \left(\gamma\Phi(0) - (\alpha - 1)\Phi'(0) \right) + (\alpha - 1) \beta \Phi^{-\alpha}(0) \\ &= -\alpha\gamma \Phi^{-\alpha}(0) \left(\Phi'(0) - \frac{\gamma\Phi(0)}{\alpha - 1} - \frac{\beta(\alpha - 1)}{\alpha\gamma} \right) \\ &= -\omega \Phi^{-\alpha}(0) < 0. \end{aligned} \quad (2.22)$$

In particular, $y'(0) < 0$. By the continuity of $y'(t)$, there exist a $t_1 > 0$ such that

$$\frac{\alpha\gamma}{\alpha - 1} y'(t) e^{\frac{\gamma t}{\alpha-1}} + (\alpha - 1) \beta y^{\alpha_1}(t) \leq 0, \quad (2.23)$$

for $0 < t \leq t_1$, which implies that

$$e^{\frac{\gamma t}{\alpha-1}} y'(t) \leq y'(0), \quad y'(t) < 0 \quad \text{and} \quad 0 < y(t) < y(0) \quad (2.24)$$

for $0 < t \leq t_1$. Substituting the first and third inequalities of (2.24) into (2.21) and using (2.22), we find

$$\begin{aligned} e^{\frac{\gamma t}{\alpha-1}} y'(t) &\leq y'(0) + \int_0^t \left(\frac{\alpha\gamma}{\alpha - 1} y'(0) + (\alpha - 1) \beta y^{\alpha_1}(0) \right) d\tau \\ &\leq y'(0) - \omega_1 t, \end{aligned} \quad (2.25)$$

where $\omega_1 = \omega \Phi^{-\alpha}(0)$. From (2.25), we can see that (2.23) is a strict inequality, that is $t_1 = T$. Substituting (2.25) into (2.21) and using (2.22), we have

$$\begin{aligned} e^{\frac{\gamma t}{\alpha-1}} y'(t) &\leq y'(0) + \int_0^t \left(\frac{\alpha\gamma}{\alpha - 1} (y'(0) - \omega_1 \tau) + (\alpha - 1) \beta y^{\alpha_1}(0) \right) d\tau \\ &\leq y'(0) - \omega_1 t - \frac{\alpha\gamma\omega_1}{\alpha - 1} \frac{t^2}{2!}. \end{aligned}$$

After n steps we obtain

$$e^{\frac{\gamma t}{\alpha-1}} y'(t) \leq y'(0) - \omega_1 \sum_{k=1}^n \left(\frac{\alpha\gamma}{\alpha-1} \right)^{k-1} \frac{t^k}{k!}. \quad (2.26)$$

Then, substituting (2.26) into (2.21), we deduce

$$\begin{aligned} & e^{\frac{\gamma t}{\alpha-1}} y'(t) \\ & \leq y'(0) + \int_0^t \left(\frac{\alpha\gamma}{\alpha-1} \left(y'(0) - \omega_1 \sum_{k=1}^n \left(\frac{\alpha\gamma}{\alpha-1} \right)^{k-1} \frac{\tau^k}{k!} \right) + (\alpha-1)\beta y^{\alpha_1}(0) \right) d\tau \\ & \leq y'(0) - \omega_1 t - \omega_1 \sum_{k=1}^n \left(\frac{\alpha\gamma}{\alpha-1} \right)^k \frac{t^{k+1}}{(k+1)!} \\ & = y'(0) - \omega_1 \sum_{k=1}^{n+1} \left(\frac{\alpha\gamma}{\alpha-1} \right)^{k-1} \frac{t^k}{k!}. \end{aligned}$$

By mathematical induction, (2.26) is satisfied for all $n \geq 2$. Let $n \rightarrow \infty$, we obtain

$$e^{\frac{\gamma t}{\alpha-1}} y'(t) \leq y'(0) - \frac{(\alpha-1)\omega_1}{\alpha\gamma} \left(e^{\frac{\alpha\gamma t}{\alpha-1}} - 1 \right), \quad (2.27)$$

that is

$$y'(t) \leq \left(y'(0) + \frac{(\alpha-1)\omega_1}{\alpha\gamma} \right) e^{-\frac{\gamma t}{\alpha-1}} - \frac{(\alpha-1)\omega_1}{\alpha\gamma} e^{\gamma t}. \quad (2.28)$$

