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BOUNDARY AND INITIAL VALUE PROBLEMS FOR SECOND-ORDER NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we consider the three-point boundary-value problem for the second order neutral functional differential equation

$$u'' + f(t, u_t, u'(t)) = 0, \quad 0 \le t \le 1,$$

with the three-point boundary condition $u_0 = \phi$, $u(1) = u(\eta)$. Under suitable assumptions on the function f we prove the existence, uniqueness and continuous dependence of solutions. As an application of the methods used, we study the existence of solutions for the same equation with a "mixed" boundary condition $u_0 = \phi, u(1) = \alpha[u'(\eta) - u'(0)]$, or with an initial condition $u_0 = \phi, u'(0) = 0$. For the initial-value problem, the uniqueness and continuous dependence of solutions are also considered. Furthermore, the paper shows that the solution set of the initial-value problem is nonempty, compact and connected. Our approach is based on the fixed point theory.

1. Introduction

Let $C = C([-r, 0]; \mathbb{R})$, with $r > 0$ is a fixed constant, be the Banach space of all continuous functions $\phi : [-r, 0] \to \mathbb{R}$, with the sup-norm $\|\phi\| = \sup\{|\phi(\theta)| : -r \le \theta \le 0\}$. For any continuous function $u : [-r, 1] \to \mathbb{R}$ and for any $t \in [0, 1]$, we denote by u_t the element of C defined by $u_t(\theta) = u(t + \theta), \theta \in [-r, 0]$. In this paper, we consider the second-order neutral functional differential equation

$$u'' + f(t, u_t, u'(t)) = 0, \quad 0 \le t \le 1, \tag{1.1}$$

where $f : [0, 1] \times C \times \mathbb{R} \to \mathbb{R}$ is continuous, with one of the following boundary conditions

$$u_0 = \phi, \quad u(1) = u(\eta) \tag{1.2}$$

$$u_0 = \phi, \quad u(1) = \alpha[u'(\eta) - u'(0)], \tag{1.3}$$

or with the initial conditions

$$u_0 = \phi, \quad u'(0) = 0, \tag{1.4}$$

where $\phi \in C$, $0 < \eta < 1$, and $\alpha \in \mathbb{R}$.

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The boundary-value problems for ordinary differential equation and for neutral functional differential equations have been studied by several authors by using the Leray-Schauder continuation theorem, Leray-Schauder nonlinear alternative, topological transversality method. We refer the reader for example to [2, 4, 5, 6, 7, 9] and references therein.

In [5], the author proved the existence of solution for the neutral FDE

$$\frac{d}{dt}[x'(t) - g(t, x_t)] = f(t, x_t, x'(t)), \quad 0 \le t \le 1,$$
$$x_0 = \phi, x(1) = \eta,$$

where $f : [0, 1] \times C \times \mathbb{R}^n \to \mathbb{R}^n, g : [0, 1] \times C \to \mathbb{R}^n$ are continuous functions, $\phi \in C, \eta \in \mathbb{R}^n$. In [9], the existence, uniqueness and continuous dependence on a real parameter α of the solution for the following problem were established

$$\big(\Lambda(t)x'(t)\big)' = f(t, x_t, x'(t)), \quad 0 \le t \le T,$$
$$x_0 = \phi, \quad Ax(T) + Bx'(T) = v,$$

where $\Lambda(t)$ is an $n \times n$ continuous matrix defined on $[0, T]$, A and B are $n \times n$ constant matrices, $v \in \mathbb{R}^n, \phi \in C = C\big([-r, 0]; \mathbb{R}^n\big)$.

Recently in [4, 7], the authors studied the boundary-value problem

$$u'' + f(t, u) = 0, \quad 0 < t < 1,$$

where $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous, with one of the following boundary conditions

$$u(0) = 0, \quad u(1) = \alpha u(\eta),$$

or

$$u'(0) = 0, \quad u(1) = \alpha u'(\eta).$$

In the base of the above papers, we shall consider the problems for FDEs (1.1), (1.2); (1.1), (1.3) and (1.1), (1.4). This paper is organized as follows. In section 2, we present some preliminaries. By using Leray-Schauder nonlinear alternative, the existence theorems of boundary-value problem (1.1)-(1.2) are given in section 3. Furthermore, the uniqueness, based on the contraction mapping principle, and continuous dependence of solution are established. In sections 4; 5, as an application of the methods which are used in the proofs of section 3, we also study the existence of solution for the equation (1.1) with a "mixed" bonundary condition (1.3) or with an initial condition (1.4). For the initial value problem (1.1)-(1.4), the uniqueness and continuous dependence of solution are also considered. From the results, based on the topological degree theory of compact vector fields, the paper shows that the solution set of the initial value problem is nonempty, compact and connected.

2. Preliminaries

We denote by $C[0, 1]$ and $C^1[0, 1]$, respectively, the Banach spaces of continuous real functions and continuously differentiable real functions on $[0, 1]$, with the norms:

$$\|u\|_0 = \sup\{|u(t)| : 0 \le t \le 1\},$$
$$\|u\|_1 = \max\{\|u\|_0, \|u'\|_0\},$$

where $\|u'\|_0 = \sup\{|u'(t)| : 0 \le t \le 1\}$, and by $L^1[0, 1]$ the space of all real functions $x(t)$ such that $|x(t)|$ is Lebesgue integrable on $[0, 1]$. The proofs of our theorems are based on the following theorems result.

Theorem 2.1 (Nonlinear Alternative of Leray-Schauder). *Let E be a Banach space and Ω be a bounded open subset of E, $0 \in \Omega$, $T : \overline{\Omega} \to E$ be a completely continuous operator. Then, either there exists $x \in \partial\Omega$ such that $Tx = \lambda x$ for some $\lambda > 1$, or there exists a fixed point $x \in \Omega$.*

The proof of the theorem above can be found in [6, Theorem 2.10].

Theorem 2.2 ([3]). *Let $(E, |\cdot|)$ be a real Banach space, D be a bounded open subset of E with boundary ∂D, closure $\overline{D}$ and $T : \overline{D} \to E$ be a completely continuous operator. Assume that T satisfies the follows conditions:*

(i) *T has no fixed points on ∂D and $\gamma(I - T, D) \neq 0$.*

(ii) *For each $\varepsilon > 0$, there is a completely continuous operator T_ε such that $|T_\varepsilon(x) - T(x)| < \varepsilon$, for all $x \in \overline{D}$, and such that for each h with $|h| < \varepsilon$, the equation $x = T_\varepsilon(x) + h$ has at most one solution in $\overline{D}$.*

Then the set of fixed points of T is nonempty, compact and connected.

The proof of the theorem above can be found in [3, theorem 48.2]. We remark that condition (i) is equivalent to the following condition.

($\tilde{\text{i}}$) T has no fixed points on ∂D and $\deg(I - T, D, 0) \neq 0$.

Because of this, if a completely continuous operator T is defined on $\overline{D}$ and has no fixed points on ∂D, then the rotation $\gamma(I-T, D)$ coincides with the Leray-Schauder degree of $I - T$ on D with respect to the origin, see [3, section 20.2].

Theorem 2.3 ([1]). *Let E, F be Banach spaces, D be an open subset of E and $f : D \to F$ be continuous. Then for each $\varepsilon > 0$, there is a mapping $f_\varepsilon : D \to F$ that is locally Lipschitz such that*

$$|f(x) - f_\varepsilon(x)| < \varepsilon, \quad \forall x \in D,$$

and $f_\varepsilon(D)$ is a subset of the closed convex hull of $f(D)$.

The proof of the above theorem can be found in [1, p. 53]. We will need the following lemmas later. The proofs of these lemmas are not difficult and we omit them.

Lemma 2.4 ([4]). *For $y \in C[0, 1]$, the problem*

$$u'' + y(t) = 0, \quad t \in (0, 1),$$
$$u(0) = 0, \quad u(1) = u(\eta),$$

with $\eta \in (0, 1)$, has a unique solution

$$u(t) = -\int_0^t (t - s)y(s)ds - \frac{t}{1-\eta}\int_0^\eta (\eta - s)y(s)ds + \frac{t}{1-\eta}\int_0^1 (1 - s)y(s)ds,$$

$t \in [0, 1]$.

