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Relaxation approximations and bounded variation estimates for some partial differential equations *

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Abstract

In this paper, we introduce a new technique for studying solutions of bounded variation for some conservation laws of first order partial differential equations and for some degenerate parabolic equations in multi-dimensional space. The connection between these two types of equations is the vanishing relaxation method.

1 Introduction

We are concerned with solutions of bounded variation and the limiting behavior of relaxation approximated solutions to the Cauchy problem for the conservation system

$$G(u)_t + \sum_{j=1}^{D} F_j(u)_{x_j} = \epsilon \Delta u \tag{1.1}$$

with initial data

$$u(x,0) = u_0(x), \tag{1.2}$$

where $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_N)^T$ is a nonnegative constant vector, $x \in \mathbb{R}^D$, $u \in \mathbb{R}^N$, D denotes the space dimension, $F_j(u) = (F_j^1(u), F_j^2(u), \dots, F_j^N(u))^T$, and $G(u) = (G^1(u), G^2(u), \dots, G^N(u))^T$ are smooth nonlinear maps from $\mathbb{R}^N$ to $\mathbb{R}^N$.

For the hyperbolic case, $\epsilon = 0$, and for the scalar equation, $N = 1$, the behavior of the unique solution of (1.1)–(1.2) have been studied in many papers; see for example [13, 29] and their references. For the interesting case $N \geq 2$, a partial list of results is as follows:
1.) For $D = 1$ and the system (1.1) is strictly hyperbolic and genuinely nonlinear in the sense of Lax [15]. When the total variation of initial data (1.2) is small, the bounded variation of solutions to (1.1)–(1.2) was obtained by using the Glimm method [8]. This method was developed by Glimm and extended by Liu

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[18] to strictly hyperbolic systems whose characteristic field is either genuinely nonlinear or linearly degenerate. The Glimm method was use by many authors in the study of arbitrary data with bounded variation for some special systems (See [27]), and for general systems of Temple type in which shock waves and rarefaction waves coincide (See [31]). The uniqueness of limits of Glimm scheme solutions was proved by Bressan [3].

One of the ideas in the Glimm's method is to use the explicit structure of the nonlinear progressing wave solutions in a single space variable, i.e. the solution of the Riemann problem constructed by Lax in [15]. However for multi-dimensional case, $D \geq 2$, even for the Riemann data, the global structure of weak solutions is not clear. See [34] for the details about the Riemann solution.
2.) For the case $N = 2$ and $D = 1$, if the system (1.1) is strictly hyperbolic and genuinely nonlinear, and the approximated solutions of (1.1) are uniformly bounded, then the global existence of L^∞ solutions for (1.1)–(1.2) was proved by DiPerna [7] using the compensated compactness ideas developed by Tartar and Murat [26, 30]. The most successful application of the theory of compensated compactness is in the study of gas dynamics system which was non-strictly hyperbolic at the vacuum line. The existence of a global solution was proved for the case of a polytropic gas [4, 16, 17]. See also [12, 20, 21] for a related system.

One of the strong restrictions on the applications of the compensated compactness is that one must construct infinite pairs of entropy-entropy flux for a given system, which makes this method work well only for systems of two equations in a single variable.
3.) The local existence in time of smooth solutions of (1.1)–(1.2) in multi-dimensional space was obtained by Kato for symmetric hyperbolic systems and sufficiently smooth data [10]. The short-time existence and stability of multi-dimensional "shock front" solutions of (1.1) with discontinuous initial data were proved by Majda under some structural hypotheses [24, 25].

