Electronic Journal of Differential Equations, Vol. 2021 (2021), No. 96, pp. 1–19.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO POROUS MEDIUM EQUATIONS WITH BOUNDARY DEGENERACY

XUTONG ZHAO, MINGJUN ZHOU, XINXIN JING

Abstract. This article concerns the asymptotic behavior of solutions to a class of one-dimensional porous medium equations with boundary degeneracy on bounded and unbounded intervals. It is proved that the degree of degeneracy, the exponents of the nonlinear diffusion, and the nonlinear source affect the asymptotic behavior of solutions. It is shown that on a bounded interval, the problem admits both nontrivial global and blowing-up solutions if the degeneracy is not strong; while any nontrivial solution must blow up if the degeneracy is strong enough. For the problem on an unbounded interval, the blowing-up theorems of Fujita type are established. The critical Fujita exponent is finite if the degeneracy is not strong, while infinite if the degeneracy is strong enough. Furthermore, the critical case is proved to be the blowing-up case if it is finite.

1. Introduction

In this article, we consider the asymptotic behavior of solutions to the following two problems

$$\frac{\partial u}{\partial t}-\frac{\partial}{\partial x}\Big(x^\lambda\frac{\partial u^m}{\partial x}\Big)=u^p,\quad (x,t)\in(0,1)\times(0,T), \tag{1.1}$$

$$\Big(x^\lambda\frac{\partial u^m}{\partial x}\Big)(0,t)=u(1,t)=0,\quad t\in(0,T), \tag{1.2}$$

$$u(x,0)=u_0(x),\quad x\in(0,1), \tag{1.3}$$

and

$$\frac{\partial u}{\partial t}-\frac{\partial}{\partial x}\Big(x^\lambda\frac{\partial u^m}{\partial x}\Big)=u^p,\quad (x,t)\in(0,+\infty)\times(0,T), \tag{1.4}$$

$$\Big(x^\lambda\frac{\partial u^m}{\partial x}\Big)(0,t)=0,\quad t\in(0,T), \tag{1.5}$$

$$u(x,0)=u_0(x),\quad x\in(0,+\infty), \tag{1.6}$$

where $p>m>1$, $\lambda>0$ and $0<T\le+\infty$. If $\lambda=0$, both (1.1) and (1.4) are porous medium equations, which have been studied extensively (see, e.g., Chapter 1 in [27]). If $m=1$, both (1.1) and (1.4) are semilinear equations which are degenerate at a portion of the lateral boundary $x=0$. Semilinear equations with

2010 *Mathematics Subject Classification.* 35K59, 35B33, 35K65.
Key words and phrases. Critical Fujita exponent; porous medium equation; boundary degeneracy.

Submitted May 29, 2021. Published December 3, 2021.

such degeneracy appear in many physical or economic models, such as the Budyko-Sellers climate model [20], a simplified Crocco-type equation coming from the study on the velocity field of a laminar flow on a flat plate [5, 7], and the Black-Scholes model coming from the option pricing problem [4]. For $m > 1$ and $\lambda > 0$, both (1.1) and (1.4) admit two kinds of degeneracy. That is to say, (1.1) and (1.4) are degenerate not only at points where $u = 0$ but also at a portion of the lateral boundary $x = 0$.

In recent years, semilinear equations with boundary degeneracy have attracted many attentions, and it was shown that the boundary degeneracy causes many essential differences. For instance, the null controllability of the system governed by

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\Big(x^\lambda \frac{\partial u}{\partial x}\Big) = h(x,t)\chi_\omega, \quad (x,t) \in (0,1) \times (0,T),$$

and related problems were studied in [2, 5, 6, 8, 10, 11, 19, 22, 26, 28], where h is the control function, ω is a subinterval of $(0,1)$, and χ_ω is the characteristic function of ω. It was proved that $\lambda = 2$ is a threshold in the sense that the system is null controllable if $0 < \lambda < 2$, while not if $\lambda \geq 2$. For another instance, the quenching phenomenon of solutions to problem

$$\begin{gathered}
\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\Big(x^\lambda \frac{\partial u}{\partial x}\Big) = f(u), \quad (x,t) \in (0,a) \times (0,T),\\
\Big(x^\lambda \frac{\partial u}{\partial x}\Big)(0,t) = u(a,t) = 0, \quad t \in (0,T),\\
u(x,0) = 0, \quad x \in (0,a)
\end{gathered}$$

was studied in [29], where $a > 0$ and $f \in C^2([0,c))$ with $c > 0$ satisfies

$$f(0) > 0, \quad f'(0) > 0, \quad f''(s) \geq 0 \text{ for } 0 < s < c, \quad \lim_{s \to c^-} f(s) = +\infty.$$

It was shown that $\lambda = 2$ is also a threshold in the sense that the critical length satisfies

$$a_* \begin{cases} > 0, & \text{if } 0 < \lambda < 2,\\ = 0, & \text{if } \lambda \geq 2. \end{cases}$$

That is to say, in the case $0 < \lambda < 2$, there is a critical length $a_* > 0$ such that the solution exists globally in time if $a < a_*$, while quenches in a finite time if $a > a_*$. As to the case $\lambda \geq 2$, the solution must quench in a finite time for each $a > 0$. In [23], it was shown that the boundary degeneracy also affects the asymptotic behavior of solutions to the semilinear problems (1.1)–(1.3) and (1.4)–(1.6) in the case $m = 1$. More precisely, for problem (1.1)–(1.3) in the case $m = 1$, there exist both nontrivial global and blowing-up solutions if $\lambda < 2$, while any nontrivial solution must blow up in a finite time if $\lambda \geq 2$. As to problem (1.4)–(1.6) in the case $m = 1$, the critical Fujita exponent is

$$p_c = \begin{cases} 3 - \lambda, & \text{if } 0 < \lambda < 2,\\ +\infty, & \text{if } \lambda \geq 2. \end{cases}$$

That is to say, in the case $0 < \lambda < 2$, any nontrivial solution must blow up in a finite time if $1 < p < 3 - \lambda$, while there are both nontrivial global and blowing-up solutions if $p > 3 - \lambda$. Whereas in the case $\lambda \geq 2$, any nontrivial solution must blow up in a finite time for $p > 1$.

The blowing-up phenomenon of solutions to nonlinear diffusion equations was first introduced in 1966 by Fujita, who proved in [13] that $p_c = 1 + 2/n$, which is called the critical Fujita exponent, is critical for the Cauchy problem of

$$\frac{\partial u}{\partial t} - \Delta u = u^p, \quad (x,t) \in \mathbb{R}^n \times (0, +\infty)$$

in the sense that any nontrivial solution blows up in a finite time if $1 < p < p_c$, whereas there exist both nontrivial global and blowing-up solutions if $p > p_c$. The critical case $p = p_c$ was proved to belong to the blowing-up case in [14, 16]. Fujita revealed an important topic of nonlinear partial differential equations. And there have been a great number of extensions of Fujita's results in several directions since then, including similar results for numerous of quasilinear parabolic equations and systems in various of geometries with nonlinear sources or nonhomogeneous boundary conditions, see the survey papers [9, 17] and the references therein, and more recent works, e.g. [1, 3, 12, 15, 18, 21, 23, 24, 25].

In this paper, we study the asymptotic behavior of solutions to the quasilinear problems (1.1)–(1.3) and (1.4)–(1.6). As mentioned above, both (1.1) and (1.4) admit two kinds of degeneracy. They are degenerate not only at points where $u = 0$ but also at a portion of the lateral boundary $x = 0$. For problem (1.1)–(1.3) in a bounded interval, it is shown that $\lambda = 2$ is a threshold in the sense that there exist both nontrivial global and blowing-up solutions to problem (1.1)–(1.3) if $\lambda < 2$, while any nontrivial solution to problem (1.1)–(1.3) must blow up in a finite time if $\lambda \geq 2$. For problem (1.4)–(1.6) in an unbounded interval, $\lambda = 2$ is also a threshold in the sense that the critical Fujita exponent is finite if $\lambda < 2$, while infinite if $\lambda \geq 2$. More precisely, it is proved that the critical Fujita exponent is

$$p_c = \begin{cases} m + 2 - \lambda, & \text{if } 0 < \lambda < 2, \\ +\infty, & \text{if } \lambda \geq 2. \end{cases}$$

That is to say, in the case $0 < \lambda < 2$, any nontrivial solution to problem (1.4)–(1.6) must blow up in a finite time if $m < p < p_c = m + 2 - \lambda$, while there are both nontrivial global and blowing-up solutions to problem (1.4)–(1.6) if $p > p_c = m+2-\lambda$. Furthermore, the critical case $p = p_c = m+2-\lambda$ belongs to the blowing-up case. Whereas in the case $\lambda \geq 2$, any nontrivial solution to problem (1.4)–(1.6) must blow up in a finite time for $p > m$. The methods used in this paper are mainly inspired by [23]. For the blowing-up of solutions to problem (1.1)–(1.3) in a bounded interval and problem (1.4)–(1.6) in an unbounded interval, we apply the methods of weighted energy estimates instead of constructing blowing-up subsolutions. The key is to choose appropriate weights and to estimate the interaction of the nonlinear degenerate diffusions and the sources. To prove the global existence of nontrivial solutions, we construct suitable self-similar supersolutions. Since (1.1) and (1.4) admit two kinds of degeneracy, some complicated estimates are needed.

