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# A scaled characteristics method for the asymptotic solution of weakly nonlinear wave equations *

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**Abstract**

We formulate a multi-scale perturbation technique to asymptotically solve weakly nonlinear hyperbolic equations. The method is based on a set of scaled characteristic coordinates. We show that this technique leads to a simplified system of ordinary differential equations describing the weakly nonlinear interaction between left and right running waves. Using this method, a uniformly valid first order solution of a prototype nonlinear equation is derived.

## 1 Introduction

In this paper we consider initial-boundary value problems for weakly non-linear hyperbolic wave equations of the form

$$u_{tt} - u_{xx} + \epsilon h(u, u_t, u_x) = 0, \quad t, x > 0, \quad 0 < \epsilon \ll 1 \tag{1}$$

$$u(x,0) = a(x); \quad u_t(x,0) = b(x); \quad u(0,t) = \rho(t),\ t, x > 0\,. \tag{2}$$

Such equations occur in many models in science and engineering (See [2, 3, 4, 6]). We assume that $a(0) = \rho(0)$, $b(0) = \rho'(0)$, that $a(x)$, $\rho(t)$ are twice continuously differentiable, $b(x)$ is continuously differentiable, and that $h$ is analytic in its arguments. One can formally write the solution of this system as two separate equations, one valid in the region $0 \leq t \leq x$ and the other valid for $t \geq x \geq 0$ [5],

$$\begin{aligned} u(x,t) &= \frac{1}{2}[a(x+t) + a(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} b(\lambda)d\lambda \\ &\quad + \frac{\epsilon}{2}\int_0^t d\tau \int_{x-(t-\tau)}^{x+(t-\tau)} h(u, u_\tau(\lambda,\tau), u_\lambda(\lambda,\tau))d\lambda, \quad 0 \leq t \leq x, \end{aligned} \tag{3}$$

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$$\begin{aligned} u(x,t) &= \rho(t-x) + \frac{1}{2}[a(t+x)+a(t-x)] + \frac{1}{2}\int_{t-x}^{t+x} b(\lambda)d\lambda \qquad (4) \\ &\quad + \frac{\epsilon}{2}\int_0^t d\tau \int_{|x-(t-\tau)|}^{x+(t-\tau)} h(u, u_\tau(\lambda,\tau), u_\lambda(\lambda,\tau))d\lambda, \quad t \ge x \ge 0. \end{aligned}$$

Using the above representation of solutions, and the assumptions on the regularity of initial and boundary conditions, one can show that a twice continuously differentiable solution $u(x,t;\epsilon)$ of the hyperbolic system exists in a rectangle $0 \le t, x \le O(\sqrt{\epsilon})$. This solution depends continuously on the initial data, and formal perturbation series expansions asymptotically converge to the solution in this rectangle.

Developing uniformly valid perturbation solutions to hyperbolic systems of the above type has been a difficult task. When the boundary condition is zero, a multi-scale perturbation scheme based on a "slow" time scale $T = \epsilon t$ has been developed [2]. For signaling problems in which the initial conditions are zero and one wishes to study the propagation of boundary data, a perturbation scheme based on a "long" distance scale $X = \epsilon x$ can be used. However, when the initial and boundary conditions are both nonzero, the forward and backward going wave components (which are perturbations of the corresponding parts of the linear wave equation) interact, leading to considerable difficulties in finding asymptotic solutions. In [1] we developed a perturbation scheme that could deal with this situation. This scheme involved a coupled set of first order partial differential equations (PDEs) in terms of two scaled variables $X = \epsilon x$ and $T = \epsilon t$, governing the interaction between forward and backward going waves in the region $t > x$. In many cases we showed that these PDEs could be explicitly solved leading to uniformly valid $O(1)$ solutions of the system. Our purpose in this paper is to present a modified perturbation scheme using a pair of scaled characteristic variables, and certain assumptions on averaged quantities, leading to coupled *ordinary differential equations* governing the interaction of left and right running waves. This makes the perturbation analysis considerably simpler. In section 3 we present comparisons of results of the present analysis with those in [1].

