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HOMOGENIZATION AND UNIFORM STABILIZATION FOR A NONLINEAR HYPERBOLIC EQUATION IN DOMAINS WITH HOLES OF SMALL CAPACITY

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Abstract. In this article we study the homogenization and uniform decay of the nonlinear hyperbolic equation

$$\partial_{tt}u_\varepsilon - \Delta u_\varepsilon + F(x,t,\partial_t u_\varepsilon, \nabla u_\varepsilon) = 0 \quad \text{in } \Omega_\varepsilon \times (0,+\infty)$$

where Ω_ε is a domain containing holes with small capacity (i. e. the holes are smaller than a critical size). The homogenization's proofs are based on the abstract framework introduced by Cioranescu and Murat [8] for the study of homogenization of elliptic problems. Moreover, uniform decay rates are obtained by considering the perturbed energy method developed by Haraux and Zuazua [10].

1. Introduction and statement main results

This paper is devoted to the study of the homogenization and uniform decay rates of the nonlinear hyperbolic equation

$$\begin{gathered} u_\varepsilon'' - \Delta u_\varepsilon + F\left(x,t,u_\varepsilon', \nabla u_\varepsilon\right) = 0 \quad \text{in } \Omega_\varepsilon \times (0,+\infty) \\ u_\varepsilon = 0 \quad \text{on } \Gamma_\varepsilon \times (0,+\infty) \\ u_\varepsilon(x,0) = u_\varepsilon^0(x); \quad u_\varepsilon'(x,0) = u_\varepsilon^1(x); \quad x \in \Omega_\varepsilon, \end{gathered} \tag{1.1}$$

where, for every $\varepsilon > 0$, Ω_ε is an open domain, locally located on one side of its smooth boundary Γ_ε, obtained by removing, from a given bounded, connected open set Ω, a set S_ε of closed subsets (the 'holes') of Ω; i. e., $\Omega_\varepsilon = \Omega \backslash S_\varepsilon$. We assume that the measure of S_ε approaches zero as the parameter ε tends to zero.

Now, we state the general hypotheses.

(A1) Assumptions on the initial data: Assume that

$$\{u_\varepsilon^0, u_\varepsilon^1\} \in D(\Omega_\varepsilon) \times D(\Omega_\varepsilon) \tag{1.2}$$

and as $\varepsilon \to 0$ we have

$$\left\{\tilde u_\varepsilon^0, \tilde u_\varepsilon^1\right\} \rightharpoonup \left\{u^0, u^1\right\} \quad \text{weakly in } H_0^1(\Omega) \cap H^2(\Omega) \times H_0^1(\Omega), \tag{1.3}$$

where the tilde on $\tilde u$ denotes the extension by zero to the whole domain Ω.

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(A2) Assumptions on $F(x,t,u',\nabla u)$: Suppose $F:\Omega\times(0,\infty)\times\mathbb{R}^{n+1}\to\mathbb{R}$ is an element of the space $C^1\left(\Omega\times(0,\infty)\times\mathbb{R}^{n+1}\right)$ and satisfies

$$|F(x,t,\xi,\zeta)|\le C_0\left(1+|\xi|^{\rho+1}+|\zeta|\right) \tag{1.4}$$

where C_0 and ρ are positive constants such that $\rho>0$ for $n=1,2$ and $0<\rho\le 2/(n-2)$ for $n\ge 3$, and $\zeta=(\zeta_1,\dots,\zeta_n)$.

Assume that there is a non-negative function $\varphi(t)$ in $W^{1,\infty}(0,\infty)\cap L^1(0,\infty)$ such that for some $\beta>0$,

$$F(x,t,\xi,\zeta)\eta\ge\beta|\xi|^\rho\xi\eta-\varphi(t)\left(1+|\eta||\zeta|\right),\quad\text{for all }\eta\in\mathbb{R}. \tag{1.5}$$

Suppose that there exist positive constants $C_1,\dots,C_n$ such that

$$|F_t(x,t,\xi,\zeta)|\le C_0(1+|\xi|^{\rho+1}+|\zeta|), \tag{1.6}$$

$$F_\xi(x,t,\xi,\zeta)\ge\beta|\xi|^\rho, \tag{1.7}$$

$$|F_{\zeta_i}(x,t,\xi,\zeta)|\le C_i\quad\text{for }i=1,\dots,n. \tag{1.8}$$

Also assume that there exists a positive constant D such that for all $\eta,\hat\eta$ in $\mathbb{R}$, one has

$$\begin{aligned}&\Big(F(x,t,\xi,\zeta)-F(x,t,\hat\xi,\hat\zeta)\Big)(\eta-\hat\eta) \qquad (1.9)\\ &\ge\beta\Big(|\xi|^\rho\xi-|\hat\xi|^\rho\hat\xi\Big)(\eta-\hat\eta)-D|\eta-\hat\eta||\zeta-\hat\zeta|.\end{aligned}$$

We assume that

$$F(x,t,0,0)=0. \tag{1.10}$$

A simple variant of the nonlinear function above is given by the example

$$F(x,t,\xi,\zeta)=\beta|\xi|^\rho\xi+\varphi(t)\sum_{i=1}^n\sin(\zeta_i).$$

Next, we make some remarks about early works concerning homogenization of distributed systems.

In the framework of homogenization of elliptic problems, Cioranescu and Murat [8] studied the problem

$$\begin{gathered}\Delta u_\varepsilon=f\quad\text{in }\Omega_\varepsilon\\ u_\varepsilon=0\quad\text{on }\Gamma_\varepsilon\end{gathered}$$

with $f\in H^{-1}(\Omega)$. They showed that for every $\varepsilon>0$ there exists a unique $u_\varepsilon\in H_0^1(\Omega_\varepsilon)$ such that

$$\tilde u_\varepsilon\rightharpoonup u\quad\text{weakly in }H_0^1(\Omega),\ \text{as }\varepsilon\to 0$$

where $\tilde u_\varepsilon$ is the extension of u_ε, by considering zero, to whole domain Ω, and u is the unique solution of the homogenized problem

$$\begin{gathered}-\Delta u+\mu u=f\quad\text{in }\Omega\\ u=0\quad\text{on }\Gamma=\partial\Omega,\end{gathered}$$

where μ is a non-negative Radon's measure which belongs to $H^{-1}(\Omega)$. This measure appears in this study and is due to the capacity's behaviour of the set S_ε when

$\varepsilon \to 0$. For this end it is necessary that we have small holes, i. e., the diameter of the holes are smaller than (or equal to) the critical diameter a_ε given by:

$$a_\varepsilon = \begin{cases} \delta_\varepsilon \exp(-C_0/\varepsilon^2) & \text{if } n = 2 \\ C_0 \varepsilon^{n/(n-2)} & \text{if } n > 2, \end{cases}$$

where C_0 is a positive constant and δ_ε is such that $\varepsilon^2 log \delta_\varepsilon \to 0$ as $\varepsilon \to 0$. From the above condition it is possible to construct an abstract framework which plays an essential role to demonstrate the results. More precisely we have:

There exists a sequence $\{w_\varepsilon, \mu_\varepsilon, \gamma_\varepsilon\}$ and $M_0 > 0$ such that

$$\begin{gathered} w_\varepsilon \in H^1(\Omega) \cap L^\infty(\Omega), \quad \|w_\varepsilon\|_{L^\infty(\Omega)} \le M_0 \quad \text{for every } \varepsilon > 0, \\ w_\varepsilon = 0 \quad \text{on } S_\varepsilon, \\ w_\varepsilon \rightharpoonup 1 \quad \text{weakly in } H^1(\Omega) \text{ as } \varepsilon \to 0, \\ -\Delta w_\varepsilon = \mu_\varepsilon - \gamma_\varepsilon \qquad \text{with } \mu_\varepsilon, \gamma_\varepsilon \in H^{-1}(\Omega), \\ \mu_\varepsilon \to \mu \text{ strongly in } H^{-1}(\Omega) \text{ and } \langle \gamma_\varepsilon, v_\varepsilon \rangle = 0 \\ \text{for every } \{v_\varepsilon\} \subset H_0^1(\Omega) \text{ with } v_\varepsilon = 0 \text{ on } S_\varepsilon\,. \end{gathered} \tag{1.11}$$

In the case above, μ will be a nonnegative constant when the diameter of the holes is the critical one. In this case, the additional term of order zero μu (so called 'terme étrange') appears in the limit equation. In [8] the authors still showed correctors results; i.e.,

$$\tilde{u}_\varepsilon = w_\varepsilon u + R_\varepsilon, \quad \text{with } R_\varepsilon \to 0 \text{ strongly in } H_0^1(\Omega).$$

An example where (1.11) is satisfied occurs when S_ε consists of periodically distributed holes of critical size. More precisely,

$$S_\varepsilon = \bigcup_{i=1}^{N_\varepsilon} T_i^\varepsilon$$

where T_i^ε are spheres of size $r_\varepsilon = a_\varepsilon$, periodically distributed (period 2ε) in each axis direction and a_ε is defined as in (1.10). In this case (1.11) holds with

$$\begin{gathered} \mu = \frac{\pi}{2}\frac{1}{C_0} \quad \text{if } n = 2, \\ \mu = \frac{\mathcal{S}_n(n-2)}{2n} C_0^{n-2} \quad \text{if } n \ge 3 \end{gathered}$$

where $\mathcal{S}_n$ is the surface of the unit sphere in $\mathbb{R}^n$.