Note that from (2.22) we have

$$y'(0) + \frac{(\alpha-1)\omega_1}{\alpha\gamma} = -\frac{(\alpha-1)^2}{\alpha\gamma} \beta y^{\alpha_1}(0). \quad (2.29)$$

Integrating (2.28) from 0 to t , we obtain

$$y(t) \leq y(0) - \frac{(\alpha-1)^3}{\alpha\gamma^2} \beta y^{\alpha_1}(0) (1 - e^{-\frac{\gamma t}{\alpha-1}}) - \frac{(\alpha-1)\omega_1}{\alpha\gamma^2} (e^{\gamma t} - 1).$$

Since $y(t) > 0$, we obtain

$$0 < \Phi^{1-\alpha}(0) - \frac{(\alpha-1)^3\beta}{\alpha\gamma^2} \Phi^{-\alpha}(0) (1 - e^{-\frac{\gamma t}{\alpha-1}}) - \frac{(\alpha-1)\omega}{\alpha\gamma^2} \Phi^{-\alpha}(0) (e^{\gamma t} - 1).$$

Hence $t < \Gamma$, where Γ is the unique solution of (2.17). \square

Remark 2.5. If $\Phi(0)$ is small, condition (2.2) is better than condition (2.16). If $\Phi(0)$ is not small, condition (2.16) is better than condition (2.2). Let us use numerical simulations to show the above conclusion. Choose $\alpha = 2$, $\beta = \gamma = 1$, $\mu = 0$, and denote T^* as the upper bound of blowup time.

- (i) Small initial data case. If $\Phi(0) = 0.4$, $\Phi'(0) = 0.88$, then (2.2) is satisfied ((2.16) is not satisfied), so the solution blows up. By (2.4), we can obtain $T^* \approx 1.62$, and the numerical $T^* \approx 0.82$.
- (ii) Large initial data case. If $\Phi(0) = 2$, $\Phi'(0) = 2.52$, then (2.16) is satisfied ((2.2) is not satisfied), so the solution blows up. We can obtain $\Gamma \approx 4.34$ by numerical method, and the numerical blowup time $T^* \approx 2.72$ when (2.1) is equality.

Next, we show another theorem to give some new sets of initial data, whose result is sharp based on numerical computations if μ is not small.

Theorem 2.6 (New initial condition II). *Suppose that $\Phi(t) \in \mathbb{C}^2$ satisfies (2.1) and*

$$\Phi'(0) > 0, \quad \mu > \max \left\{ \beta \Phi^{-1}(0), \frac{4\theta^2 + \gamma^2(\alpha - 1)^2}{2(\alpha + 1)(\alpha - 1)^2} \right\}, \quad (2.30)$$

where θ is defined in (2.3). Then $\Phi(t)$ blows up in finite time.

Proof. Multiplying (2.5) on both sides by $e^{\gamma t}$ yields

$$\frac{d}{dt} (e^{\gamma t} z'(t)) \leq (\alpha - 1) (\beta z^{\alpha_1}(t) - \mu z(t)) e^{\gamma t}. \quad (2.31)$$

Integrating (2.31) from 0 to t , we obtain

$$z'(t) \leq e^{-\gamma t} z'(0) + \int_0^t (\alpha - 1) z(\tau) (\beta z^{1/(\alpha-1)}(\tau) - \mu) e^{\gamma(\tau-t)} d\tau. \quad (2.32)$$

Since $z'(t) = (1 - \alpha)\Phi^{-\alpha}(t)\Phi'(t)$ and $z'(0) = (1 - \alpha)\Phi^{-\alpha}(0)\Phi'(0) < 0$, we have $z'(\tau) < 0$ or $z(\tau) < z(0)$ for small $t > 0$. Then

$$(\alpha - 1)z(\tau)(\beta z^{\alpha_1}(\tau) - \mu)e^{\gamma(\tau-t)} < (\alpha - 1)z(\tau)(\beta z^{1/(\alpha-1)}(0) - \mu)e^{\gamma(\tau-t)} < 0, \quad (2.33)$$

which implies that $z'(t) \leq e^{-\gamma t} z'(0)$ from (2.32) for all t such that the solution exists (also (2.33) is satisfied for such t).