This article is organized as follows. Comparison principles and well-posedness for problems (1.1)–(1.3) and (1.4)–(1.6) are established in Section 2. The asymptotic behavior of solutions to problem (1.1)–(1.3) in a bounded interval and problem (1.4)–(1.6) in an unbounded interval are studied in Section 3 and Section 4, respectively.

2. Comparison principles and well-posedness

In this section, we establish comparison principles and well-posedness for problems (1.1)–(1.3) and (1.4)–(1.6).

Since (1.1) and (1.4) admit two kinds of degeneracy, we consider weak solutions defined as follows.

Definition 2.1. Let $0 < T \le +\infty$. A nonnegative function $u \in L^\infty((0,1)\times(0,T))$ is called a subsolution (supersolution, solution) to problem (1.1)–(1.3) in $(0,T)$ if for any $0<\tau<T$, $x^{\lambda/2}\frac{\partial u^m}{\partial x} \in L^2((0,1)\times(0,\tau))$, and

$$\begin{aligned}
&\int_0^\tau \int_0^1 \Big(-u(x,t)\frac{\partial \varphi}{\partial t}(x,t) + x^\lambda \frac{\partial u^m}{\partial x}(x,t)\frac{\partial \varphi}{\partial x}(x,t)\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&\le (\ge,=) \int_0^\tau \int_0^1 u^p(x,t)\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_0^1 u_0(x)\varphi(x,0)\,\mathrm{d}x
\end{aligned}$$

holds for any $0 \le \varphi \in C^1([0,1]\times[0,\tau])$ with $\varphi(\tau,\cdot)\big|_{(0,1)} = \varphi(1,\cdot)\big|_{(0,\tau)} = 0$.

Definition 2.2. Let $0 < T \le +\infty$. A nonnegative function $u \in L^\infty((0,+\infty)\times(0,T))$ is called a subsolution (supersolution, solution) to problem (1.4)–(1.6) in $(0,T)$ if for any $0<\tau<T$, $x^{\lambda/2}\frac{\partial u^m}{\partial x} \in L^2((0,+\infty)\times(0,\tau))$, and

$$\begin{aligned}
&\int_0^\tau \int_0^{+\infty} \Big(-u(x,t)\frac{\partial \varphi}{\partial t}(x,t) + x^\lambda \frac{\partial u^m}{\partial x}(x,t)\frac{\partial \varphi}{\partial x}(x,t)\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&\le (\ge,=) \int_0^\tau \int_0^{+\infty} u^p(x,t)\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_0^{+\infty} u_0(x)\varphi(x,0)\,\mathrm{d}x
\end{aligned}$$

holds for any $0 \le \varphi \in C^1([0,+\infty)\times[0,\tau])$ vanishing at $t=\tau$ and for large x.

If u is a solution to problem (1.1)–(1.3) (or problem (1.4)–(1.6)) in $(0,+\infty)$, it is said that u is a global solution in time. Otherwise, there exists $T>0$ such that u is a solution in $(0,T)$ and satisfies

$$\begin{gathered}
\|u(\cdot,t)\|_{L^\infty(0,1)} \to +\infty, \quad \text{as } t \to T^-, \\
(\text{ or } \|u(\cdot,t)\|_{L^\infty(0,+\infty)} \to +\infty, \quad \text{as } t \to T^-),
\end{gathered}$$

and it is said that u blows up in a finite time.

We establish the comparison principle for problem (1.1)–(1.3).

Proposition 2.3. *Assume that $\underline{u}$ and $\overline{u}$ are a subsolution and a supersolution, respectively, to problem (1.1)–(1.3). then $\underline{u} \le \overline{u}$ a.e. in $(0,1)\times(0,T)$.*

Proof. Set

$$u(x,t) = \underline{u}(x,t) - \overline{u}(x,t), \quad (x,t)\in(0,1)\times(0,T).$$

Let $0<\tau<T$. For any function $0 \le \varphi \in C^1([0,1]\times[0,\tau])$ with $\varphi(\tau,\cdot)\big|_{(0,1)} = \varphi(1,\cdot)\big|_{(0,\tau)} = 0$, it follows from Definition 2.1 that

$$\begin{aligned}
&\int_0^1 u(x,\tau)\varphi(x,\tau)\,\mathrm{d}x \\
&\le \int_0^\tau \int_0^1 \Big(z\frac{\partial \varphi}{\partial t} + (\underline{u}^m - \overline{u}^m)\frac{\partial}{\partial x}\Big(x^\lambda \frac{\partial \varphi}{\partial x}\Big) + (\underline{u}^p - \overline{u}^p)\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t \qquad (2.1) \\
&= \int_0^\tau \int_0^1 z\Big(\frac{\partial \varphi}{\partial t} + a\frac{\partial}{\partial x}\Big(x^\lambda \frac{\partial \varphi}{\partial x}\Big) + ac\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

where

$$a(x,t)=\begin{cases}\frac{\underline{u}^m(x,t)-\overline{u}^m(x,t)}{\underline{u}(x,t)-\overline{u}(x,t)}, & \text{if } \underline{u}(x,t)\neq\overline{u}(x,t),\\ m\underline{u}^{m-1}(x,t), & \text{if } \underline{u}(x,t)=\overline{u}(x,t),\end{cases}$$

$$c(x,t)=\begin{cases}\frac{\underline{u}^p(x,t)-\overline{u}^p(x,t)}{\underline{u}^m(x,t)-\overline{u}^m(x,t)}, & \text{if } \underline{u}(x,t)\neq\overline{u}(x,t),\\ \frac{p}{m}\underline{u}^{m-p}(x,t), & \text{if } \underline{u}(x,t)=\overline{u}(x,t).\end{cases}$$

It is clear that $a,\, c\in L^\infty((0,1)\times(0,\tau))$ satisfying

$$0\le a(x,t)\le a_0,\quad 0\le c(x,t)\le c_0,\quad (x,t)\in(0,1)\times(0,\tau),$$

where a_0 and c_0 are two positive constants. Choosing $\{a_n\}_{n=1}^\infty$ and $\{c_n\}_{n=1}^\infty$ in $C^\infty([0,1]\times[0,\tau])$ such that

$$\frac{1}{n}\le a_n(x,t)\le a_0+\frac{1}{n},\quad 0\le c_n(x,t)\le c_0,\quad (x,t)\in(0,1)\times(0,\tau), \tag{2.2}$$

and

$$\int_0^\tau\int_0^1(a-a_n)^2\,\mathrm{d}x\,\mathrm{d}t\le\frac{1+\tau}{n^2},\quad \int_0^\tau\int_0^1(c-c_n)^2\,\mathrm{d}x\,\mathrm{d}t\le\frac{1}{n^2}. \tag{2.3}$$

For any nonnegative function $h\in C_0^\infty(0,1)$, we consider the problem

$$\frac{\partial\varphi_n}{\partial t}+a_n\frac{\partial}{\partial x}\Big(x^\lambda\frac{\partial\varphi_n}{\partial x}\Big)+a_nc_n\varphi_n=0,\quad (x,t)\in(0,1)\times(0,\tau), \tag{2.4}$$

$$\Big(x^\lambda\frac{\partial\varphi_n}{\partial x}\Big)(0,t)=\varphi_n(1,t)=0,\quad t\in(0,\tau), \tag{2.5}$$

$$\varphi_n(x,\tau)=h(x),\quad x\in(0,1). \tag{2.6}$$

The same proof as of [29, Theorem 2.2] yields that problem (2.4)–(2.6) admits a unique solution $\varphi_n\in C^\infty((0,1)\times(0,\tau))\cap C([0,1]\times[0,\tau])$ satisfying

$$0\le\varphi_n(x,t)\le\|h\|_{L^\infty(0,1)}. \tag{2.7}$$

Multiplying (2.4) by $\frac{\partial}{\partial x}\big(x^\lambda\frac{\partial\varphi_n}{\partial x}\big)$ and then integrating over $(0,1)\times(0,\tau)$ by parts, one obtains from (2.2), (2.5) and (2.6) that

$$\begin{aligned}
&\frac{1}{2}\int_0^1x^\lambda\Big|\frac{\partial\varphi_n}{\partial x}(x,0)\Big|^2\,\mathrm{d}x+\int_0^\tau\int_0^1a_n\Big|\frac{\partial}{\partial x}\Big(x^\lambda\frac{\partial\varphi_n}{\partial x}\Big)\Big|^2\,\mathrm{d}x\,\mathrm{d}t\\
&=\frac{1}{2}\int_0^1x^\lambda(h'(x))^2\,\mathrm{d}x-\int_0^\tau\int_0^1a_nc_n\varphi_n\frac{\partial}{\partial x}\Big(x^\lambda\frac{\partial\varphi_n}{\partial x}\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&\le\frac{1}{2}\int_0^1x^\lambda(h'(x))^2\,\mathrm{d}x+\frac{1}{2}\int_0^\tau\int_0^1a_nc_n^2\varphi_n^2\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\frac{1}{2}\int_0^\tau\int_0^1a_n\Big|\frac{\partial}{\partial x}\Big(x^\lambda\frac{\partial\varphi_n}{\partial x}\Big)\Big|^2\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{2.8}$$

