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TWO-DIMENSIONAL KELLER-SEGEL MODEL: OPTIMAL CRITICAL MASS AND QUALITATIVE PROPERTIES OF THE SOLUTIONS

ADRIEN BLANCHET, JEAN DOLBEAULT, BENOÎT PERTHAME

Abstract. The Keller-Segel system describes the collective motion of cells which are attracted by a chemical substance and are able to emit it. In its simplest form it is a conservative drift-diffusion equation for the cell density coupled to an elliptic equation for the chemo-attractant concentration. It is known that, in two space dimensions, for small initial mass, there is global existence of solutions and for large initial mass blow-up occurs. In this paper we complete this picture and give a detailed proof of the existence of weak solutions below the critical mass, above which any solution blows-up in finite time in the whole Euclidean space. Using hypercontractivity methods, we establish regularity results which allow us to prove an inequality relating the free energy and its time derivative. For a solution with sub-critical mass, this allows us to give for large times an "intermediate asymptotics" description of the vanishing. In self-similar coordinates, we actually prove a convergence result to a limiting self-similar solution which is not a simple reflect of the diffusion.

1. Introduction

The Keller-Segel system for chemotaxis describes the collective motion of cells, usually bacteria or amoebae, that are attracted by a chemical substance and are able to emit it. For a general introduction to chemotaxis, see [43, 41]. Various versions of the Keller-Segel system for chemotaxis are available in the literature, depending on the phenomena and scales one is interested in. We refer the reader to the very nice review papers [30, 31] and references therein. The complete Keller-Segel model is a system of two parabolic equations. In this paper, we consider only the simplified two-dimensional case and assume that the equations take the form

$$\begin{gathered}
\frac{\partial n}{\partial t} = \Delta n - \chi \nabla \cdot (n \nabla c) \quad x \in \mathbb{R}^2,\ t > 0\,, \\
-\Delta c = n \quad x \in \mathbb{R}^2,\ t > 0\,, \\
n(\cdot, t = 0) = n_0 \ge 0 \quad x \in \mathbb{R}^2\,.
\end{gathered} \tag{1.1}$$

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Here $n(x,t)$ represents the cell density, and $c(x,t)$ is the concentration of chemo-attractant which induces a drift force. A classical parameter of the system is the *sensitivity* $\chi > 0$ of the bacteria to the chemo-attractant which measures the non-linearity in the system. Here χ is a constant. Such a parameter can be removed by a scaling, to the price of a change of the total mass of the system

$$M := \int_{\mathbb{R}^2} n_0 \, dx \,.$$

In bounded domains, it is usual to impose no-flux boundary conditions. Here we are not interested in boundary effects and for this reason we are going to consider the system in the full space $\mathbb{R}^2$, without boundary conditions. There are related models in gravitation which are defined in $\mathbb{R}^3$, see, *e.g.,* [11]. The relevant case for chemotaxis is rather the two-dimensional space, although some three-dimensional versions of the model also make sense. The L^1-norm is critical in the sense that there exists a critical mass above which all solution blow-up in finite time, see [33], and below which they globally exist (see [22] for the *a priori* estimates and Theorem 1.1 below for an existence statement). The critical space is $L^{d/2}(\mathbb{R}^d)$ for $d \geq 2$, see [19, 20] and the references therein. In dimension $d = 2$, the Green kernel associated to the Poisson equation is a logarithm, namely $c = -\frac{1}{2\pi} \log |\cdot| * n$. When the Poisson interaction is replaced by a convolution kernel, it is the logarithmic singularity which is critical for the L^1-norm whatever the dimension is, see [15].

Historically the key papers for this family of models are the original contribution [34] of Keller and Segel, and a work by Patlak, [50]. The rigourous derivation of the Keller-Segel system from an interacting stochastic many-particles system has been done in [58]. Simulations of these can be found in [48]. A very interesting justification of the Keller-Segel model as a diffusion limit of a kinetic model has recently been published, see [17]. Related models with prevention of overcrowding, see [28, 12], volume effects [35, 14, 65], or involving more than one chemo-attractant have also been studied.

As conjectured by Childress and Percus [18] and Nanjundiah [47] either the solution of the complete Keller-Segel system globally exists or it blows-up in finite time, a phenomenon called *chemotactic collapse* in the literature. As we shall see, this classification is valid for the simplified Keller-Segel system (1.1). A large series of results, mostly in the bounded domain case, has been obtained by Nagai, Senba and Suzuki. Many of these results can be found in [56, 59]. Concerning blow-up phenomena, a key contribution has been brought by Herrero and Velázquez [27, 62]. Also see [42] for numerical computations.

The main tool in this paper is the *free energy*

$$F[n] := \int_{\mathbb{R}^2} n \log n \, dx - \frac{\chi}{2} \int_{\mathbb{R}^2} nc \, dx$$

which provides useful *a priori* estimates. The free energy functional is a well known tool for gravitational models, see [3, 49, 64, 10] and has been introduced for chemotactic models by T. Nagai, T. Senba and K. Yoshida in [45], by P. Biler in [7] and by H. Gajewski and K. Zacharias in [23].

Based on the logarithmic Hardy-Littlewood-Sobolev inequality in its sharp form as established in [16, 4], the free energy is bounded from below if $\chi M \leq 8\pi$, see [22]. Here we use this estimate to prove the global existence of solutions of (1.1) if $\chi M < 8\pi$. We also prove that for these solutions, the free energy is decaying and use it to study the large time behaviour of the solutions. The limiting case $\chi M = 8\pi$ has recently been studied in the radial case, see [8, 9].

The literature on the Keller-Segel model is huge and it is out of the scope of this paper to give a complete bibliography. Some additional papers will be quoted in the text. Otherwise, we suggest the interested reader to primarily refer to the surveys [52, 30, 31].

Our first main result is the following existence and regularity statement.

Theorem 1.1. *Assume that $n_0 \in L^1_+(\mathbb{R}^2, (1+|x|^2)\,dx)$ and $n_0 \log n_0 \in L^1(\mathbb{R}^2, dx)$. If $M < 8\pi/\chi$, then the Keller-Segel system* (1.1) *has a global weak non-negative solution n with initial data n_0 such that*

$$
\begin{gathered}
(1+|x|^2+|\log n|)n \in L^\infty_{\rm loc}(\mathbb{R}^+, L^1(\mathbb{R}^2))\,,\\
\int_0^t \int_{\mathbb{R}^2} n|\nabla \log n - \chi\nabla c|^2 dxdt < \infty\,,\\
\int_{\mathbb{R}^2} |x|^2 n(x,t)\,dx = \int_{\mathbb{R}^2} |x|^2 n_0(x)\,dx + 4M\big(1-\frac{\chi M}{8\pi}\big)t
\end{gathered}
$$

for $t>0$. Moreover $n \in L^\infty_{\rm loc}((\varepsilon,\infty), L^p(\mathbb{R}^2))$ for any $p \in (1,\infty)$ and any $\varepsilon > 0$, and the following inequality holds for any $t > 0$:

$$
F[n(\cdot,t)] + \int_0^t \int_{\mathbb{R}^2} n\,|\nabla(\log n) - \chi\nabla c|^2\;dx\,ds \le F[n_0]\,. \tag{1.2}
$$

This result was partially announced in [22]. Compared to [22], the main novelty is that we prove the free energy inequality (1.2) and get the hypercontractive estimate: $n(\cdot,t)$ is bounded in $L^p(\mathbb{R}^2)$ for any $p\in(1,\infty)$ and almost any $t>0$. The equation holds in the distributions sense. Indeed, writing

$$
\Delta n - \chi\nabla\cdot(n\nabla c) = \nabla\cdot[n(\nabla\log n - \chi\nabla c)]\,,
$$

we can see that the flux is well defined in $L^1(\mathbb{R}^+_{\rm loc}\times\mathbb{R}^2)$ since

$$
\begin{aligned}
&\iint_{[0,T]\times\mathbb{R}^2} n|\nabla\log n - \chi\nabla c|\,dx\,dt\\
&\le \Big(\iint_{[0,T]\times\mathbb{R}^2} n\,dx\,dt\Big)^{1/2}\Big(\iint_{[0,T]\times\mathbb{R}^2} n|\nabla\log n-\chi\nabla c|^2\,dx\,dt\Big)^{1/2} < \infty\,.
\end{aligned}
$$

Our second main result deals with large time behavior, intermediate asymptotics and convergence to asymptotically self-similar profiles given in the rescaled variables by the equation

$$
u_\infty = M\frac{e^{\chi v_\infty - |x|^2/2}}{\int_{\mathbb{R}^2} e^{\chi v_\infty-|x|^2/2}\,dx} = -\Delta v_\infty\,,\quad\text{with}\quad v_\infty = -\frac{1}{2\pi}\log|\cdot| * u_\infty\,. \tag{1.3}
$$

In the original variables, the self-similar solutions of (1.1) take the expression:

$$
\begin{aligned}
n_\infty(x,t) &:= \frac{1}{1+2t}u_\infty\left(\log(\sqrt{1+2t}), x/\sqrt{1+2t}\right)\,,\\
c_\infty(x,t) &:= v_\infty\left(\log(\sqrt{1+2t}), x/\sqrt{1+2t}\right)\,.
\end{aligned}
$$

This allows us to state our second main result, on intermediate asymptotics.

Theorem 1.2. *Under the assumptions in Theorem 1.1, there exists a solution of* (1.3) *such that*

$$\lim_{t\to\infty} \|n(\cdot,t) - n_\infty(\cdot,t)\|_{L^1(\mathbb{R}^2)} = 0\,, \quad \lim_{t\to\infty} \|\nabla c(\cdot,t) - \nabla c_\infty(\cdot,t)\|_{L^2(\mathbb{R}^2)} = 0\,.$$

This paper is organized as follows. Section 2 is devoted to the detailed proof of the existence of weak solutions with subcritical mass and without any symmetry assumption. *A priori* estimates have been derived in [22]. The point here is to establish the result with all necessary details: regularized problem, uniform estimates, passage to the limit in the regularization parameter. Compared to [22], we also establish Inequality (1.2). Proving such an inequality requires a detailed study of the regularity properties of the solutions, which is done in Section 3: By hypercontractivity methods, we prove that the solution is bounded in any L^p space for almost any positive t. Using the free energy we study in Section 4 the asymptotic behavior of the solutions for large times and prove Theorem 1.2. The main difficulty comes from the fact that the uniqueness of the solutions to (1.3) for a given $M \in (0, 8\pi/\chi)$ is not known, although many additional properties (radial symmetry, regularity, decay at infinity) of the limiting solution in the self-similar variables are known.

2. Existence for sub-critical masses

We assume that the initial data satisfies the following assumptions:

$$n_0 \in L^1_+(\mathbb{R}^2, (1+|x|^2)\,dx)\,, \quad n_0 \log n_0 \in L^1(\mathbb{R}^2, dx)\,. \tag{2.1}$$

Because of the divergence form of the right hand side of the equation for n, the total mass is conserved at least for smooth and sufficiently decay solutions

$$M := \int_{\mathbb{R}^2} n_0(x)\,dx = \int_{\mathbb{R}^2} n(x,t)\,dx\,. \tag{2.2}$$

Our purpose here is first to give a complete existence theory in the subcritical case, *i.e.* in the case

$$M < 8\pi/\chi\,.$$

This result has been announced in [22], which was dealing only with *a priori* estimates. Here, we give the proofs with all details. More precisely, we prove that under Assumption (2.1), there are only two cases:

Case 1. Solutions to (1.1) blow-up in finite time when $M > 8\pi/\chi$,
Case 2. There exists a global in time solution of (1.1) when $M < 8\pi/\chi$.

The case $M = 8\pi/\chi$ is delicate and for radial solutions, some results have been obtained recently, see [8, 9].

Our existence theory completes the partial picture established in [33]. The solution of the Poisson equation $-\Delta c = n$ is given up to an harmonic function. From the beginning, we have in mind that the concentration of the chemo-attractant is defined by

$$c(x,t) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \log|x-y|\,n(y,t)\,dy\,. \tag{2.3}$$

There are other possible solutions, which may result in significantly different qualitative behaviors, as we shall see in Section 4.2. From now on, unless it is explicitly specified, we will only consider concentrations c given by (2.3). In the following sections, 2.1, 2.2 and 2.3, we closely follow the presentation given in [22].