## 2 The multi-scale perturbation procedure

Equations (3)-(4) show that the solution of the hyperbolic system (1)-(2) splits into two parts: the region $0 \le t \le x$ where the boundary conditions do not affect the solution, and the region $t \ge x \ge 0$, where boundary conditions come into play. See Figure 1. Thus the analysis proceeds differently in these two regions.

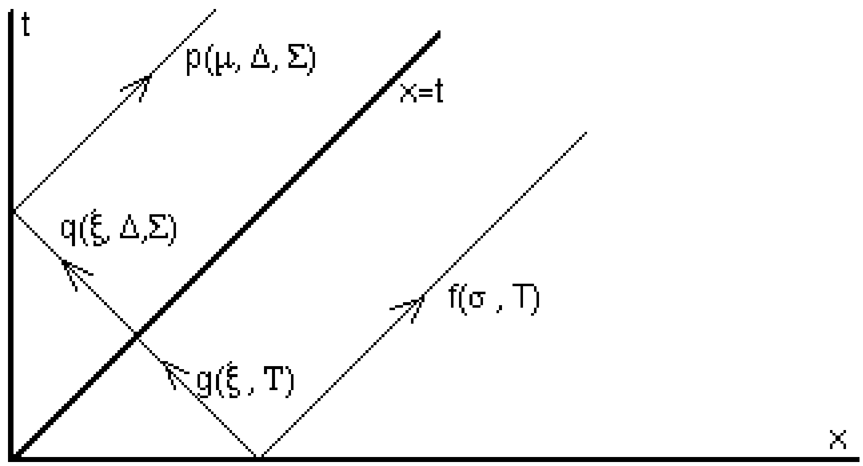


Figure 1.

### 2.1 The region $0 \le t \le x$

In $0 \le t \le x$, only initial conditions affect the solution. The asymptotics in this region is the same as in [1],[2]. The nonlinearity $h$ will have a time-cumulative effect on the solution in this region. To determine this effect, a slow time variable $T = \epsilon t$ is usually introduced, in addition to the "fast" variables $t$ and $x$. Strictly speaking, the fast variables should be relabeled, but for convenience we use $t$ and $x$ as the fast variables. We then seek a formal perturbation series solution in the form

$$u(x,t;\epsilon) = \sum_{n=0}^{N} \epsilon^n u_n(x,t,T) + O(\epsilon^{n+1}). \tag{5}$$

The derivatives are replaced by $\partial_t \longrightarrow \partial_t + \epsilon\partial_T$, $\partial_{tt} \longrightarrow \partial_{tt} + 2\epsilon\partial_{tT} + \epsilon^2\partial_{TT}$. We also introduce the characteristics $\sigma = x - t$ and $\xi = x + t$. After substituting into the original system and equating like powers of $\epsilon$, a hierarchy of initial boundary value problems are obtained:

$$-4u_{0_{\xi\sigma}} = 0 \tag{6}$$

$$-4u_{1_{\xi\sigma}} + 2u_{0_{\xi T}} - 2u_{0_{\sigma T}} + h(u_0, u_{0_\xi} - u_{0_\sigma}, u_{0_\xi} + u_{0_\sigma}) = 0 \tag{7}$$

The solution of (6) is then sought in the form

$$u_0(x,t,T) = f(\sigma,T) + g(\xi,T) \tag{8}$$

where $\sigma = x - t$ and $\xi = x + t$. When $\epsilon = 0$ we should recover the solution of the linear wave equation.

$f$ and $g$ are determined by requiring that the $O(\epsilon)$ solution, given by (7), remains bounded. We eliminate secular terms that would lead to non-uniformities at $O(\epsilon)$ level by substituting (8) into (7), and requiring that $(\frac{1}{\xi})u_{1_\xi}$ and $(\frac{1}{\sigma})u_{1_\sigma}$ must go to 0 as $\xi$ and $\sigma$ grow large. Integrating (7) with respect to $\xi$ from 0 to $M$, dividing by $M$ and taking the limit as $M \longrightarrow \infty$, we get

$$-2f_{\sigma T} + \lim_{M\to\infty} \frac{1}{M}\int_0^M h(f+g, g_\xi - f_\sigma, g_\xi + f_\sigma)d\xi = 0. \tag{9}$$

Repeating the process with respect to $\sigma$ yields

$$2g_{\xi T} + \lim_{M\to\infty} \frac{1}{M}\int_0^M h(f+g, g_\xi - f_\sigma, g_\xi + f_\sigma)d\sigma = 0. \tag{10}$$

These are a coupled set of ODEs for $f_\sigma$ and $g_\xi$ that determine the wave evolution in the region $0 \le t \le x$.