In what concerns evolution equations, Cioranescu, Donato, Murat and Zuazua [6], studied the homogenization of the linear wave equation

$$\begin{gathered} u_\varepsilon'' - \Delta u_\varepsilon = f_\varepsilon \quad \text{in } \Omega_\varepsilon \times (0,T) \\ u_\varepsilon = 0 \quad \text{on } \Gamma_\varepsilon \times (0,T) \\ u_\varepsilon(x,0) = u_\varepsilon^0(x); \quad u_\varepsilon'(x,0) = u_\varepsilon^1(x); \quad x \in \Omega_\varepsilon, \end{gathered} \tag{1.12}$$

with $\{u_\varepsilon^0, u_\varepsilon^1, f_\varepsilon\} \in H_0^1(\Omega_\varepsilon) \times L^2(\Omega_\varepsilon) \times L^1(0,T; L^2(\Omega_\varepsilon)$ and

$$\begin{gathered} \tilde{u}_\varepsilon^0 \rightharpoonup u^0 \quad \text{weakly in } H_0^1(\Omega), \\ \tilde{u}_\varepsilon^1 \rightharpoonup u^1 \quad \text{weakly in } L^2(\Omega), \\ \tilde{f}_\varepsilon \rightharpoonup f \quad \text{weakly in } L^1(0,T; L^2(\Omega)). \end{gathered}$$

They proved that $\tilde{u}_\varepsilon \rightharpoonup u$ weak-star in $L^\infty(0,T;H_0^1(\Omega))\cap W^{1,\infty}(0,T;L^2(\Omega))$, where $\tilde{u}_\varepsilon$ is the unique solution of problem (1.12), for each $\varepsilon > 0$ fixed, extended by considering zero on the holes, and u is the unique solution of the homogenized problem

$$\begin{gathered}
u'' - \Delta u + \mu u = f \quad \text{in } \Omega \times (0,T) \\
u_\varepsilon = 0 \quad \text{on } \Gamma \times (0,T) \\
u(x,0) = u^0(x); \quad u'(x,0) = u^1(x); \quad x \in \Omega,
\end{gathered}$$

and μ is a non-negative Radon's measure, which is positive when one considers holes of critical size.

Now, concerning the exact controllability of the wave equation in perforated domains, it is important to mention the work of the authors Cioranescu, Donato and Zuazua [7]. When the size of the holes is small enough, at the limit, they got the wave equation with a boundary control and when the holes are of critical size they obtained at the limit the wave equation with an additional term of order zero and two controls: the first one on the boundary and the second one an internal control.

On the other hand it is worth mentioning the papers in connection with homogenization of attractors for hyperbolic equations of the authors Fiedler and Vishik [9] as well as Pankratov and Chueshov [16]. Also, we would like to cite some papers where the damping term is, as in the present paper, in the form $G(x,t,u_t)$, as, for instance, [4, 15, 17] and references therein.

It is important to observe that from the assumption (1.3) one has $\tilde{u}_\varepsilon^0 \to u^0$ strongly in $H_0^1(\Omega)$. Consequently we deduce that: $\mu = 0$ or $u^0 = 0$. As we are interested in nontrivial initial data, we are forced to consider $\mu = 0$, which implies that the geometry of the domain Ω is such that the holes possess '*small capacity*' (i. e. the holes are smaller than the critical size); see references [6, 7] for details.

Since controllability implies stabilization, then we can expect that we can also stabilize the system (1.1) by introducing a suitable dissipative mechanism. Unfortunately the controllability is showed basically for linear problems and only for a few semi-linear problems in a very few class of nonlinearities. Even if we are dealing with homogenization results for those domains with '*small capacity*', very few is known for the nonlinear wave equation. For this reason these homogenization and stabilization results are interesting to be studied.

In what follows in this work, the geometry of the perforated domain Ω_ε, will satisfy the conditions given by (1.11) of the abstract framework introduced by Cioranescu and Murat in [8], having in mind those domains with '*small capacity*'.

Now, we are in a position to state our main result.

Theorem 1.1. *Assume that* (1.2)-(1.10) *hold. Then, supposing that* (1.11) *is assumed with* $\mu = 0$*, the unique solution* u_ε *of* (1.1) *satisfies*

$$\begin{gathered}
\tilde{u}_\varepsilon \to u \quad \textit{strongly in } C^0_{\rm loc}([0,\infty); L^2(\Omega)), \\
\tilde{u}'_\varepsilon \to u' \quad \textit{strongly in } C^0_{\rm loc}([0,\infty); L^2(\Omega)), \\
\tilde{u}_\varepsilon \rightharpoonup u \quad \textit{weak-star in } L^\infty_{\rm loc}(0,\infty; H_0^1(\Omega)), \\
\tilde{u}'_\varepsilon \rightharpoonup u' \quad \textit{weak-star in} L^\infty_{\rm loc}(0,\infty; H_0^1(\Omega)), \\
\tilde{u}''_\varepsilon \rightharpoonup u'' \quad \textit{weak-star in } L^\infty_{\rm loc}(0,\infty; L^2(\Omega)),
\end{gathered}$$

where u is the unique solution of the homogenized problem

$$\begin{gathered} u'' - \Delta u + F(x,t,u',\nabla u) = 0 \quad \text{in } \Omega \times (0,+\infty) \\ u = 0 \quad \text{on } \Gamma \times (0,+\infty) \\ u(x,0) = u^0(x); \quad u'(x,0) = u^1(x); \quad x \in \Omega. \end{gathered} \tag{1.13}$$

Defining the energy related to the homogenized problem as

$$E(t) = \frac{1}{2}|u'(t)|^2_{L^2(\Omega)} + \frac{1}{2}|\nabla u(t)|^2_{L^2(\Omega)} \tag{1.14}$$

and assuming that $\rho = 0$, and $\varphi(t) \le C_1 e^{-\gamma t}$ for all $t \ge 0$, where C_1 and γ are positive constants, we have

$$E(t) \le C e^{-\gamma_0 t}, \quad \forall t \ge 0.$$

Furthermore, supposing that

$$\varphi(t) \le \frac{k_1}{(1+t)^{(\rho+2)/\rho}}, \quad \forall t \ge 0,$$

where k_1 is a positive constant, one has

$$E(t) \le \frac{K}{(1+t)^{2/\rho}}, \quad \forall t \ge 0$$

where K is a positive constant.

Our paper is organized as follows: In section 2 we study the existence and uniqueness of problem (1.1) for each $\varepsilon > 0$ fixed. In section 3 we obtain the homogenized problem related to (1.1) making use of the abstract framework presented in (1.11) and finally in section 4 we give the proofs of the uniform decay.

2. Existence and uniqueness of solutions to problem (1.1)

In this section, we prove existence and uniqueness of solutions to problem (1.1) for each $\varepsilon > 0$ fixed, assuming that the initial data belong to the class given by (1.2) and the nonlinear function $F(x,t,\xi,\zeta)$ satisfies the hypotheses (1.4)-(1.9).

For this end, let $(\omega_\nu)_{\nu\in\mathbb{N}}$ be a basis in $H_0^1(\Omega_\varepsilon)\cap H^2(\Omega_\varepsilon)$ which is an orthornormal system for $L^2(\Omega_\varepsilon)$. Let V_m be the space generated by $\omega_1,\dots,\omega_n$ and let

$$u_{\varepsilon m}(t) = \sum_{i=1}^{m} g_{j\varepsilon m}(t)\omega_j \tag{2.1}$$

be the solution to the Cauchy problem

$$\begin{gathered} (u''_{\varepsilon m}(t), w) + (\nabla u_{\varepsilon m}(t), \nabla w) + (F(x,t,u'_{\varepsilon m}(t), \nabla u_{\varepsilon m}(t)), w) = 0 \\ \text{for all } w \in V_m, \\ u_{\varepsilon m}(0) = u^0_{\varepsilon m} \to u^0_\varepsilon \quad \text{in } H_0^1(\Omega_\varepsilon) \cap H^2(\Omega_\varepsilon) \quad \text{as } m \to \infty, \\ u'_{\varepsilon m}(0) = u^1_{\varepsilon m} \to u^1_\varepsilon \quad \text{in } H_0^1(\Omega_\varepsilon) \text{ as } m \to \infty, \end{gathered} \tag{2.2}$$

where $(\cdot,\cdot)$ is the inner product in $L^2(\Omega_\varepsilon)$. For simplicity we denote

$$|u|^2 = \int_{\Omega_\varepsilon} |u(x)|^2 dx, \quad \|u\|_p^p = \int_{\Omega_\varepsilon} |u(x)|^p dx.$$

We observe that the term $(F(x,t,u'_{\varepsilon m}(t), \nabla u_{\varepsilon m}(t)), w)$ is well defined in view of the assumption (1.4).