Lemma 2.5. *For $y \in C[0, 1]$, the "mixed" boundary-value problem*

$$u'' + y(t) = 0, \quad t \in (0, 1),$$
$$u(0) = 0, \quad u(1) = \alpha(u'(\eta) - u'(0)),$$

with $\eta \in (0, 1)$ and $\alpha \in \mathbb{R}$, has a unique solution

$$u(t) = -\int_0^t (t - s)y(s)ds - \alpha t\int_0^\eta y(s)ds + t\int_0^1 (1 - s)y(s)ds, \quad t \in [0, 1].$$

Lemma 2.6. *For $y \in C[0,1]$, the initial-value problem*

$$u'' + y(t) = 0, \quad 0 < t \le 1,$$
$$u(0) = 0, \quad u'(0) = 0,$$

has a unique solution

$$u(t) = -\int_0^t (t-s)y(s)ds, \quad t \in [0,1].$$

3. Main Results

In this section, we present our existence results for the boundary-value problem (1.1)-(1.2).

Theorem 3.1. *Let $f : [0,1] \times C \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that there exist nonnegative functions p, q, $r \in L^1[0,1]$ such that*

- (H1) $|f(t,u,v)| \le p(t)\|u\| + q(t)|v| + r(t)$, *for all* $(t,u,v) \in [0,1] \times C \times \mathbb{R}$
- (H2) $\frac{2-\eta}{1-\eta}\int_0^1 (1-s)p(s)ds + \frac{1}{1-\eta}\int_0^\eta (\eta - s)p(s)ds < 1$,
- (H3) $\int_0^1 [p(s)+q(s)]ds + \frac{1}{1-\eta}\int_0^1 (1-s)[p(s)+q(s)]ds + \frac{1}{1-\eta}\int_0^\eta (\eta-s)[p(s)+q(s)]ds < 1$.

Then the boundary-value problem (1.1)-(1.2) *has at least one solution.*

Proof. **Step 1**. Consider first the case $\phi(0) = 0$. Put

$$C_0 = \{u \in C^1[0,1] : u(0) = 0\}.$$

Then C_0 is the subspace of $C^1[0,1]$. We note that for all $u \in C_0$, $u(t) = \int_0^t u'(s)ds$, so

$$\|u\|_0 \le \|u'\|_0. \tag{3.1}$$

For a function $u \in C_0$, we define the function $\widehat{u} : [-r,1] \to \mathbb{R}$ by

$$\widehat{u}(t) = \begin{cases} \phi(t), & t \in [-r,0], \\ u(t), & t \in [0,1]. \end{cases}$$

We also note that

$$\|\widehat{u}_t\|^k \le \max\{\|u\|_0^k, \|\phi\|^k\} \le \|u\|_0^k + \|\phi\|^k, \forall t \in [0,1], k \ge 0. \tag{3.2}$$

Define the integral operator $T : C_0 \to C^1[0,1]$ by

$$\begin{aligned} Tu(t) = &-\int_0^t (t-s)f(s,\widehat{u}_s,u'(s))ds - \frac{t}{1-\eta}\int_0^\eta (\eta-s)f(s,\widehat{u}_s,u'(s))ds \\ &+ \frac{t}{1-\eta}\int_0^1 (1-s)f(s,\widehat{u}_s,u'(s))ds, \quad t \in [0,1]. \end{aligned} \tag{3.3}$$

By Lemma 2.4, it is obvious that $\overline{u}$ is a solution of the boundary-value problem (1.1)-(1.2) if and only if the operator T has a fixed point $u \in C_0$, where

$$\overline{u}(t) = \begin{cases} \phi(t), & t \in [-r,0], \\ u(t), & t \in [0,1]. \end{cases}$$

Using (H1) and (3.2), for all $u \in C_0$, for all $t \in [0,1]$, we obtain

$$\begin{aligned}
|Tu(t)| &\leq \int_0^1 (1-s)[p(s)\|\widehat{u}_s\| + q(s)|u'(s)| + r(s)]ds \\
&\quad + \frac{1}{1-\eta}\int_0^\eta (\eta - s)[p(s)\|\widehat{u}_s\| + q(s)|u'(s)| + r(s)]ds \\
&\quad + \frac{1}{1-\eta}\int_0^1 (1-s)[p(s)\|\widehat{u}_s\| + q(s)|u'(s)| + r(s)]ds \\
&\leq A_1\|u\|_0 + B_1\|u'\|_0 + C_1,
\end{aligned}$$

where

$$\begin{aligned}
A_1 &= \frac{2-\eta}{1-\eta}\int_0^1 (1-s)p(s)ds + \frac{1}{1-\eta}\int_0^\eta (\eta-s)p(s)ds, \\
B_1 &= \frac{2-\eta}{1-\eta}\int_0^1 (1-s)q(s) + \frac{1}{1-\eta}\int_0^\eta (\eta-s)q(s)ds, \\
C_1 &= \Big(\frac{2-\eta}{1-\eta}\int_0^1 (1-s)p(s)ds + \frac{1}{1-\eta}\int_0^\eta (\eta-s)p(s)ds\Big)\|\phi\| \\
&\quad + \frac{2-\eta}{1-\eta}\int_0^1 (1-s)r(s)ds + \frac{1}{1-\eta}\int_0^\eta (\eta-s)r(s)ds.
\end{aligned}$$

Hence

$$\|Tu\|_0 \leq A_1\|u\|_0 + B_1\|u'\|_0 + C_1, \quad \forall u \in C_0. \tag{3.4}$$

On the other hand,

$$\begin{aligned}
(Tu)'(t) &= -\int_0^t f(s,\widehat{u}_s,u'(s))ds - \frac{1}{1-\eta}\int_0^\eta (\eta-s)f(s,\widehat{u}_s,u'(s))ds \\
&\quad + \frac{1}{1-\eta}\int_0^1 (1-s)f(s,\widehat{u}_s,u'(s))ds, \quad t\in[0,1].
\end{aligned} \tag{3.5}$$

Similarly, it follows from (H1) and (3.2) that

$$\|(Tu)'\|_0 \leq A_2\|u\|_0 + B_2\|u'\|_0 + C_2, \quad \forall u \in C_0, \tag{3.6}$$

where

$$\begin{aligned}
A_2 &= \int_0^1 p(s)ds + \frac{1}{1-\eta}\int_0^1 (1-s)p(s)ds + \frac{1}{1-\eta}\int_0^\eta (\eta-s)p(s)ds, \\
B_2 &= \int_0^1 q(s)ds + \frac{1}{1-\eta}\int_0^1 (1-s)q(s) + \frac{1}{1-\eta}\int_0^\eta (\eta-s)q(s)ds, \\
C_2 &= \Big(\int_0^1 p(s)ds + \frac{1}{1-\eta}\int_0^1 (1-s)p(s)ds + \frac{1}{1-\eta}\int_0^\eta (\eta-s)p(s)ds\Big)\|\phi\| \\
&\quad + \int_0^1 r(s)ds + \frac{1}{1-\eta}\int_0^1 (1-s)r(s)ds + \frac{1}{1-\eta}\int_0^\eta (\eta-s)r(s)ds.
\end{aligned}$$

Put

$$A = \max\{A_1, A_2 + B_2\}. \tag{3.7}$$

From (H2)-(H3), it follows that $A_1 < 1$, $A_2 + B_2 < 1$, so $A < 1$. We now choose a constant $B > 0$ such that

$$B \geq \max\{\frac{B_1 C_2}{1 - A_2 - B_2} + C_1, C_2\}, \tag{3.8}$$

and put

$$m = \frac{B}{1-A}, \quad \Omega = \{u \in C_0 : \|u\|_1 < m\}. \tag{3.9}$$

Then Ω be a bounded open subset of C_0, $0 \in \Omega$, and $\partial\Omega = \{u \in C_0 : \|u\|_1 = m\}$. We shall show that $T : \overline{\Omega} = \Omega \cup \partial\Omega \to C^1[0,1]$ has a fixed point $u \in \Omega$ by applying Theorem 2.1.