To prove that $\Phi(t)$ blows up in finite time, we use a proof by contradiction. Assume that the solution exists for all $t > 0$. From $z'(t) < 0$, we obtain $\Phi'(t) > 0$ for all $t > 0$. We first claim that $\lim_{t \rightarrow \infty} \Phi(t) = \infty$. Otherwise, if $\lim_{t \rightarrow \infty} \Phi(t) = a < \infty$, then by monotonicity, $\lim_{t \rightarrow \infty} \Phi'(t) = \lim_{t \rightarrow \infty} \Phi''(t) = 0$. Taking limits in (2.1), we obtain $\beta a - \mu a^2 \geq 0$. Since $a > \Phi(0)$, we obtain a contradiction against (2.30). Thus, $\lim_{t \rightarrow \infty} \Phi(t) = \infty$.

Now, we discuss two cases:

- (i) There exist a $t_2 > 0$, such that $\Phi'(t_2) > \theta\Phi(t_2)/(\alpha - 1)$;
- (ii) $\Phi'(t) \leq \theta\Phi(t)/(\alpha - 1)$ for all $t > 0$.

In case (i), we can use Theorem 2.1 with $\Phi(0)$ replaced by $\Phi(t_2)$ to show that $\Phi(t)$ blows up in finite time. In case (ii), we rewrite (2.1) as

$$\Phi\Phi'' + \gamma\Phi\Phi' + \Phi'^2 \geq (\alpha + 1)\Phi'^2 + \mu\Phi^2 - \beta\Phi. \quad (2.34)$$

We denote

$$M = \frac{\gamma^2(\alpha - 1)^2(\alpha + 1)}{4\theta^2 + \gamma^2(\alpha - 1)^2}. \quad (2.35)$$

Then

$$\alpha + 1 - M = \frac{4(\alpha + 1)\theta^2}{4\theta^2 + \gamma^2(\alpha - 1)^2} > 0, \quad (2.36)$$

and by Cauchy inequality, we obtain

$$0 < \gamma\Phi\Phi' = 2\sqrt{M}\Phi' \frac{\gamma}{2\sqrt{M}}\Phi \leq M\Phi'^2 + \frac{\gamma^2}{4M}\Phi^2. \quad (2.37)$$

Substituting (2.37) into (2.34) yields

$$\frac{d}{dt}(\Phi\Phi') = \Phi\Phi'' + \Phi'^2 \geq (\alpha + 1 - M)\Phi'^2 + \left(\mu - \frac{\gamma^2}{4M}\right)\Phi^2 - \beta\Phi. \quad (2.38)$$

By the second inequality of (2.30), we can find $\varepsilon > 0$, such that

$$\mu - \frac{4\theta^2 + \gamma^2(\alpha - 1)^2}{2(\alpha + 1)(\alpha - 1)^2} - 2\varepsilon = 0. \quad (2.39)$$

Since $\lim_{t \rightarrow \infty} \Phi(t) = \infty$, we can find a $t_3 > t_2$ such that

$$\varepsilon \Phi^2(t) - \beta \Phi(t) \geq 0 \quad \text{for } t \geq t_3.$$

Integrating (2.38) from t_3 to t , we find

$$\Phi \Phi' \geq \Phi(t_3) \Phi'(t_3) + \int_{t_3}^t \left((\alpha + 1 - M) \Phi'^2(\tau) + \left(\mu - \frac{\gamma^2}{4M} - \varepsilon \right) \Phi^2(\tau) \right) d\tau.$$