2.1. **Blow-up for super-critical masses.** The case $M > 8\pi/\chi$ (Case 1) is easy to understand using moments estimates. The method is classical and has been repeatedly used for various similar problems. See for instance [51] in the similar context of the Euler-Poisson system, [32]. Concerning blow-up, we refer to [19, 20, 32] for recent references on the subject.

Following for instance [54], we can define a notion of weak solution n in the space $L^\infty_{\rm loc}\big(\mathbb{R}^+; L^1(\mathbb{R}^2)\big)$ using the symmetry in x, y for the concentration gradient, which has interest in case of blow-up. We shall say that n is a solution to (1.1) if for all test functions $\psi \in \mathcal{D}(\mathbb{R}^2)$,

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2} \psi(x) n(x,t)\,dx \\
&= \int_{\mathbb{R}^2} \Delta\psi(x) n(x,t)\,dx - \frac{\chi}{4\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} [\nabla\psi(x) - \nabla\psi(y)]\cdot\frac{x-y}{|x-y|^2} n(x,t)n(y,t)dx\,dy\,.
\end{aligned}
$$

Compared to standard distribution solutions, this is an improved concept that can handle measure solutions because the term $[\nabla\psi(x) - \nabla\psi(y)].\frac{x-y}{|x-y|^2}$ is continuous.

Lemma 2.1. *Consider a non-negative L^1 solution n to (1.1) in the above sense, on an interval $[0,T]$ and assume that n satisfies (2.2), $\int_{\mathbb{R}^2} |x|^2 n_0(x)\,dx < \infty$. Then it also satisfies*

$$
\frac{d}{dt}\int_{\mathbb{R}^2} |x|^2 n(x,t)\,dx = 4M\big(1 - \frac{\chi M}{8\pi}\big)\,.
$$

Proof. Consider a smooth function $\varphi_\varepsilon(|x|)$ with compact support that grows nicely to $|x|^2$ as $\varepsilon \to 0$. Then we use the definition of weak solutions and get

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2} \varphi_\varepsilon n\,dx \\
&= \int_{\mathbb{R}^2} \Delta\varphi_\varepsilon n\,dx - \frac{\chi}{4\pi}\int_{\mathbb{R}^2} \frac{(\nabla\varphi_\varepsilon(x) - \nabla\varphi_\varepsilon(y))\cdot(x-y)}{|x-y|^2} n(x,t)n(y,t)\,dxdy\,.
\end{aligned}
$$

Because we can always choose $\Delta\varphi_\varepsilon$ bounded and $\nabla\varphi_\varepsilon(x)$ Lipschitz continous, we deduce that

$$
\frac{d}{dt}\int_{\mathbb{R}^2} \varphi_\varepsilon n\,dx \le C\int_{\mathbb{R}^2} n_0\,dx\,,
$$

where C is some positive constant. As $\varepsilon \to 0$ we find that

$$
\int_{\mathbb{R}^2} \varphi_\varepsilon n\,dx \le c_1 + c_2 t\,,
$$

where c_1 and c_2 are two positive constants and thus

$$
\int_{\mathbb{R}^2} |x|^2 n(x,t)\,dx < \infty \quad \forall\, t \in (0,T)\,.
$$

We can pass to the limit using Lebesgue's dominated convergence theorem and thus complete the proof of Lemma 2.1. $\square$

As a consequence, we recover the statement of Case 1, namely that for $M > 8\pi/\chi$, there is a finite blow-up time T^* where solutions become singular measures.

Corollary 2.2. *Consider a non-negative solution n as in Lemma 2.1 and let $[0,T^*)$ be the maximal interval of existence. Assume that the initial data $n_0 \in L^1(\mathbb{R}^2)$ is such that $I_0 := \int_{\mathbb{R}^2} |x|^2 n_0(x)\,dx < \infty$. Then either $T^* = \infty$ or $n(\cdot,t)$ converges* *(up to extraction of sequences) as $t \to T^*$ to a measure which is not in $L^1(\mathbb{R}^2)$. If $\chi M > 8\pi$, then*

$$T^* \leq \frac{2\pi I_0}{M(\chi M - 8\pi)}\,.$$

As far as we know, it is an open question to decide whether the solutions of (1.1) with $\chi M > 8\pi$ and $I_0 = \infty$ also blow-up in finite time. Blow-up statements in bounded domains are available, see [44, 10, 29, 37, 55] and the references therein. When the solution is radially symmetric in x, the second x-moment is not needed and the blow-up profile has been explicited, namely

$$n(x,t) \to \frac{8\pi}{\chi}\delta + \tilde{n}(x,t) \quad \text{as } t \nearrow T^*\,,$$

where $\tilde{n}$ is a $L^1(\mathbb{R}^2 \times \mathbb{R}^+)$ radial function such that $t \mapsto \tilde{n}(\cdot,t)$ is measure valued, see [27, 60]. Except that solutions blow-up for large mass, in the general case very little is known on the blow-up profile (see [54] for concentrations estimates, [42] for numerical computations). Asymptotic expansions at blow-up and continuation of solutions after blow-up have been studied by Velázquez in [62, 61]. The case $\chi M = 8\pi$ has recently been investigated by Biler, Karch, Laurençot and Nadzieja in [8]. In a forthcoming paper, they prove that in the whole space case and $\chi M = 8\pi$, blow-up occurs only for infinite time, [9]. Here we will focus on the subcritical regime and prove that solutions exist and are always asymptotically vanishing for large times.

If the problem is set in dimension $d \geq 3$, the critical norm is $L^p(\mathbb{R}^d)$ with $p = d/2$. In dimension $d = 2$, the value of the mass M is therefore natural to discriminate between super- and sub-critical regimes. However, the limit of the L^p-norm is rather $\int_{\mathbb{R}^2} n \log n\, dx$ than $\int_{\mathbb{R}^2} n\, dx$, which is preserved by the evolution. This explains why it is natural to introduce the entropy, or better, as we shall see below, the *free energy.*

2.2. **The usual existence proof for not too large masses.** The usual proof of existence is due to Jäger and Luckhaus in [33]. Here we follow the variant [19, 20] which is based on the following computation. Consider the equation for n and compute $\frac{d}{dt}\int_{\mathbb{R}^2} n \log n\, dx$. Using an integration by parts and the equation for c, we obtain:

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^2} n \log n\, dx &= -4\int_{\mathbb{R}^2} \left|\nabla\sqrt{n}\right|^2 dx + \chi \int_{\mathbb{R}^2} \nabla n \cdot \nabla c\, dx \\ &= -4\int_{\mathbb{R}^2} \left|\nabla\sqrt{n}\right|^2 dx + \chi \int_{\mathbb{R}^2} n^2\, dx\,.\end{aligned}$$

This shows that two terms compete, namely the diffusion based entropy dissipation term $\int_{\mathbb{R}^2} |\nabla\sqrt{n}|^2\, dx$ and the hyperbolic production of entropy.

Thus the entropy is nonincreasing if $\chi M \leq 4C_{\rm GNS}^{-2}$, where $C_{\rm GNS} = C_{\rm GNS}^{(4)}$ is the best constant for $p = 4$ in the Gagliardo-Nirenberg-Sobolev inequality:

$$\|u\|^2_{L^p(\mathbb{R}^2)} \leq C_{\rm GNS}^{(p)} \|\nabla u\|^{2-4/p}_{L^2(\mathbb{R}^2)} \|u\|^{4/p}_{L^2(\mathbb{R}^2)} \quad \forall\, u \in H^1(\mathbb{R}^2)\,, \quad \forall\, p \in [2,\infty)\,. \tag{2.4}$$

The explicit value of $C_{\rm GNS}$ is not known but can be computed numerically (see [63]) and one finds that the entropy is nonincreasing if $\chi M \leq 4C_{\rm GNS}^{-2} \approx 1.862\cdots \times (4\pi) < 8\pi$. Such an estimate is therefore not sufficient to cover the whole range of M for global existence in the second case.

In [33] it is also shown that equiintegrability (deduced from the $n\log n$ estimate for instance) is enough to propagate any L^p initial norm. We will come back on this point in Section 2.7 and prove later that due to the regularizing effects, the solution is bounded in time with values in $L^p(\mathbb{R}^2)$ for all positive times.

2.3. **A free energy method and a priori estimates.** To obtain sharper estimates and prove a global existence result (Case 2), we use the *free energy* which has already been introduced in Section 1:

$$F[n] := \int_{\mathbb{R}^2} n\log n\,dx - \frac{\chi}{2}\int_{\mathbb{R}^2} nc\,dx\,.$$

See [7, 23, 45] in the case of a bounded domain. The first term in F is the *entropy* and the second one a *potential energy term.* Such a free energy enters in the general notion of entropies, and this is why it is sometimes referred to the method as the "entropy method", although the notion of *free energy* is physically more appropriate. See [1] for an historical review on these notions. For any solution n of (1.1), $F[n(\cdot,t)]$ is monotone nonincreasing.

Lemma 2.3. *Consider a non-negative $C^0(\mathbb{R}^+,L^1(\mathbb{R}^2))$ solution n of (1.1) such that $n(1+|x|^2)$, $n\log n$ are bounded in $L^\infty_{\rm loc}(\mathbb{R}^+,L^1(\mathbb{R}^2))$, $\nabla\sqrt{n}\in L^1_{\rm loc}(\mathbb{R}^+,L^2(\mathbb{R}^2))$ and $\nabla c\in L^\infty_{\rm loc}(\mathbb{R}^+\times\mathbb{R}^2)$. Then*

$$\frac{d}{dt}F[n(\cdot,t)] = -\int_{\mathbb{R}^2} n\left|\nabla\left(\log n\right) - \chi\nabla c\right|^2\,dx\,. \tag{2.5}$$

Following standard denomination in PDE's, we will call $\int_{\mathbb{R}^2} n|\nabla(\log n)-\chi\nabla c|^2\,dx$ the *free energy production term* or *generalized relative Fisher information.*

Proof. Because the potential energy term $\int_{\mathbb{R}^2} nc\,dx = \iint_{\mathbb{R}^2\times\mathbb{R}^2} n(x,t)n(y,t)\log|x-y|\,dxdy$ is quadratic in n, using Equation (1.1), the time derivative of $F[n(\cdot,t)]$ is given by

$$\frac{d}{dt}F[n(\cdot,t)] = \int_{\mathbb{R}^2}\Big[\Big(1+\log n-\chi c\Big)\,\nabla\cdot\Big(\frac{\nabla n}{n}-\chi\nabla c\Big)\Big]\,dx\,.$$

An integration by parts completes the proof. $\square$

¿From the representation (2.3) of the solution to the Poisson equation, we deduce that

$$\frac{d}{dt}F[n(\cdot,t)] = \frac{d}{dt}\left[\int_{\mathbb{R}^2} n\log n\,dx + \frac{\chi}{4\pi}\iint_{\mathbb{R}^2\times\mathbb{R}^2} n(x,t)n(y,t)\log|x-y|\,dxdy\right] \le 0\,.$$

On the other hand, we recall the logarithmic Hardy-Littlewood-Sobolev inequality.

Lemma 2.4 ([16, 4])**.** *Let f be a non-negative function in $L^1(\mathbb{R}^2)$ such that $f\log f$ and $f\log(1+|x|^2)$ belong to $L^1(\mathbb{R}^2)$. If $\int_{\mathbb{R}^2} f\,dx = M$, then*

$$\int_{\mathbb{R}^2} f\log f\,dx + \frac{2}{M}\iint_{\mathbb{R}^2\times\mathbb{R}^2} f(x)f(y)\log|x-y|\,dxdy \ge -\,C(M)\,, \tag{2.6}$$

with $C(M) := M(1+\log\pi-\log M)$.

This allows to prove *a priori* estimates on the two terms involved in the free energy.

Lemma 2.5. *Consider a non-negative $C^0(\mathbb{R}^+, L^1(\mathbb{R}^2))$ solution n of (1.1) such that $n(1+|x|^2)$, $n\log n$ are bounded in $L^\infty_{\rm loc}(\mathbb{R}^+, L^1(\mathbb{R}^2))$, $\int_{\mathbb{R}^2} \frac{1+|x|}{|x-y|} n(y,t)\,dy \in L^\infty\big((0,T)\times\mathbb{R}^2\big)$, $\nabla\sqrt{n} \in L^1_{\rm loc}(\mathbb{R}^+, L^2(\mathbb{R}^2))$ and $\nabla c \in L^\infty_{\rm loc}(\mathbb{R}^+\times\mathbb{R}^2)$. If $\chi M \le 8\pi$, then the following estimates hold:*

(i) Entropy:
$$M\log M - M\log[\pi(1+t)] - K \le \int_{\mathbb{R}^2} n\log n\,dx \le \frac{8\pi F_0 + \chi M C(M)}{8\pi - \chi M}$$
with $K := \max\big\{\int_{\mathbb{R}^2}|x|^2 n_0(x)\,dx, \frac{M}{2\pi}\,(8\pi-\chi M)\big\}$ *and* $F_0 := F[n_0]$.

