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EXISTENCE AND APPROXIMATION OF SOLUTIONS OF SECOND ORDER NONLINEAR NEUMANN PROBLEMS

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Abstract. We study existence and approximation of solutions of some Neumann boundary-value problems in the presence of an upper solution β and a lower solution α in the reversed order ($\alpha \geq \beta$). We use the method of quasilinearization for the existence and approximation of solutions. We also discuss quadratic convergence of the sequence of approximants.

1. Introduction

In this paper, we study existence and approximation of solutions of some second order nonlinear Neumann problem of the form

$$-x''(t) = f(t, x(t)), \quad t \in [0, 1],$$
$$x'(0) = A, \quad x'(1) = B,$$

in the presence of a lower solution α and an upper solution β with $\alpha \geq \beta$ on $[0, 1]$. We use the quasilinearization technique for the existence and approximation of solutions. We show that under suitable conditions the sequence of approximants obtained by the method of quasilinearization converges quadratically to a solution of the original problem.

There is a vast literature dealing with the solvability of nonlinear boundary-value problems with the method of upper and lower solution and the quasilinearization technique in the case where the lower solution α and the upper solution β are ordered by $\alpha \leq \beta$. Recently, the case where the upper and lower solutions are in the reversed order has also received some attention. Cabada, et al. [6, 5], Cherpion, et al. [4] have studied existence results for Neumann problems in the presence of lower and upper solutions in the reversed order. In these papers, they developed the monotone iterative technique for existence of a solution x such that $\alpha \geq x \geq \beta$.

The purpose of this paper is to develop the quasilinearization technique for the solution of the original problem in the case upper and lower solutions are in the reversed order. The main idea of the method of quasilinearization as developed by Bellman and Kalaba [3], and generalized by Lakshmikantham [9, 10], has recently been studied and extended extensively to a variety of nonlinear problems [1, 2, 7, 11, 12]. In all these quoted papers, the key assumption is that the upper and lower

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solutions are ordered with $\alpha \leq \beta$. When α and β are in the reverse order, the quasilinearization technique seems not to have studied previously.

In section 2, we discuss some basic known existence results for a solution of the BVP (2.2). The key assumption is that the function $f(t,x) - \lambda x$ is non-increasing in x for some λ. In section 3, we approximate our problem by a sequence of linear problems by the method of quasilinearization and prove that under some suitable conditions there exist monotone sequences of solutions of linear problems converging to a solution of the BVP (2.2). Moreover, we prove that the convergence of the sequence of approximants is quadratic. In section 4, we study the generalized quasilinearization method by allowing weaker hypotheses on f and prove that the conclusion of section 3 is still valid.

2. Preliminaries

We know that the linear Neumann boundary value problem

$$\begin{gathered} -x''(t) + Mx(t) = 0, \quad t \in [0,1] \\ x'(0) = 0, \quad x'(1) = 0, \end{gathered}$$

has only the trivial solution if $M \neq -n^2\pi^2$, $n \in \mathbb{Z}$. For $M \neq -n^2\pi^2$ and any $\sigma \in C[0,1]$, the unique solution of the linear problem

$$\begin{gathered} -x''(t) + Mx(t) = \sigma(t), \quad t \in [0,1] \\ x'(0) = A, \quad x'(1) = B \end{gathered} \tag{2.1}$$

is given by

$$x(t) = P_\lambda(t) + \int_0^1 G_\lambda(t,s)\sigma(s)ds,$$

where

$$P_\lambda(t) = \begin{cases} \frac{1}{\sqrt{\lambda}\sin\sqrt{\lambda}}(A\cos\sqrt{\lambda}(1-t) - B\cos\sqrt{\lambda}t), & \text{if } M = -\lambda,\ \lambda > 0, \\ \frac{1}{\sqrt{\lambda}\sinh\sqrt{\lambda}}(B\cosh\sqrt{\lambda}t - A\cosh\sqrt{\lambda}(1-t)) & \text{if } M = \lambda,\ \lambda > 0, \end{cases}$$