In case (ii), we have

$$\frac{\theta}{\alpha - 1} \Phi^2 \geq \Phi \Phi'.$$

Let

$$\begin{aligned} f(t) &= \frac{\theta}{\alpha - 1} \Phi^2(t) - \Phi(t_3) \Phi'(t_3) \\ &\quad - \int_{t_3}^t \left((\alpha + 1 - M) \Phi'^2(\tau) + \left(\mu - \frac{\gamma^2}{4M} - \varepsilon \right) \Phi^2(\tau) \right) d\tau. \end{aligned}$$

Then $f(t) \geq 0$ for $t \geq t_3$, and

$$f'(t) = \frac{2\theta}{\alpha - 1} \Phi \Phi' - (\alpha + 1 - M) \Phi'^2 - \left(\mu - \frac{\gamma^2}{4M} - \varepsilon \right) \Phi^2. \quad (2.40)$$

Using Cauchy inequality again, we obtain

$$\begin{aligned} \frac{2\theta}{\alpha - 1} \Phi \Phi' &= 2\sqrt{\alpha + 1 - M} \Phi' \frac{\theta}{(\alpha - 1)\sqrt{\alpha + 1 - M}} \Phi \\ &\leq (\alpha + 1 - M) \Phi'^2 + \frac{\theta^2}{(\alpha - 1)^2(\alpha + 1 - M)} \Phi^2. \end{aligned} \quad (2.41)$$

Note that from (2.36) and (2.35) we have

$$\frac{\theta^2}{(\alpha - 1)^2(\alpha + 1 - M)} = \frac{4\theta^2 + \gamma^2(\alpha - 1)^2}{4(\alpha + 1)(\alpha - 1)^2} = \frac{\gamma^2}{4M},$$

which combined with (2.39) yields

$$\mu - \frac{2\gamma^2}{4M} - \varepsilon = \varepsilon.$$

Substituting (2.41) into (2.40) yields

$$f'(t) \leq -\varepsilon \Phi^2,$$

which implies that $f(t) \geq 0$ is not satisfied for all $t > t_3$. This is a contradiction. Hence $\Phi(t)$ blows up in a finite time. \square

Remark 2.7. Note that $\Phi(t) \equiv \beta/\mu$ is a constant solution to (2.1). Under the condition $\mu > (4\theta^2 + \gamma^2(\alpha - 1)^2)/(2(\alpha - 1)^2(\alpha + 1))$, if $\Phi'(0) > 0$ and $\Phi(0) > \beta/\mu$, then the solution blows up in finite time. By the uniqueness, the solution of (2.1) satisfies $\Phi(t) < \beta/\mu$ if $\Phi'(0) < 0$ and $\Phi(0) < \beta/\mu$ when (2.1) is equality. Hence the result of Theorem 2.6 is sharp. To simulate this result, we choose $\alpha = 3$, $\gamma = 1$, $\beta = 1$, $\mu = 0.26$. If $\Phi'(0) = 0.01$ and $\Phi(0) = 3.8463$, then the conditions of Theorem 2.6 are satisfied, and the solution blows up (Figure 2, the solid line). If $\Phi'(0) = -0.01$ and $\Phi(0) = 3.8462$, numerical computations show that the solution will go to zero at a finite time (Figure 2, the dotted line).

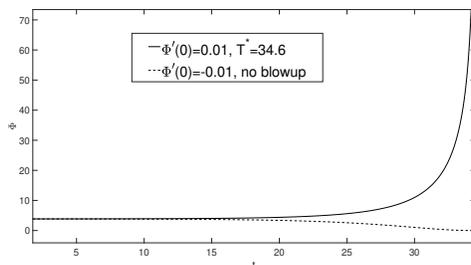


FIGURE 2. Sharp result

3. FINITE TIME BLOWUP FOR PROBLEM (1.1)-(1.3)

In this section, we apply our results established in Section 2 to prove the finite time blowup and improve the upper bound estimate of the blowup time of solutions to problem (1.1)-(1.3).