(ii) Fisher information: *For all $t>0$, with $C_1 := F_0 + \frac{\chi M}{8\pi} C(M)$ and $C_2 := \frac{\chi M - 8\pi}{8\pi}$,*
$$\begin{aligned} 0 &\le \int_0^t ds \int_{\mathbb{R}^2} n(x,s)\left|\nabla\left(\log n(x,s)\right) - \chi\nabla c(x,s)\right|^2 dx \\ &\le C_1 + C_2\big[M\log\big(\frac{\pi(1+t)}{M}\big) + K\big] \end{aligned}$$

Proof. From (2.5), with $n(\cdot) = n(\cdot,t)$ for shortness, we get that the quantity
$$(1-\theta)\int_{\mathbb{R}^2} n\log n\,dx + \theta\Big[\int_{\mathbb{R}^2} n\log n\,dx + \frac{\chi}{4\pi\theta}\iint_{\mathbb{R}^2\times\mathbb{R}^2} n(x)n(y)\log|x-y|\,dxdy\Big]$$
is bounded from above by F_0. We choose
$$\frac{\chi}{4\pi\theta} = \frac{2}{M} \quad\Longleftrightarrow\quad \theta = \frac{\chi M}{8\pi}$$
and apply (2.6):
$$(1-\theta)\int_{\mathbb{R}^2} n(x,t)\log n(x,t)\,dx - \theta C(M) \le F_0\,.$$
If $\chi M < 8\pi$, then $\theta < 1$ and
$$\int_{\mathbb{R}^2} n(x,t)\log n(x,t)\,dx \le \frac{F_0 + \theta C(M)}{1-\theta}\,.$$
This estimate proves the upper bound for the entropy. We can also see that $\int_{\mathbb{R}^2} n\log n\,dx$ is bounded from below. By Lemma 2.1,
$$\frac{1}{1+t}\int_{\mathbb{R}^2}|x|^2 n(x,t)\,dx \le K \quad \forall\, t>0\,.$$
Thus
$$\begin{aligned} \int_{\mathbb{R}^2} n(x,t)\log n(x,t) &\ge \frac{1}{1+t}\int_{\mathbb{R}^2}|x|^2 n(x,t)\,dx - K + \int_{\mathbb{R}^2} n(x,t)\log n(x,t)\,dx \\ &= \int_{\mathbb{R}^2} n(x,t)\log\Big(\frac{n(x,t)}{e^{-\frac{|x|^2}{1+t}}}\Big)\,dx - K \\ &= \int_{\mathbb{R}^2} n(x,t)\log\Big(\frac{n(x,t)}{\mu(x,t)}\Big)\,dx - M\log[\pi(1+t)] - K \\ &= \int_{\mathbb{R}^2} \frac{n(x,t)}{\mu(x,t)}\log\Big(\frac{n(x,t)}{\mu(x,t)}\Big)\mu(x,t)\,dx - M\log[\pi(1+t)] - K \end{aligned}$$

with $\mu(x,t) := \frac{1}{\pi(1+t)} \exp\big(-\frac{|x|^2}{1+t}\big)$. By Jensen's inequality,

$$\int_{\mathbb{R}^2} \frac{n(x,t)}{\mu(x,t)} \log\left(\frac{n(x,t)}{\mu(x,t)}\right) \mu(x,t)\,dx \geq X \log X,$$

where $X = \int_{\mathbb{R}^2} \frac{n(x,t)}{\mu(x,t)} \mu(x,t)\,dx = M$. This gives the lower estimate for the entropy term.

Now, from (2.5) and (2.6), we get

$$\begin{aligned}&(1-\theta)\big[M \log\big(\frac{M}{\pi(1+t)}\big) - K\big] + \theta C(M)\\ &+ \int_0^t ds \int_{\mathbb{R}^2} n(x,s)\,|\nabla(\log n(x,s)) - \chi \nabla c(x,s)|^2\, dx \leq F_0\,.\end{aligned}$$

This proves that $\sqrt{n}\,|\nabla(\log n) - \chi\nabla c|$ is bounded in $L^2_{\rm loc}(\mathbb{R}^+, L^2(\mathbb{R}^2))$ and gives the estimate on the energy. $\square$

The *a priori* upper bound on $\int_{\mathbb{R}^2} n \log n\,dx$ combined with the $|x|^2$ moment bound of Lemma 2.1 shows that $n \log n$ is bounded in $L^\infty_{\rm loc}(\mathbb{R}^+, L^1(\mathbb{R}^2))$.

Lemma 2.6. *For any $u \in L^1_+(\mathbb{R}^2)$, if $\int_{\mathbb{R}^2} |x|^2 u dx$ and $\int_{\mathbb{R}^2} u \log u dx$ are bounded from above, then $u \log u$ is uniformly bounded in $L^\infty(\mathbb{R}^+_{\rm loc}, L^1(\mathbb{R}^2))$ and*

$$\int_{\mathbb{R}^2} u|\log u|\,dx \leq \int_{\mathbb{R}^2} u\Big(\log u + |x|^2\Big)\,dx + 2\log(2\pi)\int_{\mathbb{R}^2} u\,dx + \frac{2}{e}\,.$$

Proof. Let $\bar{u} := u 1\!\!1_{\{u \leq 1\}}$ and $m = \int_{\mathbb{R}^2} \bar{u} dx \leq M$. Then

$$\int_{\mathbb{R}^2} \bar{u}\Big(\log \bar{u} + \frac{1}{2}|x|^2\Big)\,dx = \int_{\mathbb{R}^2} U \log U\,d\mu - m \log(2\pi)$$

where $U := \bar{u}/\mu$, $d\mu(x) = \mu(x)dx$ and $\mu(x) = (2\pi)^{-1} e^{-|x|^2/2}$. By Jensen's inequality,

$$\int_{\mathbb{R}^2} U \log U\,d\mu \geq \Big(\int_{\mathbb{R}^2} U\,d\mu\Big)\,\log\Big(\int_{\mathbb{R}^2} U\,d\mu\Big) = m\log m\,,$$

$$\int_{\mathbb{R}^2} \bar{u}\log\bar{u}\,dx \geq m\log\Big(\frac{m}{2\pi}\Big) - \frac{1}{2}\int_{\mathbb{R}^2}|x|^2\bar{u}\,dx \geq -\frac{1}{e} - M\log(2\pi) - \frac{1}{2}\int_{\mathbb{R}^2}|x|^2\bar{u}\,dx\,.$$

Using

$$\int_{\mathbb{R}^2} u\,|\log u|\,dx = \int_{\mathbb{R}^2} u\log u\,dx - 2\int_{\mathbb{R}^2}\bar{u}\log\bar{u}\,dx\,,$$

this completes the proof. $\square$

2.4. **Existence of weak solutions up to critical mass.** Using the informations collected in Sections 2.1, 2.2 and 2.3, in the spirit of [19], we can now state, in the subcritical case $M < 8\pi/\chi$, the following existence result of weak solutions, which is essentially the one stated without proof in [22].

Proposition 2.7. *Under Assumption* (2.1) *and $M < 8\pi/\chi$, the Keller-Segel system* (1.1) *has a global weak non-negative solution such that, for any $T > 0$,*

$$\begin{gathered}(1 + |x|^2 + |\log n|)n \in L^\infty(0,T; L^1(\mathbb{R}^2))\,,\\ \iint_{[0,T]\times\mathbb{R}^2} n|\nabla\log n - \chi\nabla c|^2 dxdt < \infty\,.\end{gathered}$$

Propostion 2.7 strongly relies on the estimates of Lemmata 2.1 and 2.5. To establish a complete proof, we need to regularize the problem (Section 2.5) and then prove that the above estimates hold uniformly with respect to the regularization procedure (Section 2.6). This allows to pass to the limit in the regularization parameter (Section 2.7) and proves the existence of a weak solution with a well defined flux. To prove Theorem 1.1, we need additional regularity properties of the solutions. This is the purpose of Section 3. Hypercontractivity and the free energy inequality (1.2) will be dealt with in Sections 3.4 and 3.3 respectively.

2.5. **A regularized model.** The goal of this section is to establish the existence of solutions for a regularized version of the Keller-Segel model, for which the logarithmic singularity of the convolution kernel $\mathcal{K}^0(z) := -\frac{1}{2\pi}\log|z|$ is appropriately truncated.

There are indeed two difficulties when dealing with $\mathcal{K}^0$. It is unbounded and has a singularity at $z=0$. First of all, the unboundedness from above of the kernel is not difficult to handle. For $R>\sqrt{e}$, $R\mapsto R^2/\log R$ is an increasing function, so that

$$0 \le \iint_{|x-y|>R} \log|x-y| n(x,t) n(y,t)\,dx\,dy \le \frac{2\log R}{R^2} M \int_{\mathbb{R}^2} |x|^2 n(x,t)\,dx\,.$$

Since $\iint_{1<|x-y|<R} \log|x-y| n(x,t)n(y,t)\,dx\,dy \le M^2 \log R$, we only need to take care of a uniform bound on

$$\iint_{|x-y|<1} \log|x-y| n^\varepsilon(x,t) n^\varepsilon(y,t)\,dx\,dy \quad \text{and} \quad \int_{\mathbb{R}^2} n^\varepsilon(x,t)\log n^\varepsilon(x,t)\,dx\,.$$

for an approximating family $(n^\varepsilon)_{\varepsilon>0}$.

The other difficulty concerning the convolution kernel $\mathcal{K}^0$ is the singularity at $z=0$. This is a much more serious difficulty that we are going to overcome by defining a truncated convolution kernel and deriving uniform estimates in Section 2.6. To do so, we first need to find solutions of the model with a truncated convolution kernel. Let $\mathcal{K}^\varepsilon$ be such that

$$\mathcal{K}^\varepsilon(z) := \mathcal{K}^1\left(\frac{z}{\varepsilon}\right)$$

where $\mathcal{K}^1$ is a radial monotone non-decreasing smooth function satisfying

$$\mathcal{K}^1(z) = \begin{cases} -\frac{1}{2\pi}\log|z| & \text{if } |z|\ge 4\,,\\ 0 & \text{if } |z|\le 1\,. \end{cases}$$

Moreover, we can assume without restriction that

$$0 \le -\nabla\mathcal{K}^1(z) \le \frac{1}{2\pi|z|}\,, \quad \mathcal{K}^1(z) \le -\frac{1}{2\pi}\log|z| \quad \text{and} \quad -\Delta\mathcal{K}^1(z)\ge 0 \tag{2.7}$$

for any $z\in\mathbb{R}^2$. Since $\mathcal{K}^\varepsilon(z)=\mathcal{K}^1(z/\varepsilon)$, we also have

$$0 \le -\nabla\mathcal{K}^\varepsilon(z) \le \frac{1}{2\pi|z|} \quad \forall\, z\in\mathbb{R}^2\,. \tag{2.8}$$

If we replace (1.1) by the regularized version

$$\begin{aligned} \frac{\partial n^\varepsilon}{\partial t} &= \Delta n^\varepsilon - \chi\nabla\cdot(n^\varepsilon\nabla c^\varepsilon) \\ c^\varepsilon &= \mathcal{K}^\varepsilon * n^\varepsilon \end{aligned} \quad x\in\mathbb{R}^2\,,\ t>0\,, \tag{2.9}$$

written in the distribution sense, then we can state the following existence result.

Proposition 2.8. *For any fixed positive ε, under Assumptions (2.1), if $n_0 \in L^2(\mathbb{R}^2)$, then for any $T > 0$ there exists $n^\varepsilon \in L^2(0,T;H^1(\mathbb{R}^2)) \cap C(0,T;L^2(\mathbb{R}^2))$ which solves (2.9) with initial data n_0.*

To prove Proposition 2.8, we will first fix a functional framework, then solve a linear problem before using it to make a fixed point argument in order to prove the existence of a solution to the regularized system (2.9).

