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PERIODIC SOLUTIONS OF A ONE DIMENSIONAL WILSON-COWAN TYPE MODEL

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Abstract. We analyze a time independent integral equation defined on a spatially extended domain which arises in the modeling of neuronal networks. In our survey, the coupling function is oscillatory and the firing rate is a smooth "heaviside-like" function. We will derive an associated fourth order ODE and establish that any bounded solution of the ODE is also a solution of the integral equation. We will then apply shooting arguments to prove that the ODE has two "1-bump" periodic solutions.

1. Introduction

In this paper we develop methods to analyze stationary solutions of the integral equation

$$u_t = -u + \int_{-\infty}^{\infty} w(x-y) f(u(y,t)) dy. \tag{1.1}$$

This equation is a Wilson-Cowan type model derived in 1972 to describe the behavior of a single layer of neurons [12]. Here, $u(x,t)$ and $f(u(x,t))$ represent the level of excitation (e.g. voltage) and the firing rate, respectively, of a neuron at position x and time t. The parameter $th \geq 0$ denotes the threshold of excitation. The term $w(x-y)$ determines the coupling between neurons at positions x and y.

In 1977, Amari [1] studied pattern formation in (1.1) for lateral inhibition type couplings. That is, w is assumed to be continuous, integrable and even, with $w(0) > 0$, and exactly one positive zero. Under the simplifying assumption that the firing rate f is a Heaviside step function, he analyzed the existence, multiplicity and stability of stationary one-bump solutions of the time independent equation

$$u = \int_{-\infty}^{\infty} w(x-y) f(u(y)) dy. \tag{1.2}$$

Equations (1.1) and (1.2) have been studied with respect to various combinations of firing rate functions and coupling functions. For example, Kishimoto and Amari [7] assume that f has a sigmoidal shape and use the Schauder Fixed Point Theorem [4] to prove the existence of a single bump stationary solution of (1.2). Ermentrout and McLeod [6] investigate the existence of traveling waves when w is strictly positive and Gaussian shaped, and f is a sigmoidal function. They use a homotopy

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argument based on the contraction mapping theorem to prove the existence of monotonic wave fronts. Subsequently, Pinto and Ermentrout [10] make use of the result in [6] and use singular perturbation methods to study wave front solutions in a related system of equations. In 1998, Ermentrout [5] gave an extensive review of theoretical methods and results.

In order to analyze more complicated solutions (e.g. multi-bump solutions), Laing et al. [9] and Coombes et al. [3] derive associated ODEs by applying Fourier Transform methods. In both cases conditions are given which show that when the integral equation (1.2) has a homoclinic orbit satisfying $u(\pm\infty) = 0$ then that solution also satisfies an associated ODE of the form

$$u'''' + q_1 u'' + h(u) = 0, \tag{1.3}$$

where q_1 is a real constant and h is a real-valued function.

Conversely, Laing et al. show that if a nonconstant solution u of (1.3) satisfies $(u, u', u'', u''') \to (0, 0, 0, 0)$ as $x \to \pm\infty$ exponentially fast, then u is also a solution of (1.2). They also give a complete numerical investigation of multi-bump homoclinic orbits, all of which are also solutions of the integral equation.

For technical reasons, the Fourier Transform argument does not necessarily apply to other classes of solutions such as

(a) periodic and aperiodic solutions, and
(b) chaotic solutions.

A fundamentally important problem is to determine whether these types of solutions are also solutions of the integral equation. Krisner [8], shows that solutions of (1.3) of the variety described above in $(a)-(b)$ are also solutions of the integral equation (1.2).

The primary goal in this paper is to develop techniques which allow us to prove the existence of periodic solutions of (1.2). In our survey the coupling function w is oscillatory shaped and the firing rate function f is a smooth step-like function. The techniques which are developed should apply to a broad range of problems. For example, applying the Fourier Transform to an integral equation studied by Bressloff [2] with non-homogeneous coupling gives rise, at least formally, to a non-autonomous partial differential equation.

The outline of the paper is as follows. In Section 2, we define our coupling and firing rate functions. These functions were originally introduced in Laing et al. [9]. We then state a previously established result which links (1.2) to a fourth order ODE. In Section 3, we define a parameter regime which gives rise to a tractable setting for our construction of periodic solutions. It is hoped that in future research we can extend our results to include a more general set of parameters. In Section 4, we state an initial value problem and prove that its solutions are even. In Section 5 we begin a rigorous analysis of the behavior of solutions of the initial value problem. We will show that the solutions are oscillatory, i.e, there exists infinitely many critical numbers. We will also show that these critical numbers are continuous with respect to the initial conditions. This analysis will lay the framework for the construction of two 1-bump periodic solutions which is contained in Section 6.

2. The Associated ODE

The primary goal in this paper is to construct periodic solutions of the time independent integral equation

$$u(x) = \int_{-\infty}^{\infty} w(x-y)f(u(y))dy, \tag{2.1}$$

where

$$w(x) = e^{-b|x|}(b\sin(|x|) + \cos(x)), \quad b > 0, \tag{2.2}$$

$$f(u) = 2e^{-r/(u-th)^2}H(u-th), \quad r > 0,\ th > 0. \tag{2.3}$$

Figure 1 depicts the essential characteristics of the functions w and f.

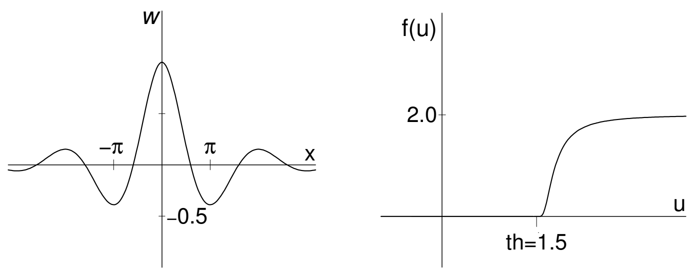

Figure 1. Left panel, example of (2.2) with $b = 0.3$. Right panel, example of (2.3) with $r = 0.05$, $th = 1.5$.

First, we state an important theorem which establishes a crucial connection between the ODE

$$u'''' - 2(b^2-1)u'' + (b^2+1)^2 u = 4b(b^2+1)f(u) \tag{2.4}$$

and the integral equation (2.1) with w defined by (2.2) and f defined by (2.3). Krisner [8] proves the following result.

Theorem 2.1. *Suppose that u is a solution of* (2.4)*, and that $u(t) = o(e^{b|t|})$ as $t \to \pm\infty$. Then u is a solution of* (2.1).

We now state an important consequence of the preceding theorem.

Corollary 2.2. *If u is a bounded solution of* (2.4)*, then u is also a solution of* (2.1).

This corollary guarantees that periodic solutions of (2.4) are also solutions of (2.1). This gives us the opportunity to employ the technique of topological shooting to prove the existence of periodic solutions.

3. Range of Parameters

The aim of this subsection is to define a range for the parameters r, b, and th that gives rise to a tractable setting for the construction of periodic solutions. Recall that b appears in the coupling function (2.2), and that r and th appear in the firing rate function (2.3). There are combinations of r, b, and th for which (2.4) does not have periodic solutions. The parameter regime that we will soon derive guarantees the existence of periodic solutions.

We begin by multiplying through (2.4) by u'. In doing so, we obtain

$$u'u'''' - 2(b^2-1)u'u'' + (b^2+1)^2u'u = 4b(b^2+1)f(u)u',$$

which leads to

$$(u'''u' - \frac{(u'')^2}{2})' - 2(b^2-1)\frac{((u')^2)'}{2} + (b^2+1)^2Q'(u) = 0, \tag{3.1}$$

where Q is defined by

$$Q(u) \equiv \int_0^u \Big(s - (\frac{4b}{b^2+1})f(s)\Big)ds. \tag{3.2}$$

The function Q will play a pivotal rule in defining our parameter regime.

An integration of equation (3.1) yields

$$u'''u' - \frac{(u'')^2}{2} - 2(b^2-1)\frac{(u')^2}{2} + (b^2+1)^2Q(u) = E \tag{3.3}$$

where E is referred to as the energy constant. We refer to (3.3) as the first integral of equation (2.4). In later sections, it will be evident that setting $E = 0$ in (3.3) will provide several technical conveniences. Thus, we will analyze the subclass of solutions of (2.4) for which

$$u'''u' - \frac{(u'')^2}{2} - (b^2-1)(u')^2 + (b^2+1)^2Q(u) = 0. \tag{3.4}$$

As previously mentioned the function Q will play a pivotal role in defining our range of parameters. Before we precisely define our parameter regime we will first acquire some intuition as to how the function Q behaves. We note from (3.2) that $Q(u) = u^2/2$ for $u \le th$. Figure 2 depicts three distinct scenarios of how the function Q can behave for $u > th$. The left panel depicts an example of a parameter choice (r, b, th) for which $Q(u) > 0$ for all $u > 0$. The middle panel shows that $Q(u) < 0$ on some interval (a, b) such that $0 < a < b < \infty$. We shall not consider combinations of (r, b, th) for which either of these two scenarios occur. The right panel of Figure 2 shows that $Q(u_s) = 0$ for some unique $u_s > 0$. We will choose our parameter regime so that Q possess this characteristic.

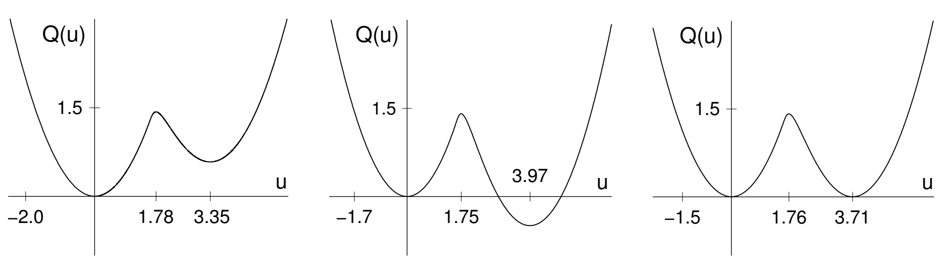

FIGURE 2. In all three graphs, we set $r = 0.05$ and $th = 1.5$. From left to right, we set $b = 1.8$, $b = 1.0$, and $b = 1.44019$. Hence, $(0.05, 1.8.1.5)$not in Λ, $(0.05, 1.0, 1.5)$ not in Λ, and $(0.05, 1.44019, 1.5)$ in Λ.