It follows from (2.8), (2.2) and (2.7) that

$$\int_0^\tau\int_0^1a_n\Big|\frac{\partial}{\partial x}\Big(x^\lambda\frac{\partial\varphi_n}{\partial x}\Big)\Big|^2\,\mathrm{d}x\,\mathrm{d}t\le M_1, \tag{2.9}$$

where M_1 is a positive constant independent of n. Taking $\varphi=\varphi_n$ in (2.1), one obtains from (2.2), (2.3), (2.7), (2.9) and the Hölder inequality that

$$\int_0^1u(x,\tau)h(x)\,\mathrm{d}x$$

$$
\begin{aligned}
&\le \int_0^\tau \int_0^1 z\Big(\frac{\partial \varphi_n}{\partial t} + a\frac{\partial}{\partial x}\Big(x^\lambda \frac{\partial \varphi_n}{\partial x}\Big) + ac\varphi_n\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^\tau \int_0^1 u(a-a_n)\frac{\partial}{\partial x}\Big(x^\lambda \frac{\partial \varphi_n}{\partial x}\Big)\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau \int_0^1 u(ac - a_n c_n)\varphi_n\,\mathrm{d}x\,\mathrm{d}t \\
&\le \|u\|_{L^\infty((0,1)\times(0,\tau))}\Big(\int_0^\tau \int_0^1 \frac{(a-a_n)^2}{a_n}\,\mathrm{d}x\,\mathrm{d}t\Big)^{1/2} \\
&\quad \times \Big(\int_0^\tau \int_0^1 a_n\Big|\frac{\partial}{\partial x}\Big(x^\lambda \frac{\partial \varphi_n}{\partial x}\Big)\Big|^2\,\mathrm{d}x\,\mathrm{d}t\Big)^{1/2} \\
&\quad + \|u\|_{L^\infty((0,1)\times(0,\tau))}\Big(\int_0^\tau \int_0^1 \varphi_n^2\,\mathrm{d}x\,\mathrm{d}t\Big)^{1/2}\Big(\int_0^\tau \int_0^1 (ac - a_n c_n)^2\,\mathrm{d}x\,\mathrm{d}t\Big)^{1/2} \\
&\le M_2\sqrt{n}\Big(\int_0^\tau \int_0^1 (a-a_n)^2\,\mathrm{d}x\,\mathrm{d}t\Big)^{1/2} + \frac{M_2}{n} \le \frac{M_3}{\sqrt{n}},
\end{aligned}
$$

where M_2 and M_3 are positive constants independent of n. Letting $n \to \infty$, one obtains that

$$
\int_0^1 u(x,\tau)h(x)\,\mathrm{d}x \le 0.
$$

Thanks to the arbitrariness of $0 \le h \in C_0^\infty(0,1)$ and $\tau \in (0,T)$, it holds that $u \le 0$ *a.e.* in $(0,1)\times(0,T)$. That is, $\underline{u} \le \overline{u}$ *a.e.* in $(0,1)\times(0,T)$. □

We turn to the local well-posedness of problem (1.1)–(1.3).

Proposition 2.4. *Assume that $0 \le u_0 \in L^\infty(0,1)$. There exists a constant $T > 0$ such that problem (1.1)–(1.3) admits a unique solution. Furthermore, $\frac{\partial u^m}{\partial t}$ in $L^2((0,1)\times(\tau,T))$ for any $\tau \in (0,T)$.*

Proof. The uniqueness follows from Proposition 2.3. Let us prove the local existence. For each integer $n \ge 1$, consider the problem

$$
\frac{\partial u_n}{\partial t} - \frac{\partial}{\partial x}\Big(\big(x+\frac{1}{n}\big)^\lambda \frac{\partial u_n^m}{\partial x}\Big) = u_n^p, \quad (x,t) \in (0,1)\times(0,T_n), \tag{2.10}
$$

$$
\frac{\partial u_n^m}{\partial x}(0,t) = 0, \quad u_n(1,t) = \frac{1}{n}, \quad t \in (0,T_n), \tag{2.11}
$$

$$
u_n(x,0) = u_{0,n}(x) + \frac{1}{n}, \quad x \in (0,1), \tag{2.12}
$$

where $u_{0,n} \in C_0^\infty(0,1)$ satisfies

$$
0 \le u_{0,n}(x) \le \|u_0\|_{L^\infty(0,1)}, \quad x \in [0,1], \tag{2.13}
$$

$$
\lim_{n\to\infty} \|u_{0,n} - u_0\|_{L^\infty(0,1)} = 0. \tag{2.14}
$$

We set

$$
\underline{u}_n(x,t) = 0, \quad \overline{u}_n(x,t) = (2M_0 - (p-1)t)^{1/(1-p)}, \quad (x,t) \in [0,1]\times[0,T],
$$

where

$$
M_0 = \frac{1}{2}(\|u_0\|_{L^\infty(0,1)} + 1)^{1-p}, \quad T = \frac{M_0}{p-1}.
$$

It is clear that $\underline{u}_n$ and $\overline{u}_n$ are a subsolution and a supersolution, respectively, to problem (2.10)–(2.12) for $T_n = T$. Thanks to the classical theory on parabolic equations, problem (2.10)–(2.12) with $T_n = T$ admits a unique solution

$u_n \in C^\infty([0,1]\times[0,T])$, and

$$0 = \underline{u}_n(x,t) \le u_n(x,t) \le \overline{u}_n(x,t) \le M_0^{1/(1-p)}, \quad (x,t)\in[0,1]\times[0,T]. \tag{2.15}$$

Multiplying (2.10) by u_n^m and then integrating over $(0,1)\times(0,T)$ by parts, one obtains

$$\begin{aligned}
&\frac{1}{m+1}\int_0^1 u_n^{m+1}(x,T)\,\mathrm{d}x + \int_0^T\int_0^1 \Big(x+\frac{1}{n}\Big)^\lambda \Big|\frac{\partial u_n^m}{\partial x}\Big|^2 \mathrm{d}x\,\mathrm{d}t \\
&= \frac{1}{m+1}\int_0^1 u_0^{m+1}\,\mathrm{d}x + \int_0^T\int_0^1 u_n^{p+m}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{2.16}$$

It follows from (2.13) and (2.16) that

$$\int_0^T\int_0^1 \Big(x+\frac{1}{n}\Big)^\lambda \Big|\frac{\partial u_n^m}{\partial x}\Big|^2 \mathrm{d}x\,\mathrm{d}t \le M_1, \tag{2.17}$$

where M_1 is a positive constant depending only on $\|u_0\|_{L^\infty(0,1)}$, m and p.

Thanks to (2.17), for $\tau\in(0,T)$, there exists $\tilde\tau\in(0,\tau)$ such that

$$\int_0^1 \big(x+\frac{1}{n}\big)^\lambda \Big|\frac{\partial u_n^m}{\partial x}(x,\tilde\tau)\Big|^2 \mathrm{d}x \le \frac{M_1}{\tau}. \tag{2.18}$$

Multiplying (2.10) by $\frac{\partial u_n^m}{\partial t}$ and then integrating over $(0,1)\times(\tilde\tau,T)$ by parts, one obtains from the Hölder inequality that

$$\begin{aligned}
&\frac{4m}{(m+1)^2}\int_{\tilde\tau}^T\int_0^1 \Big|\frac{\partial u_n^{(m+1)/2}}{\partial t}\Big|^2 \mathrm{d}x\,\mathrm{d}t \\
&= \frac12\int_0^1 \big(x+\frac{1}{n}\big)^\lambda \Big|\frac{\partial u_n^m}{\partial x}(x,\tilde\tau)\Big|^2 \mathrm{d}x - \frac12\int_0^1 \big(x+\frac{1}{n}\big)^\lambda \Big|\frac{\partial u_n^m}{\partial x}(x,T)\Big|^2 \mathrm{d}x \\
&\quad + \frac{2m}{m+1}\int_{\tilde\tau}^T\int_0^1 u_n^{p+(m-1)/2}\frac{\partial u_n^{(m+1)/2}}{\partial t}\,\mathrm{d}x\,\mathrm{d}t \\
&\le \frac12\int_0^1 \big(x+\frac{1}{n}\big)^\lambda \Big|\frac{\partial u_n^m}{\partial x}(x,\tilde\tau)\Big|^2 \mathrm{d}x + \frac{m}{2}\int_{\tilde\tau}^T\int_0^1 u_n^{2p+m-1}\,\mathrm{d}x \\
&\quad + \frac{2m}{(m+1)^2}\int_{\tilde\tau}^T\int_0^1 \Big|\frac{\partial u_n^{(m+1)/2}}{\partial t}\Big|^2 \mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{2.19}$$

It follows from (2.15), (2.18) and (2.19) that

$$\int_{\tilde\tau}^T\int_0^1 \Big|\frac{\partial u_n^{(m+1)/2}}{\partial t}\Big|^2 \mathrm{d}x\,\mathrm{d}t \le M_2,$$

which, together with (2.15), leads to

$$\int_{\tilde\tau}^T\int_0^1 \Big|\frac{\partial u_n^m}{\partial t}\Big|^2 \mathrm{d}x\,\mathrm{d}t = \frac{4m^2}{(m+1)^2}\int_{\tilde\tau}^T\int_0^1 u_n^{m-1}\Big|\frac{\partial u_n^{(m+1)/2}}{\partial t}\Big|^2 \mathrm{d}x\,\mathrm{d}t \le M_3, \tag{2.20}$$

where M_2 and M_3 are positive constants depending only on $\|u_0\|_{L^\infty(0,1)}$, m, p, and τ.

