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# REMARKS ON A CLASS OF QUASILINEAR ELLIPTIC SYSTEMS INVOLVING THE (P,Q)-LAPLACIAN

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ABSTRACT. We study the Nehari manifold for a class of quasilinear elliptic systems involving a pair of (p,q)-Laplacian operators and a parameter. We prove the existence of a nonnegative nonsemitrivial solution for the systems by discussing properties of the Nehari manifold, and so global bifurcation results are obtained. Thanks to Picone's identity, we also prove a nonexistence result.

## 1. Introduction

Consider the quasilinear elliptic boundary-value problem

$$\begin{aligned} -\Delta_p u &= \lambda a(x)|u|^{p-2}u + \lambda b(x)|u|^{\alpha-1}|v|^{\beta+1}u \\ &\quad + \frac{\mu(x)}{(\alpha+1)(\delta+1)}|u|^{\gamma-1}|v|^{\delta+1}u \quad \text{in } \Omega \\ -\Delta_q v &= \lambda d(x)|v|^{q-2}v + \lambda b(x)|u|^{\alpha+1}|v|^{\beta-1}v \\ &\quad + \frac{\mu(x)}{(\beta+1)(\gamma+1)}|u|^{\gamma+1}|v|^{\delta-1}v \quad \text{in } \Omega \\ u &= 0, \quad v = 0 \quad \text{on } \partial\Omega\,, \end{aligned} \tag{1.1}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $\lambda > 0$ is a real parameter, and $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator with $1 < p, q < N$.

Recently, many publications have appeared about semilinear and quaslinear systems which have been used in a great variety of applications. Stavrakakis and Zographopoulos [8, 9] studied existence and bifurcation results for problem (1.1) with $a(x) = d(x) \equiv 0$, using variational approach and global bifurcation theory. Fleckinger, Manasevich, Stavrakakis and de Thelin [6] and Zographopoulos [11] obtained some properties of the positive principal eigenvalue $\lambda_1$ for the unperturbed system

$$\begin{gathered} -\Delta_p u = \lambda a(x)|u|^{p-2}u + \lambda b(x)|u|^{\alpha-1}|v|^{\beta+1}u \quad \text{in } \Omega \\ -\Delta_q v = \lambda d(x)|v|^{q-2}v + \lambda b(x)|u|^{\alpha+1}|v|^{\beta-1}v \quad \text{in } \Omega \\ u = 0, \quad v = 0 \quad \text{on } \partial\Omega \end{gathered}$$

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Later, under the key condition

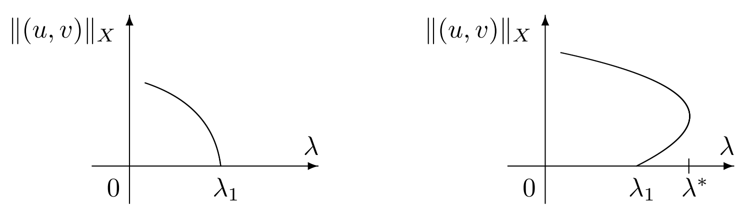
$$\int_\Omega \mu(x)|u_1|^{\gamma+1}|v_1|^{\delta+1}\mathrm{d}x < 0, \tag{1.2}$$

where $(u_1, v_1)$ is the positive normalized eigenfunction corresponding to $\lambda_1$, Drabek, Stavrakakis and Zographopoulos in [5] prove that there exists $\lambda^* > \lambda_1$ such that Problem (1.1) has two nonnegative nonsemitrival solutions wherever $\lambda \in (\lambda_1, \lambda^*)$. i.e. $\lambda = \lambda_1$ is a bifurcation point, and bifurcation is to the right when $\lambda > \lambda_1$.

In this paper, under the condition

$$\int_\Omega \mu(x)|u_1|^{\gamma+1}|v_1|^{\delta+1}\mathrm{d}x > 0, \tag{1.3}$$

we prove the existence of a nonnegative nonsemitrival solution for Problem (1.1) when $\lambda < \lambda_1$. i.e. the bifurcation is to the left. Combining this with the result of [5], we obtain global bifurcation results for Problem (1.1), for which the corresponding bifurcation diagrams are shown in Fig 1. In addition, a nonexistence result is proved by using Picone's identity when $\lambda > \lambda_1$.




(a) $\int_\Omega \mu(x)|u_1|^{\gamma+1}|v_1|^{\delta+1}\,\mathrm{d}x > 0$ b) $\int_\Omega \mu(x)|u_1|^{\gamma+1}|v_1|^{\delta+1}\,\mathrm{d}x < 0$

Figure 1. Bifurcation diagrams for Problem (1.1)

This paper is organized as follows. In section 2, we introduce notation, give some definitions, and state our basic assumptions. Section 3 is devoted to giving a detailed description of Figure 1 (a). In section 4, we prove a nonexistence result.

1.1. **Remarks.** (1) Figure 1 shows how the sign of $\int_\Omega \mu(x)|u_1|^{\gamma+1}|v_1|^{\delta+1}\mathrm{d}x$ determines the direction of bifurcation at the point $\lambda = \lambda_1$.
(2) This paper gives a complete bifurcation result for Problem (1.1) using the arguments developed in Allegretto and Huang [1] and by Brown and Zhang [4].