We now formally define our parameter regime. First, recall that each of the three variables are positive. Thus, for $X = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 > 0,\ x_2 > 0,$ and $x_3 >$

$0\}$ we define

$$\Lambda = \{(r, b, th) \in X : Q(u) = 0 \text{ has a unique positive solution}\}. \tag{3.5}$$

We now pursue deeper insight into the parameter regime described by (3.5). First, we will show that $(r, b, th) \in \Lambda$ implies that $th < 2$. Then, we will show that for each fixed $th \in (0, 2)$, there exists a continuum in (r, b) space for which $(r, b, th) \in \Lambda$. Doing so will result in valuable information about (r, b, th) and the unique $u_s > 0$ for which $Q(u_s) = 0$. This information will be used to prove that the corresponding solution of (2.4) has infinitely many critical numbers. Furthermore, our proofs will rely on sufficiently small $r > 0$. The information that we garner throughout the remainder of this section will be used to attain the "best" upper-bound on r that is possible.

Lemma 3.1. *Let $r > 0$, $b > 0$, and $th \geq 2$. Then $Q(u) > 0$ for all $u > 0$.*

Proof. First, recall that

$$Q(u) = \frac{u^2}{2} - \frac{8b}{b^2+1} \int_{th}^{u} e^{-\frac{r}{(s-th)^2}} H(s - th) ds \tag{3.6}$$

and hence, $Q(u) = u^2/2 > 0$ for $0 < u \leq th$. Thus, we will restrict our attention to $u > th$ for the remainder of the proof. It follows from (3.6) that

$$\begin{aligned} 2Q(u) &> u^2 - \frac{16b}{b^2+1}(u - th) \\ &= \big(u - \frac{8b}{b^2+1}\big)^2 + \frac{16b}{b^2+1}\big(th - \frac{4b}{b^2+1}\big) \\ &\geq \frac{16b}{b^2+1}\big(th - \frac{4b}{b^2+1}\big) \\ &\geq \frac{16b}{b^2+1}(th - 2). \end{aligned} \tag{3.7}$$

Since $th \geq 2$, then $Q(u) > 0$ for all $u > th$ follows from (3.7). This concludes the proof of the lemma. $\square$

The object of the following four lemmas is to show that for any fixed $th \in (0, 2)$, Λ contains a continuum (r, b, th). Along this continuum $b > 0$ is a two valued function of r, hence we write $b = b_r$. Proving the existence of this continuum entails finding a solution, (u, r, b), of the algebraic system

$$\begin{aligned} Q(u) &= \frac{u^2}{2} - \frac{8b}{b^2+1} \int_{th}^{u} e^{-r/(s-th)^2} ds = 0 \\ Q'(u) &= u - \frac{8b}{b^2+1} e^{-r/(u-th)^2} = 0, \end{aligned} \tag{3.8}$$

where $u > th$, $r > 0$ and $b > 0$. This system consists of two equations and three unknowns $u, r, b > 0$. To begin, we obtain $ue^{r/(u-th)^2} = \frac{8b}{b^2+1}$, directly from the second equation, and rewrite the first equation of (3.8) as

$$\frac{u^2}{2} - ue^{r/(u-th)^2} \int_{th}^{u} e^{-r/(s-th)^2} ds = 0. \tag{3.9}$$

To show that (3.8) has a solution we define and analyze the function

$$\tilde{Q}(u, r) = \frac{u^2}{2} - ue^{r/(u-th)^2} \int_{th}^{u} e^{-r/(s-th)^2} ds \tag{3.10}$$

for $u > th$. The substitution $r^{1/2}t = s - th$ transforms (3.10) into the more convenient form

$$\tilde{Q}(u,r) = \frac{u^2}{2} - ue^{r/(u-th)^2} r^{1/2} \int_0^{r^{-1/2}(u-th)} e^{-1/t^2} dt. \tag{3.11}$$

In the next lemma we determine the limiting behavior of $\tilde{Q}$ as u tends to infinity.

Lemma 3.2. *Suppose that $0 < th < 2$ and $r > 0$ are fixed. Then*

$$\lim_{u\to\infty} \tilde{Q}(u,r) = -\infty. \tag{3.12}$$

Proof. From (3.11) a calculation gives

$$\frac{\partial \tilde{Q}(u,r)}{\partial u} = r^{1/2} e^{r/(u-th)^2} \big(\frac{2ru}{(u-th)^3} - 1\big) \int_0^{r^{-1/2}(u-th)} e^{-1/t^2} dt. \tag{3.13}$$

An immediate consequence of (3.13) is that

$$\frac{\partial \tilde{Q}(u,r)}{\partial u} \to -\infty \quad \text{as } u \to \infty. \tag{3.14}$$

This proves (3.12) and concludes the proof of the lemma. □

Since $\tilde{Q}(th^+, r) = th^2/2 > 0$, (where th^+ denotes $u \to th^+$), then continuity of $\tilde{Q}$ in u and Lemma 3.2 ensure that there exists a finite $u_s(r) > th$ such that $\tilde{Q}(u_s(r), r) = 0$. Specifically, we define

$$u_s(r) = \sup\{\hat{u} > th : \tilde{Q}(u,r) > 0 \text{ for } u \in (th, \hat{u})\}. \tag{3.15}$$

Our goal is to show that $u_s(r)$ satisfies both equations in (3.8) for sufficiently small $r > 0$. For this we will need precise estimates on the location of $u_s(r)$. The first estimate is the lower bound

$$u_s(r) > 2th. \tag{3.16}$$

This bound follows immediately from the next lemma and (3.15).

Lemma 3.3. *Suppose that $0 < th < 2$ is fixed. If $th < u \le 2th$, then*

$$\tilde{Q}(u,r) > 0 \quad \text{for all } r > 0.$$

Proof. This result follows immediately upon an application of the estimate

$$\int_0^{r^{-1/2}(u-th)} e^{-1/t^2} dt < e^{-r/(u-th)^2} (r^{-1/2}(u-th)).$$

□

Next, we determine the limiting behavior of $u_s(r)$ as $r \to 0^+$.

Lemma 3.4. *Suppose that $0 < th < 2$ is fixed. Then*

$$u_s(r) \to 2th^+ \quad \text{as } r \to 0^+. \tag{3.17}$$

Proof. By (3.16) it is sufficient to show that for each $\epsilon > 0$ there exists $r_\epsilon > 0$ such that $u_s(r) < 2th + \epsilon$ for $0 < r < r_\epsilon$. This is accomplished once we prove that $\tilde{Q}(2th + \epsilon, r) < 0$ for $0 < r < r_\epsilon$. Then, since $\tilde{Q}(th^+, r) = th^2/2 > 0$, continuity of $\tilde{Q}$ in u and definition (3.15) ensure that $u_s(r) < 2th + \epsilon$ as desired.

An application of L'Hospitals reveals that

$$\lim_{r\to 0^+} \frac{\int_0^{r^{-1/2}(u-th)} e^{-1/t^2}dt}{r^{-1/2}} = \lim_{r\to 0^+} \frac{(r^{-1/2})'(u-th)e^{-r/(u-th)^2}}{(r^{-1/2})'} = u-th.$$

Hence, for fixed $u > th$ it follows that

$$\lim_{r\to 0^+} \tilde{Q}(u,r) = \frac{u^2}{2} - u(u-th) = -\frac{u}{2}(u-2th).$$

In particular, for $u = 2th+\epsilon$ we have

$$\lim_{r\to 0^+} \tilde{Q}(2th+\epsilon, r) = -\frac{2th+\epsilon}{2}\epsilon < 0.$$

This means that there exists a value $r_\epsilon > 0$ such that $\tilde{Q}(2th+\epsilon, r) < 0$ for $0 < r < r_\epsilon$. Hence, $u_s(r) < 2th+\epsilon$ as desired. □

An important consequence of Lemma 3.4 is that

$$u_s(r)e^{r/(u_s(r)-th)^2} \to 2th \quad \text{as } r \to 0^+. \tag{3.18}$$

Thus, provided that $0 < th < 2$, there exists $R > 0$ such that

$$u_s(r)e^{r/(u_s(r)-th)^2} < 4 \quad \text{for } 0 < r < R. \tag{3.19}$$

Furthermore, the function $T(b) = \frac{8b}{b^2+1}$ is strictly increasing on $(0,1)$ and strictly decreasing on $(1,\infty)$ with $T(1) = 4$ and $T(0) = T(\infty) = 0$. This and (3.19) imply that there exists a unique value $b_{r_-} \in (0,1)$ and a unique $b_{r_+} \in (1,\infty)$ such that

$$\frac{8b_{r_\pm}}{b_{r_\pm}^2+1} = u_s(r)e^{r/(u_s(r)-th)^2} \quad \text{for } 0 < r < R \tag{3.20}$$

where R is defined in (3.19).

Now, note that (3.15) and (3.20) imply

$$0 = \tilde{Q}(u_s(r), r) = \frac{u_s(r)^2}{2} - \frac{8b_{r_\pm}}{b_{r_\pm}^2+1}\int_{th}^{u_s(r)} e^{-r/(s-th)^2}ds.$$

This fact together with (3.20) shows that $(u_s(r), r, b_r)$ solves system (3.8) for $0 < r < R$. We summarize our results in the following theorem.