## 2.2 The region $t \ge x \ge 0$

In this region the boundary conditions also affect the solution. In [1], we introduced a "slow" time scale $T = \epsilon t$ and a "long" space scale $X = \epsilon x$. The resulting analysis leads to a system of coupled PDEs for the slow evolution of waves in this region. The left and right running waves interact with each other through these stretched scales.

In this paper we introduce a set of scaled characteristic variables, $\Delta = \epsilon(t+x)$ and $\Sigma = \epsilon(t-x)$, in addition to the regular "fast" characteristics $\mu = t - x$ and $\xi = t + x$ in this region. Introduction of these scaled variables enables us to make certain assumptions about $\mu-$ and $\xi-$averaged quantities that appear below. The resulting analysis, in many instances, is much simpler than previous methods.

With these new variables, the perturbation expansion becomes

$$u(x,t;\epsilon) = \sum_{n=0}^{N} \epsilon^n u_n(\xi,\mu,\Delta,\Sigma) + O(\epsilon^{n+1}). \tag{11}$$

The $O(1)$ equation is $u^0_{\xi\mu} = 0$, for which we seek a solution in the form

$$u^0 = p(\mu,\Delta,\Sigma) + q(\xi,\Delta,\Sigma). \tag{12}$$

$p$ and $q$ have to be determined subject to the conditions that when $t = x$, $q$ must equal $g$, the incoming wave in $t < x$, and when $x = 0$, the sum of $p$ and $q$ must equal $\rho(t)$. See Figure 1. In addition, we must require uniform validity of the $O(1)$ and $O(\epsilon)$ solutions – this would necessitate choosing the $\Sigma$ and $\Delta$ dependence of $p$ and $q$ to ensure that the $O(\epsilon)$ solutions remain bounded.

The $O(\epsilon)$ equation is

$$4u^1_{\xi\mu} + 4q_{\xi\Sigma} + 4p_{\mu\Delta} + h(p+q, p_\mu + q_\xi, q_\xi - p_\mu) = 0. \tag{13}$$

Integrating (13) with respect to $\mu$ from 0 to M, dividing by M and taking limit as $M \longrightarrow \infty$, we get

$$4q_{\xi\Sigma} + \lim_{M\to\infty} \frac{1}{M} \int_0^M h(p+q, p_\mu + q_\xi, q_\xi - p_\mu) d\mu = 0, \tag{14}$$

while the same process with respect to $\xi$ gives

$$4p_{\mu\Delta} + \lim_{M\to\infty} \frac{1}{M} \int_0^M h(p+q, p_\mu + q_\xi, q_\xi - p_\mu) d\xi = 0. \tag{15}$$

In calculating the $\xi$-average in (15), the $\Delta$ occurring in the argument of $q_\xi$ will be treated as $\Delta = \epsilon\xi$. This is justified because for fixed $\epsilon$, as $\xi \to \infty$, $\Delta \to \infty$, therefore the averaging process should include $\Delta$. We leave $p$ unaffected because we are interested only in the $O(1)$ contribution to the $\xi$-average, which comes from $q$. Similarly in (14), the $\Sigma$ that occurs in the argument of $p_\mu$ will be treated as $\Sigma = \epsilon\mu$ when calculating the $\mu$ integral.