## 2. Notation and hypotheses

Let $W_0^{1,p}(\Omega)$ denote the closure of the space $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_p = (\int_\Omega |\nabla u|^p \mathrm{d}x)^{1/p}$. Let $X$ denote the product space $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ equipped with the norm

$$\|(u,v)\|_X = \|u\|_p + \|v\|_q.$$

Now, we state some assumptions used in this paper.

(H1) Assume that $\alpha, \beta, \gamma, \delta$ satisfy
$$\frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1\,,$$
$$p < \gamma+1 \quad \text{or} \quad q < \delta+1, \quad \frac{\gamma+1}{p^*} + \frac{\delta+1}{q^*} < 1\,,$$
$$\frac{1}{(\alpha+1)(\delta+1)} + \frac{1}{(\beta+1)(\gamma+1)} < 1\,,$$
where $p^* = \frac{Np}{N-p}$, $q^* = \frac{Np}{N-p}$ are the well-known critical exponents.

(H2) Assume $a(x), b(x), d(x)$ are nonnegative smooth functions such that $a(x) \in L^{\frac{N}{p}}(\Omega) \cap L^\infty(\Omega)$, $b(x) \in L^{\omega_1}(\Omega) \cap L^\infty(\Omega)$, $d(x) \in L^{\frac{N}{q}}(\Omega) \cap L^\infty(\Omega)$ and
$$|\Omega_1^+| = |\{x \in \Omega : a(x) > 0\}| > 0$$
$$|\Omega_2^+| = |\{x \in \Omega : d(x) > 0\}| > 0\,,$$
where $b(x) \not\equiv 0$ and $\omega_1 = p^*q^*/[p^*q^* - (\alpha+1)q^* - (\beta+1)p^*]$.

(H3) $\mu(x)$ is a given smooth function which many change sign, and $\mu(x) \in L^{\omega_2}(\Omega) \cap L^\infty(\Omega)$, where $\omega_2 = p^*q^*/[p^*q^* - (\gamma+1)q^* - (\delta+1)p^*]$.

**Lemma 2.1** ([1, 10])**.** *There exists a number $\lambda_1 > 0$ such that*

(1)
$$\lambda_1 = \inf \frac{(\frac{\alpha+1}{p}\int_\Omega |\nabla u|^p dx + \frac{\beta+1}{q}\int_\Omega |\nabla v|^q dx)}{(\frac{\alpha+1}{p}\int_\Omega a(x)|u|^p dx + \frac{\beta+1}{q}\int_\Omega d(x)|v|^q dx + \int_\Omega b(x)|u|^{\alpha+1}|v|^{\beta+1} dx)}\,,$$
*where the infimum is taken over $(u,v) \in X$*

(2) *There exists a positive function $(u_1, v_1) \in X \cap L^\infty(\Omega)$, which is solution of the system* (1.2)

(3) *The eigenvalue $\lambda_1$ is simple in the sense that the eigenfunctions associated with it are merely a constant multiple of each other*

(4) *$\lambda_1$ is isolated, that is, there exists $\delta > 0$ such that in the interval $(\lambda_1, \lambda_1+\delta)$ there are no other eigenvalues of the system* (1.2).

**Definition 2.2.** *We say that $(u,v) \in X$ is a weak solution of Problem* (1.1) *if for all $(\varphi, \xi) \in X$,*
$$\begin{aligned}\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi dx &= \lambda(\int_\Omega a(x)|u|^{p-2}u\varphi dx + \int_\Omega b(x)|u|^{\alpha-1}|v|^{\beta+1}u\varphi dx)\\ &\quad + \frac{1}{(\alpha+1)(\delta+1)}\int_\Omega \mu(x)|u|^{\gamma-1}|v|^{\delta+1}u\varphi dx\\ \int_\Omega |\nabla v|^{q-2}\nabla v \nabla\xi dx &= \lambda(\int_\Omega a(x)|v|^{p-2}v\xi dx + \int_\Omega b(x)|u|^{\alpha+1}|v|^{\beta-1}v\xi dx)\\ &\quad + \frac{1}{(\beta+1)(\gamma+1)}\int_\Omega \mu(x)|u|^{\gamma+1}|v|^{\delta-1}v\xi dx\,.\end{aligned}$$

## 3. The case $\lambda < \lambda_1$

It is well known that Problem (1.1) has a variational structure. i.e., weak solutions of Problem (1.1) are critical points of the functional
$$I(u,v) = J(u,v) - \lambda K(u,v) - \frac{1}{(\gamma+1)(\delta+1)}M(u,v)$$

where

$$\begin{aligned}
J(u,v) &= \frac{\alpha+1}{p}\int_\Omega |\nabla u|^p \mathrm{d}x + \frac{\beta+1}{q}\int_\Omega |\nabla v|^q \mathrm{d}x,\\
K(u,v) &= \frac{\alpha+1}{p}\int_\Omega a(x)|u|^p \mathrm{d}x + \frac{\beta+1}{q}\int_\Omega d(x)|v|^q \mathrm{d}x + \int_\Omega b(x)|u|^{\alpha+1}|v|^{\beta+1}\mathrm{d}x,\\
M(u,v) &= \int_\Omega \mu(x)|u|^{\gamma+1}|v|^{\delta+1}\mathrm{d}x.
\end{aligned}$$

Clearly, $I(u,v) \in C^1(X,R)$.