We denote the norm of $L^2(\Omega)$ as $\|\cdot\|$, the inner product in $L^2(\Omega)$ as (\cdot, \cdot) , and the dual pairing between $H(\Omega)$ and $H^{-2}(\Omega)$ as $\langle \cdot, \cdot \rangle$, where

$$H(\Omega) = \begin{cases} H_0^2(\Omega), & \text{when } u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega; \\ H^2(\Omega) \cap H_0^1(\Omega), & \text{when } u = \Delta u = 0 \text{ on } \partial\Omega, \end{cases} \quad (3.1)$$

and ν is the unit outer normal. Define the energy functional of Problem (1.1)-(1.3) as follows

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 - \sum_{i=1}^n \int_{\Omega} G_i(u_{x_i}) \, dx.$$

Note that problem(1.1)-(1.3) is a special case of that in Lian et al. [9], and the existence of a local solution can be obtained from [9, Theorem 3.2]. We need the following Lemma before presenting our main results.

Lemma 3.1. *If $u \in H(\Omega)$, then $\lambda_1 \|u\| \leq \|\Delta u\|$, and the equality is satisfied if and only if $u = c\psi$, where λ_1 and ψ are the first eigenvalue and eigenfunction of $-\Delta$ with Dirichlet boundary conditions.*

Proof. By Poincare Lemma, we have $\|u\|^2 \leq \lambda_1^{-1} \|\nabla u\|^2$ and

$$\|\nabla u\|^2 = - \int_{\Omega} u \Delta u \, dx \leq \|u\| \|\Delta u\|.$$

Hence, the conclusion follows. \square

Theorem 3.2 (Blowup time estimate by Theorem 2.1). *For Problem (1.1)-(1.3), let $u_0 \in H(\Omega)$, $u_1 \in L^2(\Omega)$, $E(0) > 0$ and σ_i satisfy rm (H1) in Section 1. Assume that*

$$\xi := \sqrt{\max\{m^2 + 2(p^2 - 1)E(0)\|u_0\|^{-2} - \lambda_1^2(p-1)^2, 0\}}, \quad (3.2)$$

where p is defined in (H1), and

$$(u_0, u_1) > \frac{1}{p-1} \|u_0\|^2 (m + \xi) > 0. \quad (3.3)$$

Then the solution u blows up, and there exists a T such that

$$\lim_{t \rightarrow T^-} \|u\|^2 = +\infty$$

with

$$T < \begin{cases} \frac{1}{\xi} \ln \left(\frac{2\xi \|u_0\|^2}{(p-1)(u_0, u_1) - (m+\xi)\|u_0\|^2} + 1 \right) & \text{if } \xi > 0, \\ \frac{2\|u_0\|^2}{(p-1)(u_0, u_1) - m\|u_0\|^2} & \text{if } \xi \leq 0. \end{cases} \tag{3.4}$$

Proof. We define $\Phi(t) := \|u(t)\|^2$. Then

$$\Phi'(t) = 2\langle u, u_t \rangle, \quad \Phi''(t) = 2\langle u_{tt}, u \rangle + 2\|u_t\|^2. \tag{3.5}$$

We denote $H(t) = \|u_t\|^2$. Multiplying (1.1) by u and integrating over Ω , we obtain

$$\langle u_{tt}, u \rangle + m\langle u_t, u \rangle + \|\Delta u\|^2 - \sum_{i=1}^n \int_{\Omega} u_{x_i} \sigma_i(u_{x_i}) \, dx = 0. \tag{3.6}$$

From (3.5) and (3.6), we have

$$\frac{1}{2}\Phi''(t) + \frac{m}{2}\Phi'(t) - H(t) + \|\Delta u\|^2 = \sum_{i=1}^n \int_{\Omega} u_{x_i} \sigma_i(u_{x_i}) \, dx. \tag{3.7}$$

Multiplying (1.1) by u_t and integrating over Ω , we obtain

$$\frac{d}{dt} \left(\frac{1}{2}H(t) + \frac{1}{2}\|\Delta u\|^2 \right) + m\|u_t\|^2 = \frac{d}{dt} \left(\sum_{i=1}^n \int_{\Omega} G_i(u_{x_i}) \, dx \right). \tag{3.8}$$