By standard methods in differential equations, we can prove the existence of a solution to (2.2) on some interval $[0, t_{\varepsilon m})$. Then, this solution can be extended to the whole interval $[0, T]$; $T > 0$; by use of the first estimate below.

2.1. **A Priori estimates.** *First Estimate:* Taking $w = 2u'_{\varepsilon m}(t)$ in (2.2) and considering the assumption (1.5) one has

$$\begin{aligned} &\frac{d}{dt}\left\{|u'_{\varepsilon m}(t)|^2 + |\nabla u_{\varepsilon m}(t)|^2\right\} + 2\beta\|u'_{\varepsilon m}(t)\|_{\rho+2}^{\rho+2} \\ &\le 2\varphi(t)\int_{\Omega_\varepsilon}(1 + |u'_{\varepsilon m}||\nabla u_{\varepsilon m}|)\,dx \\ &\le 2\varphi(t)\operatorname{meas}(\Omega) + |u'_{\varepsilon m}(t)|^2 + |\nabla u_{\varepsilon m}(t)|^2. \end{aligned} \tag{2.3}$$

Integrating (2.3) over $(0, t)$, $t \in [0, t_{\varepsilon m})$, we obtain

$$\begin{aligned} &|u'_{\varepsilon m}(t)|^2 + |\nabla u_{\varepsilon m}(t)|^2 + 2\beta\int_0^t \|u'_{\varepsilon m}(s)\|_{\rho+2}^{\rho+2}ds \\ &\le |u^1_{\varepsilon m}|^2 + |u^0_{\varepsilon m}|^2 + 2\|\varphi\|_{L^1(0,\infty)}\operatorname{meas}(\Omega) + \int_0^t \left\{|u'_{\varepsilon m}(s)|^2 + |\nabla u_{\varepsilon m}(s)|^2\right\}ds. \end{aligned} \tag{2.4}$$

From (2.4), considering the convergence in (1.3) and (2.2) and employing Gronwall's lemma, we deduce

$$|u'_{\varepsilon m}(t)|^2 + |\nabla u_{\varepsilon m}(t)|^2 + 2\beta\int_0^t \|u'_{\varepsilon m}(s)\}_{\rho+2}^{\rho+2}ds \le L_1 \tag{2.5}$$

where L_1 is a positive constant independent of $m \in \mathbb{N}$, $t \in [0, T]$ and $\varepsilon > 0$.

Second Estimate: First, we prove that $u''_{\varepsilon m}(0)$ is bounded in $L^2(\Omega_\varepsilon)$ norm. Indeed, taking $w = u''_{\varepsilon m}(0)$ and $t = 0$ in (2.2), taking the assumption (1.4) into account; making use of Green's formula and Cauchy-Schwarz inequality and considering the inequality $ab \le \frac{1}{2}a^2 + \frac{1}{2}b^2$, we infer

$$|u''_{\varepsilon m}(0)|^2 \le \Big\{|\Delta u^0_{\varepsilon m}| + C_0[(\operatorname{meas}(\Omega)^{1/2} + \|u^1_{\varepsilon m}|\|_{2(\rho+1)}^{\rho+1} + |\nabla u^0_{\varepsilon m}|^2]\Big\}|u''_{\varepsilon m}(0)|.$$

From the last inequality, noting that $H^1_0(\Omega) \hookrightarrow L^{2(\rho+1)}(\Omega)$ and considering the convergence in (1.3) and (2.2) it holds that

$$|u''_{\varepsilon m}(0)|^2 \le L_2 \tag{2.6}$$

where L_2 is a positive constant independent of $t \in [0, T]$; $m \in \mathbb{N}$ and $\varepsilon > 0$.

Now, taking the derivative of (2.2) with respect to t and substituting $w = 2u''_{\varepsilon m}(t)$, it follows that

$$\begin{aligned} &\frac{d}{dt}\left\{|u''_{\varepsilon m}(t)|^2 + |\nabla u'_{\varepsilon m}(t)|^2\right\} \\ &= -2\int_{\Omega_\varepsilon} F_t\left(x, t, u'_{\varepsilon m}, \nabla u_{\varepsilon m}\right)u''_{\varepsilon m}\,dx - 2\int_{\Omega_\varepsilon} F_{u'_{\varepsilon m}}\left(x, t, u'_{\varepsilon m}, \nabla u_{\varepsilon m}\right)\left(u''_{\varepsilon m}\right)^2dx \\ &\quad - 2\sum_{i=1}^n\int_{\Omega_\varepsilon} F_{u_{\varepsilon m x_i}}\left(x, t, u'_{\varepsilon m}, \nabla u_{\varepsilon m}\right)u'_{\varepsilon m x_i}u''_{\varepsilon m}\,dx. \end{aligned}$$

From assumptions (1.6)-(1.8), taking into account the above equality and the Cauchy-Schwarz inequality, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left\{|u''_{\varepsilon m}(t)\|^2+|\nabla u'_{\varepsilon m}(t)|^2\right\}+2\beta\int_{\Omega_\varepsilon}|u'_{\varepsilon m}|^{\rho}\left(u''_{\varepsilon m}\right)^2dx\\
&\le 2C_0\Big\{\left(\operatorname{meas}(\Omega)\right)^{1/2}|u''_{\varepsilon m}(t)|+\int_{\Omega_\varepsilon}|u'_{\varepsilon m}|^{\rho/2}|u''_{\varepsilon m}\||u'_{\varepsilon m}|^{(\rho+2)/2}dx\\
&\quad+|\nabla u_{\varepsilon m}(t)||u''_{\varepsilon m}(t)|\Big\}+2n(M+1)\left\{|u''_{\varepsilon m}(t)|^2+|\nabla u'_{\varepsilon m}(t)\|^2\right\},
\end{aligned}
\tag{2.7}
$$

where $M=\max\{C_i,n\}$; $i=1,\dots,n$.

Integrating (2.7) over $(0,t)$ and making use of the inequality $ab\le\frac{1}{4\eta}a^2+\eta b^2$, for an arbitrary $\eta>0$, we deduce

$$
\begin{aligned}
&|u''_{\varepsilon m}(t)|^2+|\nabla u'_{\varepsilon m}(t)|^2+2C_0(\beta-\eta)\int_0^t\int_{\Omega_\varepsilon}|u'_{\varepsilon m}|^{\rho}\left(u''_{\varepsilon m}\right)^2dx\,ds\\
&\le|u''_{\varepsilon m}(0)|^2+|\nabla u^1_{\varepsilon m}|^2+C_0\operatorname{meas}(\Omega)T\\
&\quad+\frac{C_0}{2\eta}\int_0^t\|u'_{\varepsilon m}(s)\|^{\rho+2}_{\rho+2}ds+C_1\int_0^t\left\{|u''_{\varepsilon m}(s)|^2+|\nabla u'_{\varepsilon m}(s)|^2\right\}ds,
\end{aligned}
\tag{2.8}
$$

where $C_1=C_0+2n(M+1)$. From (2.8), (2.5), (2.6), considering the convergence in (1.3) and (2.2), choosing $\eta>0$ sufficiently small and employing Gronwall's lemma, we obtain the second estimate

$$
|u''_{\varepsilon m}(t)|^2+|\nabla u'_{\varepsilon m}(t)|^2+\int_0^t\int_{\Omega_\varepsilon}|u'_{\varepsilon m}|^{\rho}\left(u''_{\varepsilon m}\right)^2dx\,ds\le L_3
\tag{2.9}
$$

where L_3 is a positive constant independent of $t\in[0,T]$; $m\in\mathbb{N}$ and $\varepsilon>0$.