2.5.1. *Functional framework.* We will use the Aubin-Lions *compactness method,* (see [40], Ch. IV, §4 and [2], and [57] for more recent references). A simple statement goes as follows:

Lemma 2.9 (Aubin Lemma)**.** *Take $T > 0$, $p \in (1,\infty)$ and let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence of functions in $L^p(0,T;H)$ where H is a Banach space. If $(f_n)_{n\in\mathbb{N}}$ is bounded in $L^p(0,T;V)$, where V is compactly imbedded in H and $\partial f_n/\partial t$ is bounded in $L^p(0,T;V')$ uniformly with respect to $n \in \mathbb{N}$, then $(f_n)_{n\in\mathbb{N}}$ is relatively compact in $L^p(0,T;H)$.*

For our purpose, we fix $T > 0$, $p = 2$ and define $H := L^2(\mathbb{R}^2)$, $V := \{v \in H^1(\mathbb{R}^2) : \sqrt{|x|}v \in L^2(\mathbb{R}^2)\}$, V' its dual, $\mathcal{V} := L^2(0,T;V)$, $\mathcal{H} := L^2(0,T;H)$ and $\mathcal{W}(0,T) := \{v \in L^2(0,T;V) : \partial v/\partial t \in L^2(0,T;V')\}$. In this functional framework, the notion of solution we are looking for is actually more precise than in the distribution sense:

$$0 = \int_0^T \Big\{\langle n_t, \psi\rangle_{V'\times V} + \int_{\mathbb{R}^3} (\nabla n + \chi n \nabla c)\cdot\nabla\psi\,dx\Big\}dt \quad \forall \psi \in \mathcal{V}\,.$$

Notice that V is relatively compact in H, since the bound on $|x||v|^2$ in $L^1(\mathbb{R}^2)$ allows to consider only compact sets, on which compactness holds by Sobolev's imbeddings: Lemma 2.9 applies.

2.5.2. *Estimates for a linear drift-diffusion equation.* We start with the derivation of some *a priori* estimates on the solution of the linear problem

$$\frac{\partial n}{\partial t} = \Delta n - \nabla\cdot(nf) \tag{2.10}$$

for some function $f \in (L^\infty((0,T)\times\mathbb{R}^2))^2$. We assume in this section that the initial data n_0 belongs to $L^2(\mathbb{R}^2)$. By a fixed-point method, this allows us to prove the

Lemma 2.10. *Assume that (2.1) holds and consider $f \in L^\infty((0,T)\times\mathbb{R}^2)$. If $n_0 \in L^2(\mathbb{R}^2)$, for any $T > 0$, there exists $n \in \mathcal{W}(0,T)$ which solves (2.10) with initial data n_0.*

Proof. Consider the map $\mathcal{T} : L^\infty(0,T;L^1(\mathbb{R}^2)) \to L^\infty(0,T;L^1(\mathbb{R}^2))$ defined by

$$\mathcal{T}[n](\cdot,t) := G(\cdot,t) * n_0 + \int_0^t \nabla G(\cdot,t-s) * [n(\cdot,s)f(\cdot,s)]\;ds \quad \forall\,(x,t) \in [0,T]\times\mathbb{R}^2\,,$$

where $*$ denotes the space convolution. Here $G(x,t) := (4\pi t)^{-1}e^{-\frac{|x|^2}{4t}}$ is the Green function associated to the heat equation. Notice that $\|\nabla G(\cdot,s)\|_{L^1(\mathbb{R}^2)} \le Cs^{-1/2}$. We define the sequence $(n_k)_{k\in\mathbb{N}}$ by $n_{k+1} = \mathcal{T}(n_k)$ for $k \ge 1$. For any $t \in [0,T]$, we compute

$$\begin{aligned}
&\|n_{k+1}(t) - n_k(t)\|_{L^1(\mathbb{R}^2)}\\
&\le \int_{\mathbb{R}^2} \left| \int_0^t \nabla G(\cdot, t-s) * [(n_k(\cdot,s) - n_{k-1}(\cdot,s))\, f(\cdot,s)]\, ds \right| dx\\
&\le \|f\|_{L^\infty([0,T]\times\mathbb{R}^2)} \int_0^t \|\nabla G(\cdot,t-s) * (n_k(\cdot,s) - n_{k-1}(\cdot,s))\,\|_{L^1(\mathbb{R}^2)}\\
&\le \|f\|_{L^\infty([0,T]\times\mathbb{R}^2)} \int_0^t \|\nabla G(\cdot,t-s)\|_{L^1(\mathbb{R}^2)} \|n_k(s) - n_{k-1}(s)\|_{L^1(\mathbb{R}^2)}\\
&\le C\|f\|_{L^\infty([0,T]\times\mathbb{R}^2)} \sqrt{t} \|n_k - n_{k-1}\|_{L^\infty(0,t;L^1(\mathbb{R}^2))}\,.
\end{aligned}$$

For $T>0$ small enough, $(n_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $L^\infty(0,T;L^1(\mathbb{R}^2))$, which converges to a fixed point of $\mathcal{T}$. Iterating the method, we prove the existence of a solution of (2.10) on an arbitrary time interval $[0,T]$. $\square$

Next, let us establish some *a priori* estimates. The solution n is bounded in $L^\infty(0,T;L^2(\mathbb{R}^2))$ as a consequence of the following computation:

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} |n(x,t)|^2\,dx = -\int_{\mathbb{R}^2} |\nabla n(x,t)|^2\,dx + \int_{\mathbb{R}^2} \nabla n(x,t)\cdot n(x,t) f(x,t)\,dx\,.$$

The right hand side can be written $\int_{\mathbb{R}^2} a\cdot bf\,dx$ with $a := \sqrt{1/\lambda}\nabla n$ and $b := \sqrt{\lambda} n$. It is therefore bounded by $(\int_{\mathbb{R}^2} a^2\,dx + \frac{1}{4}\int_{\mathbb{R}^2} b^2\,dx)\|f\|_{L^\infty([0,T]\times\mathbb{R}^2)}$, which provides the estimate

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} |n|^2\,dx\\
&\le \Big(-1 + \frac{1}{\lambda}\|f\|_{L^\infty([0,T]\times\mathbb{R}^2)}\Big)\int_{\mathbb{R}^2} |\nabla n|^2\,dx + \frac{\lambda}{4}\|f\|_{L^\infty([0,T]\times\mathbb{R}^2)}\int_{\mathbb{R}^2} |n|^2\,dx\,.
\end{aligned}$$

In the case $\lambda = \|f\|_{L^\infty([0,T]\times\mathbb{R}^2)}$, we obtain

$$\int_{\mathbb{R}^2} |n|^2\,dx \le \int_{\mathbb{R}^2} |n_0|^2\,dx\; e^{\|f\|^2_{L^\infty([0,T]\times\mathbb{R}^2)} T/2} \quad \forall\, t\in(0,T)\,.$$

Hence n is bounded in $L^\infty(0,T;L^2(\mathbb{R}^2)) = \mathcal{H}$. Similarly, for $\lambda = \frac{3}{2}\|f\|_{L^\infty([0,T]\times\mathbb{R}^2)}$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|n(\cdot,t)\|^2_{L^2(\mathbb{R}^2)} \le -\frac{1}{3}\|\nabla n\|^2_{L^2(\mathbb{R}^2)} + \frac{3}{8}\|f\|^2_{L^\infty([0,T]\times\mathbb{R}^2)}\|n(\cdot,t)\|^2_{L^2(\mathbb{R}^2)}\,.$$

This also proves that ∇n is bounded in $L^2((0,T)\times\mathbb{R}^2)$, and n is therefore also bounded in $L^2(0,T;H^1(\mathbb{R}^2))$. Next, we need a moment estimate, which is achieved by

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2} |x|^2 n(x,t)\,dx\\
&\le 4\int_{\mathbb{R}^2} n\,dx + 2\|f\|_{L^\infty([0,T]\times\mathbb{R}^2)}\Big(\int_{\mathbb{R}^2} n\,dx\Big)^{1/2}\Big(\int_{\mathbb{R}^2} |x|^2 n(x,t)\,dx\Big)^{1/2}\,.
\end{aligned}$$

As a conclusion, this proves that $\int_{\mathbb{R}^2} |x|^2 n(x,t)\,dx$ is bounded and therefore shows that n is bounded in $\mathcal{V}$. On the other hand, $\partial n/\partial t$ is bounded in V' as can be checked by an elementary computation. We can therefore apply Aubin's Lemma (Lemma 2.9) to n:

If $(n_0^k)_{k\in\mathbb{N}}$ is a sequence of initial data with uniform bounds, then the corresponding sequence $(n^k)_{k\in\mathbb{N}}$ of solutions of (2.10) *with f replaced by f_k, for a sequence $(f^k)_{k\in\mathbb{N}}$ uniformly bounded in $(L^\infty([0,T]\times\mathbb{R}^2))^2$, is contained in a relatively compact set in $L^2(0,T;V)$.*

We will make use of this property in the next section.

2.5.3. *Existence of a solution of the regularized problem.* This section is devoted to the proof of Proposition 2.8, using a fixed point method.

Define the truncation function $h(s) := \min\{1, h_0/s\}$, for some constant $h_0 > 1$ to be fixed later and consider the map $\mathcal{T} : L^2(0,T;H) \to L^2(0,T;H)$ such that

(1) To a function $n \in L^2(0,T;H)$, we associate $\nabla c^\varepsilon := \nabla\mathcal{K}^\varepsilon * n$.
(2) With ∇c^ε, we construct the truncated function
$$f := h\left(\|\nabla c^\varepsilon\|_{L^\infty((0,T)\times\mathbb{R}^2)}\right)\nabla c^\varepsilon\,.$$
(3) The function f is bounded in $L^\infty((0,T)\times\mathbb{R}^2)$ by h_0, so we may apply Lemma 2.10 and obtain a new function $\tilde{n} =: \mathcal{T}[n]$ which solves (2.10).

The continuity of $\mathcal{T}$ is straightforward. As noticed in Section 2.5.2, we may apply the Aubin-Lions *compactness method,* which gives enough compactness to apply Schauder's fixed point theorem (Theorem 8.1 p. 199 in [39]) to a ball in $\mathcal{W}(0,T)$. Hence we obtain a solution of
$$\frac{\partial n^\varepsilon}{\partial t} = \Delta n^\varepsilon - \chi\nabla\cdot(n^\varepsilon f^\varepsilon)$$
$$f^\varepsilon = h\left(\|\nabla c^\varepsilon\|_{L^\infty((0,T)\times\mathbb{R}^2)}\right)\nabla c^\varepsilon\,,\quad c^\varepsilon = \mathcal{K}^\varepsilon * n^\varepsilon\,.$$

Assuming that $h_0 > \|\nabla\mathcal{K}^\varepsilon\|_{L^\infty(\mathbb{R}^2)}\|n_0\|_{L^1(\mathbb{R}^2)}$, we realize that n^ε is a solution of (2.9).

Notice that one can also easily prove a uniqueness result, using an appropriate Gronwall lemma. We refer for instance to [53] for similar results in a ball.

2.6. **Uniform a priori estimates.** In this section, we prove *a priori* estimates for the regularized problem which are uniform with respect to the regularization parameter ε. These estimates correspond to the formal estimates of Section 2.3.