Integrating (3.8) over $(0, t)$ and using (H1), we deduce that

$$\begin{aligned} \frac{1}{2}H(t) + \frac{1}{2}\|\Delta u\|^2 + m \int_0^t \|u_{\tau}\|^2 \, d\tau - E(0) &= \sum_{i=1}^n \int_{\Omega} G_i(u_{x_i}) \, dx \\ &\leq \frac{1}{p+1} \sum_{i=1}^n \int_{\Omega} u_{x_i} \sigma_i(u_{x_i}) \, dx. \end{aligned} \tag{3.9}$$

Substituting (3.7) into (3.9) yields

$$\begin{aligned} \frac{1}{2}H(t) + \frac{1}{2}\|\Delta u\|^2 + m \int_0^t \|u_{\tau}\|^2 \, d\tau - E(0) \\ \leq \frac{1}{p+1} \left(\frac{1}{2}\Phi''(t) + \frac{m}{2}\Phi'(t) - H(t) + \|\Delta u\|^2 \right), \end{aligned}$$

that is,

$$\frac{1}{2}\Phi''(t) + \frac{m}{2}\Phi'(t) + (p+1)E(0) \geq \frac{p+3}{2}H(t) + \frac{p-1}{2}\|\Delta u\|^2. \tag{3.10}$$

By Cauchy-Schwarz inequality, we have

$$(\Phi'(t))^2 \leq 4\Phi(t)H(t). \tag{3.11}$$

Multiplying both sides of (3.10) by $2\Phi(t)$ and using (3.11) and Lemma 3.1,

$$\begin{aligned} \Phi''(t)\Phi(t) - \frac{p+3}{4}(\Phi'(t))^2 + m\Phi'(t)\Phi(t) + 2(p+1)E(0)\Phi(t) \\ \geq (p-1)\|\Delta u\|^2\Phi(t) \\ \geq \lambda_1^2(p-1)\Phi^2(t), \end{aligned} \tag{3.12}$$

then we set

$$\alpha = \frac{p+3}{4} > 1, \quad \gamma = m \geq 0, \quad \beta = 2(p+1)E(0) > 0, \quad \mu = \lambda_1^2(p-1) > 0.$$

Then it is easy to verify that (3.12), (3.2), and (3.3) satisfy the conditions in Theorem 2.1. Hence, using Theorem 2.1, we obtain that the solution of (1.1)-(1.3) blows up in finite time, and we obtain the upper bound estimate of blowup time as (3.4). \square

Theorem 3.3 (Blowup time estimate by Theorem 2.4). *For problem (1.1)-(1.3), let $u_0 \in H(\Omega)$, $u_1 \in L^2(\Omega)$, $E(0) > 0$ and σ_i satisfy (H1). Assume that*

$$(u_0, u_1) > \frac{2m}{p-1} \|u_0\|^2 + \frac{(p^2-1)E(0)}{m(p+3)} > 0, \quad (3.13)$$

where p is defined in (H1). Then the solution to Problem (1.1)-(1.3) blows up in finite time $T \leq \Gamma$, where Γ is the unique solution of

$$\|u_0\|^2 = \frac{p-1}{(p+3)m^2} \left(\frac{(p-1)^2(p+1)E(0)}{8} (1 - e^{\frac{4m\Gamma}{1-p}}) + w(e^{m\Gamma} - 1) \right), \quad (3.14)$$

and

$$w = \frac{(p+3)m}{4} \left(2(u_0, u_1) - \frac{4m}{p-1} \|u_0\|^2 - \frac{2E(0)(p^2-1)}{(p+3)m} \right). \quad (3.15)$$

Proof. By arguments similar to those in Theorem 3.2, we obtain (3.12), which implies that

$$\Phi''(t)\Phi(t) - \frac{p+3}{4} (\Phi'(t))^2 + m\Phi'(t)\Phi(t) + 2(p+1)E(0)\Phi(t) \geq 0. \quad (3.16)$$

We set

$$\alpha = \frac{p+3}{4} > 1, \quad \gamma = m \geq 0, \quad \beta = 2(p+1)E(0) > 0, \quad \mu = 0.$$