2.2. **Analysis of the nonlinear term F.** From the assumption (1.4), there is a positive constant N such that

$$
\int_{\Omega_\varepsilon}|F\left(x,t,u'_{\varepsilon m},\nabla u_{\varepsilon m}\right)|^2dx\le N\left(1+\|u'_{\varepsilon m}(t)\|^{2(\rho+1)}_{2(\rho+1)}+|\nabla u_{\varepsilon m}(t)|^2\right).
$$

Therefore, from estimates (2.5) and (2.9) and observing that $H^1_0(\Omega)\hookrightarrow L^{2(\rho+1)}(\Omega)$, it follows that

$$
\{F\left(x,t,u'_{\varepsilon m},\nabla u_{\varepsilon m}\right)\}_{m\in\mathbb{N},\varepsilon>0}\quad\text{is bounded}\quad\text{in } L^2_{\rm loc}(0,\infty;L^2(\Omega_\varepsilon)).
\tag{2.10}
$$

Consequently, there exists a subsequence $\{u_{\varepsilon\mu}\}$ of $\{u_{\varepsilon m}\}$ (which we still denote by the same symbol) and a function χ in $L^2_{\rm loc}(0,\infty;L^2(\Omega_\varepsilon))$ such that

$$
F\left(x,t,u'_{\varepsilon\mu},\nabla u_{\varepsilon\mu}\right)\rightharpoonup\chi_\varepsilon\quad\text{weakly in } L^2_{\rm loc}(0,\infty;L^2(\Omega_\varepsilon)) \text{ as } \mu\to\infty.
\tag{2.11}
$$

From the above estimates we also deduce that there is a function $u_\varepsilon:\Omega_\varepsilon\times(0,\infty)\to\mathbb{R}$ such that

$$
u'_{\varepsilon\mu}\rightharpoonup u'_\varepsilon\quad\text{weak-star in } L^\infty_{\rm loc}(0,\infty;L^2(\Omega_\varepsilon)),\tag{2.12}
$$

$$
u_{\varepsilon\mu}\rightharpoonup u_\varepsilon\quad\text{weak-star in } L^\infty_{\rm loc}(0,\infty;H^1_0(\Omega_\varepsilon)),\tag{2.13}
$$

$$
u''_{\varepsilon\mu}\rightharpoonup u''_\varepsilon\quad\text{weak-star in } L^\infty_{\rm loc}(0,\infty;L^2(\Omega_\varepsilon)),\tag{2.14}
$$

$$
u'_{\varepsilon\mu}\rightharpoonup u'_\varepsilon\quad\text{weak-star in } L^\infty_{\rm loc}(0,\infty;H^1_0(\Omega_\varepsilon)).\tag{2.15}
$$

Moreover, making use of Aubin-Lions theorem; Lions [13, p. 57], we have

$$u_{\varepsilon\mu} \to u_\varepsilon \quad \text{strongly in } L^2_{\rm loc}(0,\infty;L^2(\Omega_\varepsilon)), \tag{2.16}$$

$$u'_{\varepsilon\mu} \to u'_\varepsilon \quad \text{strongly in } L^2_{\rm loc}(0,\infty;L^2(\Omega_\varepsilon)). \tag{2.17}$$

From the above estimates after passing to the limit, we conclude that

$$u''_\varepsilon - \Delta u_\varepsilon + \chi_\varepsilon = 0 \quad \text{in } D'(\Omega_\varepsilon \times (0,T)). \tag{2.18}$$

Since $u''_\varepsilon, \chi_\varepsilon \in L^2_{\rm loc}(0,\infty;L^2(\Omega_\varepsilon))$ from (2.18) we deduce that

$$\Delta u_\varepsilon \in L^2_{\rm loc}(0,\infty;L^2(\Omega_\varepsilon))$$

and

$$u''_\varepsilon - \Delta u_\varepsilon + \chi_\varepsilon = 0 \quad \text{in } L^2_{\rm loc}(0,\infty;L^2(\Omega_\varepsilon)). \tag{2.19}$$

Our goal is to show that

$$\chi_\varepsilon = F\left(x,t,u'_\varepsilon,\nabla u_\varepsilon\right). \tag{2.20}$$

Indeed, integrating (2.2) over (0,T) and considering $w = u_{\varepsilon\mu}(t)$, we obtain

$$\begin{aligned} &\int_0^T \left(u''_{\varepsilon\mu}(t),u_{\varepsilon\mu}(t)\right)dt + \int_0^T |\nabla u_{\varepsilon\mu}(t)|^2 dt \\ &+ \int_0^T \left(F\left(x,t,u'_{\varepsilon\mu}(t),\nabla u_{\varepsilon\mu}(t)\right),u_{\varepsilon\mu}(t)\right)dt = 0 \end{aligned} \tag{2.21}$$

Then, considering the strong convergence (2.16) and the weak ones (2.11) and (2.14), from (2.21) we obtain

$$\lim_{\mu\to\infty}\int_0^T |\nabla u_{\varepsilon\mu}(t)|^2 dt = -\int_0^T \left(u''_\varepsilon(t),u_\varepsilon(t)\right)dt - \int_0^T \left(\chi(t),u_\varepsilon(t)\right)dt. \tag{2.22}$$

Substituting (2.19) in (2.22) and applying the generalized Green formula we deduce

$$\lim_{\mu\to\infty}\int_0^T |\nabla u_{\varepsilon\mu}(t)|^2 dt = \int_0^T |\nabla u_\varepsilon(t)|^2 dt. \tag{2.23}$$

Taking into account that

$$\begin{aligned} &\int_0^T |\nabla u_{\varepsilon\mu}(t) - \nabla u_\varepsilon(t)|^2 dt \\ &= \int_0^T |\nabla u_{\varepsilon\mu}(t)|^2 dt - 2\int_0^T \left(\nabla u_{\varepsilon\mu}(t),\nabla u_\varepsilon\right)dt + \int_0^T |\nabla u_\varepsilon(t)|^2 dt, \end{aligned}$$

from (2.23) and (2.13) we deduce that $\lim_{\mu\to\infty}\int_0^T |\nabla u_{\varepsilon\mu}(t) - \nabla u_\varepsilon(t)|^2 dt = 0$, which implies that

$$\nabla u_{\varepsilon\mu} \to \nabla u_\varepsilon \quad \text{in } L^2_{\rm loc}(0,\infty;L^2(\Omega_\varepsilon)) \quad \text{as } \mu\to\infty. \tag{2.24}$$

Then, from the strong convergence (2.16), (2.17) and (2.24) we have

$$F\left(x,t,u'_{\varepsilon\mu},\nabla u_{\varepsilon\mu}\right) \to F\left(x,t,u'_\varepsilon,\nabla u_\varepsilon\right) \quad \text{a.e. in } \Omega_\varepsilon \times (0,T).$$

From the last convergence and considering (2.10) we can apply [11, Lemma 1.3] to obtain

$$F\left(x,t,u'_{\varepsilon\mu},\nabla u_{\varepsilon\mu}\right) \rightharpoonup F\left(x,t,u'_\varepsilon,\nabla u_\varepsilon\right) \quad \text{weakly as } \mu\to\infty.$$

Therefore, (2.20) is proved.

2.3. **Uniqueness.** Let u and $\hat{u}$ be two solutions of (1.1) and put $z_\varepsilon = u_\varepsilon - \hat{u}_\varepsilon$. From assumption (1.9), noting that the map $s \mapsto |s|^\rho s$ is increasing and taking (2.19) and (2.20) into account, we deduce

$$\frac{d}{dt}\big\{|z'_\varepsilon(t)|^2 + |\nabla z_\varepsilon(t)|^2\big\} \le D\big\{|z'_\varepsilon(t)|^2 + |\nabla z_\varepsilon(t)|^2\big\}. \tag{2.25}$$

Integrating (2.25) over (0,t) and employing Gronwall's lemma we conclude that $|z'_\varepsilon(t)|^2 = |\nabla z_\varepsilon(t)|^2 = 0$. Therefore, $u_\varepsilon = \hat{u}_\varepsilon$. This completes the proofs of section 2.

3. The homogenized problem

We begin this section presenting a technical result that will play an essential role to obtain the homogenized problem.

3.1. **Technical Lemma.**

Lemma 3.1. *Assume that* (1.11) *is satisfied with* $\mu = 0$*; then*

$$\lim_{\varepsilon\to 0}\int_\Omega \varphi|\nabla w_\varepsilon|^2 dx = 0; \quad \forall \varphi \in D(\Omega).$$

Proof. Let $\varphi \in D(\Omega)$. Then, from (1.11), third and fourth equations, we have $\langle -\Delta w_\varepsilon, \varphi w_\varepsilon\rangle = \langle \mu_\varepsilon - \gamma_\varepsilon, \varphi w_\varepsilon\rangle = \langle \mu_\varepsilon, \varphi w_\varepsilon\rangle$ and consequently

$$\lim_{\varepsilon\to 0}\langle -\Delta w_\varepsilon, \varphi w_\varepsilon\rangle = 0. \tag{3.1}$$

On the other hand, we deduce that

$$\langle -\Delta w_\varepsilon, \varphi w_\varepsilon\rangle = \int_\Omega \varphi|\nabla w_\varepsilon|^2 dx + \int_\Omega w_\varepsilon(\nabla w_\varepsilon \cdot \nabla\varphi)dx. \tag{3.2}$$

Now, from the third equation in (1.11), we have

$$\nabla w_\varepsilon \rightharpoonup 0 \quad \text{weakly in} L^2(\Omega) \tag{3.3}$$

and since the imbedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, we also obtain

$$w_\varepsilon \to 1 \quad \text{strongly in } L^2(\Omega). \tag{3.4}$$

Combining (3.1)-(3.4) we conclude that

$$\lim_{\varepsilon\to 0}\int_\Omega \varphi|\nabla w_\varepsilon|^2 dx = 0,$$

which concludes the proof. □

Next, we obtain the homogenized problem when $\varepsilon \to 0$, making use of the abstract framework given in (1.11) and taking into consideration the estimates obtained in section 2.