and (for $M = -\lambda$),

$$G_\lambda(t,s) = -\frac{1}{\sqrt{\lambda}\sin\sqrt{\lambda}} \begin{cases} \cos\sqrt{\lambda}(1-s)\cos\sqrt{\lambda}t, & \text{if } 0 \leq t \leq s \leq 1, \\ \cos\sqrt{\lambda}(1-t)\cos\sqrt{\lambda}s, & \text{if } 0 \leq s \leq t \leq 1, \end{cases}$$

and (for $M = \lambda$),

$$G_\lambda(t,s) = \frac{1}{\sqrt{\lambda}\sinh\sqrt{\lambda}} \begin{cases} \cosh\sqrt{\lambda}(1-s)\cosh\sqrt{\lambda}t, & \text{if } 0 \leq t \leq s \leq 1, \\ \cosh\sqrt{\lambda}(1-t)\cosh\sqrt{\lambda}s, & \text{if } 0 \leq s \leq t \leq 1, \end{cases}$$

is the Green's function of the problem. For $M = -\lambda$, we note that $G_\lambda(t,s) \leq 0$ if $0 < \sqrt{\lambda} \leq \pi/2$. Moreover, for such values of M and λ, we have, $P_\lambda(t) \leq 0$ if $A \leq 0 \leq B$, and $P_\lambda(t) \geq 0$ if $A \geq 0 \geq B$. Thus we have the following anti-maximum principle

Anti-maximum Principle. Let $-\pi^2/4 \leq M < 0$. If $A \leq 0 \leq B$ and $\sigma(t) \geq 0$, then a solution $x(t)$ of (2.1) is such that $x(t) \leq 0$. If $A \geq 0 \geq B$ and $\sigma(t) \leq 0$, then $x(t) \geq 0$.

Consider the nonlinear Neumann problem

$$\begin{gathered} -x''(t) = f(t, x(t)), \quad t \in [0,1], \\ x'(0) = A, \quad x'(1) = B, \end{gathered} \tag{2.2}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and A, $B \in \mathbb{R}$. We recall the concept of lower and upper solutions.

Definition. Let $\alpha \in C^2[0,1]$. We say that α is a lower solution of (2.2), if

$$\begin{gathered} -\alpha''(t) \le f(t, \alpha(t)), \quad t \in [0,1], \\ \alpha'(0) \ge A, \quad \alpha'(1) \le B. \end{gathered}$$

An upper solution $\beta \in C^2[0,1]$ of the BVP (2.2) is defined similarly by reversing the inequalities.

Theorem 2.1 (Upper and Lower solutions method)**.** *Let $0 < \lambda \le \pi^2/4$. Assume that α and β are respectively lower and upper solutions of* (2.2) *such that $\alpha(t) \ge \beta(t)$, $t \in [0,1]$. If $f(t,x) - \lambda x$ is non-increasing in x, then there exists a solution x of the boundary value problem* (2.2) *such that*

$$\alpha(t) \ge x(t) \ge \beta(t), \quad t \in [0,1].$$

Proof. This result is known [6] and we provide a proof for completeness. Define $p(\alpha(t), x, \beta(t)) = \min\{\alpha(t), \max\{x, \beta(t)\}\}$, then $p(\alpha(t), x, \beta(t))$ satisfies $\beta(t) \le p(\alpha(t), x, \beta(t)) \le \alpha(t)$, $x \in \mathbb{R}$, $t \in [0,1]$. Consider the modified boundary value problem

$$\begin{gathered} -x''(t) - \lambda x(t) = F(t, x(t)), \quad t \in [0,1], \\ x'(0) = A, \quad x'(1) = B, \end{gathered} \tag{2.3}$$

where

$$F(t,x) = f(t, p(\alpha(t), x, \beta(t))) - \lambda p(\alpha(t), x, \beta(t)).$$