Let $\Lambda_\lambda$ be the Nehari manifold associated with Problem (1.1). i.e.,

$$\Lambda_\lambda = \{(u,v) \in X : \langle I'(u,v),(u,v)\rangle = 0\} \tag{3.1}$$

It is clear that $\Lambda_\lambda$ is closed in $X$ and all critical points of $I(u,v)$ must lie on $\Lambda_\lambda$. So, $(u,v) \in \Lambda_\lambda$ if and only if

$$\begin{aligned}
&\int_\Omega |\nabla u|^p\,\mathrm{d}x - \lambda\int_\Omega a(x)|u|^p\,\mathrm{d}x - \lambda\int_\Omega b(x)|u|^{\alpha+1}|v|^{\beta+1}\,\mathrm{d}x\\
&= \frac{1}{(\alpha+1)(\delta+1)}\int_\Omega \mu(x)|u|^{\gamma+1}|v|^{\delta+1}\,\mathrm{d}x\\
&\int_\Omega |\nabla v|^q\,\mathrm{d}x - \lambda\int_\Omega d(x)|v|^q\,\mathrm{d}x - \lambda\int_\Omega b(x)|u|^{\alpha+1}|v|^{\beta+1}\,\mathrm{d}x\\
&= \frac{1}{(\beta+1)(\gamma+1)}\int_\Omega \mu(x)|u|^{\gamma+1}|v|^{\delta+1}\,\mathrm{d}x
\end{aligned} \tag{3.2}$$

Hence, for $(u,v) \in \Lambda_\lambda$, using $\frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1$, we have

$$I(u,v) = \Big(\frac{1}{p(\delta+1)} + \frac{1}{q(\gamma+1)} - \frac{1}{(\gamma+1)(\delta+1)}\Big)\int_\Omega \mu(x)|u|^{\gamma+1}|v|^{\delta+1}\,\mathrm{d}x \tag{3.3}$$

Now, we define the following disjoint subsets of $\Lambda_\lambda$:

$$\begin{aligned}
\Lambda_\lambda^+ &= \{(u,v)\in\Lambda_\lambda : \int_\Omega \mu(x)|u|^{\lambda+1}|v|^{\delta+1}\,\mathrm{d}x < 0\}\\
\Lambda_\lambda^0 &= \{(u,v)\in\Lambda_\lambda : \int_\Omega \mu(x)|u|^{\lambda+1}|v|^{\delta+1}\,\mathrm{d}x = 0\}\\
\Lambda_\lambda^- &= \{(u,v)\in\Lambda_\lambda : \int_\Omega \mu(x)|u|^{\lambda+1}|v|^{\delta+1}\,\mathrm{d}x > 0\}
\end{aligned}$$

Let $0 < \lambda < \lambda_1$, and consider the eigenvalue problem

$$\begin{aligned}
-\Delta_p u - \lambda(a(x)|u|^p + b(x)|u|^{\alpha+1}|v|^{\beta+1}) &= \mu_M|u|^{p-1}u \quad \Omega\\
-\Delta_q v - \lambda(d(x)|v|^q + b(x)|u|^{\alpha+1}|v|^{\beta+1}) &= \mu_M|v|^{q-1}v \quad \Omega\,.
\end{aligned} \tag{3.4}$$

Then, there exists $\mu_M > 0$ such that

$$\begin{aligned}
\int_\Omega |\nabla u|^p\,\mathrm{d}x - \lambda\int_\Omega a(x)|u|^p\,\mathrm{d}x - \lambda\int_\Omega b(x)|u|^{\alpha+1}|v|^{\beta+1}\,\mathrm{d}x \ge \mu_M\int_\Omega |u|^p\,\mathrm{d}x\\
\int_\Omega |\nabla v|^q\,\mathrm{d}x - \lambda\int_\Omega d(x)|v|^q\,\mathrm{d}x - \lambda\int_\Omega b(x)|u|^{\alpha+1}|v|^{\beta+1}\,\mathrm{d}x \ge \mu_M\int_\Omega |v|^q\,\mathrm{d}x
\end{aligned} \tag{3.5}$$

for every $(u,v) \in X$. Thus, $\Lambda_\lambda^+$ is empty, $\Lambda_\lambda^0 = \{(0,0)\}$ and $\Lambda_\lambda = \Lambda_\lambda^- \cup \{(0,0)\}$. Clearly, $I(u,v) > 0$ whenever $(u,v) \in \Lambda_\lambda^-$ and $I(u,v)$ is bounded below by on $\Lambda_\lambda^-$. i.e., $\inf_{(u,v)\in\Lambda_\lambda^-} I(u,v) \ge 0$.

**Theorem 3.1.** *Assume (H1)–(H3) and the condition* (1.3). *Then Problem* (1.1) *has a nonnegative nonsemitrivial solution for every* $\lambda \in (0,\lambda_1)$.