Then combining (3.13), (3.16), and Theorem 2.4, we obtain the proof. \square

Theorem 3.4 (Blowup based on Theorem 2.6). *For problem (1.1)-(1.3), let $u_0 \in H(\Omega)$, $u_1 \in L^2(\Omega)$, $E(0) > 0$ and σ_i satisfy (H1). Assume that $(u_0, u_1) > 0$ and*

$$\lambda_1^2(p-1) > \max \left\{ 2(p+1)E(0)\|u_0\|^{-2}, \frac{32(m+\xi)^2 + 2m^2(p-1)^2}{(p+7)(p-1)^2} \right\}, \quad (3.17)$$

where ξ and p are defined in (3.2) and (H1), respectively. Then the solution of (1.1)-(1.3) blows up in finite time.

Proof. By arguments similar those in Theorem 3.2, combining (3.12), (3.17), and Theorem 2.6, we obtain the proof. \square

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REFERENCES

- [1] L. J. An, A. Peirce; A weakly nonlinear analysis of elasto-plastic-microstructure models, *SIAM J. Appl. Math.*, **55** (1995), 136–155.
- [2] L. H. Fatori, M. A. Jorge Silva, T. F. Ma, Z. J. Yang; Long-time behavior of a class of thermoelastic plates with nonlinear strain, *J. Differential Equations*, **259** (2015), 4831–4862.
- [3] J. Han, R. Z. Xu, Y. B. Yang; Asymptotic behavior and finite time blow up for damped fourth order nonlinear evolution equation, *Asymptotic. Anal.*, **1** (2020), 1–21.
- [4] M. O. Korpusov; Non-existence of global solutions to generalized dissipative Klein-Gordon equations with positive energy, *Electron. J. Differential Equations*, **2012** (2012) no. 119, 1–10.
- [5] Y. C. Liu, R. Z. Xu; A class of fourth order wave equations with dissipative and nonlinear strain terms, *J. Differential Equations*, **244** (2008), 200–228.
- [6] Q. Lin, J. H. Shen, X. C. Wang; Critical and sup-critical initial energy finite time blowup phenomena for the fourth-order wave equations with nonlinear strain term, *Nonlinear Anal.*, **198** (2020), 111873.
- [7] Q. Lin, Y. B. Luo; Blowup phenomena for some fourth-order strain wave equations at arbitrary positive initial energy level, *Opuscula Math.*, **42** (2022), 219–238.
- [8] W. Lian, J. Wang, R. Z. Xu; Global existence and blow up of solutions for pseudo-parabolic equation with singular potential, *J. Differential Equations*, **269** (2020), 4914–4959.
- [9] W. Lian, V. D. Rădulescu, R. Z. Xu, Y. B. Yang, N. Zhao; Global well-posedness for a class of fourth order nonlinear strongly damped wave equations, *Adv. Calc. Var.*, **14** (2021), 589–611.
- [10] L. E. Payne, D. H. Sattinger; Saddle points and instability of nonlinear hyperbolic equations, *Israel J. Math.*, **22** (1975), 273–303.
- [11] D. H. Sattinger; On global solution of nonlinear hyperbolic equations, *Arch. Ration. Mech. Anal.*, **30** (1968), 148–172.
- [12] X. C. Wang, R. Z. Xu; Global existence and finite time blowup for a nonlocal semilinear pseudo-parabolic equation, *Adv. Nonlinear Anal.*, **10** (2021), 261–288.
- [13] R. Z. Xu, J. Su; Global existence and finite time blow-up for a class semilinear pseudo-parabolic equations, *J. Funct. Anal.*, **264** (2013), 2732–2763.
- [14] Z. J. Yang; Global existence, asymptotic behavior and blowup of solutions for a class of nonlinear wave equations with dissipative term, *J. Differential Equations*, **187** (2003), 520–540.
- [15] C. Yang, Y. B. Yang; Long-time behavior for fourth-order wave equations with strain term and nonlinear weak damping term, *Discrete Contin. Dyn. Syst. Ser. S*, **14** (2021), 4643–4658.

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