The equations (14) and (15) are asymptotically equivalent to a pair of partial differential equations derived in [1] to describe the $O(1)$ solution in this region:

$$2p_{\mu T} + 2p_{\mu X} + \lim_{M\to\infty} \tfrac{1}{M} \int_0^M h(p+q, p_\mu + q_\xi, q_\xi - p_\mu) d\xi = 0 \tag{16}$$

$$2q_{\xi T} - 2q_{\xi X} + \lim_{M\to\infty} \tfrac{1}{M} \int_0^M h(p+q, p_\mu + q_\xi, q_\xi - p_\mu) d\mu = 0 \tag{17}$$

One can formally derive (14) and (15) from these two equations. However the essential difference between (14)-(15) and (16)-(17) is in the way the averages are calculated in their last terms on the left side. The limiting process in (14) and (15) affects the scaled variables $\Delta$ and $\Sigma$, while the limiting process in (16) and (17) did not affect the scaled variables $X$ and $T$. Up to $O(1)$, the two sets of equations are equivalent, since the affected terms are $O(\epsilon)$.

The advantage of our approach is that the resulting equations determining the left and right running waves in the region $t \geq x$ are a set of coupled ODEs for $p_\mu$ and $q_\xi$, which can be solved in many cases. We next illustrate the procedure with examples.

## 3 Examples

We consider the prototype nonlinear equation from [1]:

$$u_{tt} - u_{xx} + \epsilon(2u_t + u(u_t - u_x)) = 0 \tag{18}$$

with the initial boundary conditions (2).

In the region $t < x$, the analysis is the same as in [1],[2], so we skip the details. The O(1) solution is determined by $u^0 = f(\sigma, T) + g(\xi, T)$ , where

$$2f_{\sigma T} - 2f_\sigma - 2ff_\sigma - 2f_\sigma < g >= 0$$

and

$$2g_{\xi T} + g_\xi = 0.$$

These can be integrated with respect to $\sigma$ and $\xi$ respectively. The arbitrary function of $T$ resulting from the integration is identically zero because the solution has to depend continuously on the initial data. Thus we get

$$2f_T - 2(1+ < g >) - f^2 = 0$$

and

$$g_T + g = 0,$$

where $< g >= \lim_{M\to\infty} \frac{1}{M} \int_0^M g(\xi, T)d\xi$. The solution, subject to the appropriate initial conditions, is then given by

$$g(\xi, T) = G(\xi)e^{-T}, \tag{19}$$

where

$$G(\xi) = \frac{1}{2}\left(a(\xi) + \int_0^\xi b(\lambda)d\lambda\right)$$

and

$$f(\sigma, T) = \frac{F(\sigma)\lambda(T)}{1 + \frac{1}{2}F(\sigma)\int_0^T \lambda(\tau)d\tau}\ , \tag{20}$$

where

$$\begin{aligned} F(\sigma) &= \tfrac{1}{2}\left(a(\sigma) - \textstyle\int_0^\sigma b(\lambda)d\lambda\right); \\ \lambda(T) &= \exp\left[-\textstyle\int_0^T (1+ < g >)d\tau\right]. \end{aligned}$$

For $t > x$, we follow the scaled characteristics approach outlined in section 2.2. The O(1) problem has a solution in the form

$$u^0 = p(\mu, \Delta, \Sigma) + q(\xi, \Delta, \Sigma).$$

The $O(\epsilon)$ equation is

$$4u^1_{\mu\xi} + 4p_{\mu\Delta} + 4q_{\xi\Sigma} + 2(p_\mu + q_\xi) + 2(p + q)p_\mu = 0.$$

Integrating this with respect to $\mu$ from 0 to $M$, dividing by $M$ and taking limit as $M \to \infty$, we get

$$q_{\xi\Sigma} + q_\xi = 0. \tag{21}$$

A similar process with $\xi$ yields

$$2p_{\mu\Delta} + p_\mu + (p^2)_\mu + < q > p_\mu = 0, \tag{22}$$

where

$$< q >= \lim_{M\to\infty} \frac{1}{M} \int_0^M q(\xi, \epsilon\xi, \Sigma) d\xi.$$

By integrating (20) with respect to $\xi$ and (21) with respect to $\mu$, we obtain a set of coupled ODEs:

$$2q_\Sigma + q = 0 \tag{23}$$

$$2p_\Delta + (1 + < q >)p + p^2 = 0 \tag{24}$$

The arbitrary functions of $\Delta$ and $\Sigma$ that result from these integrations are set to zero because the solution has to depend continuously on initial data.