*Proof.* Let $\{(u_n,v_n)\} \subset \Lambda_\lambda^-$ be a minimizing sequence; i.e., $\lim_{n\to\infty} I(u_n,v_n) = \inf_{(u,v)\in\Lambda_\lambda^-} I(u,v)$. Since

$$I(u_n,v_n) = (\frac{1}{p(\delta+1)} + \frac{1}{q(\gamma+1)} - \frac{1}{(\gamma+1)(\delta+1)}) \int_\Omega \mu(x)|u_n|^{\gamma+1}|v_n|^{\delta+1}\,\mathrm{d}x$$

using (3.2) (3.5) and $p < \gamma+1$ or $q < \delta+1$, we have

$$I(u_n,v_n) \ge \mu_M[(\frac{\alpha+1}{p} - \frac{\alpha+1}{\gamma+1}) \int_\Omega |u_n|^p\,\mathrm{d}x + (\frac{\beta+1}{q} - \frac{\beta+1}{\gamma+1}) \int_\Omega |v_n|^q\,\mathrm{d}x]\,.$$

Then $\{(u_n,v_n)\}$ is bounded in $X$, and so we may assume $(u_n,v_n) \rightharpoonup (u_0,v_0) \in X$ and $u_n \to u_0$ in $L^{\gamma+1}(\Omega)$, $v_n \to v_0$ in $L^{\delta+1}(\Omega)$.

First we claim that $\inf_{(u,v)\in\Lambda_\lambda^-} I(u,v) > 0$. Indeed, suppose $\inf_{(u,v)\in\Lambda_\lambda^-} I(u,v) = 0$. i.e. $\lim_{n\to\infty} I(u_n,v_n) = 0$, we have

$$\int_\Omega \mu(x)|u_n|^{\gamma+1}|v_n|^{\delta+1}\,\mathrm{d}x \to 0$$

and

$$\begin{aligned} &\int_\Omega |\nabla u_n|^p - \lambda a(x)|u_n|^p - \lambda b(x)|u_n|^{\alpha+1}|v_n|^{\beta+1}\,\mathrm{d}x \\ &= \frac{1}{(\alpha+1)(\delta+1)} \int_\Omega \mu(x)|u_n|^{\gamma+1}|v_n|^{\delta+1}\,\mathrm{d}x \to 0 \end{aligned} \tag{3.6}$$

$$\begin{aligned} &\int_\Omega |\nabla v_n|^q - \lambda d(x)|v_n|^q - \lambda b(x)|u_n|^{\alpha+1}|v_n|^{\beta+1}\,\mathrm{d}x \\ &= \frac{1}{(\beta+1)(\gamma+1)} \int_\Omega \mu(x)|u_n|^{\gamma+1}|v_n|^{\delta+1}\,\mathrm{d}x \to 0 \end{aligned} \tag{3.7}$$

Moreover, by [5, Lemma 2.1] the compactness of the operators $K$ implies

$$\begin{aligned} &\int_\Omega |\nabla u_n|^p - \lambda a(x)|u_n|^p - \lambda b(x)|u_n|^{\alpha+1}|v_n|^{\beta+1}\,\mathrm{d}x \\ &\to \int_\Omega |\nabla u_0|^p - \lambda a(x)|u_0|^p - \lambda b(x)|u_0|^{\alpha+1}|v_0|^{\beta+1}\,\mathrm{d}x = 0 \\ &\int_\Omega |\nabla v_n|^q - \lambda d(x)|v_n|^q - \lambda b(x)|u_n|^{\alpha+1}|v_n|^{\beta+1}\,\mathrm{d}x \\ &\to \int_\Omega |\nabla v_0|^q - \lambda d(x)|v_0|^q - \lambda b(x)|u_0|^{\alpha+1}|v_0|^{\beta+1}\,\mathrm{d}x = 0 \end{aligned}$$

From $\lambda \in (0,\lambda_1)$ and the variational characterization of $\lambda_1$, we have $(u_n,v_n) \to (u_0,v_0) = (0,0)$. Let

$$\widetilde{u}_n = \frac{u_n}{(\|u_n\|_p^p + \|v_n\|_q^q)^{1/p}}, \quad \widetilde{v}_n = \frac{v_n}{(\|u_n\|_p^p + \|v_n\|_q^q)^{\frac{1}{q}}} \tag{3.8}$$

which are bounded sequences. Indeed, we have

$$\|\widetilde{u}_n\|_p^p + \|\widetilde{v}_n\|_q^q = 1 \quad \text{for every } n \in \mathbb{N}$$

Thus, we may assume $(\widetilde{u}_n,\widetilde{v}_n) \rightharpoonup (\widetilde{u}_0,\widetilde{v}_0)$. Using that $\frac{\alpha+1}{p}+\frac{\beta+1}{q}=1$, we have

$$\int_\Omega b(x)|\widetilde{u}_n|^{\alpha+1}|\widetilde{v}_n|^{\beta+1}\,\mathrm{d}x = \int_\Omega b(x)|u_n|^{\alpha+1}|v_n|^{\beta+1}\,\mathrm{d}x \Big/ (\|u_n\|_p^p+\|v_n\|_q^q)$$

Moreover the range of exponents implies

$$\frac{\int_\Omega \mu(x)|u_n|^{\gamma+1}|v_n|^{\delta+1}\,\mathrm{d}x}{\|u_n\|_p^p+\|v_n\|_q^q} \le \frac{|\mu|_{\omega_2}|u_n|_{p^*}^{\gamma+1}|v_n|_{q^*}^{\delta+1}}{\|u_n\|_p^p+\|v_n\|_q^q} \to 0$$