Theorem 3.5. *Suppose that $0 < th < 2$ is fixed. Then, $(u_s(r), r, b_{r_\pm})$ is a solution of system (3.8) for $0 < r < R$ where R is described in (3.19), $u_s(r)$ is defined by (3.15), $0 < b_{r_-} < 1$, and $b_{r_+} > 1$ satisfies (3.20).*

In closing this subsection we note that the right panel of Figure 2 epitomizes the entire subfamily of functions Q for which $(r, b, th) \in \Lambda$. First, as illustrated in this figure, $Q(u) \geq 0$ on $(0,\infty)$ with equality at exactly one value which we denote by u_s. It is also of interest to note that Q' has exactly two positive zeros, one being u_s, and the other within the interval $(th, 2th)$.

4. Initial Conditions

In this section we define an initial value problem that gives rise to even solutions of (2.4). Thus, the periodic solutions that we construct will have the property that $u(x) = u(-x)$. This will reduce our analysis to the study of solutions on $[0,\infty)$. Furthermore, we will derive a set of initial data that continuously depend on one parameter. This will simplify our shooting method in later sections.

Even Solutions of the Associated ODE. The aim of this subsection is to provide initial conditions that give rise to symmetric solutions of Equation (2.4). Later we will confine our search for periodic solutions to a subclass of symmetric solutions.

Consider the initial-value problem (IVP):

$$\begin{gathered} u'''' - 2(b^2-1)u'' + (b^2+1)^2 u = 4b(b^2+1)f(u), \\ u(\zeta) = \alpha, \quad u'(\zeta) = 0, \quad u''(\zeta) = \beta, \quad u'''(\zeta) = 0. \end{gathered} \tag{4.1}$$

Lemma 4.1. *The solution u of (4.1) satisfies $u(\zeta - x) = u(\zeta + x)$ for all x in the domain of existence.*

Proof. Define $v_1(x) = u(\zeta+x)$ and $v_2(x) = u(\zeta-x)$. Observe that $v_2''(x) = u''(\zeta-x)$ and $v_2''''(x) = u''''(\zeta - x)$, and therefore v_1 and v_2 are solutions of (4.1) with $\zeta = 0$. Hence, $v_1 \equiv v_2$ follows by uniqueness of solutions. $\square$

Reduction to One Free Parameter. According to a standard result in ODE theory the values α, β seen in (4.1) uniquely determine the solution. We now establish a continuous relationship between α and β to show that the solution is uniquely determined by the value α.

Substituting $x = 0$ in (3.4), we solve for β to obtain $\beta = \pm(b^2+1)\sqrt{2Q(\alpha)}$. Throughout this survey, we will restrict our focus to $\alpha < 0$ and $\beta > 0$. Note that $\alpha < 0$ implies that $Q(\alpha) = \alpha^2/2$. Hence, unless stated otherwise, we will assume that u is the solution of

$$\begin{gathered} u'''' - 2(b^2-1)u'' + (b^2+1)^2 u = 4b(b^2+1)f(u), \\ u(0) = \alpha, \quad u'(0) = 0, \quad u''(0) = \beta, \quad u'''(0) = 0, \end{gathered} \tag{4.2}$$

where $\beta = -(b^2+1)\alpha$ and $\alpha < 0$.

The advantage of choosing $\alpha < 0$ is that $u(x) \le th$ on some interval $[0, M]$. The definition of our firing rate function, (2.3), yields that (4.2) has the simple solution

$$u(x) = \alpha(\cosh(bx)\cos(x) - b\sinh(bx)\sin(x))$$

so long as $u(x) \le th$.

Our intentions can now be more clearly stated. First, note that Lemma 4.1 implies that all solutions of (4.2) are even. The primary strategy is to show that there exists $\bar{x} > 0$ such that $u'(\bar{x}) = u'''(\bar{x}) = 0$. Once again, we use Lemma 4.1 to show that the solution u is symmetric about the line $x = \bar{x}$. This is the desired periodic solution.

The fact that β continuously depends on α means that solutions of (4.2) are uniquely determined by the value of α. For this reason we will denote solutions of (4.2) by $u(\cdot, \alpha)$ whenever its necessary to emphasize the initial value. Otherwise, we will simply use u to denote solutions.

Finally, we note that $\beta = -(b^2+1)\alpha$ implies that $E = 0$ in (3.3). That is, if u is a solution of (4.2), then u satisfies (3.4).

5. Critical Points

In this section we prove the existence of infinitely many critical points. We will show that u' changes sign infinitely many times regardless whether or not the maximal interval of existence is finite or infinite. We proceed by showing that the first of these critical points is continuous with respect to $u(0) = \alpha$.

Oscillatory Behavior of Solutions. Our construction of periodic solutions will begin following an analysis of the oscillatory behavior of solutions of (4.2). Lemma 4.1 ensures that u satisfies $u(x) = u(-x)$ for all $x \in [0, \omega)$ where $\omega = \omega(\alpha)$ is defined by

$$\omega(\alpha) = \sup\{\hat{x} > 0 : u(x, \alpha) \text{ exists on } [0, \hat{x})\}. \tag{5.1}$$

Our goal in this subsection is to prove the following theorem.

Theorem 5.1. *Suppose that* $(r, b, th) \in \Lambda$ *with* $r \leq \frac{th^4}{16}$*. Also, let* u *be a nontrivial solution of* (4.2) *with interval of existence* $[0, \omega)$*. Then* u' *changes sign on* (X, ω)*, for any* $X \in (0, \omega)$*.*

The condition $r \leq \frac{th^4}{16}$ is only necessary in the special case when $\omega = \infty$. Otherwise, it is not necessary to impose any restriction on the variable r.

We will prove this theorem by considering two separate cases. First, we will assume that $\omega = \infty$.

Infinite Intervals of Existence.

Theorem 5.2. *Suppose that* $(r, b, th) \in \Lambda$ *with* $r \leq \frac{th^4}{16}$*. Let* u *be a nonconstant solution of* (4.2) *which exists on an interval* $[0, \infty)$*. Then for any* $X > 0$*,* u' *changes sign on the interval* (X, ∞)*.*

The proof of Theorem 5.2 will follow several necessary lemmas. The first lemma reveals the behavior of homoclinic orbit solutions as $u \to 0$.

Lemma 5.3. *Suppose that* u *is a nontrivial solution of* (4.2) *which exists on an interval* $[0, \infty)$ *and that* $u \to 0$ *as* $x \to \infty$*. Then* u *changes sign on* (X, ∞) *for any* $X > 0$*.*

Proof. To start, suppose that $u < th$ on the interval (X_1, ∞). Hence, the equation in (4.2) is linear and homogenous, and the closed form solution is given by

$$u(x) = k_1 e^{bx}\cos(x) + k_2 e^{bx}\sin(x) + k_3 e^{-bx}\cos(x) + k_4 e^{-bx}\sin(x) \tag{5.2}$$

for some constants $k_1 - k_4$. The assumption that $u \to 0$ as $x \to \infty$ implies that $k_1 = k_2 = 0$. Thus, we rewrite (5.2) as

$$u(x) = e^{-bx}(k_3\cos(x) + k_4\sin(x)). \tag{5.3}$$

Since u is a nontrivial solution, it follows that at least one of k_3 or k_4 is nonzero. Hence, it can be seen that $k_3\cos(x) + k_4\sin(x)$ changes sign by considering sequences such as $x_n = n\pi$ and $x_n = \frac{2n+1}{2}\pi$ for sufficiently large $n \in \mathbb{Z}$. This completes the proof. $\square$

Lemma 5.4. *Assume that* u *is a monotonic solution of* (4.2) *on some interval* (X, ∞)*, and that there is a real number* s *such that* $u \to s$ *as* $x \to \infty$*. Then* $(u', u'', u''', u'''') \to (0, 0, 0, 0)$ *as* $x \to \infty$*.*

Proof. We begin by showing that $Q'(u(x)) \neq 0$ on some interval of the form $(\bar{X}, \infty)$. Since the equation in (4.2) is autonomous, then $u \to s$ as $x \to \infty$ implies that $u \equiv s$ is a constant solution. That is, $4b(b^2+1)f(s) = (b^2+1)^2 s$, or equivalently $Q'(s) = 0$. Since Q' has 3 roots, then monotonicity of u on (X, ∞) ensures that $Q'(u) \neq 0$ on $(\bar{X}, \infty)$ for some value $\bar{X} \geq X$. Therefore, we infer from $u'''' - 2(b^2 - 1)u'' = -(b^2+1)^2 Q'(u)$ that $u''' - 2(b^2-1)u'$ is monotonic on $(\bar{X}, \infty)$, and hence

$u''' - 2(b^2-1)u' \to L$ as $x \to \infty$ where L is either real or infinite. We assert that $L = 0$.

If $L > 0$, (or if $L = \infty$), then $u'' - 2(b^2-1)u \to \infty$ as $x \to \infty$. But this leads to $u \to \infty$ as $x \to \infty$ which contradicts our assumption that $u \to s$ as $x \to \infty$ where $s \in \mathbb{R}$. A similar argument can be used to show that $L < 0$ (and $L = -\infty$) is impossible. Hence, we have proved

$$u''' - 2(b^2-1)u' \to 0 \quad \text{as } x \to \infty. \tag{5.4}$$

Since $u''' - 2(b^2-1)u'$ is monotonic on $(\bar{X}, \infty)$, then (5.4) implies that $u'' - 2(b^2-1)u$ is monotonic on $(\bar{X}, \infty)$. Thus, $u'' - 2(b^2-1)u$ converges as $x \to \infty$. This fact together with our assumption that $u \to s$, where $s \in \mathbb{R}$ implies that $u'' \to 0$ as $x \to \infty$.