3.2. **A priori estimates.** From the estimates obtained in section 2, there exists a function $u : \Omega \times (0, \infty) \to \mathbb{R}$ such that

$$\tilde{u}'_\varepsilon \rightharpoonup u' \quad \text{weak-star in } L^\infty_{\rm loc}(0, \infty; L^2(\Omega)), \tag{3.5}$$

$$\tilde{u}_\varepsilon \rightharpoonup u \quad \text{weak-star in } L^\infty_{\rm loc}(0, \infty; H^1_0(\Omega)), \tag{3.6}$$

$$\tilde{u}''_\varepsilon \rightharpoonup u'' \quad \text{weak-star in } L^\infty_{\rm loc}(0, \infty; L^2(\Omega)), \tag{3.7}$$

$$\tilde{u}'_\varepsilon \rightharpoonup u' \quad \text{weak-star in } L^\infty_{\rm loc}(0, \infty; H^1_0(\Omega)). \tag{3.8}$$

From Aubin-Lions theorem, we also deduce

$$\tilde{u}_\varepsilon \to u \quad \text{strongly in } L^2_{\rm loc}(0, \infty; L^2(\Omega)), \tag{3.9}$$

$$\tilde{u}'_\varepsilon \to u' \quad \text{strongly in } L^2_{\rm loc}(0, \infty; L^2(\Omega)). \tag{3.10}$$

On the other hand, let u_{ε_1} and u_{ε_2} be two solutions of (P_ε) with initial data $\{u^0_{\varepsilon_1}, u^1_{\varepsilon_1}\}$ and $\{u^0_{\varepsilon_2}, u^1_{\varepsilon_2}\}$, respectively.

Then, considering $z_\varepsilon = u_{\varepsilon_1} - u_{\varepsilon_2}$ and repeating analogous arguments like those used to prove the uniqueness of solutions in section 2, we obtain

$$\frac{d}{dt}\big\{|z'_\varepsilon(t)|^2 + |\nabla z_\varepsilon(t)|^2\big\} \le D\big\{|z'_\varepsilon(t)|^2 + |\nabla z_\varepsilon(t)|^2\big\}.$$

Integrating the above inequality over (0,t), we infer

$$\begin{aligned} &|z'_\varepsilon(t)|^2 + |\nabla z_\varepsilon(t)|^2 \\ &\le |u^1_{\varepsilon_1} - u^1_{\varepsilon_2}|^2 + |\nabla u^0_{\varepsilon_1} - \nabla u^0_{\varepsilon_2}|^2 + D\int_0^t \big\{|z'_\varepsilon(s)|^2 + |\nabla z_\varepsilon(s)|^2\big\} ds. \end{aligned} \tag{3.11}$$

From (3.11) employing Gronwall's inequality, noting that the imbeddings $H^2(\Omega) \hookrightarrow H^1(\Omega)$ and $H^1(\Omega) \hookrightarrow L^2(\Omega)$ are compact and taking the convergence in (1.3) into account we deduce that

$$\tilde{u}_\varepsilon \to u \quad \text{strongly in } C^0([0,T]; H^1_0(\Omega)); \quad \forall T > 0, \tag{3.12}$$

$$\tilde{u}'_\varepsilon \to u' \quad \text{strongly in } C^0([0,T]; L^2(\Omega)); \quad \forall T > 0. \tag{3.13}$$

Remark. Note that in view of the strong convergence given in (3.12), it is not necessary to use the equation of the abstract framework given in (1.11). However, we decided to present the passage to the limit making use of the whole abstract framework in order to facilitate the reader's comprehension when one has, for instance, a nonlinearity given by $F(x,t,u,u')$ where the strong convergence in (3.12) is not required (see convergence in (3.9) and (3.10)).

3.3. **Passage to the limit.** Multiplying (2.19) (taking (2.20) into consideration) by $w_\varepsilon\theta\varphi$, and integrating over $Q_\varepsilon = \Omega_\varepsilon \times (0,T)$; where w_ε belongs to the abstract framework (1.11), $\theta \in D(0,T)$ and $\varphi \in D(\Omega)$, we obtain

$$\int_{Q_\varepsilon} u''_\varepsilon w_\varepsilon \varphi\theta \, dx\, dt - \int_{Q_\varepsilon} \Delta u_\varepsilon w_\varepsilon \varphi\theta \, dx\, dt + \int_{Q_\varepsilon} F\left(x,t,u_\varepsilon,\nabla u_\varepsilon\right) w_\varepsilon \varphi\theta \, dx\, dt = 0. \tag{3.14}$$

Employing Green's formula in the second term of (3.14), we deduce

$$-\int_{Q_\varepsilon} \Delta u_\varepsilon w_\varepsilon \varphi\theta \, dx\, dt = \int_{Q_\varepsilon} \nabla u_\varepsilon \cdot \theta \nabla w_\varepsilon \varphi \, dx\, dt + \int_{Q_\varepsilon} \nabla u_\varepsilon \cdot \theta w_\varepsilon \nabla\varphi dx\, dt. \tag{3.15}$$

On the other hand, we also have

$$\int_{Q_\varepsilon} \nabla u_\varepsilon \cdot \theta \nabla w_\varepsilon \varphi \, dx\, dt = -\langle \Delta w_\varepsilon, \theta u_\varepsilon \varphi \rangle - \int_{Q_\varepsilon} \nabla w_\varepsilon \cdot \theta u_\varepsilon \nabla \varphi \, dx\, dt, \tag{3.16}$$

where $\langle \cdot, \cdot \rangle$ means the duality $L^1(0,T;H^{-1}(\Omega_\varepsilon))$ and $L^\infty(0,T;H_0^1(\Omega_\varepsilon))$.

Combining (3.14)-(3.16) we arrive at

$$\begin{aligned} &\int_0^T \int_{\Omega_\varepsilon} u_\varepsilon'' w_\varepsilon \varphi \theta \, dx\, dt - \langle \Delta w_\varepsilon, \theta u_\varepsilon \varphi \rangle - \int_0^T \int_{\Omega_\varepsilon} \nabla w_\varepsilon \cdot \theta u_\varepsilon \nabla \varphi \, dx\, dt \\ &+ \int_0^T \int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \theta w_\varepsilon \nabla \varphi dx\, dt + \int_0^T \int_{\Omega_\varepsilon} F\left(x,t,u_\varepsilon,\nabla u_\varepsilon\right) w_\varepsilon \varphi \theta \, dx\, dt = 0 \end{aligned} \tag{3.17}$$

Next, we analyze the terms in (3.17).

Estimate for $I_1 := \int_0^T \int_{\Omega_\varepsilon} u_\varepsilon'' w_\varepsilon \varphi \theta \, dx\, dt$. Employing Fubini's theorem we deduce

$$I_1 = \int_0^T \int_\Omega \tilde{u}_\varepsilon'' w_\varepsilon \varphi \theta \, dx\, dt = \int_\Omega w_\varepsilon \varphi \Big(\int_0^T \theta \tilde{u}_\varepsilon'' \, dt \Big) dx. \tag{3.18}$$

From (3.4) and (3.7) we obtain

$$\lim_{\varepsilon \to 0} I_1 = \int_\Omega \varphi \Big(\int_0^T \theta u'' dt \Big) dx \quad . \tag{3.19}$$

Estimate for $I_2 := -\langle \Delta w_\varepsilon, \theta u_\varepsilon \varphi \rangle$. Consider the $\mathcal{U}_\varepsilon \in H_0^1(\Omega)$ defined by $\mathcal{U}_\varepsilon = \int_0^T \theta \tilde{u}_\varepsilon \, dt$. From the convergence (3.6) and since $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact we have

$$\begin{gathered} \mathcal{U}_\varepsilon \rightharpoonup \int_0^T \theta u \, dt \quad \text{weakly in } H_0^1(\Omega) \text{ and strongly in } L^2(\Omega), \\ \mathcal{U}_\varepsilon = 0 \quad \text{on } S_\varepsilon. \end{gathered} \tag{3.20}$$

In view of (1.11), fourth equation, $-\Delta w_\varepsilon = \mu_\varepsilon - \gamma_\varepsilon$. Then, applying Fubini's theorem one has

$$\begin{aligned} I_2 &= \langle \mu_\varepsilon - \gamma_\varepsilon, \theta u_{\varepsilon\varphi} \rangle \\ &= \big\langle \mu_\varepsilon - \gamma_\varepsilon, \big(\int_0^T \theta \tilde{u}_\varepsilon \, dt \big) \varphi \big\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \\ &= \langle \mu_\varepsilon, \mathcal{U}_\varepsilon \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \end{aligned}$$

since $\langle \gamma_\varepsilon, \varphi \mathcal{U}_\varepsilon \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = 0$. Consequently from (3.20) and (1.11), fourth equation, we infer

$$\lim_{\varepsilon \to 0} I_2 = \big\langle \mu, \big(\int_0^T \theta u \, dt \big) \varphi \big\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = 0. \tag{3.21}$$