This is equivalent to the integral equation

$$x(t) = P_\lambda(t) + \int_0^1 G_\lambda(t,s) F(s, x(s)) ds. \tag{2.4}$$

Since $P_\lambda(t)$ and $F(t, x(t))$ are continuous and bounded, this integral equation has a fixed point by the Schauder fixed point theorem. Thus, problem (2.3) has a solution. Moreover,

$$\begin{gathered} F(t, \alpha(t)) = f(t, \alpha(t)) - \lambda\alpha(t) \ge -\alpha''(t) - \lambda\alpha(t), \quad t \in [0,1], \\ F(t, \beta(t)) = f(t, \beta(t)) - \lambda\beta(t) \le -\beta''(t) - \lambda\beta(t), \quad t \in [0,1]. \end{gathered}$$

Thus, α, β are lower and upper solutions of (2.3). Further, we note that any solution $x(t)$ of (2.3) with the property $\beta(t) \le x(t) \le \alpha(t), t \in [0,1]$, is also a solution of (2.2). Now, we show that any solution x of (2.3) does satisfy $\beta(t) \le x(t) \le \alpha(t)$, $t \in [0,1]$. For this, set $v(t) = \alpha(t) - x(t)$, then $v'(0) \ge 0$, $v'(1) \le 0$. In view of the non-increasing property of the function $f(t,x) - \lambda x$ in x, the definition of lower solution and the fact that $p(\alpha(t), x, \beta(t)) \le \alpha(t)$, we have

$$\begin{aligned} &- v''(t) - \lambda v(t) \\ &= (-\alpha''(t) - \lambda\alpha(t)) - (-x''(t) - \lambda x(t)) \\ &\le (f(t, \alpha(t)) - \lambda\alpha(t)) - (f(t, p(\alpha(t), x(t), \beta(t))) - \lambda p(\alpha(t), x(t), \beta(t))) \le 0. \end{aligned}$$

By the anti-maximum principle, we obtain $v(t) \geq 0$, $t \in [0,1]$. Similarly, $x(t) \geq \beta(t)$, $t \in [0,1]$. □

Theorem 2.2. *Assume that α and β are lower and upper solutions of the boundary value problem (2.2) respectively. If $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and*

$$f(t,\alpha(t)) - \lambda\alpha(t) \leq f(t,\beta(t)) - \lambda\beta(t) \quad \text{for some } 0 < \lambda \leq \pi^2/4,\ t \in [0,1], \tag{2.5}$$

then $\alpha(t) \geq \beta(t)$, $t \in [0,1]$.

Proof. Define $m(t) = \alpha(t) - \beta(t)$, $t \in [0,1]$, then $m(t) \in C^2[0,1]$ and $m'(0) \geq 0$, $m'(1) \leq 0$. In view of (2.5) and the definition of upper and lower solution, we have

$$\begin{aligned} -m''(t) - \lambda m(t) &= (-\alpha''(t) - \lambda\alpha(t)) - (-\beta''(t) - \lambda\beta(t)) \\ &\leq (f(t,\alpha(t)) - \lambda\alpha(t)) - (f(t,\beta(t)) - \lambda\beta(t)) \leq 0. \end{aligned}$$

Thus, by anti-maximum principle, $m(t) \geq 0$, $t \in [0,1]$. □

3. Quasilinearization Technique

We now approximate our problem by the method of quasilinearization. Lets state the following assumption.

- (A1) α, $\beta \in C^2[0,1]$ are respectively lower and upper solutions of (2.2) such that $\alpha(t) \geq \beta(t)$, $t \in [0,1] = I$.
- (A2) $f(t,x)$, $f_x(t,x)$, $f_{xx}(t,x)$ are continuous on $I \times \mathbb{R}$ and are such that $0 < f_x(t,x) \leq \frac{\pi^2}{4}$ and $f_{xx}(t,x) \leq 0$ for $(t,x) \in I \times [\min\beta(t), \max\alpha(t)]$.