Since u and $u''' - 2(b^2-1)u'$ are monotonic on $(\bar{X}, \infty)$, then there exists a value $X_2 \geq \bar{X}$ such that $u(u''' - 2(b^2-1)u') \neq 0$ on the interval (X_2, ∞). But

$$u(u''' - 2(b^2-1)u') = (u''u)' - \frac{((u')^2)'}{2} - (b^2-1)(u^2)',$$

and hence $u''u - \frac{(u')^2}{2} - (b^2-1)u^2$ is monotonic on (X_2, ∞). Thus, there is an L_3 (possibly $L_3 = \pm\infty$) such that

$$u''u - \frac{(u')^2}{2} - (b^2-1)u^2 \to L_3 \quad \text{as } x \to \infty. \tag{5.5}$$

Since $u'' \to 0$ and $u \to s$ as $x \to \infty$, then $(u')^2 \to -2(L_3 + (b^2-1)s^2)$ as $x \to \infty$. This shows that u' converges as $x \to \infty$, and therefore, since s is finite, $u' \to 0$ as $x \to \infty$. From this and (5.4) it follows that $u''' \to 0$ as $x \to \infty$. Also, $u'' \to 0$ as $x \to \infty$, and $Q'(s) = 0$ implies that $u'''' = 2(b^2-1)u'' - (b^2+1)^2 Q'(u) \to 0$ as $x \to \infty$. This completes the proof. $\square$

The following lemma will be used to prove Theorem 5.2.

Lemma 5.5. *Suppose that* $(r, b, th) \in \Lambda$ *with* $r \leq th^4/16$. *Then*

$$\frac{2u_s r(b^2+1)^2}{(u_s - th)^3} - 4b^2 < 0$$

where (u_s, r, b) *is the solution to system* (3.8).

Proof. First recall the estimate (3.16), that is $u_s > 2th$. This together with our premise implies that

$$r \leq \frac{th^4}{16} < \frac{u_s(u_s - th)^3}{32} < \frac{u_s(u_s - th)^3 e^{2r/(u_s - th)^2}}{32}. \tag{5.6}$$

By (3.20) we obtain

$$\frac{b^2}{(b^2+1)^2} = \frac{u_s^2 e^{2r/(u_s-th)^2}}{64}.$$

Combining this result with (5.6) leads to

$$\frac{u_s r}{(u_s - th)^3} < \frac{u_s^2 e^{2r/(u_s-th)^2}}{32} = \frac{2b^2}{(b^2+1)^2}.$$

The desired result now follows. This concludes the proof. $\square$

Proof of Theorem 5.2. We proceed by contradiction and assume that $u' \geq 0$ on the entire interval (X, ∞) for some $X > 0$. Hence, $u \to s$ as $x \to \infty$. This yields two separate cases.

Case 1: s is finite. Because of Lemma 5.4, the first integral equation (3.4) at $x = \infty$ reduces to $(b^2+1)^2 Q(s) = 0$. Note that $(r, b, th) \in \Lambda$ implies that $Q(s) = Q'(s) = 0$. Furthermore, Lemma 5.3 guarantees that $s \neq 0$. The only other possibility is that $s > 0$.

We begin by defining $\rho = \frac{u'}{u-s}$ on (X, ∞). Then, from (3.4) we derive the equation

$$\rho''\rho + 2\rho^2\rho' - \frac{1}{2}(\rho')^2 + \frac{1}{2}\rho^4 - (b^2-1)\rho^2 + \frac{(b^2+1)^2 Q(u)}{(u-s)^2} = 0. \tag{5.7}$$

To obtain a contradiction, we analyze the limiting behavior of the solution of equation (5.7) as $x \to \infty$.

Our first claim is that we can choose $X^* \geq X$ sufficiently large so that

$$\frac{1}{2}\rho^4 - (b^2-1)\rho^2 + (b^2+1)^2 \frac{Q(u)}{(u-s)^2} > 0 \quad \text{on } (X^*, \infty). \tag{5.8}$$

To prove this note that $Q'(s) = 0$ is equivalent to $(b^2+1)s = 4bf(s)$, thus

$$f'(u) = \frac{2r}{(u-th)^3} f(u) \quad \text{leads to} \quad f'(s) = \frac{sr(b^2+1)}{2b(s-th)^3}.$$

Now, this identity together with two applications of L'Hospitals rule yields

$$\lim_{u\to s} \frac{Q(u)}{(u-s)^2} = \frac{1}{2} - \frac{2b}{b^2+1} f'(s) = \frac{1}{2} - \frac{sr}{(s-th)^3}.$$

Because $(r, b, th) \in \Lambda$ and $r \leq \frac{th^4}{16}$, Lemma 5.5 implies that

$$-4b^2 + 2(b^2+1)^2 \frac{sr}{(s-th)^3} < 0. \tag{5.9}$$

Hence,

$$\frac{1}{2}\rho^4 - (b^2-1)\rho^2 + (b^2+1)^2 \left(\frac{1}{2} - \frac{sr}{(s-th)^3}\right) > 0, \tag{5.10}$$

can be seen by noting that the left-hand side of (5.10) is quadratic in ρ^2 and the associated discriminate is the left side of (5.9). Therefore, (5.8) holds for some $X^* > 0$.

We now show that ρ is bounded on (X, ∞). Since $Q(s) = Q'(s) = 0$ for some $s > 0$, then the right panel of Figure 2 reveals that $Q'(u) < 0$ on a left neighborhood of $(s-\delta, s)$. Thus, there exists a value $\hat{X} \geq X$ such that $Q'(u) < 0$ whenever $x > \hat{X}$. But since $u'''' - 2(b^2-1)u'' = -(b^2+1)^2 Q'(u)$, then $u''' - 2(b^2-1)u'$ is increasing on the interval $(\hat{X}, \infty)$. This fact combined with Lemma 5.4 implies that $u''' - 2(b^2-1)u' < 0$ on $(\hat{X}, \infty)$. Hence, $u'' - 2(b^2-1)u \to -2(b^2-1)s^+$ as $x \to \infty$, and consequently it follows that $u'' - 2(b^2-1)(u-s) \geq 0$ on $(\hat{X}, \infty)$. From this we obtain

$$u'(u'' - 2(b^2-1)(u-s)) = \left(\frac{(u')^2}{2}\right)' - (b^2-1)\left((u-s)^2\right)' \geq 0.$$

provided that $x > \hat{X}$, and therefore

$$\frac{(u')^2}{2} - (b^2-1)(u-s)^2 \to 0^- \quad \text{as } x \to \infty. \tag{5.11}$$

But (5.11) yields

$$\frac{(u')^2}{2} - (b^2-1)(u-s)^2 < 0 \quad \text{on } \hat{X}, \infty),$$

or equivalently $\rho^2 < 2(b^2-1)$ on $(\hat{X}, \infty)$, the desired bound on ρ. Note, that $b > 1$ is an immediate consequence which we will assume that for the remainder of the proof of Case 1.

Our next assertion is that ρ' is eventually of one sign. First, recall the implication of $x > X^*$ as noted by (5.8). Now, if $\rho'(x_0) = 0$ for some $x_0 \geq X^*$, then equation (5.7) reduces to

$$\rho''\rho + \frac{1}{2}\rho^4 - (b^2-1)\rho^2 + \frac{(b^2+1)^2 Q(u)}{(u-s)^2} = 0 \quad \text{at } x = x_0. \tag{5.12}$$

Combining (5.12) with (5.8), gives $\rho''\rho < 0$ at $x = x_0$. Now the fact that $\rho \leq 0$ implies that $\rho''(x_0) > 0$ showing that if $\rho'(x_0) = 0$ for some $x_0 \geq X^*$, then $\rho' > 0$ on (x_0, ∞).

Since ρ' is of one sign on the interval (X^*, ∞), then boundedness of ρ implies that ρ converges to a finite value, which we will call s^*. This means that either

$$\lim_{x\to\infty} \rho' = 0, \quad \text{or} \quad \lim_{x\to\infty} \rho' \text{ does not exist.}$$

First suppose that $\rho' \to 0$ as $x \to \infty$. By elementary analysis we know that a sequence $\{x_n\}$ exists for which $\rho'' \to 0$ as $x_n \to \infty$. Letting $x_n \to \infty$ in equation (5.7) yields,

$$\frac{1}{2}(s^*)^4 - (b^2-1)(s^*)^2 + (b^2+1)^2\left(\frac{1}{2} - \frac{sr}{(s-th)^3}\right) = 0 \tag{5.13}$$

and this contradicts (5.10).

Now suppose that $\lim_{x\to\infty} \rho'$ does not exist. Then we can choose a sequence $\{x_n\}$ so that each x_n satisfies $\rho''(x_n) = 0$ and that $\rho'(x_n) \to 0$ as $x \to \infty$. By applying such a sequence to the left hand side of (5.7), we once again obtain (5.13) giving the desired contradiction.

Case 2: $s = \infty$**.** The proof of the case, $u \to \infty$ as $x \to \infty$, is very similar to the proof of the first case. In outline, set $\rho = \frac{u'}{u}$ and use (3.4) to obtain

$$\rho''\rho + 2\rho^2\rho' - \frac{1}{2}(\rho')^2 + \frac{1}{2}\rho^4 - (b^2-1)\rho^2 + \frac{(b^2+1)^2 Q(u)}{u^2} = 0.$$

Then show that

$$\lim_{u\to\infty} \frac{Q(u)}{u^2} = \frac{1}{2} \quad \text{and} \quad \frac{1}{2}\rho^4 - (b^2-1)\rho^2 + \frac{(b^2+1)^2}{2} > 0 \tag{5.14}$$

which yields

$$\frac{1}{2}\rho^4 - (b^2-1)\rho^2 + \frac{(b^2+1)^2 Q(u)}{u^2} > 0 \quad \text{on some interval } (X^*, \infty).$$

To prove that ρ is bounded, use the fact that $Q'(u) \to \infty$ as $x \to \infty$, to conclude that

$$u'' - 2(b^2-1)u \to -\infty \quad \text{as } x \to \infty. \tag{5.15}$$

This implies that $b > 1$. Another consequence of (5.15) is that

$$u'(u'' - 2(b^2-1)u) = \frac{((u')^2)'}{2} - (b^2-1)(u^2)' \le 0 \quad \text{for large } x, \tag{5.16}$$

and hence $\frac{(u')^2}{2} - (b^2-1)u^2 \to L^+$ for some $L < \infty$. Now to conclude that ρ is bounded, show that $L < 0$ with the possibility that $L = -\infty$ so that $\frac{(u')^2}{2} - (b^2-1)u^2 < 0$ on some interval $(\hat{X}, \infty)$. Note that $L \ge 0$ implies that $u' \to \infty$ as $x \to \infty$. This and (5.15) and (5.16) imply that $\frac{((u')^2)'}{2} - (b^2-1)(u^2)' \to -\infty$ as $x \to \infty$. Thus, we have shown that $\rho^2 \le 2(b^2-1)$ on some interval $(\hat{X}, \infty)$.