For the parabolic case, where ϵ is a fixed positive constant, (1.1) can be considered as typical degenerate parabolic equation. For instance, let $F(u) = 0$, $\epsilon = 1$ and $G(u) = u^{1/m}$, then (1.1) is equivalent to

$$w_t = \Delta w^m, \tag{1.3}$$

which is so-called porous medium equation. This equation models the non-stationary flow of a compressible Newtonian fluid in a porous medium under polytropic conditions. The value of $w \geq 0$ is proportional to the density of the fluid. It is the most typical case of degenerate parabolic equations. The study of its regularity has a long history. The optimal Hölder estimate in a single space variable was resolved by Aronson [1] many years ago, but in the multi-dimensional space, it is still an open problem. The regularity of solutions for the Cauchy problem (1.3) in one dimension and in multi-dimensions are quite different. We refer the readers to the paper [11] and the papers cited therein for the details. Some recent regularity results about the equation (1.3) can be found in [22, 23].

In this paper, we study the bounded variation solutions of the Cauchy problem (1.1)–(1.2) for the case of $\epsilon = 0$ or for the case of ϵ being a fixed positive constant. As the first one of a series, in this paper we restrict our attention to the most simple case $N = 1$. Our method is to select solutions of the Cauchy problem (1.1)–(1.2) as the singular perturbation limit of approximated solutions for the system

$$\begin{gathered} u_t + \sum_{j=1}^{D} F_j(u)_{x_j} + \frac{H(u) - v}{\tau} = \nu \Delta u \\ v_t + \frac{v - H(u)}{\tau} = \mu \Delta v, \end{gathered} \tag{1.4}$$

with initial data

$$(u(x,0), v(x,0)) = (u_0(x), v_0(x)), \tag{1.5}$$

where $\nu > 0$ and $\mu \geq 0$ are constants.

The system (1.4) itself has great interests since it can be considered as the relaxation problem which arises in many physical situations such as kinetic theory, multiphase and phase transition, viscoelasticity, river flows, traffic flows, the theory of combustion and chromatography. In the physical background, u and v are vectors. In chromatography, u_i represent the concentration of the solute in the fluid phase and v_i its concentration in the solid phase, both being expressed in moles per unit volume of their own phase. When $\tau = 0$, the equilibrium relation v_i which is usually called the adsorption isotherm is, in general, a complicated nonlinear function of u_i in which the mutual influences among different solutes are taken into account. The details of the physical backgrounds can be found in [28, 33]. It is worth while pointing that the unique solution for the system (1.4) without the parameters ν and μ was recently studied in [4] by using the Glimm method for a small initial date.

First of all, about the solutions of the Cauchy problem (1.4)–(1.5) we have the following result.

Theorem 1.1 *I.) Assume that $H(u)$ is a nondecreasing function and the initial data $(u_0(x), v_0(x))$ have compact support or vanish sufficiently fast as $|x| \to \infty$. If*

$$\iint_{\mathbb{R}^D} |(u_0(x))_{x_j}| dx \leq M, \quad \iint_{\mathbb{R}^D} |(v_0(x))_{x_j}| dx \leq M \tag{1.6}$$

for $j = 1, 2, \dots, D$, then for any fixed $\nu > 0, \mu \geq 0$ and $\tau > 0$, the solutions (u^τ, v^τ) of the Cauchy problem (1.4)–(1.5) satisfy the a-priori estimates

$$(u^\tau(x,t), v^\tau(x,t)) \to (0,0), \quad |x| \to \infty \tag{1.7}$$

for fixed $t > 0$ and

$$\iint_{\mathbb{R}^D} |(u^\tau)_{x_j}(x,t)| dx \leq M, \quad \iint_{\mathbb{R}^D} |(v^\tau)_{x_j}(x,t)| dx \leq M; \tag{1.8}$$

if

$$\iint_{\mathbb{R}^D} |(u^\tau(x,0))_t| dx \leq M, \quad \iint_{\mathbb{R}^D} |(v^\tau(x,0))_t| dx \leq M, \tag{1.9}$$

then

$$\iint_{\mathbb{R}^D} |(u^\tau)_t(x,t)|dx \le M, \quad \iint_{\mathbb{R}^D} |(v^\tau)_t(x,t)|dx \le M, \tag{1.10}$$