Lemma 2.11. *Under Assumption* (2.1)*, consider a solution n^ε of* (2.9)*. If $\chi M < 8\pi$ then, uniformly as $\varepsilon\to 0$, with bounds depending only upon $\int_{\mathbb{R}^2}(1+|x|^2)n_0\,dx$ and $\int_{\mathbb{R}^2} n_0\log n_0\,dx$, we have:*

(i) *The function $(t,x)\mapsto |x|^2 n^\varepsilon(x,t)$ is bounded in $L^\infty(\mathbb{R}^+_{\rm loc}; L^1(\mathbb{R}^2))$.*
(ii) *The functions $t\mapsto\int_{\mathbb{R}^2} n^\varepsilon(x,t)\log n^\varepsilon(x,t)\,dx$ and $t\mapsto\int_{\mathbb{R}^2} n^\varepsilon(x,t)c^\varepsilon(x,t)\,dx$ are bounded.*
(iii) *The function $(t,x)\mapsto n^\varepsilon(x,t)\log(n^\varepsilon(x,t))$ is bounded in $L^\infty(\mathbb{R}^+_{\rm loc}; L^1(\mathbb{R}^2))$.*
(iv) *The function $(t,x)\mapsto\nabla\sqrt{n^\varepsilon}(x,t)$ is bounded in $L^2(\mathbb{R}^+_{\rm loc}\times\mathbb{R}^2)$.*
(v) *The function $(t,x)\mapsto n^\varepsilon(x,t)$ is bounded in $L^2(\mathbb{R}^+_{\rm loc}\times\mathbb{R}^2)$.*
(vi) *The function $(t,x)\mapsto n^\varepsilon(x,t)\Delta c^\varepsilon(x,t)$ is bounded in $L^1(\mathbb{R}^+_{\rm loc}\times\mathbb{R}^2)$.*
(vii) *The function $(t,x)\mapsto\sqrt{n^\varepsilon}(x,t)\nabla c^\varepsilon(x,t)$ is bounded in $L^2(\mathbb{R}^+_{\rm loc}\times\mathbb{R}^2)$.*

Proof. *(i)* The integral $\int_{\mathbb{R}^2} |x|^2 n^\varepsilon(x,t)\,dx$ can be estimated as in the proof of Lemma 2.1 because $\mathcal{K}^\varepsilon$ is radial and satisfies (2.8), so

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^2}|x|^2 n^\varepsilon(x,t)\,dx &= 4M + 2\chi\iint_{\mathbb{R}^2\times\mathbb{R}^2} n^\varepsilon(x,t)n^\varepsilon(y,t)x\cdot\nabla\mathcal{K}^\varepsilon(x-y)\,dxdy\\ &= 4M + \chi\iint_{\mathbb{R}^2\times\mathbb{R}^2} n^\varepsilon(x,t)n^\varepsilon(y,t)(x-y)\nabla\mathcal{K}^\varepsilon(x-y)\,dxdy\\ &\le 4M - \frac{\chi}{2\pi}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\frac{n^\varepsilon(x,t)n^\varepsilon(y,t)}{|x-y|}\,dxdy \le 4M\,.\end{aligned}$$

(ii) We compute

$$\frac{d}{dt}\Big[\int_{\mathbb{R}^2} n^\varepsilon\log n^\varepsilon\,dx - \frac{\chi}{2}\int_{\mathbb{R}^2} n^\varepsilon c^\varepsilon\,dx\Big] = -\int_{\mathbb{R}^2} n^\varepsilon\,|\nabla(\log n^\varepsilon)-\chi\nabla c^\varepsilon|^2\,dx\,.$$

Then by (2.7) and the logarithmic Hardy-Littlewood-Sobolev inequality, see Lemma 2.4, it follows by Lemma 2.5 that both terms of the right hand side are uniformly bounded.
(iii) It is a direct consequence of Lemma 2.6.
(iv) A simple computation shows that

$$\frac{d}{dt}\int_{\mathbb{R}^2} n^\varepsilon\log n^\varepsilon\,dx \le -4\int_{\mathbb{R}^2}\left|\nabla\sqrt{n^\varepsilon}\right|^2 dx + \chi\int_{\mathbb{R}^2} n^\varepsilon\cdot(-\Delta c^\varepsilon)\,dx\,.$$

Up to the common factor χ, we can write the last term of the right hand side as

$$\int_{\mathbb{R}^2} n^\varepsilon\cdot(-\Delta c^\varepsilon)\,dx = \int_{\mathbb{R}^2} n^\varepsilon\cdot(-\Delta(\mathcal{K}^\varepsilon * n^\varepsilon))\,dx = \text{(I)}+\text{(II)}+\text{(III)}$$

with

$$\text{(I)} := \int_{n^\varepsilon<K} n^\varepsilon\cdot(-\Delta(\mathcal{K}^\varepsilon * n^\varepsilon))\,,\quad \text{(II)} := \int_{n^\varepsilon\ge K} n^\varepsilon\cdot(-\Delta(\mathcal{K}^\varepsilon * n^\varepsilon)) - \text{(III)}$$
$$\text{and}\quad \text{(III)} = \int_{n^\varepsilon\ge K}|n^\varepsilon|^2\,.$$

We define ϕ_1 such that

$$\frac{1}{\varepsilon^2}\phi_1\Big(\frac{\cdot}{\varepsilon}\Big) = -\Delta\mathcal{K}^\varepsilon\,.$$

This gives an easy estimate of (I), namely

$$\text{(I)} \le \int_{n^\varepsilon<K} K\int_{\mathbb{R}^2}\frac{1}{\varepsilon^2}\phi_1\left(\frac{x-y}{\varepsilon}\right)n^\varepsilon(y)\,dy\,dx = MK\,.$$

Notice that

$$\frac{1}{\varepsilon^2}\phi_1\Big(\frac{\cdot}{\varepsilon}\Big) = -\Delta\mathcal{K}^\varepsilon \rightharpoonup \delta \quad\text{in } \mathcal{D}'\,, \tag{2.11}$$

which heuristically explains why (II) should be small. Let us prove that this is indeed the case. By (2.7), ϕ_1 is non-negative. Using $\|\phi_1\|_{L^1(\mathbb{R}^2)}=1$, we get

$$\begin{aligned}\text{(II)} &= \int_{n^\varepsilon\ge K} n^\varepsilon(x,t)\int_{\mathbb{R}^2}\left[n^\varepsilon(x-\varepsilon y,t)-n^\varepsilon(x,t)\right]\phi_1(y)\,dy\,dx\\ &\le \int_{n^\varepsilon\ge K} n^\varepsilon(x,t)\int_{\mathbb{R}^2}\left[\sqrt{n^\varepsilon(x-\varepsilon y,t)}-\sqrt{n^\varepsilon(x,t)}\right]\sqrt{\phi_1(y)}\\ &\quad\times\left[\sqrt{n^\varepsilon(x-\varepsilon y,t)}-\sqrt{n^\varepsilon(x,t)}+2\sqrt{n^\varepsilon(x,t)}\right]\sqrt{\phi_1(y)}\,dy\,dx\,.\end{aligned}$$

By the Cauchy-Schwarz inequality and using $(a+2b)^2 \le 2a^2 + 8b^2$ we obtain

$$\begin{aligned}
(\mathrm{II}) \le \int_{n^\varepsilon \ge K} & n^\varepsilon(x,t)\Big[\|\phi_1\|_{L^\infty(\mathbb{R}^2)} \int_{1/2\le y\le 2} \Big|\sqrt{n^\varepsilon(x-\varepsilon y,t)} - \sqrt{n^\varepsilon(x,t)}\Big|^2 \, dy\Big]^{1/2} \\
&\times \Big[\int_{\mathbb{R}^2} [2\Big|\sqrt{n^\varepsilon(x-\varepsilon y,t)} - \sqrt{n^\varepsilon(x,t)}\Big|^2 + 8|n^\varepsilon(x,t)|]\phi_1(y)\, dy\Big]^{1/2} dx\,.
\end{aligned}$$

Using the Poincaré inequality,

$$\begin{aligned}
(\mathrm{II}) \le \int_{n^\varepsilon \ge K} & n^\varepsilon(x,t)\ \|\phi_1\|^{1/2}_{L^\infty(\mathbb{R}^2)} C_P \|\nabla\sqrt{n^\varepsilon}\|_{L^2(\mathbb{R}^2)} \\
&\times \Big[\sqrt{2}\|\phi_1\|^{1/2}_{L^\infty(\mathbb{R}^2)} C_P \|\nabla\sqrt{n^\varepsilon}\|_{L^2(\mathbb{R}^2)} + 2\sqrt{2}\sqrt{|n^\varepsilon(x,t)|}\|\phi_1\|^{1/2}_{L^1(\mathbb{R}^2)}\Big]\, dx\,.
\end{aligned}$$

Recall the Gagliardo-Nirenberg-Sobolev inequality (2.4):

$$\int_{n^\varepsilon \ge K} |n^\varepsilon|^2 dx \le C^2_{\mathrm{GNS}} \int_{n^\varepsilon\ge K} \Big|\nabla\sqrt{n^\varepsilon}\Big|^2 dx \int_{n^\varepsilon \ge K} n^\varepsilon \, dx\,.$$

The left hand side can therefore be made as small as desired using:

$$\int_{n^\varepsilon\ge K} n^\varepsilon\, dx \le \frac{1}{\log K}\int_{n^\varepsilon \ge K} n^\varepsilon \log n^\varepsilon \, dx \le \frac{1}{\log K}\int_{\mathbb{R}^2} n^\varepsilon |\log n^\varepsilon|\, dx =: \eta(K)\,,$$

for $K>1$, large enough. Then

$$\int_{n^\varepsilon\ge K} |n^\varepsilon|^2 dx \le \eta(K) C^2_{\mathrm{GNS}} \Big\|\nabla\sqrt{n^\varepsilon}\Big\|^2_{L^2(\mathbb{R}^2)}\,. \tag{2.12}$$

By the Cauchy-Schwarz inequality

$$\begin{aligned}
\int_{n^\varepsilon\ge K} |n^\varepsilon(x,t)|^{3/2}\, dx &\le \Big(\int_{n^\varepsilon\ge K} |n^\varepsilon| dx\Big)^{1/2}\Big(\int_{n^\varepsilon \ge K}|n^\varepsilon|^2 dx\Big)^{1/2} \\
&\le \eta(K) C_{\mathrm{GNS}}\|\nabla\sqrt{n^\varepsilon}\|_{L^2(\mathbb{R}^2)}\,.
\end{aligned}$$

From this, it follows that

$$(\mathrm{II}) + (\mathrm{III}) \le B\eta(K)\|\nabla\sqrt{n^\varepsilon}\|^2_{L^2(\mathbb{R}^2)}$$

with

$$B := C^2_{\mathrm{GNS}} + \sqrt{2}\|\phi_1\|_{L^\infty(\mathbb{R}^2)} C_P^2 + 2\sqrt{2}\|\phi_1\|^{1/2}_{L^\infty(\mathbb{R}^2)}\|\phi_1\|^{1/2}_{L^1(\mathbb{R}^2)} C_P C_{\mathrm{GNS}}\,.$$

We can choose K large enough such that $\eta(K) < 4/B$. Collecting the estimates, we have shown that

$$\frac{d}{dt}\int_{\mathbb{R}^2} n^\varepsilon \log n^\varepsilon \, dx \le MK + (-4 + B\eta(K))X(t)$$

with $X(t) := \|\nabla\sqrt{n^\varepsilon}(t)\|^2_{L^2(\mathbb{R}^2)}$, and so

$$(4-B\eta)\int_0^T X(s)\, ds \le MKT + \int_{\mathbb{R}^2} n_0 \log n_0 \, dx - \int_{\mathbb{R}^2} n^\varepsilon(x,T)\log n^\varepsilon(x,T)\, dx\,.$$

(v) It follows from the Gagliardo-Nirenberg-Sobolev inequality (2.4).
(vi) It is a straightforward consequence of *(iv)*. Notice that $-\Delta c^\varepsilon$ is non-negative as a convolution of two non-negative functions ϕ_1 and n^ε.

(vii) A computation shows that

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^2}\frac{1}{2}n^\varepsilon c^\varepsilon\,dx &= \int_{\mathbb{R}^2} c^\varepsilon\left(\Delta n^\varepsilon - \chi\nabla\cdot(n^\varepsilon\nabla c^\varepsilon)\right)\,dx\\ &= \int_{\mathbb{R}^2} n^\varepsilon\Delta c^\varepsilon dx + \chi\int_{\mathbb{R}^2} n^\varepsilon|\nabla c^\varepsilon|^2 dx\,.\end{aligned}$$

This proves that

$$\begin{aligned}&\iint_{[0,T]\times\mathbb{R}^2} n^\varepsilon|\nabla c^\varepsilon|^2\,dx\,dt\\ &\le \frac{1}{2\chi}\left|\int_{\mathbb{R}^2} n^\varepsilon c^\varepsilon\,dx - \int_{\mathbb{R}^2} n_0\,(\mathcal{K}^\varepsilon * n_0)\,dx\right| + \frac{1}{\chi}\int_0^T\int_{\mathbb{R}^2} n^\varepsilon(-\Delta c^\varepsilon)\,dx\,.\end{aligned}$$

The last term of the right hand side is controlled by *(vi)*, while the previous one is bounded by *(ii)*. □

2.7. **Passing to the limit.** All estimates of Lemma 2.11 are uniform in the limit $\varepsilon \to 0$. The fact that n_0 is assumed to be bounded in $L^2(\mathbb{R}^2)$ in Lemma 2.8 does not play any role. In this section, n_0 is assumed to satisfy Assumption (2.1) and we consider the solution n^ε of (2.9) with a non-negative initial data $n_0^\varepsilon = \min\{n_0, \varepsilon^{-1}\}$. We want to pass simultaneously to the limit as $\varepsilon \to 0$ in $n_0^\varepsilon \to n_0$ and in $\mathcal{K}^\varepsilon(z) \to \mathcal{K}^0(z) = -\frac{1}{2\pi}\log|z|$.