Proving that ρ' is eventually of one sign is practically identical to showing this property in the first case. Therefore, $\rho \to s^*$ for some $s^* > 0$. Now define sequences similar to the ones defined in the first case to arrive at limiting equations that contradict (5.14).

A similar argument can be applied to obtain a contradiction of $u' \le 0$ on (X, ∞). We now turn to initial values that lead to finite intervals of existence. That is, $\omega < \infty$ where $\omega = \omega(\alpha)$ is defined by (5.1). $\square$

Finite Intervals of Existence. In the next four technical results we show that if a solution u of (4.2) ceases to exist at $\omega < \infty$, then it cannot do so monotonically. That is, u' changes sign infinitely many times on $[0, \omega)$.

Lemma 5.6. *Suppose h is differentiable function defined on an interval (X_1, X_2) with $-\infty < X_1 < X_2 < \infty$. If $h > 0$ and $\frac{h'}{h}$ is bounded on (X_1, X_2), then h is bounded on (X_1, X_2).*

Proof. Since $(\ln(h))' = \frac{h'}{h}$, then our assumption implies $|(\ln(h))'| \le M$ for some $M > 0$. From this it can be shown that $|h| \le K$ where $K = e^{M(X_2 - X_1)}h(X_1)$. This concludes the proof of the lemma. $\square$

In the next lemma, we show that if $u \to \infty$, then it cannot do so monotonically.

Lemma 5.7. *Suppose u is a solution of (4.2) that exists on an interval $[0, \omega)$ where $0 < \omega < \infty$. Also, assume that $u' \ge 0$ on $(\omega - \delta, \omega)$ for some small $\delta > 0$. Then $u \to L < \infty$ as $x \to \omega^-$.*

Proof. Suppose for a contradiction that $u \to \infty$ as $x \to \omega^-$. We assume that $u > 1$ and that $u' \ge 0$ on $(\omega - \delta, \omega)$ by redefining δ if necessary. Hence, by defining $\rho = \frac{u'}{u}$ on $(\omega - \delta, \omega)$ we are sure that $\rho \ge 0$ and that ρ is well-defined. We will show that $\frac{\rho'}{\rho}$ is bounded on the interval $(\omega - \delta, \omega)$. Then a repeated application of Lemma 5.6 will show that u is bounded on $(\omega - \delta, \omega)$. From (3.4) we obtain the equation

$$\rho''\rho + 2\rho^2\rho' - \frac{1}{2}(\rho')^2 + \frac{1}{2}\rho^4 - (b^2-1)\rho^2 + \frac{(b^2+1)^2 Q(u)}{u^2} = 0.$$

As in the proof of Theorem 5.2, it can be shown that

$$\frac{1}{2}\rho^4 - (b^2-1)\rho^2 + \frac{(b^2+1)^2 Q(u)}{u^2} > 0$$

for $u > 0$ sufficiently large. Therefore, on the interval $(\omega - \delta, \omega)$ we have

$$\rho''\rho - (\rho')^2 + \frac{1}{2}(\rho')^2 + 2\rho^2\rho' = \rho''\rho + 2\rho^2\rho' - \frac{1}{2}(\rho')^2 < 0. \tag{5.17}$$

Dividing by ρ^2 yields

$$(\frac{\rho'}{\rho})' + \frac{1}{2}(\frac{\rho'}{\rho})^2 < -2\rho'. \tag{5.18}$$

The fact that $\rho \geq 0$ on $(\omega-\delta,\omega)$ together with (5.17) implies that ρ' is of one sign on an interval of the form $(\omega-\epsilon,\omega)$ for some $\epsilon \leq \delta$. If $\rho' \leq 0$ on $(\omega-\epsilon,\omega)$, then $\rho \geq 0$ implies that ρ is bounded on $(\omega-\epsilon,\omega)$. Then u is bounded follows from Lemma 5.6. If $\rho' > 0$ on $(\omega-\epsilon,\omega)$, then it follows from (5.18) that $h' < -\frac{1}{2}h^2 < 0$ on $(\omega-\epsilon,\omega)$ where $h = \frac{\rho'}{\rho}$. Now $h > 0$ and $h' < 0$ on $(\omega-\epsilon,\omega)$ means that h is bounded. Invoking Lemma 5.6 shows that ρ is bounded, and hence u is bounded. This completes the proof. □

Lemma 5.8. *Let u be a solution of* (4.2) *on an interval* $[0,\omega)$. *Suppose that $u' \leq 0$ on $(\omega-\delta,\omega)$ for some small $\delta > 0$. Then $u \to L > -\infty$ as $x \to \omega^-$.*

Remark: An argument similar to the one given in Lemma 5.7 can be applied to obtain this result. A simpler approach is to note $u \leq th$ results in the a linear, homogeneous equation with constant coefficients. The corresponding closed form solution is given by

$$u = k_1e^{bx}\sin(x) + k_2e^{bx}\cos(x) + k_3e^{-bx}\sin(x) + k_4e^{-bx}\cos(x)$$

for some constants $k_1 - k_4$. The result now follows very easily.

Lemma 5.9. *Let $u = u(\cdot,\alpha)$ be a solution of* (4.2) *on an interval* $[0,\omega)$. *If $u \to L \neq \pm\infty$ as $x \to \omega^-$, then $\lim_{x\to\omega^-} u^{(i)}(x)$ exists and is finite, for $i = 1,2,3$.*

Remark: The consequence of this lemma is that if $\lim_{x\to\omega^-} u$ exists and is finite, then the solution can be continued at $x = \omega$. But this contradicts our definition of ω, see (5.1), that $[0,\omega)$ is the maximal positive interval of existence of the solution u. Hence, Lemmas 5.7 and 5.8, imply that the sign of u' must change on any interval of the form $(\omega-\delta,\omega)$.

Proof of Lemma 5.9. The fact that $u \to L \neq \pm\infty$ implies that u is bounded on $[0,\omega)$. We will make repeated use of the fact that

$$\lim_{x\to\omega^-} g^{(i)}(x) = \int_0^\omega g^{(i+1)}(s)ds + g^{(i)}(0) \tag{5.19}$$

where $g \in C^{i+1}([0,\omega))$ has the property that $\lim_{x\to\omega^-} g^{(i+1)}(x)$ exists and is finite. Thus, it will be sufficient to prove that $\lim_{x\to\omega^-} u^{(iv)}(x)$ exists and is finite. It follows from (4.2) that

$$\lim_{x\to\omega^-}(u''''(x) - 2(b^2-1)u''(x)) = 4b(b^2+1)f(L) - (b^2+1)^2L. \tag{5.20}$$

Two applications of (5.19) reveals that $\lim_{x\to\omega^-}(u''(x) - 2(b^2-1)u(x)) = \hat{L}$ for some $\hat{L} \in \mathbb{R}$. Therefore, $\lim_{x\to\omega^-} u''(x) = \hat{L} + 2(b^2-1)L \in \mathbb{R}$ follows directly from our assumption that $u \to L$ as $x \to \omega^-$. This fact together with (5.20) results in $\lim_{x\to\omega^-} u^{(iv)}(x)$ exists and is finite. This concludes the proof of the lemma. □

Theorem 5.1 now follows from Lemmas 5.7-5.9 and Theorem 5.2.

Continuity of Critical Values. In this subsection we will lay the foundation of the shooting method that we use to prove the existence of periodic orbits. To accomplish this we must first assume that the conditions of Theorem 5.1 hold. Hence, solutions of (4.2) have infinitely many critical points. Furthermore, $\alpha < 0$ and $\beta > 0$ implies that the first critical point of $u(\cdot, \alpha)$ is a local maximum. We formally denote the first critical value of $u(\cdot, \alpha)$ by

$$\xi(\alpha) = \sup\{x > 0 : u'(\cdot, \alpha) > 0 \text{ on } (0, x)\}. \tag{5.21}$$

The primary goal of this subsection is to prove that ξ continuously depends on α. The following general lemma will assist us in accomplishing this task.

Lemma 5.10. *Suppose that $u(x, \alpha_*)$ is a nonconstant solution of (4.2) such that $u'(x_*, \alpha_*) = u''(x_*, \alpha_*) = 0 \neq u'''(x_*, \alpha_*)$ for some $x_* > 0$ and some $\alpha_* \in \mathbb{R}$. Then for any $\epsilon > 0$ such that*

$$u'''(x, \alpha_*) \neq 0 \quad \text{on } [x_* - \epsilon, x_* + \epsilon] \tag{5.22}$$

it follows that

(i) *$u''(x_* - x, \alpha_*)u''(x_* + x, \alpha_*) < 0$ on $[-\epsilon, \epsilon]$.*

In addition, assume that $\{\alpha_n\}$ is a sequence such that

$$\alpha_n \to \alpha_* \quad \text{as } n \to \infty, \tag{5.23}$$

and that $u(x, \alpha_n)$ is a nonconstant solution of (4.2) for each $n \geq 1$. Then there exists $N > 0$ such that

(ii) *$u'''(x, \alpha_n)u'''(x, \alpha_*) > 0$ on $[x_* - \epsilon, x_* + \epsilon]$,*
(iii) *$u''(x_* - \epsilon, \alpha_n)u''(x_* + \epsilon, \alpha_n) < 0$, and*
(iv) *there exists a unique $\tau_n \in (x_* - \epsilon, x_* + \epsilon)$ such that $u''(\tau_n, \alpha_n) = 0$*

for all $n \geq N$. Furthermore, it also follows that

(v) *$\tau_n \to x_*$ as $n \to \infty$.*

Proof. (i) Note that (5.22) implies that $u''(x, \alpha_*)$ is monotonic on $[x_* - \epsilon, x_* + \epsilon]$. Hence, (i) follows from the premise $u''(x_*, \alpha_*) = 0$.