Using (3.6) and (3.7), we obtain

$$\begin{gathered}
\int_\Omega |\nabla\widetilde{u}_n|^p - \lambda a(x)|\widetilde{u}_n|^p - \lambda b(x)|\widetilde{u}_n|^{\alpha+1}|\widetilde{v}_n|^{\beta+1}\,\mathrm{d}x \to 0\,,\\
\int_\Omega |\nabla\widetilde{v}_n|^q - \lambda d(x)|\widetilde{v}_n|^q - \lambda b(x)|\widetilde{u}_n|^{\alpha+1}|\widetilde{v}_n|^{\beta+1}\,\mathrm{d}x \to 0\,.
\end{gathered}$$

Following the argument used on $\{(\widetilde{u}_n,\widetilde{v}_n)\}$ above, for $\{(u_n,v_n)\}$ we have

$$\begin{aligned}
&\int_\Omega |\nabla\widetilde{u}_n|^p - \lambda a(x)|\widetilde{u}_n|^p - \lambda b(x)|\widetilde{u}_n|^{\alpha+1}|\widetilde{v}_n|^{\beta+1}\,\mathrm{d}x\\
&\to \int_\Omega |\nabla\widetilde{u}_0|^p - \lambda a(x)|\widetilde{u}_0|^p - \lambda b(x)|\widetilde{u}_0|^{\alpha+1}|\widetilde{v}_0|^{\beta+1}\,\mathrm{d}x = 0\,,\\
&\int_\Omega |\nabla\widetilde{v}_n|^q - \lambda d(x)|\widetilde{v}_n|^q - \lambda b(x)|\widetilde{u}_n|^{\alpha+1}|\widetilde{v}_n|^{\beta+1}\,\mathrm{d}x\\
&\to \int_\Omega |\nabla\widetilde{v}_0|^q - \lambda d(x)|\widetilde{v}_0|^q - \lambda b(x)|\widetilde{u}_0|^{\alpha+1}|\widetilde{v}_0|^{\beta+1}\,\mathrm{d}x = 0\,,
\end{aligned}$$

and $(\widetilde{u}_n,\widetilde{v}_n) \to (\widetilde{u}_0,\widetilde{v}_0)=(0,0)$ in $X$, which contradict $\|(\widetilde{u}_n,\widetilde{v}_n)\|_X=1$, for every $n\in\mathbb{N}$.

Now we show that $(u_n,v_n)\to(u_0,v_0)$ in $X$. Suppose otherwise, then $\|u_0|_p < \liminf_{n\to\infty}\|u_n\|_p$, $\|v_0\|_p < \liminf_{n\to\infty}\|v_n\|_q$, and

$$\begin{aligned}
&\int_\Omega |\nabla u_0|^p - \lambda\int_\Omega a(x)|u_0|^p - \lambda\int_\Omega b(x)|u_0|^{\alpha+1}|v_0|^{\beta+1}\,\mathrm{d}x\\
&< \liminf_{n\to\infty}\int_\Omega |\nabla u_n|^p - \lambda a(x)|u_n|^p - \lambda b(x)|u_n|^{\alpha+1}|v_n|^{\beta+1}\,\mathrm{d}x = 0\\
&\int_\Omega |\nabla v_0|^q - \lambda d(x)|v_0|^q - \lambda b(x)|u_0|^{\alpha+1}|v_0|^{\beta+1}\,\mathrm{d}x\\
&< \liminf_{n\to\infty}\int_\Omega |\nabla v_n|^q - \lambda d(x)|v_n|^q - \lambda b(x)|u_n|^{\alpha+1}|v_n|^{\beta+1}\,\mathrm{d}x = 0\,.
\end{aligned}$$

Since $\lambda\in(0,\lambda_1)$ and $(u_0,v_0)\not\equiv(0,0)$, we have

$$\begin{gathered}
\int_\Omega |\nabla u_0|^p - \lambda a(x)|u_0|^p - \lambda b(x)|u_0|^{\alpha+1}|v_0|^{\beta+1}\,\mathrm{d}x > 0\,,\\
\int_\Omega |\nabla v_0|^q - \lambda d(x)|v_0|^q - \lambda b(x)|u_0|^{\alpha+1}|v_0|^{\beta+1}\,\mathrm{d}x > 0
\end{gathered}$$

which is a contradiction. Hence $(u_n,v_n)\to(u_0,v_0)$ in $X$. □

From [4, Theorem 2.3], $(u_0,v_0)$ is a local minimizer on $\Lambda_\lambda^-$ and $(u_0,v_0)\notin\Lambda_\lambda^0=\{(0,0)\}$, then $(u_0,v_0)$ is a critical point of $I(u,v)$. This solution is nonnegative due to the fact that $I(|u|,|v|)=I(u,v)$, and it is also nonsemitrivial by [5, Lemma 2.5].

**Theorem 3.2.** *Assume (H1)–(H3) and the condition* (1.3), *if* $\lambda_n \to \lambda_1^-$ *and* $(u_n, v_n)$ *is a minimizer of* $I(u_n, v_n)$ *on* $\Lambda_\lambda^-$, *then* $(u_n, v_n) \to (0, 0)$.