Estimate for $I_3 := \int_0^T \int_{\Omega_\varepsilon} \nabla w_\varepsilon \cdot \theta u_\varepsilon \nabla \varphi \, dx\, dt$. ¿From Fubini's theorem we deduce

$$I_3 = \int_\Omega \nabla w_\varepsilon \cdot \big(\int_0^T \theta \tilde{u}_\varepsilon dt \big) \nabla \varphi \, dx. \tag{3.22}$$

Taking (3.3) and (3.20) into account,from (3.22) it holds that

$$\lim_{\varepsilon \to 0} I_3 = 0. \tag{3.23}$$

Estimate for $I_4 := \int_0^T \int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \theta w_\varepsilon \nabla\varphi dx\, dt$. Analogously, employing Fubini's theorem it follows that

$$I_4 = \int_\Omega w_\varepsilon \nabla\varphi \cdot \nabla \left(\int_0^T \theta \tilde{u}_\varepsilon dt \right) dx. \tag{3.24}$$

Considering (3.4) and (3.20) from (3.24) we conclude

$$\lim_{\varepsilon\to 0} I_4 = \int_\Omega \nabla\varphi \cdot \nabla\big(\int_0^T \theta u\, dt\big) dx. \tag{3.25}$$

Estimate for $I_5 := \int_0^T \int_{\Omega_\varepsilon} F(x,t,u_\varepsilon, \nabla u_\varepsilon) w_\varepsilon \varphi\theta\, dx\, dt$. Analogously considering Fubini's theorem and in view of assumption (1.10) we can write

$$I_5 = \int_\Omega w_\varepsilon \varphi\big(\int_0^T \theta F(x,t,\tilde{u}_\varepsilon', \nabla\tilde{u}_\varepsilon) dt\big) dx. \tag{3.26}$$

On the other hand, from the convergence (3.12), (3.13) and making use of Lion's lemma, we deduce that

$$F(x,t,\tilde{u}_\varepsilon', \nabla\tilde{u}_\varepsilon) \rightharpoonup F(x,t,u',\nabla u) \quad \text{weakly in } L^2(0,T;L^2(\Omega)). \tag{3.27}$$

Then, from (3.4), (3.26) and (3.27) we conclude

$$\lim_{\varepsilon\to 0} I_5 = \int_\Omega \varphi\big(\theta \int_0^T F(x,t,u',\nabla u) dt\big) dx. \tag{3.28}$$

Combining (3.17), (3.19), (3.21), (3.23), (3.25) and (3.28) we deduce

$$\langle \int_\Omega u''\varphi dx, \theta\rangle + \langle \int_\Omega \nabla u \cdot \nabla\varphi dx, \theta\rangle + \langle \int_\Omega F(x,t,u',\nabla u)\varphi dx, \theta\rangle = 0, \tag{3.29}$$

where $\langle\cdot,\cdot\rangle$ means the duality $D'(0,T)$, $D(0,T)$, for all $\varphi \in D(\Omega)$ and for all $\theta \in D(0,T)$. Then, since $D(\Omega)$ is dense in $H_0^1(\Omega)$ we obtain

$$(u''(t),v) + (\nabla u(t),v) + (F(x,t,u'(t),\nabla u(t)), v) = 0 \quad \text{in } D'(0,T) \tag{3.30}$$

for all $v \in H_0^1(\Omega)$, where $(\cdot,\cdot)$ is the inner product in $L^2(\Omega)$.

The uniqueness of solutions follows considering analogous arguments like those ones used to prove (2.25).

4. Uniform Decay Rates

In this section we establish uniform rates of decay (exponential and algebraic) for the homogenized problem

$$\begin{gathered} u'' - \Delta u + F(x,t,u',\nabla u) = 0 \quad \text{in } \Omega\times(0,T) \\ u = 0 \quad \text{on } \Gamma\times(0,+\infty) \\ u(x,0) = u^0(x); \quad u'(x,0) = u^1(x); \quad x\in\Omega. \end{gathered} \tag{4.1}$$

Since $E(t) = \lim_{\varepsilon\to 0} E_\varepsilon(t)$, it is sufficient to prove that problem (1.1) decays exponentially or polynomially independently of ε. In other words, it is enough to prove that there exist positive constants C, γ_0, k_1, k_2 and k_3 independent of $\varepsilon > 0$ and such that

$$E_\varepsilon(t) \le Ce^{-\gamma_0 t} \quad \text{or} \quad E_\varepsilon(t) \le k_2 \frac{k_3 + 2k_1}{(1+t)^{2/\rho}} \tag{4.2}$$

for all $t \ge 0$ and for all $\varepsilon > 0$.

Remark 2. It is important to observe that when $\mu > 0$ and supposing that one could be able to homogenize the problem under consideration, an useful alternative to derive uniform decay rates for the energy

$$E^{\mu}(t) = \frac{1}{2}\left(|u'(t)|_{L^2(\Omega)^2 + |\nabla u(t)|^2_{L^2(\Omega)}} + |u(t)|^2_{L^2(\Omega,\mu)}\right)$$

would be making use of the lower semi-continuity of the energy, or, more precisely, to consider the following estimate:

$$E^{\mu} \leq \liminf_{\varepsilon \to 0} E_{\varepsilon}(t), \quad \text{for all } t \geq 0.$$

In order to obtain (4.2), we consider the following auxiliary lemmas.

Lemma 4.1. *Let E be a real C^1 positive function satisfying*

$$E'(t) \leq -C_0 E(t) + C_1 e^{-\gamma t} \tag{4.3}$$

where C_0, C_1 and γ are positive constants. Then, there exists γ_0 such that

$$E(t) \leq \left(E(0) + (2C_1)/\gamma\right) e^{-\gamma_0 t}. \tag{4.4}$$

Proof. Let $F(t) = E(t) + \frac{2C_1}{\gamma} e^{-\gamma t}$. Then

$$F'(t) = E'(t) - 2C_1 e^{-\gamma t} \leq -C_0 E(t) - C_1 e^{-\gamma t} \leq -\gamma_0 F(t),$$

where $\gamma_0 = \min\{C_0, \frac{\gamma}{2}\}$. Integrating the last inequality over $(0,t)$, we have

$$F(t) \leq F(0) e^{-\gamma_0 t} \quad \text{implies} \quad E(t) \leq C_2 e^{-\gamma_0 t},$$

where $C_2 = E(0) + \frac{2C_1}{\gamma}$. This completes the proof. ☐

Lemma 4.2. *Let E be a real C^1 positive function satisfying*

$$E'(t) \leq -k_0 [E(t)]^{\frac{\rho+2}{2}} + \frac{k_1}{(1+t)^{\frac{\rho+2}{\rho}}} \tag{4.5}$$

where $0 < \rho < 2$, and k_0 and k_1 are positive constants. Then, there exists $k_2 > 0$ such that

$$E(t) \leq k_2 \frac{\frac{\rho}{2} E(0) + 2k_1}{(1+t)^{2/\rho}}. \tag{4.6}$$

Proof. Consider $h(t) = \frac{2k_1}{\frac{\rho}{2}(1+t)^{\rho/2}}$ and set $g(t) = E(t) + h(t)$. We have

$$\begin{aligned} g'(t) = E'(t) - \frac{2k_1}{(1+t)^{\frac{\rho+2}{\rho}}} &\leq -k_0 \big\{ [E(t)]^{\frac{\rho+2}{2}} + \frac{k_1}{k_0 (1+t)^{\frac{\rho+2}{\rho}}} \big\} \\ &\leq -k_0 \big\{ [E(t)]^{\frac{\rho+2}{2}} + (\frac{1}{\rho})^{\frac{\rho+2}{2}} \frac{1}{k_0 k_1^{\rho/2}} [h(t)]^{\frac{\rho+2}{2}} \big\}. \end{aligned}$$

Let $a_0 = \min\{1, (\frac{1}{\rho})^{\frac{\rho+2}{2}} \frac{1}{k_0 k_1^{\rho/2}}\}$. Then,

$$g'(t) \leq -k_0 a_0 \big\{ [E(t)]^{\frac{\rho+2}{2}} + [h(t)]^{\frac{\rho+2}{2}} \big\}.$$

Since there exists a positive constant a_1 such that

$$[E(t) + h(t)]^{\frac{\rho+2}{2}} \leq a_1 \big\{ [E(t)]^{\frac{\rho+2}{2}} + [h(t)]^{\frac{\rho+2}{2}} \big\}$$

we conclude that

$$g'(t) \leq -\frac{k_0 a_0}{a_1} [g(t)]^{\frac{\rho+2}{2}}.$$

Integrating the last inequality over (0,t), we deduce

$$g(t) \le \frac{(\frac{2}{\rho})^{2/\rho} g(0)}{\{\frac{2}{\rho} + \frac{k_0 a_0}{a_1}[g(0)]^{\rho/2} t\}} \le \frac{(\frac{2}{\rho})^{\frac{2-\rho}{\rho}}[\frac{2}{\rho}E(0) + 2k_1]}{a_2^{2/\rho}(1+t)^{2/\rho}},$$

where $a_2 = \min\{\frac{2}{\rho}, \frac{k_0 a_0}{a_1}[g(0)]^{\rho/2}\}$. Considering $k_2 = \frac{1}{a_2}(\frac{2}{\rho a_2})^{\frac{2-\rho}{\rho}}$, it follows the desired result. □

For short notation, we will omit the parameter ε on the energy

$$E_\varepsilon(t) = \frac{1}{2}\big(|u_\varepsilon'(t)|^2_{L^2(\Omega_\varepsilon)} + |\nabla u_\varepsilon(t)|^2_{L^2(\Omega_\varepsilon)}\big) \tag{4.7}$$

having in mind that the constants obtained can not depend on ε. Inspired in the work of Haraux and Zuazua [10] let us define the Liapunov functional

$$\psi(t) = [E(t)]^{\rho/2}\,(u'(t), u(t)) \tag{4.8}$$

where $(\cdot,\cdot)$ is the inner product in $L^2(\Omega_\varepsilon)$.