Lemma 2.12. *Assume that n_0 satisfies Assumption* (2.1) *and consider the solution n^ε of* (2.9) *with a non-negative initial data $n_0^\varepsilon = \min\{n_0, \varepsilon^{-1}\}$. Then up to the extraction of a sequence ε_k of ε converging to 0, n_k^ε converges to a function n solution of* (1.1) *in the distribution sense. Furthermore the flux $n|\nabla(\log n) - \chi\nabla c|$ is bounded in $L^1([0,T)\times\mathbb{R}^2)$.*

Proof. Assertion *(vii)* of Lemma 2.11 allows to give a sense to the equation in the limit $\varepsilon \searrow 0$. The term which is difficult to handle is $n^\varepsilon\nabla c^\varepsilon$. It is first of all bounded in $L^1((0,T)\times\mathbb{R}^2)$ uniformly with respect to ε, as shown by the Cauchy-Schwarz inequality:

$$\begin{aligned}\Big(\iint_{[0,T]\times\mathbb{R}^2} n^\varepsilon\,|\nabla c^\varepsilon|\,dx\,dt\Big)^2 &\le \iint_{[0,T]\times\mathbb{R}^2} n^\varepsilon\,dx\,dt\iint_{[0,T]\times\mathbb{R}^2} n^\varepsilon\,|\nabla c^\varepsilon|^2\,dx\,dt\\ &= MT\iint_{[0,T]\times\mathbb{R}^2} n^\varepsilon\,|\nabla c^\varepsilon|^2\,dx\,dt\,,\end{aligned}$$

where the last term is controlled according to *(vii)* of Lemma 2.11.

Actually, $n^\varepsilon\nabla c^\varepsilon$ converges to $n\nabla c$ in the sense of distributions. By the Gagliardo-Nirenberg-Sobolev inequality (2.4), for any $p > 2$, for $t \in \mathbb{R}^+$ a.e.,

$$\int_{\mathbb{R}^2} |n^\varepsilon|^{p/2}\,dx \le \Big(C_{\rm GNS}^{(p)}\Big)^{p/2} M\Big(\int_{\mathbb{R}^2}\left|\nabla\sqrt{n^\varepsilon}\right|^2\,dx\Big)^{\frac{p}{2}-1}\,,$$

which proves that n^ε is bounded in $L^q(\mathbb{R}^+_{\rm loc}\times\mathbb{R}^2)$ for any $p/2 = q \in [1,+\infty)$, and that, up to the extraction of a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ which converges to 0, n^{ε_k} weakly converges to n in any $L^q_{\rm loc}(\mathbb{R}^+\times\mathbb{R}^2)$, $q\geq 1$. Next,

$$\begin{aligned}\nabla c^\varepsilon_k-\nabla c=&-\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{x-y}{|x-y|^2}\,(n^\varepsilon_k(y,t)-n(y,t))\;dy\\&+\int_{|x-y|\leq 2\varepsilon_k}\Big(\frac{1}{\varepsilon\,k}\nabla\mathcal{K}^1\big(\frac{x-y}{\varepsilon\quad k}\big)+\frac{x-y}{2\pi|x-y|^2}\Big)\,n^\varepsilon_k(y,t)\,dy\,.\end{aligned}$$

Since $\frac{1}{\varepsilon\,k}\nabla\mathcal{K}^1(\frac{z}{\varepsilon\,k})+\frac{z}{2\pi|z|^2}$ can be bounded by $\frac{1}{2\pi|z|}$, all terms converge to 0 for almost any $(x,t)\in\mathbb{R}^2\times\mathbb{R}^+$ and the convergence of n^ε_k to n is strong in $L^q_{\rm loc}(\mathbb{R}^+\times\mathbb{R}^2)$ for any $q\in(2,\infty)$, which is enough to prove that

$$n^\varepsilon_k\nabla c^\varepsilon_k\rightharpoonup n\nabla c\quad\text{in }\mathcal{D}'(\mathbb{R}^+\times\mathbb{R}^2)\,.$$

As a consequence, we also get by weak semi-continuity that

$$\begin{gathered}\iint_{[0,T]\times\mathbb{R}^2}n|\nabla c|^2\,dx\,dt\leq\liminf_{\varepsilon_k\to 0}\iint_{[0,T]\times\mathbb{R}^2}n^\varepsilon_k|\nabla c^\varepsilon_k|^2\,dx\,dt\,,\\\iint_{[0,T]\times\mathbb{R}^2}n^2\,dx\,dt\leq\liminf_{\varepsilon_k\to 0}\iint_{[0,T]\times\mathbb{R}^2}|n^\varepsilon_k|^2\,dx\,dt\,.\end{gathered}$$

Since the functional $n\mapsto\int_{\mathbb{R}^2}|\nabla\sqrt{n}|^2\,dx$ is convex, we also get

$$\iint_{[0,T]\times\mathbb{R}^2}|\nabla\sqrt{n}|^2\,dx\,dt\leq\liminf_{\varepsilon_k\to 0}\iint_{[0,T]\times\mathbb{R}^2}|\nabla\sqrt{n^\varepsilon_k}|^2\,dx\,dt\,.$$

The proof of the convexity goes as follows. Let $n(\tau)=n_0+\tau\nu$, $\tau>0$. Then

$$\frac{d^2}{d\tau^2}\int_{\mathbb{R}^2}\Big|\nabla\sqrt{n(\tau)}\Big|^2\,dx\Big|_{\tau=0}=\frac{1}{2n_0^3}\int_{\mathbb{R}^2}\big|\nu\nabla\sqrt{n_0}-n_0\nabla\sqrt{\nu}\big|^2\,dx\geq 0\,.$$

See [5, 6] for more details. Now, since

$$\begin{aligned}&\iint_{[0,T]\times\mathbb{R}^2}n^\varepsilon_k|\nabla(\log n^\varepsilon_k)-\chi\nabla c^\varepsilon_k|^2\,dx\,dt\\&=4\iint_{[0,T]\times\mathbb{R}^2}|\nabla\sqrt{n^\varepsilon_k}|^2\,dx\,dt-2\chi\iint_{[0,T]\times\mathbb{R}^2}|n^\varepsilon_k|^2\,dx\,dt\\&\quad+\chi^2\iint_{[0,T]\times\mathbb{R}^2}n^\varepsilon_k|\nabla c^\varepsilon_k|^2\,dx\,dt\end{aligned}$$

is bounded uniformly with respect to ε_k by (2.5),

$$\iint_{[0,T]\times\mathbb{R}^2}n|\nabla(\log n)-\chi\nabla c|^2\,dx\,dt$$

is also finite. Notice that this is not enough to prove that (2.5) holds if n is a solution of (1.1), even with an inequality instead of the equality. This is however enough to prove that the flux $n|\nabla(\log n)-\chi\nabla c|$ is bounded in $L^1([0,T)\times\mathbb{R}^2)$, simply by using the Cauchy-Schwarz inequality. This concludes the proof of Lemma 2.12. $\square$

As a consequence of the approximation procedure and of Lemma 2.12, we have also proved Proposition 2.7. To establish Inequality (1.2) in Theorem 1.1, we only need to prove that

$$\iint_{[0,T]\times\mathbb{R}^2}n^2\,dx\,dt=\liminf_{\varepsilon_k\to 0}\iint_{[0,T]\times\mathbb{R}^2}|n^\varepsilon_k|^2\,dx\,dt\,,$$

but this requires some additional work on the regularity properties of the solutions of (1.1).

3. Free energy inequality and regularity properties

In this section, we give some additional regularity properties of the solutions when $\chi M < 8\pi$.

3.1. Weak regularity results.

The following result is due to Goudon, see [25].

Theorem 3.1 ([25]). *Let $n^\varepsilon : (0,T)\times\mathbb{R}^N \to \mathbb{R}$ be such that for almost all $t \in (0,T)$, $n^\varepsilon(t)$ belongs to a weakly compact set in $L^1(\mathbb{R}^N)$ for almost any $t \in (0,T)$. If $\partial_t n^\varepsilon = \sum_{|\alpha|\le k} \partial_x^\alpha g_\varepsilon^{(\alpha)}$ where, for any compact set $K \subset \mathbb{R}^n$,*

$$\limsup_{|E|\to 0} \Big(\sup_{\varepsilon>0} \iint_{E\times K} |g_\varepsilon^{(\alpha)}| \,dt\,dx \Big) = 0\,,$$

where the supremum is taken over set $E \subset \mathbb{R}$ which are measurable, then $(n^\varepsilon)_{\varepsilon>0}$ is relatively compact in $C^0([0,T];L^1_{\rm weak}(\mathbb{R}^N)$.

This result immediately applies to the solution of (2.9).

Corollary 3.2. *Let n^ε be a solution of (2.9) with initial data $n_0^\varepsilon = \min\{n_0, \varepsilon^{-1}\}$ such that $n_0(1+|x|^2+|\log n_0|) \in L^1(\mathbb{R}^2)$. If n is a solution of (1.1) with initial data n_0, such that, for a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ with $\lim_{k\to\infty} \varepsilon_k = 0$, $n^{\varepsilon_k} \rightharpoonup n$ in $L^1((0,T)\times\mathbb{R}^2)$, then n belongs to $C^0(0,T;L^1_{\rm weak}(\mathbb{R}^2))$.*

Proof. We are able to apply Theorem 3.1 to n^ε where $g_\varepsilon^{(1)} := -\chi n^\varepsilon \nabla c^\varepsilon = -\chi\sqrt{n^\varepsilon}\cdot\sqrt{n^\varepsilon}\nabla c^\varepsilon$ and $g_\varepsilon^{(2)} := n^\varepsilon$. Notice indeed that as a consequence of Lemma 3.4, we have, uniformly with respect to ε,

$$\limsup_{t_1\to t_2} \sup_\varepsilon g_\varepsilon^{(1)} \le \chi \limsup_{t_1\to t_2} M(t_2-t_1) \int_{t_1}^{t_2}\int_{\mathbb{R}^2} n^\varepsilon |\nabla c^\varepsilon|^2 \,dx\,ds = 0\,,$$

$$\limsup_{t_1\to t_2} \sup_\varepsilon g_\varepsilon^{(2)} \le \limsup_{t_1\to t_2} \int_{t_1}^{t_2}\int_{\mathbb{R}^2} n^\varepsilon dx = 0\,.$$

□

3.2. L^p uniform estimates.

Here we prove that if the initial data n_0 is bounded in $L^p(\mathbb{R}^2)$, then it is also the case for the solution $n(\cdot,t)$ for any finite positive time t. By uniform, we mean estimates that hold up to $t = 0$.