Thanks to (2.15), (2.17) and (2.20), there exists a subsequence of $\{u_n\}_{n=1}^\infty$, denoted by itself for convenience, and three functions $u,\xi,\zeta\in L^\infty((0,1)\times(0,T))$ such that

$$0\le u(x,t)\le M_0^{1/(1-p)}, \quad (x,t)\in(0,1)\times(0,T), \tag{2.21}$$

$$\int_0^T \int_0^1 x^\lambda \left|\frac{\partial \xi}{\partial x}\right|^2 \mathrm{d}x\,\mathrm{d}t \le M_1, \tag{2.22}$$

$$\int_\tau^T \int_0^1 \left|\frac{\partial \xi}{\partial t}\right|^2 \mathrm{d}x\,\mathrm{d}t \le M_3, \quad \tau \in (0,T), \tag{2.23}$$

and

$$u_n \rightharpoonup u,\ u_n^m \rightharpoonup \xi \text{ and } u_n^p \rightharpoonup \zeta \text{ weakly } * \text{ in } L^\infty((0,1)\times(0,T)) \text{ as } n\to\infty, \tag{2.24}$$

$$u_n^m \to \xi \text{ in } L^2((\varepsilon,1)\times(\varepsilon,T)) \text{ as } n\to\infty \text{ for any } 0<\varepsilon<\min\{1,T\}, \tag{2.25}$$

$$\big(x+\frac{1}{n}\big)^{\lambda/2}\frac{\partial u_n^m}{\partial x} \rightharpoonup x^{\lambda/2}\frac{\partial \xi}{\partial x} \text{ weakly in } L^2((0,1)\times(0,T)) \text{ as } n\to\infty, \tag{2.26}$$

$$\frac{\partial u_n^m}{\partial t} \rightharpoonup \frac{\partial \xi}{\partial t} \text{ weakly in } L^2((0,1)\times(\varepsilon,T)) \text{ as } n\to\infty \text{ for any } 0<\varepsilon<T. \tag{2.27}$$

It follows from (2.24) and (2.25) that

$$\xi(x,t)=u^m(x,t), \quad \zeta(x,t)=u^p(x,t), \quad (x,t)\in(0,1)\times(0,T). \tag{2.28}$$

Owing to (2.21)–(2.28), one can prove that u is a solution to problem (1.1)–(1.3), and $\frac{\partial u^m}{\partial t}\in L^2((0,1)\times(\tau,T))$ for any $\tau\in(0,T)$. □

For problem (1.4)–(1.6) in an unbounded interval, the comparison principle can be proved similarly to Proposition 2.3, and one needs some estimates in an unbounded interval (see, e.g., [23]). As to the local existence of solutions to problem (1.4)–(1.6), one can use the problems in bounded intervals to approximate the problem in an unbounded interval (see, e.g., [23]). We state these results without proof.

Proposition 2.5. *Assume that $\underline{u}$ and $\overline{u}$ are a subsolution and a supersolution, respectively, to problem (1.4)–(1.6). then $\underline{u}\le\overline{u}$ a.e. in $(0,+\infty)\times(0,T)$.*

Proposition 2.6. *Assume that $0\le u_0\in L^\infty(0,+\infty)\cap L^1(0,+\infty)$. There exists a constant $T>0$ such that problem (1.4)–(1.6) admits a unique solution. Furthermore, $\frac{\partial u^m}{\partial t}\in L^2((0,+\infty)\times(\tau,T))$ for any $\tau\in(0,T)$.*

3. Problems in a bounded interval

In this section, we investigate the asymptotic behavior of solutions to problem (1.1)–(1.3) in a bounded interval.

Theorem 3.1. *Let $\lambda\ge 2$. For any nontrivial $0\le u_0\in L^\infty(0,1)$, the solution to problem (1.1)–(1.3) must blow up in a finite time.*

Proof. For $0<\delta<1$, we set

$$\eta_\delta(x)=\begin{cases}\frac{\lambda-1}{\delta}2^{\lambda-1-\delta}x^{-\delta}-\frac{\lambda-1-\delta}{\delta}2^{\lambda-1}-1, & \text{if } x\in(0,1/2),\\ x^{1-\lambda}-1, & \text{if } x\in[1/2,1).\end{cases}$$

It follows from the proof of [23, Theorem 2.2] that $0\le\eta_\delta\in C^{1,1}([0,1])$, and there exists two constants M_1, $M_2>0$ depending only on λ but independent of δ such that

$$(x^\lambda\eta_\delta'(x))'\ge -M_1\delta\eta_\delta(x), \quad 0<x<1, \tag{3.1}$$

$$\int_0^1 \eta_\delta(x)\mathrm{d}x\le M_2. \tag{3.2}$$

Assume that u is a global solution of problem (1.1)–(1.3), and denote

$$w_\delta(t) = \int_0^1 u(x,t)\eta_\delta(x)\mathrm{d}x, \quad t \ge 0.$$

For $0 < \varepsilon < 1/2$, choose $\xi_\varepsilon \in C^\infty([0,1])$ to satisfy

$$\xi_\varepsilon(x) = \begin{cases} 0, & \text{if } x \in [0,\varepsilon], \\ 1, & \text{if } x \in [2\varepsilon, 1], \end{cases}$$

and

$$0 \le \xi_\varepsilon(x) \le 1, \quad 0 \le \xi_\varepsilon'(x) \le \frac{2}{\varepsilon}, \quad |\xi_\varepsilon''(x)| \le \frac{4}{\varepsilon^2}, \quad 0 \le x \le 1.$$

Thanks to Definition 2.1 and Proposition 2.4, one obtains

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u(x,t)\xi_\varepsilon(x)\eta_\delta(x)\mathrm{d}x \\ &= \int_0^1 u^m(x,t)(x^\lambda(\xi_\varepsilon(x)\eta_\delta(x))')'\mathrm{d}x + \int_0^1 u^p(x,t)\xi_\varepsilon(x)\eta_\delta(x)\mathrm{d}x \\ &= \int_0^1 u^m(x,t)\xi_\varepsilon(x)(x^\lambda\eta_\delta'(x))'\mathrm{d}x \\ &\quad + \int_\varepsilon^{2\varepsilon} u^m(x,t)\xi_\varepsilon'(x)(2x^\lambda\eta_\delta'(x) + \lambda x^{\lambda-1}\eta_\delta(x))\mathrm{d}x \\ &\quad + \int_\varepsilon^{2\varepsilon} u^m(x,t)x^\lambda\xi_\varepsilon''(x)\eta_\delta(x)\mathrm{d}x + \int_0^1 u^p(x,t)\xi_\varepsilon(x)\eta_\delta(x)\mathrm{d}x, \quad t > 0. \end{aligned} \tag{3.3}$$

Letting $\varepsilon \to 0^+$ in (3.3), it follows from $\lambda \ge 2$ and (3.1) that

$$\begin{aligned} w_\delta'(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u(x,t)\eta_\delta(x)\mathrm{d}x \\ &= \int_0^1 u^m(x,t)(x^\lambda\eta_\delta'(x))'\mathrm{d}x + \int_0^1 u^p(x,t)\eta_\delta(x)\mathrm{d}x \\ &\ge -M_1\delta\int_0^1 u^m(x,t)\eta_\delta(x)\mathrm{d}x + \int_0^1 u^p(x,t)\eta_\delta(x)\mathrm{d}x, \quad t > 0. \end{aligned} \tag{3.4}$$

Since $p > m > 1$, it follows from the Hölder inequality and (3.2) that

$$\begin{aligned} \int_0^1 u^m(x,t)\eta_\delta(x)\mathrm{d}x &\le \Big(\int_0^1 \eta_\delta(x)\mathrm{d}x\Big)^{(p-m)/p}\Big(\int_0^1 u^p(x,t)\eta_\delta(x)\mathrm{d}x\Big)^{m/p} \\ &\le M_2^{(p-m)/p}\Big(\int_0^1 u^p(x,t)\eta_\delta(x)\mathrm{d}x\Big)^{m/p}, \quad t > 0, \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \int_0^1 u(x,t)\eta_\delta(x)\mathrm{d}x &\le \Big(\int_0^1 \eta_\delta(x)\mathrm{d}x\Big)^{(p-1)/p}\Big(\int_0^1 u^p(x,t)\eta_\delta(x)\mathrm{d}x\Big)^{1/p} \\ &\le M_2^{(p-1)/p}\Big(\int_0^1 u^p(x,t)\eta_\delta(x)\mathrm{d}x\Big)^{1/p}, \quad t > 0, \end{aligned} \tag{3.6}$$