(a) First, T is continuous. Indeed, for each $u_0 \in \overline{\Omega}$, let $\{u_n\}$ be a sequence in $\overline{\Omega}$ such that $\lim_{n\to\infty} u_n = u_0$. For all $t \in [0,1]$, from (3.3), we get

$$\begin{aligned} Tu_n(t) - Tu_0(t) = &-\int_0^t (t-s)\Big[f(s,(\widehat{u}_n)_s,u_n'(s)) - f(s,(\widehat{u}_0)_s,u_0'(s))\Big]ds \\ &- \frac{t}{1-\eta}\int_0^\eta (\eta-s)\Big[f(s,(\widehat{u}_n)_s,u_n'(s)) - f(s,(\widehat{u}_0)_s,u_0'(s))\Big]ds \\ &+ \frac{t}{1-\eta}\int_0^1 (1-s)\Big[f(s,(\widehat{u}_n)_s,u_n'(s)) - f(s,(\widehat{u}_0)_s,u_0'(s))\Big]ds. \end{aligned}$$

Put $D = \{(\widehat{u}_n)_s : s \in [0,1], n = 0,1,2,\dots\}$, then D is compact in C. Since $f : [0,1] \times C \times \mathbb{R} \to \mathbb{R}$ is continuous, f is uniformly continuous on the compact subset $[0,1] \times D \times [-m,m]$. This implies that, for all $\varepsilon > 0$, there exists $\delta > 0$ such that for each (s_1,ϕ_1,ν_1), $(s_2,\phi_2,\nu_2) \in [0,1] \times D \times [-m,m]$,

$$\begin{gathered} |s_1 - s_2| < \delta, \quad \|\phi_1 - \phi_2\| < \delta, \quad |\nu_1 - \nu_2| < \delta \\ \Rightarrow \quad |f(s_1,\phi_1,\nu_1) - f(s_2,\phi_2,\nu_2)| < \frac{\varepsilon}{2\beta}, \end{gathered}$$

with $\beta = 1 + \frac{2}{1-\eta} > 0$. Since $\lim_{n\to\infty} u_n = u_0$ in $\overline{\Omega}$, with respect to $\|\cdot\|_1$, there exists n_0 such that for all $n \ge n_0$,

$$\|(\widehat{u}_n)_s - (\widehat{u}_0)_s\| < \delta, |u_n'(s) - u_0'(s)| < \delta, \quad \forall s \in [0,1].$$

On the other hand, for all $s \in [0,1]$, $\big(s,(\widehat{u}_n)_s,u_n'(s)\big), \big(s,(\widehat{u}_0)_s,u_0'(s)\big) \in [0,1] \times D \times [-m,m]$, therefore, for all $n \ge n_0$,

$$\begin{aligned} |Tu_n(t) - Tu_0(t)| &\le (1 + \frac{2}{1-\eta})\int_0^1 \Big|f(s,(\widehat{u}_n)_s,u_n'(s)) - f(s,(\widehat{u}_0)_s,u_0'(s))\Big|ds \\ &< (1 + \frac{2}{1-\eta})\frac{\varepsilon}{2\beta} = \frac{\varepsilon}{2}, \quad \forall t \in [0,1]. \end{aligned}$$

Similarly

$$|(Tu_n)'(t) - (Tu_0)'(t)| < \frac{\varepsilon}{2}, \quad \forall t \in [0,1].$$

This implies that for all $n \ge n_0$,

$$\|Tu_n - Tu_0\|_1 = \max\Big\{\|Tu_n - Tu_0\|_0, \|(Tu_n)' - (Tu_0)'\|_0\Big\} \le \frac{\varepsilon}{2} < \varepsilon.$$

(b) Next, we show that $T(\overline{\Omega})$ is relatively compact. Let $\{Tu_n\}$ be a bounded sequence of $T(\overline{\Omega})$, corresponding $\{u_n\} \subset \overline{\Omega}$, we shall show that $\{Tu_n\}$ contains a convergence subsequence in $C^1[0,1]$, with respect to $\|.\|_1$. The proof of this fact is obtained as follows. For all n, it follows from (3.4), (3.6), (3.9) that

$$\begin{aligned} \|Tu_n\|_0 &\le A_1\|u_n\|_0 + B_1\|u_n'\|_0 + C_1 \le A_1 m + B_1 m + C_1, \\ \|(Tu_n)'\|_0 &\le A_2\|u_n\|_0 + B_2\|u_n'\|_0 + C_2 \le A_2 m + B_2 m + C_2. \end{aligned}$$

Hence, the sequences $\{Tu_n\}, \{(Tu_n)'\}$ are uniformly bounded. On the other hand, combining (3.3), (3.5), (3.9) and (H1), for all n, for all $t_1, t_2 \in [0,1]$, we have

$$
\begin{aligned}
&|Tu_n(t_1) - Tu_n(t_2)| \\
&\leq \Big| \int_{t_1}^{t_2} (1-s)[(m+\|\phi\|)p(s) + mq(s) + r(s)]ds \Big| \\
&\quad + \frac{1}{1-\eta}\Big(\int_0^{\eta} (\eta - s)[(m+\|\phi\|)p(s) + mq(s) + r(s)]ds \Big)|t_1 - t_2| \\
&\quad + \frac{1}{1-\eta}\Big(\int_0^{1} [(m+\|\phi\|)p(s) + mq(s) + r(s)]ds \Big)|t_1 - t_2| \\
&\leq K_1|t_1 - t_2|,
\end{aligned}
$$

$$
\begin{aligned}
|(Tu_n)'(t_1) - (Tu_n)'(t_2)| &\leq \Big| \int_{t_1}^{t_2} [(m+\|\phi\|)p(s) + mq(s) + r(s)]ds \Big| \\
&\leq K_2|t_1 - t_2|,
\end{aligned}
$$

where K_1, K_2 are independent of t_1, t_2 and n. This implies that the sequences $\{Tu_n\}, \{(Tu_n)'\}$ are equi-continuous. By using the Ascoli-Arzela theorem, we have $\{Tu_n\}$, $\{(Tu_n)'\}$ are relatively compact in $C[0,1]$. Therefore, there exists a subsequence $\{u_{n_k}\} \subset \{u_n\}$, such that

$$
Tu_{n_k} \to u \quad \text{and} \quad (Tu_{n_k})' \to v, \quad \text{as} \quad k \to \infty,
$$

with respect to $\|.\|_0$. Then u is differentiable and $u' = v$, so $Tu_{n_k} \to u$, as $k \to \infty$, in $C^1[0,1]$, with respect to $\|.\|_1$. Thus T is completely continuous.
(c) Finally, suppose that there exists $u^* \in \partial\Omega$, such that $T(u^*) = \lambda u^*$, for some $\lambda > 1$. Then, we have the following set is bounded

$$
\{u^* \in \partial\Omega : T(u^*) = \lambda u^*, \lambda > 1\}.
$$

Indeed, it follows from (3.6) that

$$
\|(u^*)'\|_0 = \frac{1}{\lambda}\|(Tu^*)'\|_0 \leq \|(Tu^*)'\|_0 \leq A_2\|u^*\|_0 + B_2\|(u^*)'\|_0 + C_2. \tag{3.10}
$$

Combining (3.1), (3.10), we get

$$
(1 - A_2 - B_2)\|(u^*)'\|_0 \leq C_2.
$$

Since $A_2 + B_2 < 1$, this implies that

$$
\|(u^*)'\|_0 \leq M, \tag{3.11}
$$

where $M = C_2/(1-A_2-B_2)$ is a constant. Thus, combining (3.1), (3.4), (3.6)-(3.8), (3.10) and (3.11), we obtain

$$
\begin{aligned}
\|Tu^*\|_0 &\leq A_1\|u^*\|_0 + B_1\|(u^*)'\|_0 + C_1 \\
&\leq A_1\|u^*\|_0 + B_1 M + C_1 \\
&\leq A\|u^*\|_0 + B, \\
\|(Tu^*)'\|_0 &\leq A_2|u^*\|_1 + B_2\|u^*\|_1 + C_2 \\
&\leq A\|u^*\|_1 + B.
\end{aligned} \tag{3.12}
$$

Consequently

$$
\lambda\|u^*\|_1 = \|Tu^*\|_1 \leq A\|u^*\|_1 + B,
$$

which implies

$$\lambda m \le Am + B \quad \text{or} \quad \lambda \le A + \frac{B}{m}, \quad \text{i.e. } \lambda \le 1,$$

this contradicts $\lambda > 1$. The proof of step 1 is complete.