(ii) By (5.22), (5.23), and the fact that solutions are continuous with respect to the initial data over compact sets, we can choose $N > 0$ sufficiently large so that

$$u'''(x, \alpha_n)u'''(x, \alpha_*) > 0 \quad \text{on } [x_* - \epsilon, x_* + \epsilon] \tag{5.24}$$

whenever $n \geq N$. This concludes (ii).

(iii) From part (i) it follows that $u''(x_* \pm \epsilon, \alpha_*) \neq 0$. Because $[0, x_* + \epsilon]$ is compact, it immediately follows from (5.23), and continuity of solutions with respect to initial conditions, that $N > 0$ can be chosen to satisfy

$$u''(x_* - \epsilon, \alpha_n)u''(x_* - \epsilon, \alpha_*) > 0 \quad \text{and} \quad u''(x_* + \epsilon, \alpha_n)u''(x_* + \epsilon, \alpha_*) > 0$$

for $n \geq N$. Thus, as a consequence of part (i) we have that

$$u''(x_* - \epsilon, \alpha_n)u''(x_* + \epsilon, \alpha_n) < 0 \tag{5.25}$$

for all $n \geq N$ as desired.

(iv) Choose $N > 0$ so that (5.24) and (5.25) hold. Because of (5.25) there exists an intermediate value $x_* - \epsilon < \tau_n < x_* + \epsilon$ such that $u''(\tau_n, \alpha_n) = 0$ for all $n \geq N$. Because of (5.24), each $u''(x, \alpha_n)$ is strictly monotonic on the interval $[x_* - \epsilon, x_* + \epsilon]$. Hence, τ_n is the unique zero of $u''(x, \alpha_n)$ on $(x_* - \epsilon, x_* + \epsilon)$. This concludes part (iv).

(v) This follows from (iv) and the fact that ϵ is any arbitrary small positive number that satisfies (5.22). $\square$

In the next theorem we show that ξ is a continuous function of α. Since our construction of periodic solutions are based on $\alpha < 0$ and $\beta > 0$ we will prove continuity of ξ for $\alpha < 0$.

Theorem 5.11. *The function ξ as defined in* (5.21) *is a continuous function of $\alpha < 0$.*

Remark: We will assume that u is a nonconstant solution of (4.2) to ensure the existence of ξ.

Proof of Theorem 5.11. First, note that if $u''(\xi(\alpha_*), \alpha_*) \neq 0$, then continuity of ξ at $\alpha = \alpha_*$ follows directly from the Implicit Function Theorem. Thus, we will assume that $u''(\xi(\alpha_*), \alpha_*) = 0$ throughout the remainder of the proof.

Suppose that $\alpha_* < 0$, and that $\beta_* = -(b^2 + 1)\alpha_*$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence such that $\alpha_n \to \alpha_*$ as $n \to \infty$. Without loss of generality, assume that $\alpha_n < 0$ for all n. This ensures that $\beta_n = -(b^2 + 1)\alpha_n > 0$ and $u(x, \alpha_n)$ is a nonconstant solution of (4.2).

Let $\epsilon > 0$. We begin by showing that there exists $N > 0$ such that $\xi(\alpha_n) > \xi(\alpha_*) - \epsilon$ whenever $n \geq N$. Specifically, we will show that $u'(x, \alpha_n) > 0$ on $(0, \xi(\alpha_*) - \epsilon]$ whenever $n \geq N$. Following this, we will show that $\xi(\alpha_n) < \xi(\alpha_*) + \epsilon$ for all $n \geq N$.

To begin, note that $u''(0, \alpha_*) = \beta_* > 0$, so that we can choose $\delta > 0$ sufficiently small to guarantee that $u''(x, \alpha_*) > 0$ on the interval $[0, \delta]$. For technical purposes let $\epsilon > 0$ be small enough to guarantee that $\xi(\alpha_*) - \epsilon > \delta$. We now define $I_1 = [\delta, \xi(\alpha_*) - \epsilon]$, $I_2 = [0, \delta]$, and $m_j = \min_{x \in I_j} u^{(j)}(x, \alpha_*)$ for $j = 1, 2$. The fact that $\xi(\alpha_*)$ is the first positive zero of $u'(x, \alpha_*)$, ensures that $m_1 > 0$. Because of continuity of solutions with respect to the initial conditions we can choose $N > 0$ so that

$$|u^{(j)}(x, \alpha_n) - u^{(j)}(x, \alpha_*)| \leq \frac{m_j}{2} \quad \text{on } I_j,$$

for all $n \geq N$. It now follows from our choice of m_j that $u^{(j)}(x, \alpha_n) \geq \frac{m_j}{2} > 0$ on I_j. Hence, $u'(x, \alpha_n) > 0$ on $(0, \xi(\alpha_*) - \epsilon]$ now follows from the fact that $u'(0, \alpha_n) = 0$ for all n. This proves that $\xi(\alpha_n) > \xi(\alpha_*) - \epsilon$ whenever $n \geq N$.

We now prove that there exists $N > 0$ such that $\xi(\alpha_n) < \xi(\alpha_*) + \epsilon$ whenever $n \geq N$. For a contradiction, assume that there exists $\epsilon > 0$ and a sequence $\{\alpha_n\}_{n=1}^{\infty}$, such that $\alpha_n \to \alpha_*$ as $n \to \infty$, and $\xi(\alpha_n) \geq \xi(\alpha_*) + \epsilon$. For ease of notation we write $u_*^{(i)} = u^{(i)}(\xi(\alpha_*), \alpha_*)$ for $i = 0, 1, 2, 3$.

Our first claim is that $Q'(u_*) = 0$ and $u_*''' > 0$. Substituting $u_*'' = u_*' = 0$ into the first integral equation (3.4) reveals that $Q(u_*) = 0$. Recall that $(r, b, th) \in \Lambda$, means that $Q(u) \geq 0$ for all $u \in \mathbb{R}$, and therefore

$$Q'(u_*) = u_* - \frac{4b}{b^2 + 1} f(u_*) = 0.$$

It remains to show that $u_*''' > 0$. Since u is not a constant solution of (4.2) and $Q'(u_*) = u_*' = u_*'' = 0$, then uniqueness of solutions implies that $u_*''' \neq 0$. Note that $u_*''' < 0$, leads to $u'(x, \alpha_*) < 0$ in a left neighborhood of $\xi(\alpha_*)$. This contradicts the fact that $u'(x, \alpha_*) > 0$ on $(0, \xi(\alpha_*))$. Hence, $u_*''' > 0$ as desired.

We now have all the conditions of Lemma 5.10. By property (iv) of Lemma 5.10, there exists $N > 0$ such that for every $n \geq N$, there is a unique
$\tau_n \in (\xi(\alpha_*) - \epsilon, \xi(\alpha_*) + \epsilon)$ so that $u''(\tau_n, \alpha_n) = 0$. Once again we simplify our notation and write $u_n^{(i)} = u^{(i)}(\tau_n, \alpha_n)$.

Our next claim is that $N > 0$ can be chosen so that $u_n'(u_n''' - (b^2-1)u_n') > 0$ for each $n \geq N$. By property (v) of Lemma 5.10, we know that $\tau_n \to \xi(\alpha_*)$ as $n \to \infty$, and hence $|u^{(i)}(\tau_n, \alpha_*) - u_*^{(i)}| \to 0$ as $\tau_n \to \xi(\alpha_*)$. This combined with the fact that solutions are continuous with respect to their initial conditions implies that $|u_n^{(i)} - u_*^{(i)}| \to 0$ as $\tau_n \to \xi(\alpha_*)$. Hence, $u_n''' - (b^2-1)u_n' \to u_*''' > 0$ as $n \to \infty$. follows immediately from the fact that $u_n' \to u_*' = 0$. Therefore, if necessary, we can redefine $N > 0$ so that $u_n''' - (b^2-1)u_n' > 0$ for all $n \geq N$. Now, $u_n' > 0$ is a result of the fact that $\xi(\alpha_n) \geq \xi(\alpha_*) + \epsilon > \tau_n$. Therefore, $u_n'(u_n''' - (b^2-1)u_n') > 0$ whenever $n \geq N$ as desired.

The desired contradiction now follows immediately upon substitution of $u_n^{(i)}$ into (3.4) giving
$$u_n'(u_n''' - (b^2-1)u_n') + (b^2+1)^2 Q(u_n) = 0 \tag{5.26}$$
since $u_n'' = 0$. The fact that $Q(u_n) \geq 0$ and $u_n'(u_n''' - (b^2-1)u_n') > 0$ for $n \geq N$ makes (5.26) impossible. This concludes the proof of the theorem. □

6. Periodic Solutions

We now highlight our scheme to find periodic solutions (4.2). We begin by recalling Lemma 4.1 of Section 2. This lemma implies that the corresponding solution u satisfies
$$u(x,\alpha) = u(-x,\alpha) \quad \text{for all } x \in [0,\omega) \tag{6.1}$$
where $\omega = \omega(\alpha)$ is defined by (5.1). Our approach is to use the method of topological shooting to show that there exists a value $\alpha < 0$, such that
$$u'''(\xi(\alpha),\alpha) = u'(\xi(\alpha),\alpha) = 0$$
where $\xi(\alpha)$ is defined in (5.21). Then we invoke Lemma 4.1 once again to obtain
$$u(x - \xi(\alpha),\alpha) = u(x + \xi(\alpha),\alpha) \quad \text{for all } x \in \mathbb{R}. \tag{6.2}$$
Because of (6.1) and (6.2) we see that u is symmetric about $x = 0$ and $x = \xi(\alpha)$. The resulting solution $u(\cdot,\alpha)$ is referred to as a "1-bump" periodic solution of (4.2).