The coupled ODEs (23) and (24) have to be solved subject to the conditions that when $t = x$, $q$ equals $g$, the backward-going component in $t < x$, and when $x = 0$, the sum of $p$ and $q$ should equal $\rho(t)$, the boundary condition. One can compute these solutions readily:

$$q(\xi, \Delta, \Sigma) = g(\xi, \tfrac{\Delta}{2})e^{-\Sigma/2} \tag{25}$$

$$p(\mu, \Delta, \Sigma) = \Big( (\tfrac{1}{2k} + \tfrac{1}{\rho(\mu) - q(\mu,\Sigma,\Sigma)})e^{-k(\Sigma-\Delta)/2} - \tfrac{1}{2k} \Big)^{-1} \tag{26}$$

where $k(\Sigma; \epsilon) = 1 + < q >$. We now numerically compare the solutions obtained in this paper, (25)-(26), with those obtained in [1].

Figure 2 plots the $O(1)$ solution at $t = 5$, using equations (25)-(26) and the solution derived in [1] by solving the PDEs resulting from (16)-(17). We used the set of initial-boundary conditions

$$u(x, 0) = \sin(3x) \tag{27}$$

$$u_t(x, 0) = 0 \tag{28}$$

$$u(0, t) = \tfrac{2t}{1+t^2} \sin t\,. \tag{29}$$

In this case $< q >= 0$ in equation (26). The two plots are indistinguishable. Figure 3. plots the same solutions at $x = 8$.

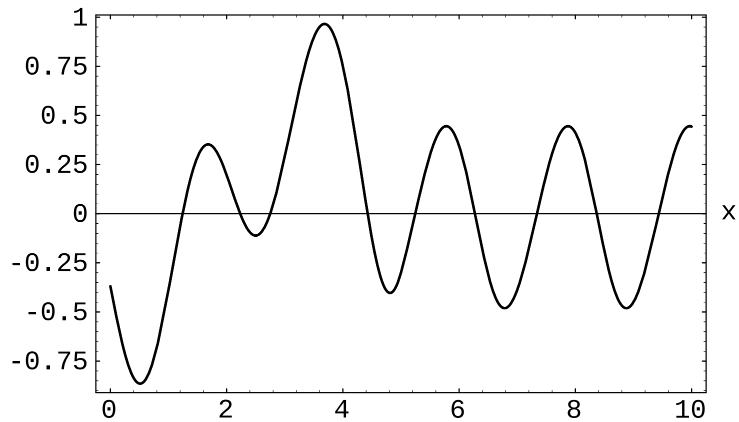


Figure 2. Waveform at t=5, Epsilon =0.1

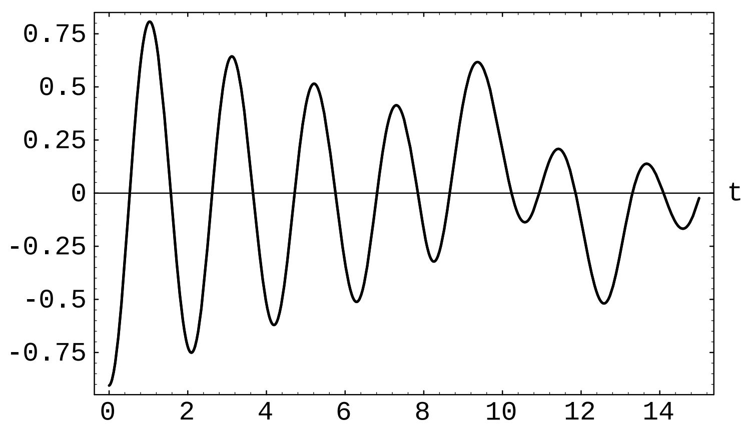


Figure 3. Waveform at x=5,Epsilon =0.1

Figure 4 plots the solutions at $t = 5$ for $u(x, 0) = \sin^2(3x)$, and the other conditions as in (28)-(29). In this case $< q > \neq 0$. We plotted the solution in the region $x > 5$ where interaction between left and right going waves take place. In Figure 5 are solutions at $x = 8$. Note that each of figures 2-5 contains two plots which are indistinguishable except in figure 5.