Theorem 3.1. *Under assumptions (A1)-(A2), there exists a monotone sequence $\{w_n\}$ of solutions converging uniformly and quadratically to a solution of the problem (2.2).*

Proof. Taylor's theorem and the condition $f_{xx}(t,x) \leq 0$ imply that

$$f(t,x) \leq f(t,y) + f_x(t,y)(x-y), \tag{3.1}$$

for (t,x), $(t,y) \in I \times [\min\beta(t), \max\alpha(t)]$. Define

$$F(t,x,y) = f(t,y) + f_x(t,y)(x-y), \tag{3.2}$$

$x, y \in \mathbb{R}$, $t \in I$. Then, $F(t,x,y)$ is continuous and satisfies the relations

$$\begin{aligned} f(t,x) &\leq F(t,x,y) \\ f(t,x) &= F(t,x,x), \end{aligned} \tag{3.3}$$

for (t,x), $(t,y) \in I \times [\min\beta(t), \max\alpha(t)]$. Let $\lambda = \max\{f_x(t,x) : (t,x) \in I \times [\min\beta(t), \max\alpha(t)]\}$, then $0 < \lambda \leq \frac{\pi^2}{4}$. Now, set $w_0 = \beta$ and consider the linear problem

$$\begin{gathered} -x''(t) - \lambda x(t) = F(t, p(\alpha(t), x(t), w_0(t)), w_0(t)) - \lambda p(\alpha(t), x(t), w_0(t)), \quad t \in I, \\ x'(0) = A, \quad x'(1) = B. \end{gathered} \tag{3.4}$$

This is equivalent to the integral equation

$$\begin{aligned} &x(t) \\ &= P_\lambda(t) + \int_0^1 G_\lambda(t,s)\big[F(s, p(\alpha(s), x(s), w_0(s)), w_0(s)) - \lambda p(\alpha(s), x(s), w_0(s))\big]\,ds. \end{aligned}$$

Since $P_\lambda(t)$ and $F(t,p(\alpha,x,w_0),w_0)-\lambda p(\alpha,x,w_0)$ are continuous and bounded, this integral equation has a fixed point w_1 (say) by the Schauder fixed point theorem. Moreover,

$$\begin{aligned}
&F(t,p(\alpha(t),w_0(t),w_0(t)),w_0(t)) - \lambda p(\alpha(t),w_0(t),w_0(t))\\
&= f(t,w_0(t)) - \lambda w_0(t)\\
&\le -w_0''(t) - \lambda w_0(t), \quad t \in I,
\end{aligned}$$

and

$$\begin{aligned}
&F(t,p(\alpha(t),\alpha(t),w_0(t)),w_0(t)) - \lambda p(\alpha(t),\alpha(t),w_0(t))\\
&\ge f(t,\alpha(t)) - \lambda\alpha(t)\\
&\ge -\alpha''(t) - \lambda\alpha(t), \quad t \in I.
\end{aligned}$$

This implies that α, w_0 are lower and upper solutions of (3.4). Now, we show that

$$w_0(t) \le w_1(t) \le \alpha(t) \text{ on } I.$$

For this, set $v(t) = w_1(t) - w_0(t)$, then the boundary conditions imply that $v'(0) \ge 0$, $v'(1) \le 0$. Further, in view of the condition $f_x(t,x) \le \lambda$ for $(t,x) \in I \times [\min\beta(t), \max\alpha(t)]$ and (3.2), we have

$$\begin{aligned}
-v''(t) - \lambda v(t) &= (-w_1''(t) - \lambda w_1(t)) - (-w_0''(t) - \lambda w_0(t))\\
&\le (f_x(t,w_0(t)) - \lambda)(p(\alpha(t),w_1(t),w_0(t)) - w_0(t)) \le 0.
\end{aligned}$$

Thus, by anti-maximum principle, we obtain $v(t) \ge 0$, $t \in I$. Similarly, $\alpha(t) \ge w_1(t)$. Thus,

$$w_0(t) \le w_1(t) \le \alpha(t), \quad t \in I. \tag{3.5}$$

In view of (3.3) and the fact that w_1 is a solution of (3.4) with the property (3.5), we have

$$\begin{gathered}
-w_1''(t) = F(t,w_1(t),w_0(t)) \ge f(t,w_1(t))\\
w_1'(0) = A,\ w_1'(1) = B,
\end{gathered} \tag{3.6}$$

which implies that w_1 is an upper solution of (2.2).