*Proof.* First we show that $\{(u_n, v_n)\}$ is bounded in $X$. Suppose not, then we may assume without loss of generality that $\|u_n\|_p \to \infty$, $\|v_n\|_q \to \infty$, as $n \to \infty$. Let $(\widetilde{u}_n, \widetilde{v}_n)$ are the sequence introduced by (3.8). The boundedness of $(\widetilde{u}_n, \widetilde{v}_n)$ implies $(\widetilde{u}_n, \widetilde{v}_n) \rightharpoonup (\widetilde{u}_0, \widetilde{v}_0)$ in $X$. Then

$$
\begin{aligned}
&\int_\Omega |\nabla \widetilde{u}_n|^p - \lambda_n a(x)|\widetilde{u}_n|^p - \lambda_n b(x)|\widetilde{u}_n|^{\alpha+1}|\widetilde{v}_n|^{\beta+1}\,\mathrm{d}x \\
&\to \int_\Omega |\nabla \widetilde{u}_0|^p - \lambda_1 a(x)|\widetilde{u}_0|^p - \lambda_1 b(x)|\widetilde{u}_0|^{\alpha+1}|\widetilde{v}_0|^{\beta+1}\,\mathrm{d}x = 0 \\
&\int_\Omega |\nabla \widetilde{v}_n|^q - \lambda_n d(x)|\widetilde{v}_n|^q - \lambda_n b(x)|\widetilde{u}_n|^{\alpha+1}|\widetilde{v}_n|^{\beta+1}\,\mathrm{d}x \\
&\to \int_\Omega |\nabla \widetilde{v}_0|^q - \lambda_1 d(x)|\widetilde{v}_0|^q - \lambda_1 b(x)|\widetilde{u}_0|^{\alpha+1}|\widetilde{v}_0|^{\beta+1}\,\mathrm{d}x = 0\,.
\end{aligned}
$$

Since $(u_n, v_n)$ is a minimizer of $I(u_n, v_n)$ on $\Lambda_\lambda^-$, we have

$$
I(u_n, v_n) = \Big(\frac{1}{p(\delta+1)} + \frac{1}{q(\gamma+1)} - \frac{1}{(\gamma+1)(\delta+1)}\Big) \int_\Omega \mu(x)|u_n|^{\gamma+1}|v_n|^{\delta+1}\,\mathrm{d}x \to 0
$$

Thus, we must have $(\widetilde{u}_n, \widetilde{v}_n) \to (\widetilde{u}_0, \widetilde{v}_0) \not\equiv (0, 0)$ and $\widetilde{u}_0 = k^p u_1, \widetilde{v} = k^q v_1$ for some positive constant $k$, it is easy to see

$$
\lim_{n\to\infty} (\|u_n\|_p^p + \|v_n\|_q^q)\Big(\frac{\gamma+1}{p} + \frac{\delta+1}{q} - 1\Big) \int_\Omega \mu(x)|\widetilde{u}_n|^{\gamma+1}|\widetilde{v}_n|^{\delta+1}\,\mathrm{d}x = 0\,.
$$

Hence $\lim_{n\to\infty} \int_\Omega \mu(x)|\widetilde{u}_n|^{\gamma+1}|\widetilde{v}_n|^{\delta+1}\,\mathrm{d}x = \int_\Omega \mu(x)|\widetilde{u}_0|^{\gamma+1}|\widetilde{v}_0|^{\delta+1}\,\mathrm{d}x$, it follows that $k = 0$. But as $\|\widetilde{u}_0\|_p^p + \|\widetilde{v}_0\|_q^q = 1$, that is impossible. Hence $\{(u_n, v_n)\}$ is bounded.

Thus we may assume $(u_n, v_n) \rightharpoonup (u_0, v_0)$ in $X$. Then, using the same argument on $(u_n, v_n)$ as used on $(\widetilde{u}_n, \widetilde{v}_v)$. It follows that $(u_n, v_n) \to (0, 0)$, and so the proof is complete. □

We remark that the two theorems above give a rather detailed description of the bifurcation diagram in Figure 1(a).

## 4. The case $\lambda > \lambda_1$

In this section, we prove a nonexistence result for Problem (1.1) by using the Picone identity.

**Lemma 4.1** (Picone identity [1])**.** *Let* $v > 0$, $u \geq 0$ *be differentiable, and let*

$$
L(u, v) = |\nabla u|^p + (p-1)\frac{u^p}{v^p}|\nabla v|^p - p\frac{u^p}{v^{p-1}}\nabla u \nabla v|\nabla v|^{p-2}
$$

$$
R(u, v) = |\nabla u|^p - \nabla\Big(\frac{u^p}{v^{p-1}}\Big)|\nabla v|^{p-2}\nabla v
$$

*Then* $L(u, v) = R(u, v) \geq 0$.

Moreover, $L(u, v) = 0$ a.e. on $\Omega$ if and only if $\nabla(\frac{u}{v}) = 0$ a.e. on $\Omega$. For the next theorem we will assume

(H3') $u(x)$ is a nonnegative smooth function, and $\mu(x) \in L^{\omega_2}(\Omega) \cap L^\infty(\Omega)$, where $\omega_2 = p^* q^*/[p^* q^* - (\gamma+1)q^* - (\delta+1)p^*]$

**Theorem 4.2.** *Assume (H1), (H2), (H3') and Condition* (1.3). *Then Problem* (1.1) *has no nonnegative nonsemitrivial solution, for every* $\lambda > \lambda_1$.