where M denotes a positive constant which is independent of ν, μ and τ.
II.) If the conditions in I.) are satisfied and the space dimension is $D = 1$, then the Cauchy problem (1.4)–(1.5) has a unique classical solution (u^τ, v^τ) on $\mathbb{R} \times [0,T]$ for any even time T, which satisfies (1.8), (1.10) and the boundedness estimates

$$|u^\tau(x,t)| \le M, \quad |v^\tau(x,t)| \le M; \tag{1.11}$$

If the space dimension $D \ge 2$ and there exist two large constants M_1, M_2 such that $H(M_1) = M_2, |u_0(x)| \le M_1, |v_0(x)| \le M_2$, then the above boundedness estimate (1.11) is true.

From the estimates given in (1.8), (1.10), we immediately have the following theorem.

Theorem 1.2 *Under the conditions of Theorem 1.1, if $v_0(x) = H(u_0(x))$, then there exists a subsequence (still denoted by $(u^\tau(x,t), v^\tau(x,t))$) such that*

$$(u^\tau(x,t), v^\tau(x,t)) \to (u(x,t), v(x,t)) \quad a.e. \tag{1.12}$$

as ν, μ, τ go to zero and the limit function (u,v) satisfies $v = H(u)$, a.e. and u is a generalized solution of the scalar equation

$$(u + H(u))_t + \sum_{j=1}^{D} F_j(u)_{x_j} = 0, \tag{1.13}$$

$$u(x,0) = u_0(x) \tag{1.14}$$

which satisfies the BV estimates

$$|u(x,t)| \le M, \quad \iint_{\mathbb{R}^D} |u_x(x,t)|dx \le M, \quad \iint_{\mathbb{R}^D} |u_t(x,t)|dx \le M. \tag{1.15}$$

Theorem 1.3 *Assume that $0 \le w_0(x) \le M$, $m > 1$, $\iint_{\mathbb{R}^D} |w_0(x)_{x_j}|dx$, and $\iint_{\mathbb{R}^D} |\Delta(w_0^m(x))|dx$ are bounded. Then the unique weak solution for the Cauchy problem*

$$w_t = \Delta w^m \tag{1.16}$$

$$w(x,0) = w_0(x) \tag{1.17}$$

satisfies the bounded variation estimates

$$|w(x,t)| \le M, \quad \iint_{\mathbb{R}^D} |w(x,t)_{x_j}|dx \le M, \quad \iint_{\mathbb{R}^D} |w(x,t)_t|dx \le M. \tag{1.18}$$

The proofs of the above theorems are given in the next section.

2 Proofs of Theorems

Proof of Theorem 1.1 The behavior (1.7) of the solutions can be seen from the proof of the existence of a local solution; see [29] for the details. To prove (1.8), we differentiate (1.4) with respect to y, where $y = x_l$ for $l = 1, 2, \dots, D$, and multiply by $\tau \operatorname{sgn}(u_y)$ in the first equation, and by $\tau \operatorname{sgn}(v_y)$ in the second equation. We obtain

$$\begin{aligned} \tau|u_y|_t + \tau \sum_{j=1}^{D} (F_j'(u)|u_y|)_{x_j} + (H'(u)|u_y| - \operatorname{sgn}(u_y)v_y) = \nu\tau sgn u_y)\Delta(u_y), \\ \tau|v_y|_t + (|v_y| - H'(u)u_y \operatorname{sgn}(v_x)) = \tau\mu \operatorname{sgn}(v_y)\Delta(v_y). \end{aligned} \tag{2.1}$$

Adding the above two equations, we obtain

$$\begin{aligned} \tau(|u_y| + |v_y|)_t + \tau \sum_{j=1}^{D} (F_j'(u)|u_y|)_{x_j} + (H'(u)|u_y| + |v_y|)(1 - \operatorname{sgn}(v_y)\operatorname{sgn}(u_y)) \\ = \tau(\nu \operatorname{sgn} u_y)\Delta(u_y) + \mu \operatorname{sgn}(v_y)\Delta(v_y)). \end{aligned} \tag{2.2}$$