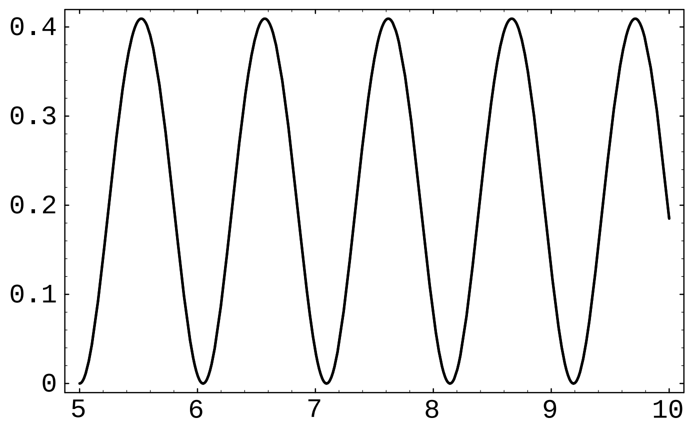


Figure 4. Waveform at t = 5, Epsilon = 0.1

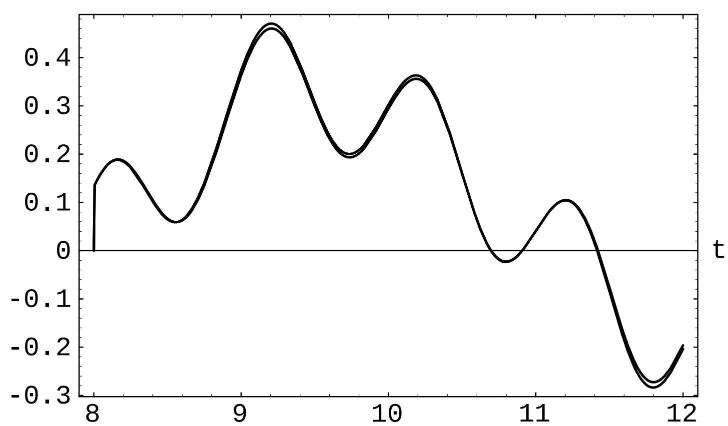


Figure 5. Waveform at x=8, Epsilon =0.1

## 4 Conclusion

We presented a method to develop uniformly valid asymptotic approximations to weakly nonlinear wave equations that is considerably simpler than previously available methods. In principle, the procedure could be carried out to higher order, although we presented only first order approximations. The new method offers a way to study the nonlinear interaction of left and right running waves through a set of coupled ordinary differential equations. This should be compared with previous methods that required solving coupled sets of partial differential equations. Our method used a pair of "slow" characteristic variables $\Delta = \epsilon(t+x)$ and $\Sigma = \epsilon(t-x)$ and certain assumptions on the averages of forward and backward going wave amplitudes to simplify the analysis. This simplification will hopefully help analyze a variety of weakly nonlinear wave equations that occur in applications such as gas dynamics and acoustics ([2],[4]), in which the boundary and initial data are non-zero.

## References

[1] S. C. Chikwendu and C. V. Easwaran, *Multiple-scale solution of initial-boundary value problems for weakly nonlinear wave equations on the semi-infinite line*. SIAM J. Appl. Math. **52**(1992), pp. 964 – 958.

[2] J. Kevorkian and J. D. Cole, *Multiple Scale and Singular Perturbation Methods*. Springer Verlag, New York, 1996.

[3] C. J. Myerscough, *A simple model for the growth of wind-induced oscillations in overhead lines*. J. Sound and Vibration **39**(1975), pp. 503 – 517.

[4] A. H. Nayfeh, *A comparison of perturbation methods for nonlinear hyperbolic waves*. In *Singular Perturbations and Asymptotics*, R.E. Mayer and S.U.Parter, Eds, Academic Press, NY 1980, pp 223-276.

[5] A. N. Tychonov and A. A. Samarski, *Partial Differential Equations of Mathematical Physics*. Holden-Day, San Francisco, 1964.

[6] G.B. Whitham, *Linear and Non-Linear Waves*. Wiley-Interscience, New York, 1974.

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