Proposition 4.3. *There exists $L > 0$, independent of ε, such that $E(t) \le L$ for all $t \ge 0$.*

Proof. From (2.19), (2.20) and (1.5) we deduce

$$E'(t) \le -\beta\|u'(t)\|^{\rho+2}_{\rho+2} + \varphi(t)\int_{\Omega_\varepsilon}(1 + |u'||\nabla u|)\,dx \le \varphi(t)\,\mathrm{meas}(\Omega) + \varphi(t)E(t). \tag{4.9}$$

Multiplying both sides of the above inequality by $e^{-\int_0^t \varphi(s)ds}$, it follows that

$$\Big(E(t)e^{-\int_0^t \varphi(s)ds}\Big)' \le \varphi(t)\,\mathrm{meas}(\Omega). \tag{4.10}$$

Integrating (4.10) over $(0,t)$, we obtain

$$E(t) \le E(0)e^{\int_0^\infty \varphi(s)ds} + e^{\int_0^t \varphi(s)ds}\Big(\int_0^\infty \varphi(s)ds\Big)\,\mathrm{meas}(\Omega). \tag{4.11}$$

Considering the convergence in (1.3) we deduce that

$$E_\varepsilon(0) \le K; \quad \forall \varepsilon > 0 \tag{4.12}$$

where $K = K(|\nabla u^0|_{L^2(\Omega)}, |u^1|_{L^2(\Omega)})$. Combining (4.11) and (4.12), we obtain $E(t) \le L$ for all $t \ge 0$, where

$$L = e^{\int_0^\infty \varphi(s)ds}\Big(K + \int_0^\infty \varphi(s)ds\ \mathrm{meas}(\Omega)\Big), \tag{4.13}$$

which concludes the proof □

Proposition 4.4. *There exists $\lambda > 0$, independent of ε, such that*

$$|\psi(t)| \le \lambda L^{\rho/2}E(t); \quad \forall t \ge 0.$$

Proof. From (4.8) we deduce

$$|\psi(t)| \le [E(t)]^{\rho/2}|u'(t)|\lambda|\nabla u(t)|$$

where $\lambda > 0$ comes from the Poincaré inequality in Ω; i.e.,

$$|u_\varepsilon(t)|_{L^2(\Omega_\varepsilon)} = |\tilde{u}_\varepsilon(t)|_{L^2(\Omega)} \le \lambda|\nabla\tilde{u}_\varepsilon(t)|_{L^2(\Omega)} = \lambda|\nabla u_\varepsilon(t)|_{L^2(\Omega_\varepsilon)}. \tag{4.14}$$

The above inequalities and Proposition 4.3 yield

$$|\psi(t)| \le \lambda L^{\rho/2}E(t),$$

which completes the proof. □

Proposition 4.5. *Assume that $\rho = 0$ and that there exist C_1 and γ positive constants such that*
$$\varphi(t) \le C_1 e^{-\gamma t} \quad \forall t \ge 0. \tag{4.15}$$
Then, (4.3) *holds where C_0 is a positive constant independent of ε. Now, considering that there exists $k_1 > 0$ such that*
$$\varphi(t) \le \frac{k_1}{(1+t)^{\frac{\rho+2}{\rho}}} \quad \forall t \ge 0, \tag{4.16}$$
then (4.5) *holds where k_0 is a positive constant independent of ε.*

Proof. From (4.9), we have that
$$E'(t) - \varphi(t) \int_{\Omega_\varepsilon} (1 + |u'||\nabla u|)\, dx \le 0. \tag{4.17}$$
Computing the derivative of (4.8) with respect to t and substituting $u'' = \Delta u - F(x, t, u', \nabla u)$, from (1.5) and making use of Green formula we deduce
$$\begin{aligned}\psi'(t) \le{}& \frac{\rho}{2} [E(t)]^{\frac{\rho-2}{2}} E'(t)\,(u'(t), u(t)) \\ &+ [E(t)]^{\rho/2} \big\{ -|\nabla u(t)|^2 - \beta\,(|u'(t)|^\rho u'(t), u(t)) \\ &+ \varphi(t) \int_{\Omega_\varepsilon} (1 + |u'||\nabla u|)\, dx + |u'(t)|^2 \big\}.\end{aligned} \tag{4.18}$$
Adding and subtracting the term
$$\frac{\rho}{2} [E(t)]^{\frac{\rho-2}{2}} \varphi(t) \int_{\Omega_\varepsilon} (1 + |u'||\nabla u|)\, dx\, (u'(t), u(t))$$
in (4.8), we infer
$$\begin{aligned}\psi'(t) \le{}& -\frac{\rho}{2} [E(t)]^{\frac{\rho-2}{2}} (u'(t), u(t)) \big[\varphi(t) \int_{\Omega_\varepsilon} (1 + |u'||\nabla u|)\, dx - E'(t)\big] \\ &+ \frac{\rho}{2} [E(t)]^{\frac{\rho-2}{2}} \varphi(t) \int_{\Omega_\varepsilon} (1 + |u'||\nabla u|)\, dx\, (u'(t), u(t)) \\ &+ [E(t)]^{\rho/2} \big\{ -|\nabla u(t)|^2 - \beta\,(|u'(t)|^\rho u'(t), u(t)) \\ &+ \varphi(t) \int_{\Omega_\varepsilon} (1 + |u'||\nabla u|)\, dx + |u'(t)|^2 \big\}.\end{aligned} \tag{4.19}$$
Observe that from (4.14) and taking Proposition 4.3 into account, we can write
$$|\,(u'(t), u(t))\,| \le \lambda E(t) \le \lambda L. \tag{4.20}$$
Then, combining (4.17), (4.19) and (4.20) we deduce
$$\begin{aligned}\psi'(t) \le{}& \frac{\rho \lambda L^{\rho/2}}{2} \big[\varphi(t) \int_{\Omega_\varepsilon} (1 + |u'||\nabla u|)\, dx - E'(t)\big] \\ &+ \frac{\rho \lambda L^{\rho/2}}{2} \varphi(t) \int_{\Omega_\varepsilon} (1 + |u'||\nabla u|)\, dx \\ &+ [E(t)]^{\rho/2} \big\{ -|\nabla u(t)|^2 - \beta\,(|u'(t)|^\rho u'(t), u(t)) \\ &+ \varphi(t) \int_{\Omega_\varepsilon} (1 + |u'||\nabla u|)\, dx + |u'(t)|^2 \big\}.\end{aligned} \tag{4.21}$$

Therefore,

$$\begin{aligned}\psi'(t) \le & -\rho\lambda L^{\rho/2} E'(t) + \rho\lambda L^{\rho/2} (\text{meas}(\Omega) + L)\, \varphi(t) \\ & + [E(t)]^{\rho/2} \big\{ -|\nabla u(t)|^2 - \beta\, (|u'(t)|^\rho u'(t), u(t)) \\ & + (\text{meas}(\Omega) + L)\, \varphi(t) + |u'(t)|^2 \big\}.\end{aligned} \tag{4.22}$$

Estimate for $I_1 := \beta\,(|u'(t)|^\rho u'(t), u(t))$. Making use of Hölder inequality having in mind that $\frac{\rho+1}{\rho+2} + \frac{1}{\rho+2} = 1$, we deduce

$$|I_1| \le \beta \|u'(t)\|_{\rho+2}^{\rho+1} \|u(t)\|_{\rho+2}. \tag{4.23}$$

Now, since $H_0^1(\Omega) \hookrightarrow L^{\rho+2}(\Omega)$, we have

$$\|u_\varepsilon(t)\|_{L^{\rho+2}(\Omega_\varepsilon)} = \|\tilde{u}_\varepsilon\|_{L^{\rho+2}(\Omega)} \le \xi \|\tilde{u}_\varepsilon(t)\|_{H_0^1(\Omega)} = \xi \|u_\varepsilon(t)\|_{H_0^1(\Omega_\varepsilon)}. \tag{4.24}$$