As mentioned in previous sections we will assume that α and β are related by
$$\alpha < 0, \quad \beta = (b^2+1)\sqrt{2Q(\alpha)} = -(b^2+1)\alpha > 0. \tag{6.3}$$

Theorem 6.1. *Suppose that $(r,b,th) \in \Lambda$ with $r \leq \frac{th^4}{16}$, and that α, β satisfy* (6.3). *Then, there exists $\alpha^* < \alpha_* < 0$ with $\beta_* = -\sqrt{2}(b^2+1)\alpha_*$ and $\beta^* = -\sqrt{2}(b^2+1)\alpha^*$ such that $u(\cdot,\alpha^*)$ and $u(\cdot,\alpha_*)$ are 1-bump periodic solutions of* (4.2). *Moreover, we can choose α^* and α_* so that*
$$th < \|u(\cdot,\alpha_*)\|_\infty < u_s < \|u(\cdot,\alpha^*)\|_\infty \tag{6.4}$$
where $Q(u_s) = 0$.

Remark: To prove Theorem 6.1 we will use the notation ξ as defined by (5.21). Note that the conditions of Theorem 5.1 are restated for the sake of ensuring that $\xi(\alpha)$ exists. We will also make use of Theorem 5.11 where it was shown that ξ is a continuous function of α. These important results lay the framework for the topological shooting argument that will be used to prove Theorem 6.1. First, we will obtain a precise qualitative description of the solution $u(\cdot,\alpha)$ of (4.2) for small $|\alpha|$. Afterwards, we will analyze $u(\cdot,\alpha)$ for large $|\alpha|$.

Small negative α. We begin by analyzing the behavior of $u(x,\alpha)$ for $\alpha \in [\alpha_{th}, 0)$, where

$$\alpha_{th} \equiv -th\,\mathrm{sech}(b\pi). \tag{6.5}$$

The fact that $\alpha < 0$ implies that the solution $u(x,\alpha)$ of (4.2) has the closed form

$$u(x,\alpha) = c_1 e^{-bx}\cos(x) + c_2 e^{-bx}\sin(x) + c_3 e^{bx}\cos(x) + c_4 e^{bx}\sin(x) \tag{6.6}$$

where $c_1 - c_4$ are real constants. In particular, this formula holds as long as $u < th$. With $\beta = -(b^2+1)\alpha$, we use Mathematica to determine the precise values of $c_1 - c_4$. In doing so we find that (6.6) can be written as

$$u(x,\alpha) = \alpha\cosh(bx)\cos(x) - b\alpha\sinh(bx)\sin(x). \tag{6.7}$$

Repeated differentiation of (6.7) leads to

$$u'(x,\alpha) = -(b^2+1)\alpha\cosh(bx)\sin(x), \tag{6.8}$$

$$u''(x,\alpha) = -(b^2+1)\alpha(b\sinh(bx)\sin(x) + \cosh(bx)\cos(x)), \tag{6.9}$$

$$u'''(x,\alpha) = -(b^2+1)\alpha((b^2-1)\cosh(bx)\sin(x) + 2b\sinh(bx)\cos(x)). \tag{6.10}$$

We will use equations (6.7)–(6.10) to prove the following result.

Theorem 6.2. *Suppose that $\alpha_{th} \le \alpha < 0$. Then $u(x,\alpha)$ has the following properties:*

(i) *$u(x,\alpha) < th$ and $u'(x,\alpha) > 0$ on $(0,\pi)$,*
(ii) *$\xi(\alpha) = \pi$,*
(iii) *$0 < u(\xi(\alpha),\alpha) < th$ if $\alpha_{th} < \alpha < 0$*
(iv) *$u(\xi(\alpha_{th}),\alpha_{th}) = th$,*
(v) *$u''(\xi(\alpha),\alpha) < 0$, and $u'''(\xi(\alpha),\alpha) < 0$.*

Proof. (i) To prove that $u(x,\alpha) < th$ on $(0,\pi)$ we will use (6.7). That is we will show that $\alpha(\cosh(bx)\cos(x) - b\sinh(bx)\sin(x)) < th$ on the interval $(0,\pi)$. First note that

$$\alpha(\cosh(bx)\cos(x) - b\sinh(bx)\sin(x))' = -(b^2+1)\alpha\cosh(bx)\sin(x) > 0 \tag{6.11}$$

on the interval $(0,\pi)$. Hence,

$$\alpha(\cosh(bx)\cos(x) - b\sinh(bx)\sin(x)) < -\alpha\cosh(b\pi) \le -\alpha_{th}\cosh(b\pi) \tag{6.12}$$

on the interval $(0,\pi)$. By (6.5) it follows that $-\alpha_{th}\cosh(b\pi) = th$. Hence, by (6.12) we have that

$$\alpha(\cosh(bx)\cos(x) - b\sinh(bx)\sin(x)) < th \tag{6.13}$$

on the interval $(0,\pi)$. Because of (6.11) and (6.13) we conclude from (6.7) and (6.8) that $u(x,\alpha) < th$ and $u'(x,\alpha) > 0$ on $(0,\pi)$. This completes the proof of part (i).

Note that Property (i) implies that the closed form solutions, (6.7)–(6.10), can be applied on the interval $[0,\pi)$. Properties (ii)–(v) can easily be verified by applying the closed form solutions. $\square$

Large negative α. For large negative values it is equally important that we establish properties of $u(\cdot,\alpha)$ as $\alpha\to-\infty$. We begin with the transformation
$$-\alpha U_\alpha(x)=u(x,\alpha) \tag{6.14}$$
and study U_α as $\alpha\to-\infty$. Since $u(\cdot,\alpha)$ is a solution of (4.2), and $\beta=-(b^2+1)\alpha$, then U_α satisfies
$$\begin{gathered} v''''-2(b^2-1)v''+(b^2+1)^2v=\frac{4b(b^2+1)f(-\alpha v)}{-\alpha}\\ v(0)=-1,\quad v'(0)=0,\quad v''(0)=b^2+1,\quad v'''(0)=0. \end{gathered} \tag{6.15}$$
Because f is a bounded function, (6.15) becomes
$$\begin{gathered} V''''-2(b^2-1)V''+(b^2+1)^2V=0\\ V(0)=-1,\quad V'(0)=0,\quad V''(0)=b^2+1,\quad V'''(0)=0 \end{gathered} \tag{6.16}$$
as $\alpha\to-\infty$. Note that the ODE in (6.16) is linear, homogeneous, and has constant coefficients. Also notice that the solution $\hat U$ of (6.16), is identical to the solution $u(\cdot,-1)$ of (4.2) so long as $u(x,-1)<th$. Hence, the closed form of $\hat U$ is given by (6.7)–(6.10) with -1 in place of α. For easy reference, we now state several important characteristics of $\hat U$ in the following lemma.

Lemma 6.3. *The solution $\hat U$ of (6.16) satisfies*

(i) $\hat U(x)<\hat U(\pi)=\cosh(b\pi)$ *and* $\hat U'(x)>\hat U'(\pi)=0$ *for* $x\in(0,\pi)$*,*
(ii) $\hat U''(\pi)=-(b^2+1)\cosh(b\pi)<0$ *and* $\hat U'''(\pi)=-2b(b^2+1)\sinh(b\pi)<0$*.*

Proof. This result follows immediately from the fact that $\hat U$ and its derivatives are given by the closed form formulas (6.7) – (6.10) with $\alpha=-1$. □

We now determine the limiting value of $\xi(\alpha)$ as $\alpha\to-\infty$.

Lemma 6.4. *Suppose that $(r,b,th)\in\Lambda$ and that α,β satisfy (6.3). Then*
$$\xi(\alpha)\to\pi\quad \text{as } \alpha\to-\infty, \tag{6.17}$$

Proof. Fix $\epsilon>0$. We will show that there exists a value $\tilde\alpha<0$ so that

(i) $U'_\alpha(x)>0$ on $(0,\pi-\epsilon]$, and
(ii) $U'_\alpha(x_\alpha)=0$ for some $x_\alpha\in(\pi-\epsilon,\pi+\epsilon)$ whenever $\alpha<\tilde\alpha$.

By Lemma 6.3 we see that $x=\pi$ is the first positive critical value of $\hat U$ and that $\hat U''(\pi)<0$. For technical purposes we will assume that $\epsilon>0$ is small enough to guarantee that $\hat U''(x)<0$ on $[\pi-\epsilon,\pi+\epsilon]$. Thus, if $X_\pm=\pi\pm\epsilon$, then
$$\hat U'(X_-)>0\quad\text{and}\quad \hat U'(X_+)<0. \tag{6.18}$$
Next, we observe that problem (6.15) is a regular perturbation of problem (6.16) for large negative values of α. Thus, $(U_\alpha,U'_\alpha,U''_\alpha,U'''_\alpha)\to(\hat U,\hat U',\hat U'',\hat U''')$ uniformly on compact sets as $\alpha\to-\infty$. Specifically, $U'_\alpha(X_\pm)\to\hat U'(X_\pm)$ as $\alpha\to-\infty$. This and (6.18) imply that there exists $\tilde\alpha<0$ such that
$$U'_\alpha(X_-)>0\quad\text{and}\quad U'_\alpha(X_+)<0 \tag{6.19}$$
whenever $\alpha<\tilde\alpha$. Finally, (6.19) implies that $U'_\alpha(x_\alpha)=0$ for some $x_\alpha\in(\pi-\epsilon,\pi+\epsilon)$. Thus, (6.14) implies that $u'(x_\alpha,\alpha)=0$. This proves (ii).