Step 2. The case $\phi(0) \neq 0$. By the transformation $v = u - \phi(0)$, the boundary-value problem (1.1)-(1.2) reduces to the boundary-value problem

$$\begin{gathered} v'' + f(t, v_t + \phi(0), v'(t)) = 0, \quad 0 \le t \le 1, \\ v_0 = \phi - \phi(0) \equiv \widetilde{\phi}, \quad v(1) = v(\eta), \end{gathered}$$

with $\widetilde{\phi} \in C$ and $\widetilde{\phi}(0) = 0$. By step 1, this boundary-value problem has at least one solution. Step 2 follows and Theorem 3.1 is proved. $\square$

Theorem 3.2. *Let $f : [0,1] \times C \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that there exist nonnegative functions p, q, $r \in L^1[0,1]$ and reals constants $k, l \in [0,1]$ such that (H2) holds and*

- ($\tilde{\text{H}}$1) $|f(t,u,v)| \le p(t)\|u\|^k + q(t)|v|^l + r(t)$, *for all* $(t,u,v) \in [0,1] \times C \times \mathbb{R}$,
- ($\tilde{\text{H}}$3) $Q(k)A_2 + Q(l)B_2 < 1$,

where

$$\begin{aligned} A_2 &= \int_0^1 p(s)ds + \frac{1}{1-\eta}\int_0^1 (1-s)p(s)ds + \frac{1}{1-\eta}\int_0^\eta (\eta - s)p(s)ds, \\ B_2 &= \int_0^1 q(s)ds + \frac{1}{1-\eta}\int_0^1 (1-s)q(s) + \frac{1}{1-\eta}\int_0^\eta (\eta - s)q(s)ds, \end{aligned}$$

and

$$Q(\mu) = \begin{cases} 0, & 0 \le \mu < 1, \\ 1, & \mu = 1. \end{cases}$$

Then the boundary-value problem $(1.1) - (1.2)$ *has at least one solution.*

Proof. It is obvious that the Theorem 3.1 is a special case of this theorem with $k = l = 1$. Here, we consider only the case $\phi(0) = 0$ and let the subspace C_0, the function $\widehat{u}$ and the operator T be defined as in Theorem 3.1. Using ($\tilde{\text{H}}$1) and (3.2), for all $u \in C_0$ and all $t \in [0,1]$, we have

$$\begin{aligned} |Tu(t)| &\le \int_0^1 (1-s)[p(s)\|\widehat{u}_s\|^k + q(s)|u'(s)|^l + r(s)]ds \\ &\quad + \frac{1}{1-\eta}\int_0^\eta (\eta - s)[p(s)\|\widehat{u}_s\|^k + q(s)|u'(s)|^l + r(s)]ds \\ &\quad + \frac{1}{1-\eta}\int_0^1 (1-s)[p(s)\|\widehat{u}_s\|^k + q(s)|u'(s)|^l + r(s)]ds \\ &\le A_1\|u\|_0^k + B_1\|u'\|_0^l + C_3, \end{aligned}$$

where A_1 and B_1 as in Theorem 3.1, and

$$\begin{aligned} C_3 = &\Big(\frac{2-\eta}{1-\eta}\int_0^1 (1-s)p(s)ds + \frac{1}{1-\eta}\int_0^\eta (\eta - s)p(s)ds\Big)\|\phi\|^k \\ &+ \frac{2-\eta}{1-\eta}\int_0^1 (1-s)r(s)ds + \frac{1}{1-\eta}\int_0^\eta (\eta - s)r(s)ds. \end{aligned}$$

It follows that for all $u \in C_0$,

$$\|Tu\|_0 \le A_1\|u\|_0^k + B_1\|u'\|_0^l + C_3. \tag{3.13}$$

Similarly, for all $u \in C_0$, we obtain

$$\begin{aligned}\|(Tu)'\|_0 &\le A_2\|u\|_0^k + B_2\|u'\|_0^l + C_4\\ &\le A_2\|u'\|_0^k + B_2\|u'\|_0^l + C_4,\end{aligned} \tag{3.14}$$

where A_2 and B_2 are as above and

$$\begin{aligned}C_4 =&\Big(\int_0^1 p(s)ds + \frac{1}{1-\eta}\int_0^1 (1-s)p(s)ds + \frac{1}{1-\eta}\int_0^\eta (\eta-s)p(s)ds\Big)\|\phi\|^k\\ &+\int_0^1 r(s)ds + \frac{1}{1-\eta}\int_0^1 (1-s)r(s)ds + \frac{1}{1-\eta}\int_0^\eta (\eta-s)r(s)ds.\end{aligned}$$

Clearly, as the proof of the Theorem 3.1, if we show the boundedness of the following set

$$\{u^* \in \partial\Omega : T(u^*) = \lambda u^*, \lambda > 1\}, \tag{3.15}$$

then, combining the assume (H2), the proof of Theorem 3.2 will be completely. That is proved as follows.

Suppose that there exists $u^* \in \partial\Omega$ such that $T(u^*) = \lambda u^*$ for some $\lambda > 1$. We consider three cases.

Case 1: $0 \le k < 1$, $0 \le l < 1$. If $\|(u^*)'\|_0 > 1$, then from (3.14), we have

$$\|(Tu^*)'\|_0 \le (A_2 + B_2)\|(u^*)'\|_0^h + C_4, \tag{3.16}$$

where $h = \max\{k, l\}$. It follows that

$$\|(u^*)'\|_0 = \frac{1}{\lambda}\|(Tu^*)'\|_0 \le \|(Tu^*)'\|_0 \le (A_2 + B_2)\|(u^*)'\|_0^h + C_4. \tag{3.17}$$

Here, let us note that if $K \ge 0$, $H > 0$, $0 \le \beta < 2$ are given constants, then there exists a constant $C > 0$ such that

$$Kx^\beta \le \frac{Hx^2}{2} + C, \quad \forall x \ge 0. \tag{3.18}$$

Hence, with $x = \sqrt{\|(u^*)'\|_0}$, $K = A_2 + B_2$, $\beta = 2h, H = 1$, the inequality (3.18) implies that

$$(A_2 + B_2)\|(u^*)'\|_0^h + C_4 \le \frac{1}{2}\|(u^*)'\|_0 + C_4 + C.$$

Combining the above inequalities,

$$\|(u^*)'\|_0 \le \frac{1}{2}\|(u^*)'\|_0 + C_4 + C \quad \text{or} \quad \|(u^*)'\|_0 \le 2C_4 + 2C.$$

We can choose C such that $2C_4 + 2C > 1$; therefore,

$$\|(u^*)'\|_0 \le 2C_4 + 2C,$$

although $\|(u^*)'\|_0 \le 1$ or $\|(u^*)'\|_0 > 1$. Thus, in case 1, there exists a positive constant $\widetilde{M} = 2C_4 + 2C$, such that

$$\|(u^*)'\|_0 \le \widetilde{M}. \tag{3.19}$$

Case 2: $k = 1$, $0 \le l < 1$. From (3.14), we have

$$\|(Tu^*)'\|_0 \le A_2\|(u^*)'\|_0 + B_2\|(u^*)'\|_0^l + C_2,$$

where $C_4 = C_2$, since $k = 1$. So we have

$$(1 - A_2)\|(u^*)'\|_0 \le B_2\|(u^*)'\|_0^l + C_2.$$

Clearly, from ($\tilde{\mathrm{H}}$3), $A_2 < 1$. Using (3.18) again, with $x = \sqrt{\|(u^*)'\|_0}$, $K = B_2$, $\beta = 2l$, $H = 1 - A_2$, we get

$$B_2\|(u^*)'\|_0^l + C_2 \le \frac{1}{2}(1 - A_2)\|(u^*)'\|_0 + C_2 + \widetilde{C},$$

and so

$$(1 - A_2)\|(u^*)'\|_0 \le \frac{1}{2}(1 - A_2)\|(u^*)'\|_0 + C_2 + \widetilde{C} \Leftrightarrow \|(u^*)'\|_0 \le \frac{2C_2 + 2\widetilde{C}}{1 - A_2},$$

where $\widetilde{C}$ is a positive constant. We deduce that (3.19) also holds in the second case, in which $\widetilde{M} = \frac{2C_2+2\widetilde{C}}{1-A_2}$.

Case 3: $0 \le k < 1$, $l = 1$. We conclude from the hypothesis ($\tilde{\mathrm{H}}$3) that $B_2 < 1$, hence that it is similar to the above cases, (3.19) also holds. Therefore, Theorem 3.2 is proved. □

Now, we present the uniqueness of the solution of the boundary-value problem (1.1)-(1.2).

Theorem 3.3. *Let $f : [0, 1] \times C \times \mathbb{R} \to \mathbb{R}$ be continuous function and satisfy on $[0, 1] \times C \times \mathbb{R}$ the Lipschitz condition*

$$|f(t, u, v) - f(t, \widetilde{u}, \widetilde{v})| \le \theta(\|u - \widetilde{u}\| + |v - \widetilde{v}|),$$

for some positive constant θ. If $2(1+\frac{2}{1-\eta})\theta < 1$, then there exists a unique solution of (1.1)-(1.2).