*Proof.* On the contrary, let $u_n \in C_0^\infty(\Omega), v_n \in C_0^\infty(\Omega)$. We apply Picone's identity to the functions $u_n, u$ and $v_n, v$, to obtain

$$0 \le \int_\Omega |\nabla u_n|^p \,\mathrm{d}x + \int_\Omega \frac{u_n^p}{u^{p-1}} \Delta_p u \,\mathrm{d}x \tag{4.1}$$

$$0 \le \int_\Omega |\nabla v_n|^q \,\mathrm{d}x + \int_\Omega \frac{v_n^q}{v^{q-1}} \Delta_q v \,\mathrm{d}x \tag{4.2}$$

Using that $\frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1$, then multiplying (4.1) by $\frac{\alpha+1}{p}$ and (4.2) by $\frac{\beta+1}{q}$, and then adding, we obtain

$$\begin{aligned}
&\frac{\alpha+1}{p} \int_\Omega |\nabla u_n|^p \,\mathrm{d}x + \frac{\beta+1}{q} \int_\Omega |\nabla v_n|^q \,\mathrm{d}x \\
&- \frac{\alpha+1}{p} \int_\Omega \lambda a(x) u_n^p \,\mathrm{d}x - \frac{\beta+1}{q} \int_\Omega \lambda d(x) v_n^q \,\mathrm{d}x \\
&\ge \frac{\alpha+1}{p} \int_\Omega \lambda b(x) u_n^p u^{\alpha+1-p} v^{\beta+1} \,\mathrm{d}x + \frac{\beta+1}{q} \int_\Omega \lambda b(x) v_n^q |u|^{\alpha+1} v^{\beta+1-q} \,\mathrm{d}x \\
&\quad + \frac{1}{p(\delta+1)} \int_\Omega \mu(x) u_n^p u^{\gamma+1-p} v^{\delta+1} \,\mathrm{d}x + \frac{1}{q(\gamma+1)} \int_\Omega \mu(x) v_n^q u^{\gamma+1} v^{\delta+1-q} \,\mathrm{d}x
\end{aligned} \tag{4.3}$$

Now, put $\theta_1 = (\alpha+1)(\beta+1)/q$ and $\theta_2 = (\alpha+1)(\beta+1)/p$, then

$$u_n^{\alpha+1} v_n^{\beta+1} = u_n^{\alpha+1} v_n^{\beta+1} \frac{v^{\theta_2}}{u^{\theta_1}} \frac{u^{\theta_1}}{v^{\theta_2}} \le \frac{\alpha+1}{p} u_n^p u^{\alpha+1-p} v^{\beta+1} + \frac{\beta+1}{q} v_n^q u^{\alpha+1} v^{\beta+1-q}$$

Since $\lambda > 0$ and $b(x) \ge 0$, we obtain

$$\begin{aligned}
&\lambda \int_\Omega b(x) u_n^{\alpha+1} v_n^{\beta+1} \,\mathrm{d}x \\
&\le \frac{\alpha+1}{p} \int_\Omega \lambda b(x) u_n^p u^{\alpha+1-p} v^{\beta+1} \,\mathrm{d}x + \frac{\beta+1}{q} \int_\Omega \lambda b(x) v_n^q u^{\alpha+1} v^{\beta+1-q} \,\mathrm{d}x
\end{aligned} \tag{4.4}$$

Using that $\frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1$ and $\frac{1}{(\alpha+1)(\delta+1)} + \frac{1}{(\beta+1)(\gamma+1)} < 1$, we obtain

$$\frac{\gamma+1}{p} + \frac{\delta+1}{q} > (\alpha+1)(\beta+1) > 1\,.$$

Then

$$u_n^{\gamma+1} v_n^{\delta+1} < \frac{\gamma+1}{p} u_n^p u^{\gamma+1-p} v^{\delta+1} + \frac{\delta+1}{q} v_n^q u^{\gamma+1} v^{\delta+1-q}\,.$$

Since $\mu(x) \ge 0$, we have

$$\begin{aligned}
&\frac{1}{p(\delta+1)} \int_\Omega \mu(x) u_n^p u^{\gamma+1-p} v^{\delta+1} \,\mathrm{d}x + \frac{1}{\gamma+1} \int_\Omega \mu(x) v_n^q u^{\gamma+1} v^{\delta+1-q} \,\mathrm{d}x \\
&= \frac{1}{(\gamma+1)(\delta+1)} \Big[ \frac{\gamma+1}{p} \int_\Omega \mu(x) u_n^p u^{\gamma+1-p} v^{\delta+1} \,\mathrm{d}x \\
&\quad + \frac{\delta+1}{q} \int_\Omega \mu(x) v_n^q u^{\gamma+1} v^{\delta+1-q} \,\mathrm{d}x \Big] \\
&\ge \frac{1}{(\gamma+1)(\delta+1)} \int_\Omega \mu(x) u_n^{\gamma+1} v_n^{\delta+1} \,\mathrm{d}x
\end{aligned} \tag{4.5}$$