Now, consider the problem

$$\begin{gathered}
-x''(t) - \lambda x(t) = F(t,p(\alpha(t),x(t),w_1(t)),w_1(t)) - \lambda p(\alpha(t),x(t),w_1(t)), \quad t \in I,\\
x'(0) = A, \quad x'(1) = B.
\end{gathered} \tag{3.7}$$

Denote by w_2 a solution of (3.7). In order to show that

$$w_1(t) \le w_2(t) \le \alpha(t), \quad t \in I, \tag{3.8}$$

set $v(t) = w_2(t) - w_1(t)$, then $v'(0) = 0$, $v'(1) = 0$. Further, in view of (3.2) and the condition $f_x(t,x) \le \lambda$ for $(t,x) \in I \times [\min\beta(t), \max\alpha(t)]$, we obtain

$$\begin{aligned}
&-v''(t) - \lambda v(t)\\
&\le F(t,p(\alpha(t),w_2(t),w_1(t)),w_1(t)) - \lambda p(\alpha(t),w_2(t),w_1(t)) - (f(t,w_1(t)) - \lambda w_1(t))\\
&\le (f_x(t,w_1(t)) - \lambda)(p(\alpha(t),w_2(t),w_1(t)) - w_1(t)) \le 0, \quad t \in I.
\end{aligned}$$

Hence $w_2(t) \ge w_1(t)$ follows from the anti-maximum principle. Similarly, we can show that $w_2(t) \le \alpha(t)$ on I.

Continuing this process, we obtain a monotone sequence $\{w_n\}$ of solutions satisfying

$$w_0(t) \le w_1(t) \le w_2(t) \le \cdots \le w_n(t) \le \alpha(t), \quad t \in I, \tag{3.9}$$

where, the element w_n of the sequence $\{w_n\}$ that for $t \in I$, satisfies

$$-x''(t) - \lambda x(t) = F(t, p(\alpha(t), x(t), w_{n-1}(t)), w_{n-1}(t)) - \lambda p(\alpha(t), x(t), w_{n-1}(t)),$$
$$x'(0) = A, \quad x'(1) = B.$$

That is,

$$-w_n''(t) = F(t, w_n(t), w_{n-1}(t)), \quad t \in I,$$
$$w_n'(0) = A, \quad w_n'(1) = B.$$

Employing the standard argument [8], it follows that the convergence of the sequence is uniform. If $x(t)$ is the limit point of the sequence, since F is continuous, we have

$$\lim_{n\to\infty} F(t, w_n(t), w_{n-1}(t)) = F(t, x(t), x(t)) = f(t, x(t))$$

which implies that, x is a solution the boundary value problem (2.2).

Now, we show that the convergence of the sequence is quadratic. For this, set $e_n(t) = x(t) - w_n(t)$, $t \in I$, $n \in \mathbb{N}$, where x is a solution of (2.2). Note that, $e_n(t) \geq 0$ on I and $e_n'(0) = 0$, $e_n'(1) = 0$. Let $\rho = \min\big\{f_x(t, x) : (t, x) \in I \times [\min \beta(t), \max \alpha(t)]\big\}$, then $0 < \rho < \frac{\pi^2}{4}$. Using Taylor's theorem and (3.2), we obtain