Proposition 3.3. *Assume that (2.1) and $M < 8\pi/\chi$ hold. If n_0 is bounded in $L^p(\mathbb{R}^2)$ for some $p > 1$, then any solution n of (1.1) is bounded in $L^\infty_{\rm loc}(\mathbb{R}^+, L^p(\mathbb{R}^2))$.*

Proof. We formally compute

$$\begin{aligned}
\frac{1}{2(p-1)}\frac{d}{dt}\int_{\mathbb{R}^2} |n(x,t)|^p\,dx &= -\frac{2}{p}\int_{\mathbb{R}^2} |\nabla(n^{p/2})|^2\,dx + \chi\int_{\mathbb{R}^2} \nabla(n^{p/2})\cdot n^{p/2}\cdot\nabla c\,dx\\
&= -\frac{2}{p}\int_{\mathbb{R}^2} |\nabla(n^{p/2})|^2\,dx + \chi\int_{\mathbb{R}^2} n^p(-\Delta c)\,dx\\
&= -\frac{2}{p}\int_{\mathbb{R}^2} |\nabla(n^{p/2})|^2\,dx + \chi\int_{\mathbb{R}^2} n^{p+1}\,dx\,.
\end{aligned}$$

Using the following Gagliardo-Nirenberg-Sobolev inequality:

$$\int_{\mathbb{R}^2} |v|^{2(1+1/p)}\,dx \le K_p \int_{\mathbb{R}^2} |\nabla v|^2\,dx \int_{\mathbb{R}^2} |v|^{2/p}\,dx\,,$$

or equivalently, with $n = v^{2/p}$,

$$\int_{\mathbb{R}^2} |n|^{p+1}\,dx \le K_p \int_{\mathbb{R}^2} |\nabla(n^{p/2})|^2\,dx \int_{\mathbb{R}^2} |n|\,dx\,,$$

we get the estimate

$$\frac{1}{2(p-1)}\frac{d}{dt}\int_{\mathbb{R}^2} n^p\,dx \le \int_{\mathbb{R}^2} |\nabla(n^{p/2})|^2\,dx\Big(-\frac{2}{p}+K_p\chi M\Big)\,,$$

which proves the decay of $\int_{\mathbb{R}^2} n^p\,dx$ if $M < \frac{2}{pK_p\chi}$. Otherwise, we can rely on the entropy estimate to get a bound: Let $K > 1$ be a constant, to be chosen later.

$$\int_{\mathbb{R}^2} n^p\,dx = \int_{n\le K} n^p\,dx + \int_{n>K} n^p\,dx\,.$$

The first term of the right hand side is bounded by $K^{p-1}M$. Concerning the second one, define first

$$M(K) := \int_{n>K} n\,dx\,.$$

Using the fact that $|n\log n|$ is bounded in $L^\infty(\mathbb{R}^+_{\mathrm{loc}}; L^1(\mathbb{R}^2))$, we can estimate $M(K)$ by

$$M(K) \le \frac{1}{\log K}\int_{n>K} n\log n\,dx \le \frac{1}{\log K}\int_{\mathbb{R}^2} |n\log n|\,dx$$

and choose it arbitrarily small on any given time interval $(0,T)$. Following [33], compute now

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2}(n-K)_+^p\,dx + \frac{4}{p}(p-1)\int_{\mathbb{R}^2}|\nabla((n-K)_+^{p/2})|^2\,dx\\
&= p\int_{\mathbb{R}^2}(n-K)_+^{p-1}\,[\Delta n - \chi\nabla(n\nabla c)]\,dx + \frac{4}{p}(p-1)\int_{\mathbb{R}^2}|\nabla((n-K)_+^{p/2})|^2\,dx\\
&= -p\chi\int_{\mathbb{R}^2}(n-K)_+^{p-1}\,[\nabla(n-K)\cdot\nabla c + n\Delta c]\,dx\\
&= -\chi\int_{\mathbb{R}^2}(n-K)_+^{p-1}\,[(n-K)_+(-\Delta c) - pn(-\Delta c)]\,dx\\
&= (p-1)\chi\int_{\mathbb{R}^2}(n-K)_+^{p+1}\,dx + (2p-1)\chi K\int_{\mathbb{R}^2}(n-K)_+^p\,dx\\
&\quad + p\chi K^2\int_{\mathbb{R}^2}(n-K)_+^{p-1}\,dx
\end{aligned}$$

The term involving $\int_{\mathbb{R}^2}(n-K)_+^{p-1}\,dx$ can be estimated as follows:

$$\begin{gathered}
\int_{\mathbb{R}^2}(n-K)_+^{p-1}\,dx = \int_{K<n\le K+1}(n-K)_+^{p-1}\,dx + \int_{n>K+1}(n-K)_+^{p-1}\,dx\,,\\
\int_{K<n\le K+1}(n-K)_+^{p-1}\,dx \le \int_{K<n\le K+1} 1\,dx \le \frac{1}{K}\int_{K<n\le K+1} n\,dx \le \frac{M}{K}\,,\\
\int_{n>K+1}(n-K)_+^{p-1}\,dx \le \int_{n>K+1}(n-K)_+^p\,dx \le \int_{\mathbb{R}^2}(n-K)_+^p\,dx\,.
\end{gathered}$$

By choosing K sufficiently large, we obtain

$$-\frac{4}{p}(p-1)\int_{\mathbb{R}^2}|\nabla((n-K)_+^{p/2})|^2\,dx+(p-1)\chi\int_{\mathbb{R}^2}(n-K)_+^{p+1}\,dx\leq 0$$

using again the Gagliardo-Nirenberg-Sobolev inequality but with M replaced by $M(K)$, small. Summarizing, for a fixed interval $(0,T)$ with T arbitrarily large, we have found K such that

$$\frac{d}{dt}\int_{\mathbb{R}^2}(n-K)_+^p\,dx\leq C_1\int_{\mathbb{R}^2}(n-K)_+^p\,dx+C_2$$

for some positive constants C_1 and C_2. A Gronwall estimate shows that $\int_{\mathbb{R}^2}(n-K)_+^p\,dx$ is finite on $(0,T)$.

To justify this estimate, one has as above to establish it for the regularized problem and then pass to the limit. This is purely technical but not difficult and we leave it to the reader.

To conclude, we still need to check that the bound on $\int_{\mathbb{R}^2}(n-K)_+^p\,dx$ is enough to control $\int_{n>K}n^p\,dx$. Using the estimate

$$x^p\leq\Big(\frac{\lambda}{\lambda-1}\Big)^{p-1}(x-1)^p$$

for any $x\geq\lambda>1$, we get

$$\begin{aligned}\int_{n>K}n^p\,dx&=\int_{K<n\leq\lambda K}n^p\,dx+\int_{n>\lambda K}n^p\,dx\\&\leq(\lambda K)^{p-1}M+\Big(\frac{\lambda}{\lambda-1}\Big)^{p-1}K^p\int_{n>\lambda K}\Big(\frac{n}{K}-1\Big)^p\,dx\\&\leq(\lambda K)^{p-1}M+\Big(\frac{\lambda}{\lambda-1}\Big)^{p-1}\int_{\mathbb{R}^2}(n-K)_+^p\,dx\,.\end{aligned}$$

□

Notice that very similar estimates have been derived, without the knowledge of the optimal bound $\chi M<8\pi$, by Jäger and Luckhaus in [33] in $\mathbb{R}^d$, $d=2$ (also see [19, 20] if $d\geq 2$), by working directly in an L^p-framework, instead of the free energy framework.

3.3. **The free energy inequality in a regular setting.** Using the *a priori* estimates of the previous section for $p=2+\varepsilon$, we can prove that the free energy inequality (1.2) holds.

Lemma 3.4. *Let n_0 be in a bounded set in $L^1_+(\mathbb{R}^2,(1+|x|^2)dx)\cap L^{2+\varepsilon}(\mathbb{R}^2,dx)$, for some $\varepsilon>0$, eventually small. Then n_0 satisfies Assumption* (2.1)*, the solution n of* (1.1) *found in Theorem 1.1, with initial data n_0, is in a compact set in $L^2(\mathbb{R}^+_{\rm loc}\times\mathbb{R}^2)$ and moreover the free energy production estimate* (1.2) *holds:*

$$F[n]+\int_0^t\Big(\int_{\mathbb{R}^2}n\,|\nabla(\log n)-\chi\nabla c|^2\,dx\Big)\,ds\leq F[n_0]\,.$$

Proof. We split the proof in three steps.
First Step: n is bounded in $L^2(\mathbb{R}^+_{\rm loc}\times\mathbb{R}^2)$. To apply Theorem 1.1, we need to prove that $n_0\log n_0$ is integrable. By Hölder's inequality we have

$$5\|u\|_{L^q(\mathbb{R}^2)}\leq\|u\|^\alpha_{L^p(\mathbb{R}^2)}\|u\|^{1-\alpha}_{L^r(\mathbb{R}^2)}$$

with $\alpha = \frac{p}{q}\frac{r-q}{r-p}$, $p \le q \le r$. Take the logarithm of both sides:

$$\alpha \log\Big(\frac{\|u\|_{L^q(\mathbb{R}^2)}}{\|u\|_{L^p(\mathbb{R}^2)}}\Big) + (\alpha-1)\log\Big(\frac{\|u\|_{L^r(\mathbb{R}^2)}}{\|u\|_{L^q(\mathbb{R}^2)}}\Big) \le 0\,.$$

Since this inequality trivializes to an equality when $q = p$, we may differentiate it with respect to q at $q = p$ and get that for any $u \in L^p(\mathbb{R}^d) \cap L^r(\mathbb{R}^d)$, $1 \le p < r < +\infty$, we have

$$\int u^p \log\Big(\frac{|u|}{\|u\|_{L^p(\mathbb{R}^2)}}\Big)\,dx \le \frac{r}{r-p}\|u\|^p_{L^p(\mathbb{R}^2)} \log\Big(\frac{\|u\|_{L^r(\mathbb{R}^2)}}{\|u\|_{L^p(\mathbb{R}^2)}}\Big)\,.$$

With $u = n_0$, $p = 1$ and $r = 2+\varepsilon$, by applying Lemma 2.6, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^2} n_0\big|\log n_0\big|\,dx\\ &\le \frac{M}{1+\varepsilon}\big[(2+\varepsilon)\log(\|n_0\|_{L^{2+\varepsilon}(\mathbb{R}^2)}) - \log M + 2\log(2\pi)\big] + \int_{\mathbb{R}^2}|x|^2 n_0\,dx + \frac{2}{e} < \infty\,.\end{aligned}$$

Since $n_0 \in L^1 \cap L^{2+\varepsilon}(\mathbb{R}^2)$, by Hölder's inequality, n_0 is initially in any $L^q(\mathbb{R}^2)$ for all $q \in [1, 2+\varepsilon]$, and as a special case in $L^2(\mathbb{R}^2)$:

$$\|n_0\|^2_{L^2(\mathbb{R}^2)} \le \|n_0\|^{\varepsilon/(1+\varepsilon)}_{L^1(\mathbb{R}^2)}\ \|n_0\|^{1/(1+\varepsilon)}_{L^{2+\varepsilon}(\mathbb{R}^2)}\,.$$

Hence by Theorem 1.1, the solution n of (1.1) is bounded in the space $L^\infty(\mathbb{R}^+_{\rm loc}; L^1 \cap L^{2+\varepsilon}(\mathbb{R}^2))$. As a special case n is bounded in $L^2(\mathbb{R}^+_{\rm loc} \times \mathbb{R}^2)$.
Second Step: ∇n is bounded in $L^2(\mathbb{R}^+_{\rm loc} \times \mathbb{R}^2)$. The following computation

$$\frac{d}{dt}\int_{\mathbb{R}^2} n^2\,dx = -2\int_{\mathbb{R}^2}|\nabla n|^2\,dx + 2\chi\int_{\mathbb{R}^2}\nabla n\cdot n\nabla c\,dx$$

shows that $X := \|\nabla n\|_{L^2((0,T)\times\mathbb{R}^2)}$ satisfies the estimate

$$2X^2 - 2\chi\|n\nabla c\|_{L^\infty(0,T;L^2(\mathbb{R}^2))}X \le \|n\|^2_{L^\infty(0,T;L^2(\mathbb{R}^2))} + \|n_0\|^2_{L^2(\mathbb{R}^2)}\,.$$

This implies that X is bounded if $\|n\nabla c\|_{L^\infty(0,T;L^2(\mathbb{R}^2))}$ is bounded. Let us prove that this is indeed the case. The drift force term takes the form

$$\nabla c(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{x-y}{|x-y|^2}n(y,t)\,dy\,.$$

Since $n_0 \in L^{2+\varepsilon}(\mathbb{R}^2)$, by Theorem 1.1, the solution n is bounded in the space $L^\infty(\mathbb{R}^+_{\rm loc}; L^{2+\varepsilon}(\mathbb{R}^2))$. As a consequence of the Hardy-Littlewood-Sobolev inequality (see below), for any $(p_1, q_1) \in (2,+\infty)\times(1,2)$ such that $\frac{1}{p_1} = \frac{1}{q_1} - \frac{1}{2}$, there exists a constant $C = C(p_1) > 0$ such that for almost any $t > 0$,

$$\|\nabla c(\cdot,t)\|_{L^{p_1}(\mathbb{R}^2)} \le C\|n(\cdot,t)\|_{L^{q_1}(\mathbb{R}^2)}\,.$$

We can indeed evaluate $\|f * |\cdot|^{-\lambda}\|_{L^{p_1}(\mathbb{R}^d)}$ by

$$\|f * |\cdot|^{-\lambda}\|_{L^{p_1}(\mathbb{R}^d)} = \sup_{g\in L^{q_1}(\mathbb{R}^d),\|g\|_{L^{q_1}(\mathbb{R}^d)}\le 1}\int_{\mathbb{R}^d}\big(f * |\cdot|^{-\lambda}\big)\ g\,dx$$

with $\frac{1}{p_1} + \frac{1}{q_1} = 1$. The right hand side is bounded, up to a multiplicative constant, by $\|f\|_{L^p(\mathbb{R}^2)}$ according to the Hardy-Littlewood-Sobolev inequality, if $\frac{1}{p} + \frac{1}{q_1} + \frac{\lambda}{d} = 2$ and $0 < \lambda < d$. This inequality, see, *e.g.,* [38], indeed states that: For all

$f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$, $1 < p,\, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} + \frac{\lambda}{d} = 2$ and $0 < \lambda < d$, there exists a constant $C = C(p, q, \lambda) > 0$ such that

$$\Big| \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{|x-y|^\lambda} f(x) g(y)\, dx\, dy \Big| \leq C \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)} \,.$$

Applied with $\lambda = 1$, $d = 2$, this proves that $\|n \nabla c\|_{L^\infty(0,T;L^2(\mathbb{R}^2))}$ is bounded.