Substituting (3.5) and (3.6) into (3.4) we obtain

$$
\begin{aligned}
& w_\delta'(t) \\
& \geq -M_1\delta\Big(\int_0^1 \eta_\delta(x)\mathrm{d}x\Big)^{(p-m)/p}\Big(\int_0^1 u^p(x,t)\eta_\delta(x)\mathrm{d}x\Big)^{m/p} \\
& \quad + \int_0^1 u^p(x,t)\eta_\delta(x)\mathrm{d}x \\
& \geq \Big(\int_0^1 u^p(x,t)\eta_\delta(x)\mathrm{d}x\Big)^{m/p}\Big(-M_1M_2^{(p-m)/p}\delta \\
& \quad + \Big(\int_0^1 u^p(x,t)\eta_\delta(x)\mathrm{d}x\Big)^{(p-m)/p}\Big) \\
& \geq M_2^{-m(p-1)/p}w^m(t)\Big(-M_1M_2^{(p-m)/p}\delta + M_2^{-(p-1)(p-m)/p}w^{p-m}(t)\Big),
\end{aligned}
\tag{3.7}
$$

for $t > 0$. Owing to the nontriviality of u_0, it holds that

$$
\mu = \inf_{0<\delta<1} w_\delta(0) = \inf_{0<\delta<1}\int_0^1 u_0(x)\eta_\delta(x)\mathrm{d}x > 0.
$$

We take

$$
\delta = \frac{1}{2}\min\{\mu^{p-m}M_1^{-1}M_2^{-(p-m)},\, 1\}.
$$

Then

$$
M_1M_2^{(p-m)/p}\delta \leq \frac{1}{2}M_2^{-(p-1)(p-m)/p}w_\delta^{p-m}(0).
$$

Hence one obtains from (3.7) that

$$
w_\delta'(t) \geq \frac{1}{2}M_2^{-(p-1)}w_\delta^p(t), \quad t > 0.
$$

Since $p > 1$, there exists $T_* > 0$ such that

$$
w_\delta(t) = \int_0^1 u(x,t)\eta_\delta(x)\mathrm{d}x \to +\infty \quad \text{as } t \to T_*^-,
$$

which leads to

$$
\lim_{t\to T_*^-}\|u(\cdot,t)\|_{L^\infty(0,1)} \to +\infty.
$$

That is to say, u blows up in a finite time. $\square$

Theorem 3.2. *Let $0 < \lambda < 2$. There exist both nontrivial global and blowing-up solutions to problem* (1.1)–(1.3).

Proof. First, we prove the existence of the global solution for sufficiently small $0 \leq u_0 \in L^\infty(0,1)$. We set

$$
\overline{u}(x,t) = (t+\tau)^{-\alpha}U((t+\tau)^{-\beta}x), \quad 0 \leq x \leq 1,\ t \geq 0,
\tag{3.8}
$$

where

$$
\alpha = \frac{1}{p-1}, \quad \beta = \frac{p-m}{(2-\lambda)(p-1)},
$$

and $\tau > 1$ will be determined later. One can verify that if $0 \leq U \in C^{0,1}(0,\tau^{-\beta})$ with $U^m \in C^{1,1}(0,\tau^{-\beta})$ satisfies

$$
(r^\lambda(U^m)'(r))' + \alpha U(r) + \beta rU'(r) + U^p(r) \leq 0, \quad 0 < r < \tau^{-\beta},
$$

then $\overline{u}$ given by (3.8) is a supersolution to (1.1). We take

$$U(r)=\frac{1}{(2-\lambda)^{1/m}}\Big(\frac{1}{\tau^{(p-m)/(p-1)}}-r^{2-\lambda}\Big)^{1/m},\quad 0\le r\le\tau^{-\beta}. \tag{3.9}$$

A direct calculation shows that

$$\begin{aligned}
&(r^\lambda(U^m)'(r))'+\alpha U(r)+\beta rU'(r)+U^p(r)\\
&=-1+\alpha U(r)-\frac{\beta}{m}r^{1-m}U^{1-m}(r)+U^p(r)\\
&\le -1+\alpha U(r)+U^p(r)\\
&\le -1+\frac{1}{\tau^{1/m}}\Big(\frac{\alpha}{(2-\lambda)^{1/m}}+\frac{1}{(2-\lambda)^{p/m}}\Big).
\end{aligned}$$

Therefore, $\overline{u}$ is a supersolution to (1.1) for $\tau\ge\tau_0$, where

$$\tau_0=2^{m-1}\Big(\frac{\alpha^m}{(2-\lambda)}+\frac{1}{(2-\lambda)^p}\Big)+1.$$

It is noted that $\lim_{r\to0^+}r^\lambda(U^m)'(r)=0$. Hence $\overline{u}$ is a supersolution to problem (1.1)–(1.3) if

$$u_0(x)\le\overline{u}(x,0),\quad 0<x<1 \tag{3.10}$$

for some $\tau\ge\tau_0$. It follows from Proposition 2.3 that problem (1.1)–(1.3) admits a global solution if $0\le u_0\in L^\infty(0,1)$ satisfies (3.10) for some $\tau\ge\tau_0$.

Below we prove that the solution to problem (1.1)–(1.3) must blow up if $0\le u_0\in L^\infty(0,1)$ is sufficiently large. We set

$$\eta(x)=\begin{cases}2, & 0<x\le 1/2,\\ 1+\cos(2x-1)\pi, & 1/2<x\le 1.\end{cases}$$

It is clear that $\eta\in C^{1,1}([0,1])$ satisfies $\eta(1)=0$, $\eta'(0)=0$, and

$$(x^\lambda\eta(x))'\ge-4\pi^2\eta(x),\quad 1/2<x\le1.$$

Assume that u is a global solution to problem (1.1)–(1.3). We denote

$$w(t)=\int_0^1u(x,t)\eta(x)\mathrm{d}x,\quad t\ge0.$$

It follows from the Hölder inequality that

$$\begin{aligned}
\int_0^1u^m(x,t)\eta(x)\mathrm{d}x&\le\Big(\int_0^1\eta(x)\mathrm{d}x\Big)^{(p-m)/p}\Big(\int_0^1u^p(x,t)\eta(x)\mathrm{d}x\Big)^{m/p}\\
&\le 2\Big(\int_0^1u^p(x,t)\eta(x)\mathrm{d}x\Big)^{m/p},\quad t>0,
\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}
w(t)&\le\Big(\int_0^1\eta(x)\mathrm{d}x\Big)^{(p-1)/p}\Big(\int_0^1u^p(x,t)\eta(x)\mathrm{d}x\Big)^{1/p}\\
&\le 2\Big(\int_0^1u^p(x,t)\eta(x)\mathrm{d}x\Big)^{1/p},\quad t>0.
\end{aligned} \tag{3.12}$$

It follows from Definition 2.1, (3.11) and (3.12) that

$$\begin{aligned} & w'(t) \\ & = \int_0^1 u^m(x,t)(x^\lambda \eta'(x))' \mathrm{d}x + \int_0^1 u^p(x,t)\eta(x)\mathrm{d}x \\ & \geq -4\pi^2 \int_0^1 u^m(x,t)\eta(x)\mathrm{d}x + \int_0^1 u^p(x,t)\eta(x)\mathrm{d}x \\ & \geq -8\pi^2 \Big(\int_0^1 u^p(x,t)\eta(x)\mathrm{d}x \Big)^{m/p} + \int_0^1 u^p(x,t)\eta(x)\mathrm{d}x \\ & \geq \Big(\int_0^1 u^p(x,t)\eta(x)\mathrm{d}x \Big)^{m/p} \Big(-8\pi^2 + \Big(\int_0^1 u^p(x,t)\eta(x)\mathrm{d}x \Big)^{(p-m)/p} \Big) \\ & \geq 2^{-m} w^m(t) \Big(-8\pi^2 + 2^{-(p-m)} w^{p-m}(t) \Big), \quad t > 0. \end{aligned} \tag{3.13}$$

If u_0 is sufficiently large so that

$$w^{p-m}(0) = \Big(\int_0^1 u_0(x)\eta(x)\mathrm{d}x \Big)^{p-m} \geq 2^{p-m+4}\pi^2, \tag{3.14}$$

from (3.13) one obtains

$$w'(t) \geq 2^{-(p+1)} w^p(t), \quad t > 0.$$

Since $p > 1$, there exists $T_* > 0$ such that

$$w(t) = \int_0^1 u(x,t)\eta(x)\mathrm{d}x \to +\infty \quad \text{as } t \to T_*^-,$$

which leads to

$$\lim_{t \to T_*^-} \|u(\cdot,t)\|_{L^\infty(0,1)} \to +\infty.$$

That is to say, u blows up in a finite time. □

4. Problems in an unbounded interval

In this section, we investigate the asymptotic behavior of solutions to problem (1.4)–(1.6) in an unbounded interval. As a corollary of Theorem 3.1 and Proposition 2.3, one obtains the following result.

Theorem 4.1. *Let $\lambda \geq 2$. For any nontrivial $0 \leq u_0 \in L^\infty(0,+\infty) \cap L^1(0,+\infty)$, the solution to problem (1.4)–(1.6) must blow up in a finite time.*

Below we deal with the case that $0 < \lambda < 2$.