Proof. Let S be the space of continuous functions $u : [-r, 1] \to \mathbb{R}$ such that u is continuously differentiable on $[0, 1]$ and $u_0 = \phi$. We define

$$d(u, v) = \max\big\{ \max_{0\le t\le 1} |u(t) - v(t)|, \max_{0\le t\le 1} |u'(t) - v'(t)|\big\}. \tag{3.20}$$

Then S is a completely metrizable space with the distance function d. By Lemma 2.4, for each $u \in S$, the problem

$$\begin{gathered} x'' + f(t, u_t, u'(t)) = 0, \quad 0 \le t \le 1, \\ x(0) = \phi(0), \quad x(1) = x(\eta), \end{gathered} \tag{3.21}$$

has a unique solution on $[0, 1]$ which is defined as

$$\begin{aligned} x(t) =&\phi(0) - \int_0^t (t - s)f(s, u_s, u'(s))ds - \frac{t}{1-\eta}\int_0^\eta (\eta - s)f(s, u_s, u'(s))ds \\ &+ \frac{t}{1-\eta}\int_0^1 (1 - s)f(s, u_s, u'(s))ds, \quad t \in [0, 1]. \end{aligned}$$

We define $\widetilde{u} \in S$, by $\widetilde{u}(t) = x(t)$ on $[0, 1]$ and $\widetilde{u}_0 = \phi$. Therefore, the mapping $P : S \to S$ is defined by

$$P(u) = \widetilde{u}, \quad u \in S.$$

For any $u, v \in S$, we put $w = \widetilde{u} - \widetilde{v}$. Then w satisfies

$$\begin{gathered} w'' + f(t, u_t, u'(t)) - f(t, v_t, v'(t)) = 0, \quad 0 \le t \le 1, \\ w_0 = 0, \quad w(1) = w(\eta). \end{gathered} \tag{3.22}$$

It follows that for all $t \in [0,1]$, we have

$$\begin{aligned} |w(t)| &\le \int_0^1 |f(s,u_s,u'(s)) - f(t,v_s,v'(s))|ds \\ &\quad + \frac{1}{1-\eta}\int_0^\eta |f(s,u_s,u'(s)) - f(t,v_s,v'(s))|ds \\ &\quad + \frac{1}{1-\eta}\int_0^1 |f(s,u_s,u'(s)) - f(t,v_s,v'(s))|ds \\ &\le K\theta \int_0^1 \big(\|u_s - v_s\| + |u'(s) - v'(s)|\big)ds \\ &\le K\theta\Big(\max_{0\le t\le 1}|u(t)-v(t)| + \max_{0\le t\le 1}|u'(t)-v'(t)|\Big), \end{aligned} \tag{3.23}$$

where $K = 1 + \frac{2}{1-\eta}$. Similarly,

$$\begin{aligned} |w'(t)| &\le K\int_0^1 |f(s,u_s,u'(s)) - f(t,v_s,v'(s))|ds \\ &\le K\theta\Big(\max_{0\le t\le 1}|u(t)-v(t)| + \max_{0\le t\le 1}|u'(t)-v'(t)|\Big). \end{aligned} \tag{3.24}$$

By the definition of d, we have

$$\begin{aligned} d(\widetilde{u},\widetilde{v}) &= \max\big\{\max_{0\le t\le 1}|\widetilde{u}(t)-\widetilde{v}(t)|, \max_{0\le t\le 1}|\widetilde{u}'(t)-\widetilde{v}'(t)|\big\} \\ &\le K\theta\Big(\max_{0\le t\le 1}|u(t)-v(t)| + \max_{0\le t\le 1}|u'(t)-v'(t)|\Big) \\ &\le 2K\theta d(u,v). \end{aligned}$$

Since $2K\theta = 2(1+\frac{2}{1-\eta})\theta < 1$, we deduce that P is the contraction mapping. Therefore there exists a unique $u \in S$ such that $P(u) = u$. This implies that u is the unique solution of the boundary-value problem (1.1)-(1.2). Then Theorem 3.3 is proved. □

We remark that Theorem 3.3 remains valid if we consider the boundary-value problem

$$\begin{gathered} u'' + f(t,u_t,u'(t),\lambda), \quad 0 \le t \le 1, \\ u_0 = \phi, \quad u(1) = u(\eta), \end{gathered} \tag{3.25}$$

where λ is a real parameter and

$$|f(t,u,v,\lambda) - f(t,\widetilde{u},\widetilde{v},\lambda)| \le \theta(\|u-\widetilde{u}\| + |v-\widetilde{v}|), \tag{3.26}$$

on $[0,1]\times C\times\mathbb{R}\times\mathbb{R}$ for some positive constant θ, with

$$2(1+\frac{2}{1-\eta})\theta < 1. \tag{3.27}$$

In other words, by Theorem 3.3, if (3.26) , (3.27) hold then the boundary-value problem (3.25) has a unique solution $u(t) = u(t,\lambda)$ for each λ. We will show that the solution of (3.25) depends continuously on the parameter λ if

$$|f(t,u,v,\lambda_1) - f(t,u,v,\lambda_2)| \le L|\lambda_1 - \lambda_2|, \tag{3.28}$$

for some positive constant L, for all λ_1, λ_2.

Theorem 3.4. *Let $f : [0,1]\times C\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a continuous function. If* (3.26) *-*(3.28) *hold then the solution of* (3.25) *depends continuously on λ.*

Proof. Let $u(t) = u(t,\lambda_1)$ and $v(t) = v(t,\lambda_2)$ be solutions of (3.25) with $\lambda = \lambda_1$ and $\lambda = \lambda_2$, respectively. It follows from (3.23), (3.24) and (3.28) that for all $t \in [0,1]$,

$$\begin{aligned}
|u(t)-v(t)| &\le K\int_0^1 |f(s,u_s,u'(s),\lambda_1) - f(t,v_s,v'(s),\lambda_2)|ds \\
&\le K\int_0^1 |f(s,u_s,u'(s),\lambda_1) - f(t,v_s,v'(s),\lambda_1)|ds \\
&\quad + K\int_0^1 |f(s,v_s,v'(s),\lambda_1) - f(t,v_s,v'(s),\lambda_2)|ds \\
&\le K\theta\Big(\max_{0\le t\le 1}|u(t)-v(t)| + \max_{0\le t\le 1}|u'(t)-v'(t)|\Big) + KL|\lambda_1-\lambda_2|,
\end{aligned}$$

$$|u'(t)-v'(t)| \le K\theta\Big(\max_{0\le t\le 1}|u(t)-v(t)| + \max_{0\le t\le 1}|u'(t)-v'(t)|\Big) + KL|\lambda_1-\lambda_2|,$$

where $K = 1 + \frac{2}{1-\eta}$. Thus, in the completely metrizable space (S,d) which is defined as above, we have

$$\begin{aligned}
d(u,v) &= \max\big\{\max_{0\le t\le 1}\max|u(t)-v(t)|, \max_{0\le t\le 1}\max|u'(t)-v'(t)|\big\} \\
&\le K\theta\Big(\max_{0\le t\le 1}|u(t)-v(t)| + \max_{0\le t\le 1}|u'(t)-v'(t)|\Big) + KL|\lambda_1-\lambda_2| \\
&\le 2K\theta d(u,v) + KL|\lambda_1-\lambda_2|.
\end{aligned}$$

By (3.27), we have $2K\theta < 1$, so

$$d(u,v) \le \frac{KL}{1-2K\theta}|\lambda_1-\lambda_2|.$$

Thus, the solution of (3.25) depends continuously on the parameter λ. The proof of Theorem 3.4 is complete. $\square$

4. Application for the "mixed" boundary value problem

Now, we present our existence results for the solution to the boundary-value problem (1.1)-(1.3). Based on lemma 2.5, the proofs for the following theorems are similar to that of the section 3.

Theorem 4.1. *Let $f : [0,1]\times C\times\mathbb{R} \to \mathbb{R}$ be a continuous function and assume there exist nonnegative functions p, q, $r \in L^1[0,1]$ such that*

- (M1) $|f(t,u,v)| \le p(t)\|u\| + q(t)|v| + r(t)$, *for all* $(t,u,v) \in [0,1]\times C\times\mathbb{R}$
- (M2) $2\int_0^1(1-s)p(s)ds + |\alpha|\int_0^\eta p(s)ds < 1$,
- (M3) $\int_0^1(2-s)[p(s)+q(s)]ds + |\alpha|\int_0^\eta[p(s)+q(s)]ds < 1$.