Integrating (2.2) on $\mathbb{R}^D \times [0, T_1]$ for a fixed time T_1, and noticing (1.7), we have

$$\iint_{\mathbb{R}^D} (|u_y(x,t)| + |v_y(x,t)|)dx \iint_{\mathbb{R}^D} (|(u_0(x))_y| + |(v_0(x))_y|)dx, \tag{2.3}$$

which is the estimate (1.8). Similarly we can get

$$\iint_{\mathbb{R}^D} (|u_t(x,t)| + |v_t(x,t)|)dx \le \iint_{\mathbb{R}^D} (|(u(x,0))_t| + |(v(x,0))_t|)dx, \tag{2.4}$$

which is the estimate (1.10). So part I) of Theorem 1.1 is proved.

For the one dimension case $D = 1$ in II) of Theorem 1.1, we have

$$|u| = \Big| \int_{-\infty}^{x} u_x dx \Big| \le \int_R |u_x| dx \le M, \tag{2.5}$$

which imply the existence of global solutions in time for the Cauchy problem (1.4),(1.5). Using the condition $H(M_1) = M_2$ given in II), a maximum principle applying to (1.4) gives the boundedness estimate (1.11), which again implies the global existence of solutions for the Cauchy problem (1.4),(1.5). So Theorem 1.1 is proved.

Proof of Theorem 1.2 Let $\mu = 0$ in (1.4). Then if the total variation of $u_0(x)$ is bounded, we can smooth $u_0(x)$ by a molifier such that $\nu\Delta u_0(x)$ is L^1 bounded. Then from the first equation in (1.4) and $v_0(x) = H(u_0(x))$, $u^\tau(x,0)_t$ is also L^1 bounded; and from the second equation in (1.4), $v^\tau(x,0)_t = 0$.

From the second equation in (1.4) and the the estimate in (1.10), we have

$$\iint_{\mathbb{R}^D} |H(u^\tau) - v^\tau| dx \le \tau M. \tag{2.6}$$

So there exists a subsequence (u^{τ_k}, v^{τ_k}) such that u^{τ_k} converges to a function u as τ, ν tend to zero and so v^{τ_k} converges to $H(u)$ from (2.6), where u satisfies the estimates (1.8),(1.10). Adding two equations in (1.4) together, we have

$$(u^\tau + v^\tau)_t + \sum_{j=1}^{D} F_j(u^\tau)_{x_j} = \nu \Delta u^\tau. \tag{2.7}$$

which gives (1.13) in the sense of distributions as τ and ν tend to zero. So Theorem 1.2 is proved.

Proof of Theorem 1.3 Let $F_j = 0, \mu = 0$ and $\nu = c$ be a fixed positive constant in (1.4). Then (1.4) is equivalent to

$$\begin{aligned} u_t + \frac{H(u) - v}{\tau} &= c\Delta u \\ v_t + \frac{v - H(u)}{\tau} &= 0, \end{aligned} \tag{2.8}$$

with the initial data

$$(u_0(x), v_0(x)) = (w_0^m(x), H(w_0^m(x))), \tag{2.9}$$

where $H(u) = cu^{1/m} - u$. Let $0 \le u_0(x) \le (\frac{c}{m})^{\frac{m}{m-1}} = U_+$ and $V_+ = H(U_+)$, then $H'(u) \ge 0$ for $u \in [0, U_+]$. From the conditions in Theorem 1.3, $\iint_{\mathbb{R}^D} |w_{0y}| dx$ is bounded, where y denotes an x_j, $j = 1, 2, \dots D$, then the integral $\iint_{\mathbb{R}^D} |(w_0^m)_y| dx$ is also bounded since $m > 1$ and the boundedness of w_0 follows. Thus $\iint_{\mathbb{R}^D} |u_{0y}| + |v_{0y}| dx \le$ is bounded. Furthermore if $\iint_{\mathbb{R}^D} |\Delta(w_0^m)| dx$ is bounded, then $\iint_{\mathbb{R}^D} |u_t(x,0)| dx$ is bounded from the first equation in (2.8) and $v_t(x,0) = 0$ from the second in (2.8). Therefore from the conclusions given in II) in Theorem 1.1, we have the following estimates for the solutions (u^τ, v^τ) of the Cauchy problem (2.8),(refe27)