Theorem 4.2. *Let $0 < \lambda < 2$ and $m < p < p_c = m + 2 - \lambda$. For any nontrivial $0 \leq u_0 \in L^\infty(0,+\infty) \cap L^1(0,+\infty)$, the solution to problem (1.4)–(1.6) must blow up in a finite time.*

Proof. For $R > 0$, set

$$\eta_R(x) = \begin{cases} 1, & 0 < x \leq R, \\ \frac{1}{2}\big(1 + \cos\frac{(x-R)\pi}{R}\big), & R < x < 2R, \\ 0, & x \geq 2R. \end{cases}$$

It follows from the proof of [23, Lemma 2.1] that $\eta_R \in C^{1,1}([0,+\infty)$ satisfies

$$(x^\lambda \eta_R(x))' \geq -2^\lambda \pi^2 R^{\lambda-2} \eta_R(x), \quad x > 0.$$

Assume that u is a global solution to problem (1.4)–(1.6). We denote

$$w_R(t)=\int_0^{+\infty}u(x,t)\eta_R(x)\mathrm{d}x,\quad t\ge 0. \tag{4.1}$$

Thanks to Definition 2.2 and Proposition 2.6, one obtains that u satisfies

$$\begin{aligned} w_R'(t)&=\int_0^{+\infty}u^m(x,t)(x^\lambda\eta_R'(x))'\mathrm{d}x+\int_0^{+\infty}u^p(x,t)\eta_R(x)\mathrm{d}x\\ &\ge -2^\lambda\pi^2R^{\lambda-2}\int_0^{+\infty}u^m(x,t)\eta_R(x)\mathrm{d}x+\int_0^{+\infty}u^p(x,t)\eta_R(x)\mathrm{d}x, \end{aligned}\tag{4.2}$$

for $t>0$. It follows from the Hölder inequality that

$$\begin{aligned} &\int_0^{+\infty}u^m(x,t)\eta_R(x)\mathrm{d}x\\ &\le\Big(\int_0^{+\infty}\eta_R(x)\mathrm{d}x\Big)^{(p-m)/p}\Big(\int_0^{+\infty}u^p(x,t)\eta_R(x)\mathrm{d}x\Big)^{m/p}\\ &\le (2R)^{(p-m)/p}\Big(\int_0^{+\infty}u^p(x,t)\eta_R(x)\mathrm{d}x\Big)^{m/p},\quad t>0, \end{aligned}\tag{4.3}$$

and

$$\begin{aligned} &\int_0^{+\infty}u(x,t)\eta_R(x)\mathrm{d}x\\ &\le\Big(\int_0^{+\infty}\eta_R(x)\mathrm{d}x\Big)^{(p-1)/p}\Big(\int_0^{+\infty}u^p(x,t)\eta_R(x)\mathrm{d}x\Big)^{1/p}\\ &\le 2R^{(p-1)/p}\Big(\int_0^{+\infty}u^p(x,t)\eta_R(x)\mathrm{d}x\Big)^{1/p},\quad t>0. \end{aligned}\tag{4.4}$$

We substitute (4.3) and (4.4) into (4.2) to obtain

$$\begin{aligned} w_R'(t)&\ge -2^\lambda\pi^2R^{\lambda-2}\Big(\int_0^{+\infty}\eta_R(x)\mathrm{d}x\Big)^{(p-m)/p}\Big(\int_0^{+\infty}u^p(x,t)\eta_R(x)\mathrm{d}x\Big)^{m/p}\\ &\quad+\int_0^{+\infty}u^p(x,t)\eta_R(x)\mathrm{d}x\\ &\ge\Big(\int_0^{+\infty}u^p(x,t)\eta_R(x)\mathrm{d}x\Big)^{m/p}\Big(-2^\lambda\pi^2R^{\lambda-1-m/p}\\ &\quad+\Big(\int_0^{+\infty}u^p(x,t)\eta_R(x)\mathrm{d}x\Big)^{(p-m)/p}\Big)\\ &\ge 2^{-m}R^{-m(p-1)/p}w_R^m(t)\Big(-2^\lambda\pi^2R^{\lambda-1-m/p}\\ &\quad+2^{-(p-m)}R^{-(p-1)(p-m)/p}w_R^{p-m}(t)\Big),\quad t>0. \end{aligned}\tag{4.5}$$

It is noted that $p<m+2-\lambda$, $w_R(0)$ is nondecreasing with respect to $R\in(0,+\infty)$, and $\sup_{R>0}w_R(0)>0$. Therefore, there exists $R>0$ suitably large such that

$$2^{p-m+\lambda}\pi^2R^{p-(m+2-\lambda)}\le\frac{1}{2}w_R^{p-m}(0). \tag{4.6}$$

It follows from that (4.4)–(4.6) that

$$w_R'(t)\ge 2^{-(p+1)}R^{-(p-1)}w_R^p(t),\quad t>0.$$

Since $p>1$, there exists $T_*>0$ such that

$$w_R(t)=\int_0^{+\infty}u(x,t)\eta_R(x)\mathrm{d}x\to+\infty\quad\text{as } t\to T_*^-,$$

which leads to

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}\to+\infty\quad\text{as } t\to T_*^-.$$

That is to say, u blows up in a finite time. $\square$

Theorem 4.3. *Assume that $0<\lambda<2$ and $p>p_c=m+2-\lambda$. There exist both nontrivial global and blowing-up solutions to problem* (1.4)–(1.6).

Proof. Owing to Theorem 3.2 and Proposition 2.3, problem (1.4)–(1.6) admits blowing-up solutions. Below we prove the existence of nontrivial global solutions to problem (1.4)–(1.6). We set

$$\overline{u}(x,t)=(t+1)^{-\alpha}U((t+1)^{-\beta}x),\quad x\ge 0,\ t\ge 0, \tag{4.7}$$

where

$$\alpha=\frac{1}{p-1},\quad \beta=\frac{p-m}{(2-\lambda)(p-1)}.$$

It follows from $p>m+2-\lambda$ that $\alpha<\beta$. One can verify that if $0\le U\in C^{0,1}(0,+\infty)$ with $U^m\in C^{1,1}(0,+\infty)$ satisfies

$$(r^\lambda(U^m)'(r))'+\alpha U(r)+\beta rU'(r)+U^p(r)\le 0,\quad r>0,$$

then $\overline{u}$ given by (4.7) is a supersolution to (1.4). We take

$$U(r)=(l^{(2-\lambda)/2}-Ar^{2-\lambda})_+^{1/(m-1)},\quad r\ge 0, \tag{4.8}$$

where

$$\frac{(m-1)\alpha}{m(2-\lambda)}<A<\frac{(m-1)\beta}{m(2-\lambda)},\quad 0<l<l_0$$

with l_0 satisfying

$$l_0^{(2-\lambda)(p-1)/(2m-2)}=\frac{m(2-\lambda)A}{m-1}-\alpha. \tag{4.9}$$

For $0<r<A^{-1/(2-\lambda)}l$, one has

$$\begin{aligned}
&(r^\lambda(U^m)'(r))'+\alpha U(r)+\beta rU'(r)+U^p(r)\\
&=\frac{m(2-\lambda)^2A^2}{(m-1)^2}r^{2-\lambda}U^{2-m}(r)-\frac{m(2-\lambda)A}{m-1}U(r)+\alpha U(r)\\
&\quad-\frac{(2-\lambda)A\beta}{m-1}r^{2-\lambda}U^{2-m}(r)+U^p(r)\\
&=\frac{(2-\lambda)A}{m-1}\Big(\frac{m(2-\lambda)A}{m-1}-\beta\Big)r^{2-\lambda}U^{2-m}(r)\\
&\quad+\Big(\alpha-\frac{m(2-\lambda)A}{m-1}\Big)U(r)+U^p(r)\\
&\le U(r)\Big(\alpha-\frac{m(2-\lambda)A}{m-1}+(l^{(2-\lambda)/2}-Ar^{2-\lambda})^{(p-1)/(m-1)}\Big)\\
&\le U(r)\Big(\alpha-\frac{m(2-\lambda)A}{m-1}+l^{(2-\lambda)(p-1)/(2m-2)}\Big),\quad 0<r<A^{-1/(2-\lambda)}l.
\end{aligned} \tag{4.10}$$

Thanks to (4.9) and (4.10), $\bar{u}$ is a supersolution to (1.4) if $0 < l < l_0$. It is noted that $\lim_{r\to 0^+} r^\lambda (U^m)'(r) = 0$. Hence $\overline{u}$ is a supersolution to problem (1.4)–(1.6) if

$$u_0(x) \le \overline{u}(x,0), \quad x > 0$$

for some $0 < l < l_0$. Thanks to Proposition 2.5, problem (1.4)–(1.6) admits nontrivial global solutions. □

Below we investigate the critical case $p = p_c = m + 2 - \lambda$.

Lemma 4.4. *Assume that $0 < \lambda < 2$, $p = p_c = m + 2 - \lambda$ and $0 < \theta < \frac{p-m}{p-1}$. Let u be a nontrivial global solution to problem* (1.4)–(1.6)*. Then for any $R > 0$,*

$$w_R(t) \le 2^{3/(2-\lambda)}\pi^{2/(2-\lambda)}, \quad t > 0, \tag{4.11}$$

$$w_R'(t) \ge -2^{\lambda-m+2m/(2-\lambda)}\pi^{2+2m/(2-\lambda)}R^{\lambda-m-1}, \quad t > 0, \tag{4.12}$$

$$\begin{aligned} w_R'(t) \ge R^{\lambda-m-1}w_R^{m-\theta}(t)\Big(&-2^{\lambda-m+\theta}\pi^2\Big(\int_R^{2R} u(x,t)\eta_R(x)\mathrm{d}x\Big)^\theta \\ &+ 2^{\lambda-m-2}w_R^{2-\lambda+\theta}(t)\Big), \quad t > 0, \end{aligned} \tag{4.13}$$

where $w_R(t)$ is the function defined by (4.1)*.*

Proof. Since $p = p_c = m + 2 - \lambda$, one obtains from (4.5) that

$$\begin{aligned} w_R'(t) &\ge 2^{-m}R^{-m(p_c-1)/p_c}w_R^m(t)\Big(-2^\lambda\pi^2R^{\lambda-1-m/p_c} \\ &\quad + 2^{-(p_c-m)}R^{-(p_c-1)(p_c-m)/p_c}w_R^{p_c-m}(t)\Big) \\ &= -2^{\lambda-m}\pi^2R^{\lambda-m-1}w_R^m(t) + 2^{\lambda-m-2}R^{\lambda-m-1}w_R^{m+2-\lambda}(t), \quad t > 0. \end{aligned} \tag{4.14}$$

If (4.11) is not true, there exists $t_0 > 0$ such that

$$2^{\lambda-m+1}\pi^2 \le 2^{\lambda-m-2}w_R^{2-\lambda}(t_0).$$

Then, it follows from (4.14) that

$$w_R'(t) \ge 2^{\lambda-m-3}R^{\lambda-m-1}w_R^{m+2-\lambda}(t), \quad t > t_0,$$

which leads to that u blows up in a finite time as the discussion at the end of the proof of Theorem 4.2. Therefore, (4.11) holds.