Then the boundary-value problem (1.1)-(1.3) *has at least one solution.*

Proof. We first consider the case $\phi(0) = 0$ and let the subspace C_0, the functions $\widehat{u}$ be defined as in Theorem 3.1. Define the integral operator $T : C_0 \to C^1[0,1]$ by

$$\begin{aligned}
Tu(t) = &-\int_0^t (t-s)f(s,\widehat{u}_s,u'(s))ds - \alpha t\int_0^\eta f(s,\widehat{u}_s,u'(s))ds \\
&+ t\int_0^1(1-s)f(s,\widehat{u}_s,u'(s))ds, t\in[0,1].
\end{aligned} \tag{4.1}$$

Using (M1) and (3.2), it follows that

$$\|Tu\|_0 \le a_1\|u\|_0 + b_1\|u'\|_0 + c_1, \quad \forall u \in C_0, \tag{4.2}$$

where

$$\begin{aligned}
a_1 &= 2\int_0^1 (1-s)p(s)ds + |\alpha|\int_0^\eta p(s)ds,\\
b_1 &= 2\int_0^1 (1-s)q(s)ds + |\alpha|\int_0^\eta q(s)ds,\\
c_1 &= \Big(2\int_0^1 (1-s)p(s)ds + |\alpha|\int_0^\eta p(s)ds\Big)\|\phi\|\\
&\quad + 2\int_0^1 (1-s)r(s)ds + |\alpha|\int_0^\eta r(s)ds.
\end{aligned}$$

Also using (M1) and (3.2), we obtain

$$\|(Tu)'\|_0 \le a_2\|u\|_0 + b_2\|u'\|_0 + c_2, \quad \forall u \in C_0, \tag{4.3}$$

where

$$\begin{aligned}
a_2 &= \int_0^1 (2-s)p(s)ds + |\alpha|\int_0^\eta p(s)ds,\\
b_2 &= \int_0^1 (2-s)q(s)ds + |\alpha|\int_0^\eta q(s)ds,\\
c_2 &= \Big(\int_0^1 (2-s)p(s)ds + |\alpha|\int_0^\eta p(s)ds\Big)\|\phi\|\\
&\quad + \int_0^1 (2-s)r(s)ds + |\alpha|\int_0^\eta r(s)ds.
\end{aligned}$$

As in the proof of the theorems 3.1, 3.2, we conclude from (4.2), (4.1) and (M3) that the following set is bounded

$$\{u^* \in \partial\Omega : T(u^*) = \lambda u^*, \lambda > 1\}. \tag{4.4}$$

Hence that, combining the assumption (M2) and the continuity of f, T has a fixed point $u \in C_0$. In the case $\phi(0) \neq 0$, by the transformation $v = u - \phi(0)$, we can rewrite the boundary-value problem (1.1)-(1.3) in the form

$$\begin{gathered}
v'' + f(t, v_t + \phi(0), v'(t)) = 0, \quad 0 \le t \le 1,\\
v_0 = \phi - \phi(0) \equiv \widetilde{\phi}, \quad v(1) = \alpha[v'(\eta) - v'(0)] - \phi(0),
\end{gathered}$$

in which $\widetilde{\phi} \in C$ and $\widetilde{\phi}(0) = 0$. Here, we also consider the subspace C_0 and for a function $v \in C_0$, we define the function $\widehat{v} : [-r, 1] \to \mathbb{R}$ by

$$\widehat{v}(t) = \begin{cases} \widetilde{\phi}(t), & t \in [-r, 0],\\ v(t), & t \in [0, 1]. \end{cases}$$

Consider the operator $\widetilde{T} : C_0 \to C^1[0, 1]$ defined by

$$\begin{aligned}
\widetilde{T}v(t) = &-\int_0^t (t-s)f(s, \widehat{v}_s + \phi(0), v'(s))ds - \alpha t\int_0^\eta f(s, \widehat{v}_s + \phi(0), v'(s))ds\\
&- \phi(0)t + t\int_0^1 (1-s)f(s, \widehat{v}_s + \phi(0), v'(s))ds, \quad t \in [0, 1].
\end{aligned}$$

Then, we can prove in a similar manner as above that $\widetilde{T}$ has a fixed point $v \in C_0$. This completes the proof of Theorem 4.1. $\square$

Theorem 4.2. *Let $f : [0,1] \times C \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that there exist nonnegative functions p, q, $r \in L^1[0,1]$ and reals constants $k, l \in [0,1]$ such that (M2) holds and*

($\tilde{\text{M}}1$) $|f(t,u,v)| \le p(t)\|u\|^k + q(t)|v|^l + r(t)$, *for all* $(t,u,v) \in [0,1] \times C \times \mathbb{R}$

($\tilde{\text{M}}3$) $Q(k)a_2 + Q(l)b_2 < 1$,

where

$$a_2 = \int_0^1 (2-s)p(s)ds + |\alpha| \int_0^\eta p(s)ds,$$
$$b_2 = \int_0^1 (2-s)q(s)ds + |\alpha| \int_0^\eta q(s)ds,$$

and the function $Q(\mu)$ is defined as in the Theorem 3.2. Then the boundary-value problem (1.1)-(1.3) *has at least one solution.*

The proof for the above theorem is similar to that of the Theorem 3.2 and is omitted.

5. Application for the initial value problem

First, by the same method as in section 3, combining Lemma 2.6, we also establish the following results for the existence, uniqueness, continuous dependence on a real parameter of the solution to the IVP (1.1)-(1.4).

Theorem 5.1. *Let $f : [0,1] \times C \times \mathbb{R} \to \mathbb{R}$ be continuous function and there exist nonnegative functions p, q, $r \in L^1[0,1]$ such that*

(I1) $|f(t,u,v)| \le p(t)\|u\| + q(t)|v| + r(t)$, *for all* $(t,u,v) \in [0,1] \times C \times \mathbb{R}$,

(I2) $\int_0^1 p(s)ds + \int_0^1 q(s)ds < 1$.

Then the (1.1)-(1.4) *has at least one solution.*

We remark that the above theorem may be a special case of [5, Corollary 4.2] which is stated there without proving.

Proof of the Theorem 5.1. Here, we consider only the case $\phi(0) = 0$ and let the subspace C_0, and the function $\widehat{u}$ defined as in Theorem 3.1. Define the integral operator $T : C_0 \to C^1[0,1]$ by

$$Tu(t) = -\int_0^t (t-s)f(s, \widehat{u}_s, u'(s))ds, \quad t \in [0,1]. \tag{5.1}$$

Using (I1), (3.2) and (5.1), for all $u \in C_0$, we obtain

$$\|Tu\|_0 \le \widetilde{A}_1\|u\|_0 + \widetilde{B}_1\|u'\|_0 + \widetilde{C}_1, \tag{5.2}$$

where

$$\widetilde{A}_1 = \int_0^1 (1-s)p(s)ds, \quad \widetilde{B}_1 = \int_0^1 (1-s)q(s)ds,$$
$$\widetilde{C}_1 = \|\phi\| \int_0^1 (1-s)p(s)ds + \int_0^1 (1-s)r(s)ds,$$

and

$$\|(Tu)'\|_0 \le \widetilde{A}_2\|u\|_0 + \widetilde{B}_2\|u'\|_0 + \widetilde{C}_2, \tag{5.3}$$

where

$$\widetilde{A}_2 = \int_0^1 p(s)ds, \quad \widetilde{B}_2 = \int_0^1 q(s)ds,$$
$$\widetilde{C}_2 = \|\phi\| \int_0^1 p(s)ds + \int_0^1 r(s)ds.$$

It is easy to see that

$$\widetilde{A}_1 \le \widetilde{A}_2, \quad \widetilde{B}_1 \le \widetilde{B}_2, \quad \widetilde{C}_1 \le \widetilde{C}_2.$$