Then, from (4.23), (4.24) and making use of Young's inequality, we conclude

$$|I_1| \le \beta\xi \|u'(t)\|_{\rho+2}^{\rho+1} |\nabla u(t)| \le \frac{(\rho+1)\,(\beta\xi)^{\frac{\rho+2}{\rho+1}}}{\eta^{\frac{1}{\rho+1}}} \|u'(t)\|_{\rho+2}^{\rho+2} + \frac{\eta}{\rho+2} |\nabla u(t)|^{\rho+2},$$

where $\eta > 0$ is an arbitrary positive constant. On the other hand, from Proposition 4.3 one has

$$|\nabla u(t)|^{\rho+2} \le 2^{\rho/2} L^{\rho/2} |\nabla u(t)|^2.$$

Then,

$$|I_1| \le \frac{(\rho+1)\,(\beta\xi)^{\frac{\rho+2}{\rho+1}}}{\eta^{\frac{1}{\rho+1}}} \|u'(t)\|_{\rho+2}^{\rho+2} + \eta \frac{2^{\rho/2} L^{\rho/2}}{\rho+2} |\nabla u(t)|^2. \tag{4.25}$$

Combining (4.22) and (4.25) choosing $\eta = \frac{\rho+2}{2^{\frac{\rho+2}{2}} L^{\rho/2}}$, we infer

$$\begin{aligned}\psi'(t) \le & -\rho\lambda L^{\rho/2} E'(t) + \rho\lambda L^{\rho/2} (\text{meas}(\Omega) + L)\, \varphi(t) \\ + \quad & [E(t)]^{\rho/2} \big\{ -\frac{1}{2} |\nabla u(t)|^2 + M \|u'(t)\|_{\rho+2}^{\rho+2} (\text{meas}(\Omega) + L)\, \varphi(t) + |u'(t)|^2 \big\},\end{aligned} \tag{4.26}$$

where

$$M = \frac{(\rho+1)\,(\beta\xi)^{(\rho+2)/(\rho+1)}}{(\rho+2)\big(\frac{\rho+2}{2^{\frac{\rho+2}{2}} L^{\rho/2}}\big)^{1/(\rho+1)}}. \tag{4.27}$$

Now, from (4.26), Proposition 4.3 and (4.9), we obtain

$$\begin{aligned}\psi'(t) \le & -\Big(\rho\lambda + M\beta^{-1} L^{\rho/2}\Big) E'(t) + M\beta^{-1} L^{\rho/2} \varphi(t) \int_{\Omega_\varepsilon} (1 + |u'||\nabla u|)\, dx \\ & - \frac{1}{2} [E(t)]^{\rho/2} |\nabla u(t)|^2 + [E(t)]^{\rho/2} |u'(t)|^2 + L^{\rho/2} (\text{meas}(\Omega) + L)\, \varphi(t).\end{aligned} \tag{4.28}$$

Consequently

$$\begin{aligned}\psi'(t) \le & -\Big(\rho\lambda + M\beta^{-1} L^{\rho/2}\Big) E'(t) - \frac{1}{2} [E(t)]^{\rho/2} |\nabla u(t)|^2 \\ & + [E(t)]^{\rho/2} |u'(t)|^2 + N\varphi(t),\end{aligned} \tag{4.29}$$

where $N = L^{\rho/2} (\text{meas}(\Omega) + L) \big(1 + M\beta^{-1}\big)$ Defining the perturbed energy by

$$E_\tau(t) = (1 + \tau R) E(t) + \tau\psi(t); \quad \tau > 0, \tag{4.30}$$

where $R = \rho\lambda + M\beta^{-1}L^{\rho/2}$, from Proposition 4.4 we deduce

$$|E_\tau(t) - E(t)| \le \tau \left(R + \lambda L^{\rho/2}\right) E(t). \tag{4.31}$$

Setting $C_1 = R + \lambda L^{\rho/2}$, considering $\tau \in (0, 1/2C_1]$, we deduce

$$\frac{1}{2}E(t) \le E_\tau(t) \le 2E(t); \quad \forall t \ge 0, \tag{4.32}$$

which implies

$$2^{-\frac{\rho+2}{2}}[E(t)]^{\frac{\rho+2}{2}} \le [E_\tau(t)]^{\frac{\rho+2}{2}} \le 2^{\frac{\rho+2}{2}}[E(t)]^{\frac{\rho+2}{2}}. \tag{4.33}$$

On the other hand, taking the derivative of (4.30) with respect to t taking (4.29) into account, it holds that

$$E'_\tau(t) \le E'(t) - \frac{\tau}{2}[E(t)]^{\rho/2}\left|\nabla u(t)|^2 + \tau[E(t)]^{\rho/2}|u'(t)\right|^2 + \tau N\varphi(t).$$

The last inequality and (4.9) yield

$$E'_\tau(t) \le -\beta\|u'(t)\|^{\rho+2}_{\rho+2} - \frac{\tau}{2}[E(t)]^{\rho/2}|\nabla u(t)|^2 + \tau[E(t)]^{\rho/2}|u'(t)|^2 + \tau N^*\varphi(t), \tag{4.34}$$

where $N^* = N + \operatorname{meas}(\Omega) + L$. Having in mind that

$$-\frac{1}{2}|\nabla u(t)|^2 = \frac{1}{2}|u'(t)|^2 - \frac{1}{2}E(t) \tag{4.35}$$

and noting that $L^{\rho+2}(\Omega) \hookrightarrow L^2(\Omega)$, from (4.34) we deduce

$$E'_\tau(t) \le -\beta\theta^{-(\rho+2)}|u'(t)|^{\rho+2} - \frac{\tau}{2}[E(t)]^{\frac{\rho+2}{2}} + \frac{3}{2}\tau[E(t)]^{\rho/2}|u'(t)|^2 + \tau N^*\varphi(t), \tag{4.36}$$

where θ comes from the inequality

$$|u'_\varepsilon(t)|_{L^2(\Omega_\varepsilon)} = |\tilde{u}'_\varepsilon(t)|_{L^2(\Omega)} \le \theta\|\tilde{u}'_\varepsilon(t)\|_{L^{\rho+2}(\Omega)} = \theta\|u'_\varepsilon(t)\|_{L^{\rho+2}(\Omega_\varepsilon)}. \tag{4.37}$$

However, since $\frac{\rho}{\rho+2} + \frac{2}{\rho+2} = 1$, the Hölder inequality yields

$$\begin{aligned} {}^{\rho/2}|u'(t)|^2 &\le \frac{\rho}{\rho+2}\left(\eta[E(t)]^{\rho/2}\right)^{\frac{\rho+2}{\rho}} + \frac{2}{\rho+2}\left(\frac{1}{\eta}|u'(t)|^2\right)^{\frac{\rho+2}{2}} \\ &\le \eta^{\frac{\rho+2}{\rho}}[E(t)]^{\frac{\rho+2}{2}} + \frac{1}{\eta^{\frac{\rho+2}{2}}}|u'(t)|^{\rho+2}, \end{aligned} \tag{4.38}$$

where η is an arbitrary positive constant. Then, from (4.36) and (4.38) we obtain

$$E'_\tau(t) \le -\big(\beta\theta^{-(\rho+2)} - \frac{3\tau}{2}\frac{1}{\eta^{\frac{\rho+2}{2}}}\big)|u'(t)|^{\rho+2} - \frac{\tau}{2}\left(1 - 3\eta^{\frac{\rho+2}{\rho}}\right)[E(t)]^{\frac{\rho+2}{2}} + \tau N^*\varphi(t). \tag{4.39}$$

Choosing η sufficiently small in order to have $\zeta = 1 - 3\eta^{\frac{\rho+2}{\rho}} > 0$ and τ small enough to have

$$\beta\theta^{-(\rho+2)} - \frac{3\tau}{2}\frac{1}{\eta^{\frac{\rho+2}{2}}} \ge 0,$$

from (4.39) we conclude that

$$E'_\tau(t) \le -\frac{\tau\zeta}{2}[E(t)]^{\frac{\rho+2}{2}} + \tau N^*\varphi(t). \tag{4.40}$$

At this point, we have to divide our proof into two parts, namely,

(A) If $\rho > 0$ and $\varphi(t)$ verifies (4.16). Then, combining (4.33) and (4.40), we obtain

$$E'_\tau(t) \leq -k_0^*[E_\tau(t)]^{\frac{\rho+2}{2}} + \frac{k_1^*}{(1+t)^{\frac{\rho+2}{\rho}}},$$

where k_0^*, k_1^* are positive constants independent of ε. So, from Lemma 4.2 and considering (4.33), the decay in (4.6) holds.

(B) If $\rho = 0$ and $\varphi(t)$ verifies (4.15). Again, combining (4.33) and (4.40) we obtain

$$E'_\tau(t) \leq -C_0^* E_\tau(t) + C_1^* e^{-\gamma t}$$

where C_0^* and C_1^* are positive constants independent of ε.

Now, from Lemma 4.1 and taking (4.33) into account, (4.4) holds. This completes the proof of Theorem 1.1. □

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