It follows from (4.14) and the Hölder inequality that

$$\begin{aligned} w_R'(t) &\ge 2^{\lambda-m-2}R^{\lambda-m-1}\Big(-4\pi^2w_R^m(t) + w_R^{m+2-\lambda}(t)\Big) \\ &\ge 2^{\lambda-m-2}R^{\lambda-m-1}\Big(-\frac{m}{m+2-\lambda}w_R^{m+2-\lambda}(t) \\ &\quad - \frac{2-\lambda}{m+2-\lambda}(4\pi^2)^{(m+2-\lambda)/(2-\lambda)} + w_R^{m+2-\lambda}(t)\Big) \\ &\ge -2^{\lambda-m+2m/(2-\lambda)}\pi^{2+2m/(2-\lambda)}R^{\lambda-m-1}, \quad t > 0, \end{aligned}$$

which is (4.12).

Below we prove (4.13). For $t > 0$, it follows from the Hölder inequality that

$$\begin{aligned} &\int_R^{2R} u^m(x,t)\eta_R(x)\mathrm{d}x \\ &\le \Big(\int_R^{2R}\eta_R(x)\mathrm{d}x\Big)^{(p_c-m-\theta(p_c-1))/p_c}\Big(\int_R^{2R}u(x,t)\eta_R(x)\mathrm{d}x\Big)^\theta \end{aligned}$$

$$
\begin{aligned}
&\times \Big(\int_R^{2R} u^{p_c}(x,t)\eta_R(x)\mathrm{d}x \Big)^{(m-\theta)/p_c} \\
&\le R^{(p_c-m-\theta(p_c-1))/p_c} \Big(\int_R^{2R} u(x,t)\eta_R(x)\mathrm{d}x \Big)^{\theta} \Big(\int_0^{+\infty} u^{p_c}(x,t)\eta_R(x)\mathrm{d}x \Big)^{(m-\theta)/p_c},
\end{aligned}
$$

which, together with (4.2), yield

$$
\begin{aligned}
w_R'(t) \ge &\Big(\int_0^{+\infty} u^{p_c}(x,t)\eta_R(x)\mathrm{d}x \Big)^{(m-\theta)/p_c} \\
&\times \Big(-2^{\lambda}\pi^2 R^{\lambda-2+(p_c-m-\theta(p_c-1))/p_c} \Big(\int_R^{2R} u(x,t)\eta_R(x)\mathrm{d}x \Big)^{\theta} \\
&+ \Big(\int_0^{+\infty} u^{p_c}(x,t)\eta_R(x)\mathrm{d}x \Big)^{(p_c-m+\theta)/p_c} \Big).
\end{aligned}
\tag{4.15}
$$

Since $p_c = m+2-\lambda$, it follows that

$$
\lambda - 2 + \frac{p_c - m - \theta(p_c-1)}{p_c} = -\frac{(m+1-\lambda)(2-\lambda+\theta)}{m+2-\lambda}.
$$

Then from (4.4) and (4.15) it follows that

$$
\begin{aligned}
w_R'(t) &\ge 2^{\theta-m} R^{-(m+1-\lambda)(m-\theta)/(m+2-\lambda)} w_R^{m-\theta} \\
&\quad \times \Big(-2^{\lambda}\pi^2 R^{-(m+1-\lambda)(2-\lambda+\theta)/(m+2-\lambda)} \Big(\int_R^{2R} u(x,t)\eta_R(x)\mathrm{d}x \Big)^{\theta} \\
&\quad + 2^{\lambda-2-\theta} R^{-(m+1-\lambda)(2-\lambda+\theta)/(m+2-\lambda)} w_R^{2-\lambda+\theta} \Big) \\
&\ge R^{\lambda-m-1} w_R^{m-\theta}(t) \Big(-2^{\lambda-m+\theta}\pi^2 \Big(\int_R^{2R} u(x,t)\eta_R(x)\mathrm{d}x \Big)^{\theta} \\
&\quad + 2^{\lambda-m-2} w_R^{2-\lambda+\theta}(t) \Big), \quad t>0.
\end{aligned}
$$

This is (4.13). ∎

Theorem 4.5. *Assume that $0<\lambda<2$ and $p = p_c = m+2-\lambda$. For any nontrivial $0 \le u_0 \in L^\infty(0,+\infty) \cap L^1(0,+\infty)$, the solution to problem* (1.4)–(1.6) *must blow up in a finite time.*

Proof. Assumed that u is a global solution to problem (1.4)–(1.6). We denote

$$
\varLambda = \sup_{R>0,\, t>0} w_R(t) = \sup_{t>0} \int_0^{+\infty} u(x,t)\mathrm{d}x. \tag{4.16}
$$

The nontriviality of u and (4.11) yield $0 < \varLambda < +\infty$. Fix $\varepsilon_0 \in (0,\varLambda)$ and $M_0 > 0$ such that

$$
2^{\theta+3}\pi^2(\varepsilon_0 + M_0)^{\theta} \le (\varLambda - \varepsilon_0)^{p_c - m + \theta}, \tag{4.17}
$$

where $0 < \theta < \frac{p-m}{p-1}$ is a constant. By (4.16), there exist $t_0 > 0$ and $R_0 > 0$ such that

$$
w_{R_0}(t_0) \ge \varLambda - \varepsilon_0. \tag{4.18}
$$

For any $t \ge t_0$, it follows from (4.12) with $R = R_0$ and (4.18) that

$$
\begin{aligned}
w_{R_0}(t) &\ge w_{R_0}(t_0) - 2^{\lambda-m+2m/(2-\lambda)}\pi^{2+2m/(2-\lambda)} R_0^{\lambda-m-1}(t-t_0) \\
&\ge \varLambda - \varepsilon_0 - 2^{\lambda-m+2m/(2-\lambda)}\pi^{2+2m/(2-\lambda)} R_0^{\lambda-m-1}(t-t_0), \quad t > t_0,
\end{aligned}
$$

which, together with (4.16), leads to

$$\begin{aligned}
&\int_{2R_0}^{4R_0} u(x,t)\eta_{2R_0}(x)\mathrm{d}x \\
&\le \int_0^{+\infty} u(x,t)\mathrm{d}x - w_{R_0}(t) \\
&\le \varepsilon_0 + 2^{\lambda-m+2m/(2-\lambda)}\pi^{2+2m/(2-\lambda)}R_0^{\lambda-m-1}(t-t_0), \quad t \ge t_0.
\end{aligned} \tag{4.19}$$

Choosing $R = 2R_0$ in (4.13) yields

$$\begin{aligned}
w'_{2R_0}(t) \ge (2R_0)^{\lambda-m-1}w_{2R_0}^{m-\theta}(t)\Big(&-2^{\lambda-m+\theta}\pi^2\Big(\int_{2R_0}^{4R_0} u(x,t)\eta_R(x)\mathrm{d}x\Big)^\theta \\
&+ 2^{\lambda-m-2}w_{2R_0}^{2-\lambda+\theta}(t)\Big), \quad t > t_0,
\end{aligned}$$

which together with (4.16)–(4.19) implies

$$w'_{2R_0}(t) \ge 2^{2(\lambda-m-2)}R_0^{\lambda-m-1}(\Lambda-\varepsilon_0)^{m+2-\lambda}, \quad t_0 < t < t_1, \tag{4.20}$$

where

$$t_1 = t_0 + 2^{m-\lambda-2m/(2-\lambda)}\pi^{-2-2m/(2-\lambda)}M_0R_0^{1+m-\lambda}.$$

It follows from (4.18) and(4.20) that

$$\begin{aligned}
w_{2R_0}(t_1) &\ge w_{2R_0}(t_0) + 2^{2(\lambda-m-2)}R_0^{\lambda-m-1}(\Lambda-\varepsilon_0)^{m+2-\lambda}(t_1-t_0) \\
&\ge \Lambda - \varepsilon_0 + \sigma_0,
\end{aligned} \tag{4.21}$$

where

$$\sigma_0 = 2^{\lambda-m-4-2m/(2-\lambda)}\pi^{-2-2m/(2-\lambda)}M_0(\Lambda-\varepsilon_0)^{m+2-\lambda}.$$