This implies from (I2) and (5.2), (5.3) that the following set is bounded

$$\{u^* \in \partial\Omega : T(u^*) = \lambda u^*, \lambda > 1\}. \tag{5.4}$$

Choose the constants $\widetilde{A}, \widetilde{B}, \widetilde{m}$ as follows

$$\widetilde{A} = \max\{\widetilde{A}_1, \widetilde{A}_2 + \widetilde{B}_2\} = \widetilde{A}_2 + \widetilde{B}_2, \tag{5.5}$$

by (I2), we have $\widetilde{A}_2 + \widetilde{B}_2 < 1$, so $\widetilde{A} < 1$,

$$\widetilde{B} > \max\{\frac{\widetilde{B}_1 \widetilde{C}_2}{1 - \widetilde{A}} + \widetilde{C}_1, \widetilde{C}_2\}, \tag{5.6}$$

clearly, $\widetilde{B} > 0$. Put

$$\Omega = \{u \in C_0 : \|u\|_1 < \widetilde{m}\}, \quad \text{with } \widetilde{m} = \frac{\widetilde{B}}{1 - \widetilde{A}}. \tag{5.7}$$

Clearly, Ω is a bounded open subset of C_0, $0 \in \Omega$, and $\partial\Omega = \{u \in C_0 : \|u\|_1 = \widetilde{m}\}$. Then, we can prove that the operator $T : \overline{\Omega} = \Omega \cup \partial\Omega \to C^1[0,1]$ is completely continuous and there is not $u^* \in \partial\Omega$ such that $T(u^*) = \lambda u^*$, for some $\lambda > 1$. By using theorem 2.1, T has a fixed point $u \in \Omega$. The proof of Theorem 5.1 is complete. □

Theorem 5.2. *Let $f : [0,1] \times C \times \mathbb{R} \to \mathbb{R}$ be continuous function. Assume that there exist nonnegative functions p, q, $r \in L^1[0,1]$ and reals constants $k, l \in [0,1]$ such that*

- ($\tilde{\mathrm{I}}$1) $|f(t,u,v)| \le p(t)\|u\|^k + q(t)|v|^l + r(t)$, *for all* $(t,u,v) \in [0,1] \times C \times \mathbb{R}$
- ($\tilde{\mathrm{I}}$2) $Q(k)\int_0^1 p(s)ds + Q(l)\int_0^1 q(s)ds < 1$,

where the function $Q(\mu)$ is defined as in the Theorem 3.2. Then (1.1)-(1.4) *has at least one solution.*

Theorem 5.3. *Let $f : [0,1] \times C \times \mathbb{R} \to \mathbb{R}$ be continuous function and satisfy on $[0,1] \times C \times \mathbb{R}$ the Lipschitz condition*

$$|f(t,u,v) - f(t,\widetilde{u},\widetilde{v})| \le \theta(\|u - \widetilde{u}\| + |v - \widetilde{v}|),$$

for some positive constant θ. If $2\theta < 1$, then there exists a unique solution to (1.1)-(1.4).

Now, we consider the problem

$$\begin{gathered} u'' + f(t, u_t, u'(t), \lambda), \quad 0 \le t \le 1, \\ u_0 = \phi, \quad u'(0) = 0, \end{gathered} \tag{5.8}$$

where λ is a real parameter and

$$|f(t,u,v,\lambda) - f(t,\widetilde{u},\widetilde{v},\lambda)| \le \theta(\|u - \widetilde{u}\| + |v - \widetilde{v}|), \tag{5.9}$$

on $[0,1]\times C\times\mathbb{R}\times\mathbb{R}$ for some positive constant θ, with

$$2\theta<1, \tag{5.10}$$

$$|f(t,u,v,\lambda_1)-f(t,u,v,\lambda_2)|\le L|\lambda_1-\lambda_2|, \tag{5.11}$$

for some positive constant L, for all λ_1, λ_2.

Theorem 5.4. *Let $f:[0,1]\times C\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be continuous function. If* (5.9)-(5.11) *hold, then the solution to* (5.8) *depends continuously on λ.*

The proofs of Theorems 5.2–5.4 are similar to that of Theorems 3.2–3.4, respectively, let us omit them.

Next, we shall show that the solution set of (1.1)-(1.4) is nonempty, compact and connected. To this end, we need the following result.

Proposition 5.5. *Let $f:[0,1]\times C\times\mathbb{R}\to\mathbb{R}$ be continuous and locally Lipschitz with respect to $C\times\mathbb{R}$, i.e. for every $(t_0,u_0,v_0)\in[0,1]\times C\times\mathbb{R}$, there exist positive constants δ,ρ,σ and $\theta\ge 0$ such that*

$$|f(t,u,v)-f(t,\widetilde{u},\widetilde{v})|\le\theta(\|u-\widetilde{u}\|+|v-\widetilde{v}|),$$

for some positive constant θ, for all $t\in[0,1]$, $(u,v),(\widetilde{u},\widetilde{v})\in C\times\mathbb{R}$, with

$$|t-t_0|\le\delta,\quad \|u-u_0\|\le\rho,\quad \|\widetilde{u}-u_0\|\le\rho,\quad |v-v_0|\le\sigma, |\widetilde{v}-v_0|\le\sigma.$$

Then (1.1)-(1.4) *has at most a solution.*

Proof. Suppose that (1.1)-(1.4) have two solutions $u(t),v(t)$ on $[-r,1]$. Then

$$u(t)=v(t),\quad \text{for all } t\in[-r,0].$$

We shall show that $u(t)=v(t)$, for all $t\in[-r,1]$. Put

$$b=\max\big\{\tau:u(t)=v(t),\forall t\in[-r,\tau]\big\}. \tag{5.12}$$

Clearly, $b\ge0$. Thus $0\le b\le1$. We suppose by contradiction that $b<1$. Since f is locally lipschitz, for $(b,u_b,u'(b))\in[0,1]\times C\times\mathbb{R}$, there exist real numbers δ,ρ,σ and $\theta\ge0$ such that

$$|f(t,\widetilde{u}_1,\widetilde{v}_1)-f(t,\widetilde{u}_2,\widetilde{v}_2)|\le\theta\big(\|\widetilde{u}_1-\widetilde{u}_2\|+|\widetilde{v}_1-\widetilde{v}_2|\big),$$

for all $t\in[0,1]$, $(\widetilde{u}_1,\widetilde{v}_1)$, $(\widetilde{u}_2,\widetilde{v}_2)\in C\times\mathbb{R}$, with $|t-b|\le\delta$,

$$\|\widetilde{u}_1-u_b\|\le\rho,\quad \|\widetilde{u}_2-u_b\|\le\rho,\quad |\widetilde{v}_1-u'(b)|\le\sigma,\quad |\widetilde{v}_2-u'(b)|\le\sigma.$$

Note that $u_b=v_b$, $u'(b)=v'(b)$ and $b+\delta\le1$.

For each fixed $u\in C([-r,1];\mathbb{R})$ which is continuously differentiable on $[0,1]$, since the mappings

$$s\mapsto u_s,\quad s\mapsto u'(s)\quad\text{with } s\in[0,1],$$

are continuous, so there exists $\delta'>0$ with $\delta'<\delta$ and $2\theta\delta'<1$, such that

$$\|u_s-u_b\|\le\rho,\quad \|v_s-u_b\|\le\rho,\quad |u'(s)-u'(b)|\le\sigma,\quad |v'(s)-u'(b)|\le\sigma,$$

for all $s\in[b,b+\delta']$.

Let S_b be the space of continuous functions $x:[-r,b+\delta']\to\mathbb{R}$ which are continuously differentiable on $[b,b+\delta']$ with $x_b=u_b$. We define

$$d_b(x,y)=\max\Big\{\max_{b\le t\le b+\delta'}|x(t)-y(t)|,\max_{b\le t\le b+\delta'}|x'(t)-y'(t)|\Big\}.$$

Then S_b is a completely metrizable space with the distance function d_b. It is easy to see that $\overline{u} = u|_{[-r,b+\delta']} \in S_b$ and $\overline{v} = v|_{[-r,b+\delta']} \in S_b$. Put $w = \overline{u} - \overline{v}$, then w satisfies

$$\begin{gathered} w'' + f(t, \overline{u}_t, \overline{u}'(t)) - f(t, \overline{v}_t, \overline{v}'(t)) = 0, \quad b \le t \le b + \delta', \\ w_b = 0, \quad w'(b) = 0. \end{gathered} \tag{5.13}$$

It follows that for all $t \in [b, b + \delta']$, we have

$$\begin{aligned} |w(t)| &\le \int_b^t (1-s)|f(s, \overline{u}_s, \overline{u}'(s)) - f(s, \overline{v}_s, \overline{v}'(s))|ds \\ &\le \theta \int_b^t \big(\|\overline{u}_s - \overline{v}_s\| + |\overline{u}'(s) - \overline{v}'(s)|\big) ds \\ &\le \theta\delta' \Big(\max_{b \le t \le b+\delta'} |\overline{u}(t) - \overline{v}(t)| + \max_{b \le t \le b+\delta'} |\overline{u}'(t) - \overline{v}'(t)| \Big). \end{aligned}$$