$$\begin{aligned}
&- e_n''(t)\\
&= -x''(t) + w_n''(t) = f(t, x(t)) - F(t, w_n(t), w_{n-1}(t))\\
&= f(t, w_{n-1}(t)) + f_x(t, w_{n-1}(t))(x(t) - w_{n-1}(t)) + \frac{f_{xx}(t, \xi(t))}{2!}(x(t) - w_{n-1}(t))^2\\
&\quad - [f(t, w_{n-1}(t)) + f_x(t, w_{n-1}(t))(w_n(t) - w_{n-1}(t))]\\
&= f_x(t, w_{n-1}(t))e_n(t) + \frac{f_{xx}(t, \xi(t))}{2!} e_{n-1}^2(t)\\
&\geq \rho e_n(t) + \frac{f_{xx}(t, \xi(t))}{2!}\|e_{n-1}\|^2, \quad t \in I
\end{aligned} \tag{3.10}$$

where, $w_{n-1}(t) < \xi(t) < x(t)$. Thus, by comparison results, the error function e_n satisfies $e_n(t) \leq r(t)$, $t \in I$, where r is the unique solution of the boundary-value problem

$$\begin{gathered}
-r''(t) - \rho r(t) = \frac{f_{xx}(t, \xi(t))}{2!}\|e_{n-1}\|^2, \quad t \in I\\
r'(0) = 0, \quad r'(1) = 0,
\end{gathered} \tag{3.11}$$

and

$$r(t) = \int_0^1 G_\rho(t, s)\frac{f_{xx}(t, \xi(s))}{2!}\|e_{n-1}\|^2 ds \leq \delta\|e_{n-1}\|^2,$$

where $\delta = \max\{\frac{1}{2}|G_\rho(t, s) f_{xx}(s, x)| : (t, x) \in I \times [\min \beta(t), \max \alpha(t)]\}$. Thus $\|e_n\| \leq \delta\|e_{n-1}\|^2$. □

Remark 3.2. In (A2), if we replace the concavity assumption $f_{xx}(t, x) \leq 0$ on $I \times [\min \beta(t), \max \alpha(t)]$ by the convexity assumption $f_{xx}(t, x) \geq 0$ on $I \times [\min \beta(t), \max \alpha(t)]$. Then we have the relations

$$f(t, x) \geq F(t, x, y)$$
$$f(t, x) = F(t, x, x),$$

for $x, y \in [\min \beta(t), \max \alpha(t)]$, $t \in [0,1]$, instead of (3.3) and we obtain a monotonically nonincreasing sequence

$$\alpha(t) \geq w_1(t) \geq w_2(t) \geq \cdots \geq w_n(t) \geq \beta(t), \quad t \in I,$$

of solutions of linear problems which converges uniformly and quadratically to a solution of (2.2).

4. Generalized quasilinearization technique

Now we introduce an auxiliary function ϕ to relax the concavity(convexity) conditions on the function f and hence prove results on the generalized quasilinearization. Let

- (B1) α, $\beta \in C^2(I)$ are lower and upper solutions of (2.2) respectively, such that $\alpha(t) \geq \beta(t)$ on I.
- (B2) $f \in C^2(I \times \mathbb{R})$ and is such that $0 < f_x(t,x) \leq \frac{\pi^2}{4}$ for $(t,x) \in I \times [\min \beta(t), \max \alpha(t)]$ and
$$\frac{\partial^2}{\partial^2 x}(f(t,x) + \phi(t,x)) \leq 0$$
on $I \times [\min \beta(t), \max \alpha(t)]$, for some function $\phi \in C^2(I \times \mathbb{R})$ satisfies $\phi_{xx}(t,x) \leq 0$ on $I \times [\min \beta(t), \max \alpha(t)]$.

Theorem 4.1. *Under assumptions (B1)-(B2), there exists a monotone sequence $\{w_n\}$ of solutions converging uniformly and quadratically to a solution of the problem* (2.2).