To show that $x_\alpha = \xi(\alpha)$, we need to prove that $U'_\alpha(x) > 0$ on $(0, \pi - \epsilon]$ whenever $\alpha < \tilde{\alpha}$. Since $U''_\alpha(0) = \hat{U}''(0) > 0$, then we can choose $0 < \delta < \pi - \epsilon$ and $\tilde{\alpha} < 0$ so that $\hat{U}''(x) > 0$ on $[0, \delta]$, and

$$|U_\alpha^{(j)}(x) - \hat{U}^{(j)}(x)| \le \min_{I_j} \frac{\hat{U}^{(j)}(x)}{2} \quad \text{on } I_j \text{ whenever } \alpha < \tilde{\alpha} \tag{6.20}$$

where $I_1 = [\delta, \pi - \epsilon]$, and $I_2 = [0, \delta]$. We note that (6.20) guarantees that $U''_\alpha(x) > 0$ on $[0, \delta]$ and $U'_\alpha(x) > 0$ on $[\delta, \pi - \epsilon]$ for all $\alpha < \tilde{\alpha}$. Hence, $U'_\alpha(x) > 0$ on $(0, \pi - \epsilon]$ whenever $\alpha < \tilde{\alpha}$ as desired. This concludes (i) as well as the proof of the lemma. □

The closed form solution $\hat{U}$ of the limiting initial value problem (6.16) provided vital information in our proof of Lemma 6.4. We will continue to use $\hat{U}$ to prove the next lemma. The objective of the following lemma is to determine the limiting values of $u(\xi(\alpha), \alpha)$, and $u'''(\xi(\alpha), \alpha)$.

Lemma 6.5. *Suppose that $u(\cdot, \alpha)$ is a solution of* (4.2) *where α, β are related by* (6.3). *Also, let $(r, b, th) \in \Lambda$. Then*

(a) *$u(\xi(\alpha), \alpha) \to \infty$ as $\alpha \to -\infty$, and*
(b) *$u'''(\xi(\alpha), \alpha) \to -\infty$ as $\alpha \to -\infty$.*

Proof. (a) It follows from classical ODE theory that $|U_\alpha(x) - \hat{U}(x)| \to 0$ uniformly as $\alpha \to -\infty$ for all $x \in [0, \pi + 1]$. Lemma 6.4 ensures that there exists a value $\hat{\alpha} < 0$ such that $\xi(\alpha) \in [0, \pi + 1]$ whenever $\alpha < \hat{\alpha}$. Thus,

$$|U_\alpha(\xi(\alpha)) - \hat{U}(\xi(\alpha))| \to 0 \quad \text{as } \alpha \to -\infty. \tag{6.21}$$

Another consequence of Lemma 6.4 is that $|\hat{U}(\xi(\alpha)) - \hat{U}(\pi)| \to 0$ as $\alpha \to -\infty$. Combining this fact with (6.21) yields

$$|U_\alpha(\xi(\alpha)) - \hat{U}(\pi)| \to 0 \quad \text{as } \alpha \to -\infty. \tag{6.22}$$

By part (i) of Lemma 6.3, we know that $\hat{U}(\pi) = \cosh(b\pi) > 0$. This fact together with (6.22) leads to $u(\xi(\alpha), \alpha) = -\alpha U_\alpha(\xi(\alpha)) \to \infty$ as $\to -\infty$ as desired. This proves (a). The proof of (b) is done in similar fashion. □

Proof of Theorem 6.1. We will show that $u'''(\xi(\alpha^*), \alpha^*) = u'''(\xi(\alpha_*), \alpha_*) = 0$, and $u(\xi(\alpha_*), \alpha_*) < u_s < u(\xi(\alpha^*), \alpha^*)$ for some $\alpha^* < \alpha_* < 0$. Throughout this proof, we will rely on the results of Theorem 6.2 which asserted that

$$u(\xi(\alpha), \alpha) \le th, \quad \text{and} \quad u'''(\xi(\alpha), \alpha) < 0 \quad \text{for all } 0 > \alpha \ge \alpha_{th}. \tag{6.23}$$

We will use the set $S = \{\alpha < 0 : u(\xi(\alpha), \alpha) = u_s\}$ to help us obtain the estimate (6.4). We will show that S is a non-empty, closed, and bounded set. To see that S is bounded we note that α_{th} is an upper-bound as a consequence of (6.23) and the fact that $th < u_s$. We deduce from part (a) of Lemma 6.5 that S is bounded below. Hence S is a bounded set.

To show that $S \neq \emptyset$ we define $\phi(\alpha) = u(\xi(\alpha), \alpha)$. By standard ODE theory u is a jointly continuous function of (x, α). Hence, the fact that ξ is a continuous function of α implies that ϕ is a continuous function of α. By (6.23), we know that $\phi(\alpha_{th}) \le th < u_s$. It follows from Lemma 6.5 that there exists $\bar{\alpha} < \alpha_{th}$ such that $\phi(\bar{\alpha}) > u_s$. Continuity of ϕ guarantees the existence of an intermediate value, $\alpha_0 \in (\bar{\alpha}, \alpha_{th})$, such that $\phi(\alpha_0) = u_s$. Thus, $S \neq \emptyset$ follows.

To see that S is a closed set we consider a sequence $\{\alpha_n\}$ where each $\alpha_n \in S$ and $\alpha_n \to \alpha'$ as $n \to \infty$. We must show that $\alpha' \in S$. Since u is jointly continuous in (x, α) and ξ is continuous in α, then $u(\xi(\alpha_n), \alpha_n) \to u(\xi(\alpha'), \alpha')$ as $\alpha_n \to \alpha'$. But $u(\xi(\alpha_n), \alpha_n) = u_s$ for each n, hence $u(\xi(\alpha'), \alpha') = u_s$. Thus, it follows that S is closed.

Now, define $\alpha_a = \sup S$. Because S is a closed set we conclude that $\alpha_a \in S$, i.e, $u(\xi(\alpha_a), \alpha_a) = u_s$. Substituting $u^{(i)}(\xi(\alpha_a), \alpha_a)$ into (3.4) we find that $u''(\xi(\alpha_a), \alpha_a) = 0$. Since $u \equiv u_s$ is a constant solution of (4.2) and $u(\xi(\alpha_a), \alpha_a) = u_s, u'(\xi(\alpha_a), \alpha_a) = 0$, it follows that $u'''(\xi(\alpha_a), \alpha_a) \neq 0$. As demonstrated in the proof of Theorem 5.11, $u'''(\xi(\alpha_a), \alpha_a) > 0$. This fact together with (6.23) implies that there exists $\alpha_* \in (\alpha_a, \alpha_{th})$ such that $u'''(\xi(\alpha_*), \alpha_*) = 0$. Since $\alpha_* > \alpha_a$, then the definition α_a implies that $u(\xi(\alpha_*), \alpha_*) < u_s$. This establishes the first half of (6.4).

In a similar fashion, we define $\alpha_b = \inf S$ and conclude that there exists $\alpha^* \in (-\infty, \alpha_b)$ such that $u(\xi(\alpha^*), \alpha^*) > u_s$ and $u'''(\xi(\alpha^*), \alpha^*) = 0$. This completes the proof of Theorem 6.1. $\square$

Theorem 6.1 can easily be applied to prove the existence of two periodic solutions with $\alpha > 0$ and $\beta < 0$. This is because solutions of (2.4) are translation invariant. For example, we can define $\bar{\alpha} = u(\xi(\alpha^*), \alpha^*)$ and $\bar{\beta} = -(b^2 + 1)\sqrt{2Q(\bar{\alpha})}$ to obtain a 1-bump periodic solution satisfying $u_s < ||u(\cdot, \bar{\alpha})||_\infty$.

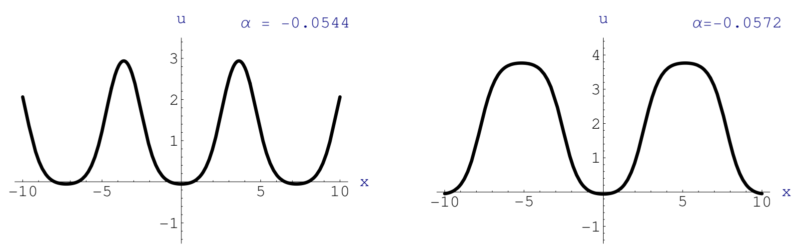

FIGURE 3. Periodic solutions of (4.2). Parameters are $r = 0.05$, $b = 1.4402$, and $th = 1.5$.

We used the software package Mathematica to obtain the two 1-bump periodic solutions seen in Figure 3. Numerical experimentation suggests that these periodic solutions are highly sensitive to the value of α, and probably do not represent *stable* stationary states of the integral equation.

Conclusion. In this paper we have analyzed a subclass of stationary solutions of (1.1). In previous studies, (see [3, 9]), the Fourier transform was applied to both sides of (1.2) to obtain a fourth order ODE. Then ODE methods were implemented to obtain a thorough numerical investigation of homoclinic orbit solutions. For technical reasons, the Fourier transform does not give rise to other types of interesting solutions such as periodic, heteroclinic, or chaotic solutions. The fundamental aim of this paper was to use the results of Krisner [8] to prove that (1.1) does have periodic solutions. In fact, under the parameter regime derived in Section 5, it was shown in Section 6 that (1.1) has two stationary 1-bump periodic solutions.

A natural extension of this result would be to find other classes of periodic solutions. As previously defined, a 1-bump periodic solution has the property that $u'(\xi(\alpha), \alpha) = u'''(\xi(\alpha), \alpha) = 0$ where $\xi(\alpha)$ is defined to be the first positive critical number of the solution u. Suppose we denote $\eta(\alpha)$ to be the second positive critical number of u. It would be interesting to see if (1.1) has a stationary "2-bump" periodic solution, i.e., a solution that satisfies $u'''(\eta(\alpha),\alpha)=0$ but $u'''(\xi(\alpha),\alpha)\neq 0$.

Lastly, inspired by the work of Amari [1], an analytical proof of the existence of N-bump homoclinic orbit solutions would be very desirable. That is, given a fixed positive threshold value, th say, there exists N disjoint intervals, $I_1 \dots I_n$, for which $u(x) > th$ if and only if $x \in I_j$.

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