Similarly,

$$\begin{aligned} |w'(t)| &\le \int_b^t |f(s, \overline{u}_s, \overline{u}'(s)) - f(s, \overline{v}_s, \overline{v}'(s))|ds \\ &\le \theta\delta' \Big(\max_{b \le t \le b+\delta'} |\overline{u}(t) - \overline{v}(t)| + \max_{b \le t \le b+\delta'} |\overline{u}'(t) - \overline{v}'(t)| \Big). \end{aligned}$$

By the definition of the distance d_b, we have

$$\begin{aligned} d_b(\overline{u}, \overline{v}) &= \max \Big\{ \max_{b \le t \le b+\delta'} |\overline{u}(t) - \overline{v}(t)|, \max_{b \le t \le b+\delta'} |\overline{u}'(t) - \overline{v}'(t)| \Big\} \\ &\le \theta\delta' \Big(\max_{b \le t \le b+\delta'} |\overline{u}(t) - \overline{v}(t)| + \max_{b \le t \le b+\delta'} |\overline{u}'(t) - \overline{v}'(t)| \Big) \\ &\le 2\theta\delta' d_b(\overline{u}, \overline{v}). \end{aligned}$$

Since $2\theta\delta' < 1$, we deduce that $d_b(\overline{u}, \overline{v}) = 0$ i.e. $\overline{u} = \overline{v}$. Therefore,

$$u(t) = v(t), \quad \forall t \in [-r, b + \delta'].$$

This leads to a contradiction with the definition of b in (5.12). Then the proof is complete. □

From Theorems 5.1, 5.2 and Proposition 5.5, we obtain the following corollary.

Corollary 5.6. *Let $f : [0,1] \times C \times \mathbb{R} \to \mathbb{R}$ be a continuous function and locally Lipschitz with respect to $C \times \mathbb{R}$. Assume that there exist nonnegative functions p, q, $r \in L^1[0,1]$ and reals constants $k, l \in [0,1]$ such that*

- ($\tilde{\text{I}}$1) $|f(t, u, v)| \le p(t)\|u\|^k + q(t)|v|^l + r(t)$, *for all* $(t, u, v) \in [0,1] \times C \times \mathbb{R}$
- ($\tilde{\text{I}}$2) $Q(k) \int_0^1 p(s)ds + Q(l) \int_0^1 q(s)ds < 1$,

where the function $Q(\mu)$ is defined as in the Theorem 3.2. Then (1.1)-(1.4) *has a unique solution.*

By the above results and applying Theorems 2.2, 2.3, we have the following theorem.

Theorem 5.7. *Let $f : [0,1] \times C \times \mathbb{R} \to \mathbb{R}$ be continuous function and satisfy the conditions (I1)-(I2) or ($\tilde{\text{I}}$1)-($\tilde{\text{I}}$2). Then the solution set of the IVP* (1.1)-(1.4) *is nonempty, compact and connected.*

Proof. **Step 1**. *The case* $\phi(0) = 0$. We again consider the subspace C_0, the function $\widehat{u}$ and the operator T, which are defined as in Theorem 5.1 As above, $T : \overline{\Omega} = \Omega \cup \partial\Omega \to C^1[0,1]$ is completely continuous, where

$$\Omega = \{u \in C_0 : \|u\|_1 < \widetilde{m}\}, \quad \widetilde{m} = \frac{\widetilde{B}}{1-\widetilde{A}}.$$

According to Theorems 5.1-5.2, it is obvious that the fixed point set of T is nonempty. Furthermore, it is compact and connected. Indeed, First, for all $u \in \overline{\Omega}$, it follows from (5.2), (5.3), (5.6) and (5.7), that

$$\|Tu\|_1 \le \widetilde{A}\ \widetilde{m} + \widetilde{C}_2,$$
$$\widetilde{m} = \frac{\widetilde{B}}{1-\widetilde{A}} > \frac{\widetilde{C}_2}{1-\widetilde{A}}, \quad \text{i.e. } \widetilde{A}\widetilde{m} + \widetilde{C}_2 < \widetilde{m}.$$

Therefore, $\|Tu\|_1 < \widetilde{m}$. Then we obtain

$$T(\overline{\Omega}) \subset \Omega.$$

On the other hand, Ω is convex, so

$$\deg(I - T, \Omega, 0) \neq 0.$$

Obviously, T has no fixed points on $\partial\Omega$.

Next, the function $f : [0,1] \times C \times \mathbb{R} \to \mathbb{R}$ is continuous function, by Theorem 2.3, for each $\varepsilon > 0$, there is a mapping $f_\varepsilon : [0,1] \times C \times \mathbb{R} \to \mathbb{R}$ that is locally Lipschitz with respect to $C \times \mathbb{R}$, such that

$$|f(t,u,v) - f_\varepsilon(t,u,v)| \le \frac{\varepsilon}{2}, \quad \forall (t,u,v) \in [0,1] \times C \times \mathbb{R}. \tag{5.14}$$

Clearly, f_ε is continuous. Moreover, by f satisfies the conditions (I1)-(I2) or ($\tilde{\text{I}}$1)-($\tilde{\text{I}}$2), it follows from (5.14) that f_ε satisfies the conditions (I1)-(I2) or ($\tilde{\text{I}}$1)-($\tilde{\text{I}}$2). Let $T_\varepsilon : \overline{\Omega} \to C^1[0,1]$ be defined by

$$T_\varepsilon u(t) = -\int_0^t (t-s) f_\varepsilon(s, \widehat{u}_s, u'(s))ds, \quad t \in [0,1]. \tag{5.15}$$

It is easy to check that T_ε is completely continuous and

$$\|T(u) - T_\varepsilon(u)\|_1 \le \frac{\varepsilon}{2} < \varepsilon, \quad \forall u \in \overline{\Omega}. \tag{5.16}$$

Finally, we need prove that for each $h \in \overline{\Omega}$ with $\|h\|_1 < \varepsilon$, the equation

$$u = T_\varepsilon(u) + h, \tag{5.17}$$

has at most one solution. Suppose that u_1, u_2 are two solutions of (5.17). Put

$$w_1 = \widehat{u}_1 - \widehat{h}, \quad w_2 = \widehat{u}_2 - \widehat{h},$$

where

$$\widehat{h}(t) = \begin{cases} \phi(t), & t \in [-r,0], \\ h(t), & t \in [0,1], \end{cases} \qquad \widehat{u}_i(t) = \begin{cases} \phi(t), & t \in [-r,0], \\ u_i(t), & t \in [0,1], \end{cases}$$

$i = 1,2$. Then w_1, w_2 are two solutions of the problem

$$\begin{gathered} w'' + f_\varepsilon\big(t, w_t + \widehat{h}_t, w'(t) + h'(t)\big) = 0, \quad 0 \le t \le 1, \\ w_0 = 0, \quad w'(0) = 0. \end{gathered} \tag{5.18}$$

This implies from Proposition 5.5 that the problem (5.18) has at most one solution, so

$$w_1 = w_2, \quad \text{i.e. } u_1 = u_2.$$

It follows that (5.17) has at most one solution.

Applying Theorem 2.2, we have the fixed point set of T is nonempty, compact and connected. Thus, so is the solution set of (1.1)-(1.4). The step 1 is complete.

Step 2. *The case* $\phi(0) \neq 0$. By the transformation $v = u - \phi(0)$, the IVP (1.1)-(1.4) can be rewritten in the form

$$\begin{gathered} v'' + f\big(t, v_t + \phi(0), v'(t)\big) = 0, \quad 0 \le t \le 1, \\ v_0 = \phi - \phi(0) \equiv \widetilde{\phi}, \quad v'(0) = 0. \end{gathered} \tag{5.19}$$

in which $\widetilde{\phi} \in C$ and $\widetilde{\phi}(0) = 0$. By the step 1, we can prove without difficulty that the solution set of (5.19) is nonempty, compact and connected. In this proof, when f satisfies the conditions ($\tilde{\text{I}}$1)-($\tilde{\text{I}}$2), the inequality (3.18) is used again. Consequently, the solution set of (1.1)-(1.4) is nonempty, compact and connected. Theorem 5.7 is proved. $\square$

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