Proof. Define $F : I \times \mathbb{R} \to \mathbb{R}$ by

$$F(t,x) = f(t,x) + \phi(t,x). \tag{4.1}$$

Then, in view of (B2), we have $F(t,x) \in C^2(I \times \mathbb{R})$ and

$$F_{xx}(t,x) \leq 0 \quad \text{on } I \times [\min \beta(t), \max \alpha(t)], \tag{4.2}$$

which implies

$$f(t,x) \leq F(t,y) + F_x(t,y)(x-y) - \phi(t,x), \tag{4.3}$$

for (t,x), $(t,y) \in I \times [\min \beta(t), \max \alpha(t)]$. Using Taylor's theorem on ϕ, we obtain

$$\phi(t,x) = \phi(t,y) + \phi_x(t,y)(x-y) + \frac{\phi_{xx}(t,\eta)}{2!}(x-y)^2,$$

where x, $y \in \mathbb{R}$, $t \in I$ and η lies between x and y. In view of (B2), we have

$$\phi(t,x) \leq \phi(t,y) + \phi_x(t,y)(x-y), \tag{4.4}$$

for (t,x), $(t,y) \in I \times [\min \beta(t), \max \alpha(t)]$ and

$$\phi(t,x) \geq \phi(t,y) + \phi_x(t,y)(x-y) - \frac{M}{2}\|x-y\|^2, \tag{4.5}$$

for (t,x), $(t,y) \in I \times [\min \beta(t), \max \alpha(t)]$, where

$$M = \max\{|\phi_{xx}(t,x)| : (t,x) \in I \times [\min \beta(t), \max \alpha(t)]\}.$$

Using (4.5) in (4.3), we obtain

$$f(t,x) \leq f(t,y) + f_x(t,y)(x-y) + \frac{M}{2}\|x-y\|^2, \tag{4.6}$$

for (t,x), $(t,y) \in I \times [\min \beta(t), \max \alpha(t)]$. Define

$$F^*(t,x,y) = f(t,y) + f_x(t,y)(x-y) + \frac{M}{2}\|x-y\|^2, \tag{4.7}$$

for $t \in I$, x, $y \in \mathbb{R}$. then, $F^*(t,x,y)$ is continuous and for (t,x), $(t,y) \in I \times [\min \beta(t), \max \alpha(t)]$, satisfies the following relations

$$\begin{aligned} f(t,x) &\le F^*(t,x,y) \\ f(t,x) &= F^*(t,x,x). \end{aligned} \tag{4.8}$$

Now, we set $\beta = w_0$ and consider the Neumann problem

$$\begin{gathered} -x''(t) - \lambda x(t) = F^*(t, p(\alpha(t), x(t), w_0(t)), w_0(t)) - \lambda p(\alpha(t), x(t), w_0(t)), \quad t \in I, \\ x'(0) = A, \quad x'(1) = B, \end{gathered} \tag{4.9}$$

where λ and p are the same as defined in Theorem 3.1. Since $F^*(t, p(\alpha, x, w_0), w_0) - \lambda p(\alpha, x, w_0)$ is continuous and bounded, it follows that the problem (4.9) has a solution. Also, we note that any solution x of (4.9) which satisfies

$$w_0(t) \le x(t) \le \alpha(t), \quad t \in I, \tag{4.10}$$

is a solution of

$$\begin{gathered} -x''(t) = F^*(t, x(t), w_0(t)), \quad t \in I, \\ x'(0) = A, \quad x'(1) = B, \end{gathered}$$

and in view of (4.8), $F^*(t, x(t), w_0(t)) \ge f(t, x(t))$. It follows that any solution x of (4.9) with the property (4.10) is an upper solution of (2.2). Now, set $v(t) = \alpha(t) - x(t)$, where x is a solution of (4.9), then $v'(0) \ge 0$, $v'(1) \le 0$. Moreover, using (B2) and (4.8), we obtain

$$\begin{aligned} &- v''(t) - \lambda v(t) \\ &= (-\alpha''(t) - \lambda\alpha(t)) - (-x''(t) - \lambda x(t)) \\ &\le (f(t,\alpha(t)) - \lambda\alpha(t)) - [F^*(t, p(\alpha(t), x(t), w_0(t)), w_0(t)) - \lambda p(\alpha(t), x(t), w_0(t))] \\ &\le (f(t,\alpha(t)) - \lambda\alpha(t)) - [f(t, p(\alpha(t), x(t), w_0(t))) - \lambda p(\alpha(t), x(t), w_0(t))] \le 0. \end{aligned}$$