Combining (4.3), (4.4) and (4.5), we have

$$\begin{aligned} &\frac{\alpha+1}{p}\int_\Omega |\nabla u_n|^p\,\mathrm{d}x+\frac{\beta+1}{q}\int_\Omega |\nabla v_n|^q\,\mathrm{d}x-\frac{\alpha+1}{p}\int_\Omega \lambda a(x)u_n^p\,\mathrm{d}x\\ &-\frac{\beta+1}{q}\int_\Omega \lambda d(x)v_n^q\,\mathrm{d}x-\lambda\int_\Omega b(x)u_n^{\alpha+1}v_n^{\beta+1}\,\mathrm{d}x\\ &>\frac{1}{(\gamma+1)(\delta+1)}\int_\Omega \mu(x)u_n^{\gamma+1}v_n^{\delta+1}\,\mathrm{d}x \end{aligned}\tag{4.6}$$

Let $(u_n,v_n)$ converge to $(u_1,v_1)\in X$, then

$$\begin{aligned} &\frac{\alpha+1}{p}\int_\Omega |\nabla u_n|^p\,\mathrm{d}x+\frac{\beta+1}{q}\int_\Omega |\nabla v_n|^q\,\mathrm{d}x-\frac{\alpha+1}{p}\int_\Omega \lambda a(x)u_n^p\,\mathrm{d}x\\ &-\frac{\beta+1}{q}\int_\Omega \lambda d(x)v_n^q\,\mathrm{d}x-\lambda\int_\Omega b(x)u_n^{\alpha+1}v_n^{\beta+1}\,\mathrm{d}x\\ &\to\frac{\alpha+1}{p}\int_\Omega |\nabla u_1|^p\,\mathrm{d}x+\frac{\beta+1}{q}\int_\Omega |\nabla v_1|^q\,\mathrm{d}x-\frac{\alpha+1}{p}\int_\Omega \lambda a(x)u_1^p\,\mathrm{d}x\\ &-\frac{\beta+1}{q}\int_\Omega \lambda d(x)v_1^q\,\mathrm{d}x-\lambda\int_\Omega b(x)u_1^{\alpha+1}v_1^{\beta+1}\,\mathrm{d}x\,,\\ &\frac{1}{(\gamma+1)(\delta+1)}\int_\Omega \mu(x)u_n^{\gamma+1}v_n^{\delta+1}\,\mathrm{d}x\to\frac{1}{(\gamma+1)(\delta+1)}\int_\Omega \mu(x)u_1^{\gamma+1}v_1^{\delta+1}\,\mathrm{d}x \end{aligned}$$

From the variational characterization of $\lambda_1$ and $\lambda>\lambda_1$, we have

$$\begin{aligned} &\frac{\alpha+1}{p}\int_\Omega |\nabla u_1|^p\,\mathrm{d}x+\frac{\beta+1}{q}\int_\Omega |\nabla v_1|^q\,\mathrm{d}x-\frac{\alpha+1}{p}\int_\Omega \lambda a(x)u_1^p\,\mathrm{d}x\\ &-\frac{\beta+1}{q}\int_\Omega \lambda d(x)v_1^q\,\mathrm{d}x-\lambda\int_\Omega b(x)u_1^{\alpha+1}v_1^{\beta+1}\,\mathrm{d}x<0\,. \end{aligned}$$

Since $\int_\Omega \mu(x)u_1^{\gamma+1}v_1^{\delta+1}\,\mathrm{d}x>0$, we have $\frac{1}{(\gamma+1)(\delta+1)}\int_\Omega \mu(x)u_1^{\gamma+1}v_1^{\delta+1}\,\mathrm{d}x>0$, which is a contradiction that completes the proof. $\square$

## References

[1] W. Allegretto and Y. Huang; *A Picone's identity for the p-Laplacian and applications.* Nonlinear Analysis TMA, 32(1998), 819-830.

[2] H. Berestycki, I. Capuzzo Dolcetta and L. Nirenberg; *Variational methods for indefinite superlinear homogeneous elliptic problems.* Nonlinear Differential Equations Appl., 2(1995), 553-572.

[3] P. A. Binding, P. Drabek and Y. Huang; *On Neumann boundary value problems for some quasilinear elliptic equations.* Electron. J. Differential Equations, 1997(1997), No. 5, 1-11.

[4] K. J. Brown and Y. Zhang; *The Nehari manifold for a semilinear elliptic equation with a sign-changing weight function.* J. Differential Equations, 193(2003), 481-499.

[5] P. Drabek, N. M. Stavrakakis and N. B. Zographopoulos; *Multiple nonsemitrivial solutions for quasilinear elliptic systems.* Differential and Integral Equations, 16(2003), 1519-1531.

[6] J. Fleckinger, R. Mansevich, N. M. Stravrakakis and F. de Thelin; *Principal eigenvalues for some quasilinear elliptic equations on $R^N$.* Adv. Diff. Equ., 2(1997), 981-1003.

[7] Z. Nehari; *On a class of nonlinear second order differential equations.* Trans. Amer. Math. Soc., 95(1960), 101-123.

[8] N. M. Stavrakakis and N.B. Zographopoulos; *Existence results for some quasilinear elliptic systems in $R^N$.* Electron. J. Differential Equations, 1999 (1999), No. 39, 1-15.

[9] N. M. Stavrakakis and N. B. Zographopoulos; *Global bifurcation results for some quasilinear elliptic systems on $R^N$*, Adv. Diff. Equ., 8(2003), 315-336.

[10] M. Willem, *Minimax Theorems.* Birkhauser. Boston, 1996.

[11] N. B. Zographopoulos, *On the isolation of the principal eigenvalue for a p-laplacian systems.* Appl. Math. Lett., to appear.

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