Applying (4.12) with $R = 2R_0$ and (4.21), one obtains

$$\begin{aligned}
w_{2R_0}(t) &\ge w_{2R_0}(t_1) - 2^{\lambda-m+2m/(2-\lambda)}\pi^{2+2m/(2-\lambda)}(2R_0)^{\lambda-m-1}(t-t_1) \\
&\ge \Lambda - \varepsilon_0 - 2^{\lambda-m+2m/(2-\lambda)}\pi^{2+2m/(2-\lambda)}(2R_0)^{\lambda-m-1}(t-t_1), \quad t > t_1,
\end{aligned}$$

which, together with (4.16) with $R = 2R_0$, leads to

$$\begin{aligned}
&\int_{4R_0}^{8R_0} u(x,t)\eta_{4R_0}(x)\mathrm{d}x \\
&\le \int_0^{+\infty} u(x,t)\mathrm{d}x - w_{2R_0}(t) \\
&\le \varepsilon_0 + 2^{\lambda-m+2m/(2-\lambda)}\pi^{2+2m/(2-\lambda)}(2R_0)^{\lambda-m-1}(t-t_1), \quad t \ge t_1.
\end{aligned} \tag{4.22}$$

Taking $R = 4R_0$ in (4.13), one obtains

$$\begin{aligned}
w'_{4R_0}(t) \ge (4R_0)^{\lambda-m-1}w_{4R_0}^{m-\theta}(t)\Big(&-2^{\lambda-m+\theta}\pi^2\Big(\int_{4R_0}^{8R_0} u(x,t)\eta_R(x)\mathrm{d}x\Big)^\theta \\
&+ 2^{\lambda-m-2}w_{4R_0}^{2-\lambda+\theta}(t)\Big), \quad t > t_1,
\end{aligned} \tag{4.23}$$

Thanks to (4.12)–(4.23), one obtains

$$w'_{4R_0}(t) \ge 2^{2(\lambda-m-2)}(2R_0)^{\lambda-m-1}(\Lambda-\varepsilon_0)^{m+2-\lambda}, \quad t_1 < t < t_2, \tag{4.24}$$

where

$$t_2 = t_1 + 2^{m-\lambda-2m/(2-\lambda)}\pi^{-2-2m/(2-\lambda)}M_0(2R_0)^{1+m-\lambda}.$$

It follows from (4.12) and (4.24) that

$$w_{4R_0}(t_2) \geq w_{4R_0}(t_1) + \sigma_0 \geq w_{2R_0}(t_1) + \sigma_0 \geq \mathit{\Lambda} - \varepsilon_0 + 2\sigma_0.$$

Repeating the procedure in turn, one obtains that for any positive integer i,

$$w_{2^i R_0}(t_i) \geq w_{2^i R_0}(t_{i-1}) + \sigma_0 \geq w_{2^{i-1} R_0}(t_{i-1}) + \sigma_0 \geq \mathit{\Lambda} - \varepsilon_0 + i\sigma_0,$$

where

$$t_{i+1} = t_i + 2^{m-\lambda-2m/(2-\lambda)} \pi^{-2-2m/(2-\lambda)} M_0 (2^{i-1} R_0)^{1+m-\lambda}.$$

Therefore,

$$\sup_{t>0} \int_0^{+\infty} u(x,t) \mathrm{d}x = +\infty,$$

which contradicts (4.16). □

Acknowledgments. This research was supported by National Key R & D Program of China (No. 2020YFA0714101), and the National Natural Science Foundation of China (Nos. 11925105, 11871133 and 12001227).

References

[1] J. Aguirre, M. Escobedo; *On the blow-up of solutions of a convective reaction diffusion equation*, Proc. Roy. Soc. Edinburgh Sect. A, 123 (3) (1993), 433–460.

[2] F. Alabau-Boussouira, P. Cannarsa, G. Fragnelli; *Carleman estimates for degenerate parabolic operators with applications to null controllability*, J. Evol. Equ., 6(2) (2006), 161–204.

[3] D. Andreucci, G. Cirmi, S. Leonardi, A. Tedeev; *Large time behavior of solutions to the Neumann problem for a quasilinear second order degenerate parabolic equation in domains with noncompact boundary*, J. Differential Equations, 174 (2001), 253–288.

[4] F. Black, M. Scholes; *The pricing of options and corporate liabilities*, J. Polit. Econ., 81(3) (1973), 637–654.

[5] P. Cannarsa, P. Martinez, J. Vancostenoble; *Persistent regional null controllability for a class of degenerate parabolic equations*, Commun. Pure Appl. Anal., 3(4) (2004), 607–635.

[6] P. Cannarsa, P. Martinez, J. Vancostenoble; *Carleman estimates for a class of degenerate parabolic operators*, SIAM J. Control Optim., 47(1) (2008), 1–19.

[7] P. Cannarsa, P. Martinez, J. Vancostenoble; *Null controllability of degenerate heat equations*, Adv. Differential Equations, 10(2) (2005), 153–190.

[8] P. Cannarsa, L. de Teresa; *Controllability of 1-D coupled degenerate parabolic equations*, Electron. J. Differential Equations, 2009 (2009), Paper No. 73, 21 pp.

[9] K. Deng, H. Levine; *The role of critical exponents in blow-up theorems: the sequel*, J. Math. Anal. Appl., 243(1) (2000), 85–126.

[10] R. Du, J. Eichhorn, Q. Liu, C. Wang; *Carleman estimates and null controllability of a class of singular parabolic equations*, Adv. Nonlinear Anal., 8(1) (2019), 1057–1082.

[11] R. Du, C. Wang; *Null controllability of a class of systems governed by coupled degenerate equations*, Appl. Math. Lett., 26(1) (2013), 113–119.

[12] M. Fira, B. Kawohl; *Large time behavior of solutions to a quasilinear parabolic equation with a nonlinear boundary condition*, Adv. Math. Sci. Appl., 11(1) (2001), 113–126.

[13] H. Fujita; *On the blowing up of solutions of the Cauchy problem for* $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo Sect. I, 13 (1966), 109–124.

[14] K. Hayakawa; *On nonexistence of global solutions of some semilinear parabolic equations*, Proc. Japan Acad., 49 (1973), 503–525.

[15] X. Jing, Y. Nie, C. Wang; *Asymptotic behavior of solutions to coupled semilinear parabolic systems with boundary degeneracy*, Electron. J. Differential Equations, 2021 (2021), Paper No. 67, 17 pp.

[16] K. Kobayashi, T. Siaro, H. Tanaka; *On the blowing up problem for semilinear heat equations*, J. Math. Soc. Japan, 29 (1977), 407–424.

[17] H. Levine; *The role of critical exponents in blow-up theorems*, SIAM Rev., 32(2) (1990), 262–288.

[18] H. Levine, Q. Zhang; *The critical Fujita number for a semilinear heat equation in exterior domains with homogeneous Neumann boundary values*, Proc. Roy. Soc. Edinburgh Sect. A, 130(3) (2000), 591–602.
[19] P. Martinez, J. Vancostenoble; *Carleman estimates for one-dimensional degenerate heat equations*, J. Evol. Equ., 6(2) (2006), 325–362.
[20] G. North, L. Howard, D. Pollard, B. Wielicki; *Variational formulation of Budyko-Sellers climate models*, J. Atmos. Sci., 36 (1979), 255–259.
[21] R. Suzuki; *Existence and nonexistence of global solutions to quasilinear parabolic equations with convection*, Hokkaido Math. J., 27(1) (1998), 147–196.
[22] C. Wang; *Approximate controllability of a class of semilinear systems with boundary degeneracy*, Journal of Evolution Equations, 10(1) (2010), 163–193.
[23] C. Wang; *Asymptotic behavior of solutions to a class of semilinear parabolic equations with boundary degeneracy*, Proc. Amer. Math. Soc., 141(9) (2013), 3125–3140.
[24] C. Wang, S. Zheng, Z. Wang; *Critical Fujita exponents for a class of quasilinear equations with homogeneous Neumann boundary data*, Nonlinearity, 20(6) (2007), 1343–1359.
[25] C. Wang, S. Zheng; *Critical Fujita exponents of degenerate and singular parabolic equations*, Proc. Roy. Soc. Edinburgh Sect. A, 136(2) (2006), 415–430.
[26] C. Wang, R. Du; *Carleman estimates and null controllability for a class of degenerate parabolic equations with convection terms*, SIAM J. Control Optim., 52(3) (2014), 1457–1480.
[27] Z. Wu, J. Zhao, J. Yin, H. Li; *Nonlinear Diffusion Equations World Scientific*, 2001.
[28] J. Xu, C. Wang, Y. Nie; *Carleman estimate and null controllability of a cascade degenerate parabolic system with general convection terms*, Electron. J. Differential Equations, 2018 (2018), no. 195, 20 pp.
[29] M. Zhou, C. Wang, Y. Nie; *Quenching of solutions to a class of semilinear parabolic equations with boundary degeneracy*, Journal of Mathematical Analysis and Applications, 421(1) (2015), 59–74.

Xutong Zhao
School of Mathematics, Jilin University, Changchun 130012, China
Email address: 847692570@qq.com

Mingjun Zhou (corresponding author)
School of Mathematics, Jilin University, Changchun 130012, China
Email address: zhoumingjun@jlu.edu.cn

Xinxin Jing
School of Mathematics, Jilin University, Changchun 130012, China
Email address: 1776043712@qq.com