Hence, by anti-maximum principle, $\alpha(t) \ge x(t)$, $t \in I$. Similarly, $w_0(t) \le x(t)$, $t \in I$. Continuing this process we obtain a monotone sequence $\{w_n\}$ of solutions satisfying

$$w_0(t) \le w_1(t) \le w_2(t) \le w_3(t) \le \dots \le w_{n-1}(t) \le w_n(t) \le \alpha(t), \quad t \in I.$$

The same arguments as in Theorem 3.1, shows that the sequence converges to a solution x of the boundary value problem (2.2).

Now we show that the convergence of the sequence of solutions is quadratic. For this, we set $e_n(t) = x(t) - w_n(t)$, $t \in I$, where x is a solution of the boundary-value problem (2.2). Note that, $e_n(t) \ge 0$ on I and, $e_n'(0) = 0$, $e_n'(1) = 0$. Using Taylor's theorem, (4.4) and the fact that $\|w_n - w_{n-1}\| \leq \|e_{n-1}\|$, we obtain

$$
\begin{aligned}
& -e_n''(t) \\
&= -x''(t) + w_n''(t) \\
&= (F(t, x(t)) - \phi(t, x(t))) - F^*(t, w_n(t), w_{n-1}(t)) \\
&= F(t, w_{n-1}(t)) + F_x(t, w_{n-1}(t))(x(t) - w_{n-1}(t)) + \frac{F_{xx}(t, \xi(t))}{2}(x(t) - w_{n-1}(t))^2 \\
&\quad - [\phi(t, w_{n-1}(t)) + \phi_x(t, w_{n-1}(t))(x(t) - w_{n-1}(t))] \\
&\quad - [f(t, w_{n-1}(t)) + f_x(t, w_{n-1}(t))(w_n(t) - w_{n-1}(t)) + \frac{M}{2}\|w_n - w_{n-1}\|^2] \\
&= f_x(t, w_{n-1}(t))e_n(t) + \frac{F_{xx}(t, \xi(t))}{2}e_{n-1}^2(t) - \frac{M}{2}\|w_n - w_{n-1}\|^2 \\
&\geq f_x(t, w_{n-1}(t))e_n(t) - (\frac{|F_{xx}(t, \xi(t))|}{2} + \frac{M}{2})\|e_{n-1}\|^2 \\
&\geq \rho e_n(t) - Q\|e_{n-1}\|^2, \quad t \in I,
\end{aligned}
$$

where, $w_{n-1}(t) \leq \xi(t) \leq x(t)$,

$$
Q = \max\{\frac{|F_{xx}(t, x)|}{2} + \frac{M}{2} : (t, x) \in I \times [\min \beta(t), \max \alpha(t)]\}
$$

and ρ is defined as in Theorem 3.1. Thus, by comparison results $e_n(t) \leq r(t)$, $t \in I$, where r is a unique solution of the linear problem

$$
\begin{gathered}
-r''(t) - \rho r(t) = -Q\|e_{n-1}\|^2, \quad t \in I \\
r'(0) = 0, \quad r'(1) = 0,
\end{gathered}
$$

and

$$
r(t) = Q\int_0^1 |G_\rho(t, s)|\|e_{n-1}\|^2 ds \leq \sigma\|e_{n-1}\|^2,
$$

where $\sigma = Q \max\{|G_\rho(t, s)| : (t, s) \in I \times I\}$. Thus $\|e_n\| \leq \sigma\|e_{n-1}\|^2$. □

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