$$\begin{gathered} 0 \le u^\tau \le U_+, \quad 0 \le v^\tau \le V_+ \\ \iint_{\mathbb{R}^D} |u_y^\tau(x,t)| dx \le M, \quad \iint_{\mathbb{R}^D} |v_y^\tau(x,t)| dx \le M \\ \iint_{\mathbb{R}^D} |u_t^\tau(x,t)| dx \le M, \quad \iint_{\mathbb{R}^D} |v_t^\tau(x,t)| dx \le M. \end{gathered} \tag{2.10}$$

Using the second equation in (2.8), we have

$$\iint_{\mathbb{R}^D} |v^\tau - H(u^\tau)| dx \le \tau M. \tag{2.11}$$

The estimates in (2.10),(2) imply the convergence of the relaxation solutions (u^τ, v^τ) as the relaxation parameter τ goes to zero. Let the limit be (u, v). Then $v = H(u)$, a.e. from (2). Adding two equations in (2.8) together, we obtain

$$(c(u^\tau)^{1/m})_t + (v^\tau - A(u^\tau))_t = c\Delta u^\tau. \tag{2.12}$$

This implies that (let $u^{1/m} = w$)

$$w_t = \Delta w^m. \tag{2.13}$$

Since $v = H(u)$, a.e. and $H(u) = cu^{1/m} - u$, then from (2.10), $w = u^{1/m}$ satisfies the estimates in (1.18). Theorem 1.3 is proved.

Remark 1 The first strong and local regularity L^p estimate of w_t for the solution w of the porous media equation (31) was obtained by Bénilan in [2]. Here we extended the estimates to the whole $\mathbb{R}^D$ space both for w_t and for w_{x_i}.

Remark 2 In Theorem 1.3, we only consider the bounded variation solution for the most typical degenerate parabolic model, i.e. the porous medium equation. In fact, from its proof, we can see that the results are still true for more general degenerate parabolic equation of the form

$$G(w)_t + F(x, t, w, w_{x_i}) = \Delta w, \tag{2.14}$$

where the nonlinear function $G(w)$ is smooth and $G'(w) > c > 0$ for a constant c. For instance, the singular nonlinear partial differential equation

$$\begin{gathered} \beta(u(x,t))_t \ni \Delta u(x,t), \quad (x,t) \in Q = \Omega \times (0,T), \\ u(x,t) = 0, \quad (x,t) \in \partial Q = \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{gathered} \tag{2.15}$$

where Ω is a bounded smooth domain in $\mathbb{R}^D (D \geq 1), u_0$ is a given smooth function, and β is the multivalued mapping

$$\beta(x) = \begin{cases} ax - 1, & x \leq 0 \quad (a > 0) \\ (-1, 1), & x = 0, \\ bx + 1, & x \geq 0 \quad (b > 0). \end{cases} \tag{2.16}$$

Equations (2.15), (2.16) are a formulation of the classical two-phase Stefan problem, describing the flow of heat within a substance (say water) which changes phase (melts or freezes) at the temperature zero. The constants a and b denote the respective thermal conductivities in the ice and water regions, and the jump in β at zero corresponds to the latent heat of fusion. The temperature is

$$T = \begin{cases} u/b, & u > 0, \\ 0, & u = 0, \\ u/a, & u < 0. \end{cases} \tag{2.17}$$

The continuity of a unique weak solution of (2.15),(2.16) for all $D \geq 1$ was obtained by Caffarelli and Evan [5]. Here if we omit the region Ω and consider the solution in whole space, then the solution for the Cauchy problem of two-phase Stefan problem has BV bounded from Theorem 1.3.

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