Applying this estimate with $p_1 = 2(1 + 2/\varepsilon)$ and $q_1 = 2 - \varepsilon/(1+\varepsilon)$, and using Hölder's inequality, we can write

$$\begin{aligned}\|n(\cdot,t)\nabla c(\cdot,t)\|_{L^2(\mathbb{R}^2)} &\leq \|n(\cdot,t)\|_{L^{2+\varepsilon}(\mathbb{R}^2)} \|\nabla c(\cdot,t)\|_{L^{p_1}(\mathbb{R}^2)} \\ &\leq C \|n(\cdot,t)\|_{L^{2+\varepsilon}(\mathbb{R}^2)} \|n(\cdot,t)\|_{L^{q_1}(\mathbb{R}^2)} \,.\end{aligned}$$

which is bounded as $q_1 \in [1, 2+\varepsilon]$. Thus, if n is a solution of (1.1), $n\nabla c$ is bounded in $L^\infty(\mathbb{R}^+_{\rm loc}; L^2(\mathbb{R}^2))$.

Third Step: Compactness. As a consequence of Hölder's inequality with $p := (1+\varepsilon)/\varepsilon$, $q := 1+\varepsilon$:

$$\int_{\mathbb{R}^2} |x|^{\frac{2\varepsilon}{1+\varepsilon}} n^2\, dx = \int_{\mathbb{R}^2} (n|x|^2)^{\frac{\varepsilon}{1+\varepsilon}} \cdot n^{\frac{2+\varepsilon}{1+\varepsilon}}\, dx \leq \Big(\int_{\mathbb{R}^2} n|x|^2\, dx \Big)^{\frac{\varepsilon}{1+\varepsilon}} \Big(\int_{\mathbb{R}^2} n^{2+\varepsilon}\, dx \Big)^{\frac{1}{1+\varepsilon}} \,,$$

the function $(x,t) \mapsto |x|^{\frac{\varepsilon}{1+\varepsilon}} n$ is bounded in $L^\infty(\mathbb{R}^+_{\rm loc}; L^2(\mathbb{R}^2))$. The imbedding of the set $V := \{u \in H^1 \cap L^1_+(\mathbb{R}^2) : |x|^{\frac{\varepsilon}{1+\varepsilon}} u \in L^1(\mathbb{R}^2)\}$ into $L^2(\mathbb{R}^2) =: H$ is compact and by the Aubin-Lions compactness method (see Lemma 2.9) as in Section 2.5, it results that n belongs to a compact set of $L^2(\mathbb{R}^+_{\rm loc} \times \mathbb{R}^2)$.

Let $(n_k)_{k\in\mathbb{N}} := (n^{\varepsilon_k})_{k\in\mathbb{N}}$ be an approximating sequence defined as in the proof of Theorem 1.1. Compared to the results of Lemma 2.12, we have

$$\begin{gathered}\iint_{[0,T]\times\mathbb{R}^2} |\nabla n|^2\, dx\, dt \leq \liminf_{k\to\infty} \iint_{[0,T]\times\mathbb{R}^2} |\nabla n_k|^2\, dx\, dt \,, \\ \iint_{[0,T]\times\mathbb{R}^2} n|\nabla c|^2\, dx\, dt \leq \liminf_{k\to\infty} \iint_{[0,T]\times\mathbb{R}^2} n_k|\nabla c_k|^2\, dx\, dt \,, \\ \iint_{[0,T]\times\mathbb{R}^2} n^2\, dx\, dt = \liminf_{k\to\infty} \iint_{[0,T]\times\mathbb{R}^2} |n_k|^2\, dx\, dt \,,\end{gathered}$$

where the only difference lies in the last equality, a consequence of the above compactness result. This proves the free energy estimate using

$$\begin{aligned}&\iint_{[[0,T]\times\mathbb{R}^2} n\, |\nabla(\log n) - \chi\nabla c|^2\, dx\, dt \\ &= 4 \iint_{[[0,T]\times\mathbb{R}^2} |\nabla\sqrt{n}|^2\, dx\, dt + \chi^2 \iint_{[[0,T]\times\mathbb{R}^2} n|\nabla c|^2\, dx\, dt - 2\chi \iint_{[[0,T]\times\mathbb{R}^2} n^2\, dx\, dt \,.\end{aligned}$$

□

3.4. **Hypercontractivity.** Much more regularity can actually be achieved as follows. All computations are easy to justify for smooth solutions with sufficient decay at infinity. Up to a regularization step, the final estimates certainly hold if the initial data is bounded in $L^\infty(\mathbb{R}^2)$, which is the case for the regularized problem of Section 2.5 with truncated initial data $n_0^\varepsilon = \min\{n_0, \varepsilon^{-1}\}$. However, we will see that the $L^\infty(\mathbb{R}^+_{\rm loc}; L^p(\mathbb{R}^2))$-estimates hold for any $p > 1$ independently of ε, so that we may pass to the limit and get the result for any solution of (1.1) with initial data satisfying only (2.1) and $\chi M < 8\pi$. To simplify the presentation of the method, we will therefore do the computations only at a formal level, for smooth solutions which behave well at infinity.

Theorem 3.5. *Consider a solution n of* (1.1) *with initial data n_0 satisfying* (2.1) *and $\chi M < 8\pi$. Then for any $p \in (1,\infty)$, there exists a continuous function h_p on $(0,\infty)$ such that for almost any $t>0$, $\|n(\cdot,t)\|_{L^p(\mathbb{R}^2)} \le h_p(t)$.*

Notice that unless n_0 is bounded in $L^p(\mathbb{R}^2)$, $\lim_{t\to 0_+} h_p(t) = +\infty$. Such a result is called an *hypercontractivity* result, see [26], since to an initial data which is originally in $L^1(\mathbb{R}^2)$ but not in $L^p(\mathbb{R}^2)$, we associate a solution which at almost any time $t>0$ is in $L^p(\mathbb{R}^2)$ with p arbitrarily large.

Proof. Fix $t>0$ and $p\in(1,\infty)$, and consider $q(s) := 1+(p-1)\frac{s}{t}$, so that $q(0)=1$ and $q(t)=p$. Exactly as in the proof of Theorem 1.1, for an arbitrarily small $\eta>0$ given in advance, we can find $K>1$ big enough such that $M(K) := \sup_{s\in(0,t)} \int_{n>K} n(\cdot,s)\,dx$ is smaller than η. It is indeed sufficient to notice that

$$\int_{n>K} n(\cdot,s)\,dx \le \frac{1}{\log K}\int_{n>K} n(\cdot,s)\log n(\cdot,s)\,dx \le \frac{1}{\log K}\int_{\mathbb{R}^2} |n(\cdot,s)\log n(\cdot,s)|\,dx\,.$$

Since $\chi M < 8\pi$, $n|\log n|$ is bounded in $L^\infty(0,t;L^1(\mathbb{R}^2))$. This proves that for K big enough, we may assume

$$\int_{\mathbb{R}^2} (n-K)_+\,dx \le \eta\,,$$

for an arbitrarily small $\eta>0$. Next, we define

$$F(s) := \Big[\int_{\mathbb{R}^2} (n-K)_+^{q(s)}(x,s)\,dx\Big]^{1/q(s)}$$

for the function $s\mapsto q(s)$ defined above. A derivation with respect to s gives

$$F'F^{q-1} = \frac{q'}{q^2}\int_{\mathbb{R}^2} (n-K)_+^q \log\Big(\frac{(n-K)_+^q}{F^q}\Big) + \int_{\mathbb{R}^2} n_t (n-K)_+^{q-1}\,.$$

If n is a solution to (1.1), then

$$\int_{\mathbb{R}^2} (n-K)_+^{q-1} n_t\,dx = -4\,\frac{q-1}{q^2}\int_{\mathbb{R}^2} |\nabla((n-K)_+^{q/2})|^2\,dx + \chi\,\frac{q-1}{q}\int_{\mathbb{R}^2} (n-K)_+^{q+1}\,dx\,,$$

and we get

$$\begin{aligned} F'F^{q-1} = {}& \frac{q'}{q^2}\int_{\mathbb{R}^2} (n-K)_+^q \log\Big(\frac{(n-K)_+^q}{F^q}\Big) - 4\frac{q-1}{q^2}\int_{\mathbb{R}^2} |\nabla((n-K)_+^{q/2})|^2 \\ & + \chi\frac{(q-1)}{q}\int_{\mathbb{R}^2} (n-K)_+^{q+1}\,. \end{aligned}$$

Using the assumption $q'\ge 0$, we can apply the logarithmic Sobolev inequality [26]

$$\int_{\mathbb{R}^2} v^2 \log\Big(\frac{v^2}{\int_{\mathbb{R}^2} v^2\,dx}\Big)\,dx \le 2\sigma\int_{\mathbb{R}^2} |\nabla v|^2\,dx - (2+\log(2\pi\sigma))\int_{\mathbb{R}^2} v^2\,dx$$

for any $\sigma>0$, and the Gagliardo-Nirenberg-Sobolev inequality

$$\int_{\mathbb{R}^2} |v|^{2(1+1/q)}\,dx \le \mathcal{K}(q)\|\nabla v\|^2_{L^2(\mathbb{R}^2)}\int_{\mathbb{R}^2} |v|^{2/q}\,dx\,, \quad \forall\, q\in[2,\infty)$$

to $v := (n-K)_+^{q/2}$, and obtain

$$F'F^{q-1} \le \Big(\frac{2\sigma q'}{q^2} - 4\frac{q-1}{q^2} + \chi\,\frac{q-1}{q}\mathcal{K}(q)\eta\Big)\|\nabla v\|^2_{L^2(\mathbb{R}^2)} - \frac{q'}{q^2}(2+\log(2\pi\sigma))F^q\,.$$

With the specific choice of $\sigma := (q-1)/q'$ and provided η is chosen small enough in order that

$$-2\frac{q-1}{q^2} + \chi\,\frac{q-1}{q}\sup_{r\in(1,p)}[\mathcal{K}(r)]\,\eta \le 0\,,$$

this shows that

$$\frac{F'}{F} \le -\frac{q'}{q^2}(2+\log(2\pi\sigma)) =: G(t)\,.$$

The function G is integrable on $(0,t)$, which proves that $F(t)$ can be bounded in terms of $F(0)$. □

3.5. **The free energy inequality for weak solutions.** As a consequence of Lemma 3.4 and Theorem 3.5, we have the following result.

Corollary 3.6. *Let $(n^k)_{k\in\mathbb{N}}$ be a sequence of solutions of* (1.1) *with initial data n_0^k satisfying Assumption* (2.1) *with uniform corresponding bounds. For any $t_0>0$, $T\in\mathbb{R}^+$ such that $0<t_0<T$, $(n^k)_{k\in\mathbb{N}}$ is relatively compact in $L^2((t_0,T)\times\mathbb{R}^2)$, and if n is the limit of $(n^k)_{k\in\mathbb{N}}$, then n is a solution of* (1.1) *such that the free energy inequality* (1.2) *holds.*

Proof. By Theorem 3.5, for $t>t_0>0$, n^k is bounded in $L^\infty(t_0,t;L^{2+\varepsilon}(\mathbb{R}^2))$, for any $\varepsilon>0$. We can therefore apply Lemma 3.4 with initial data $n^k(\cdot,t_0)$ at $t=t_0$:

$$F[n^k(\cdot,t)] + \int_{t_0}^t \left(\int_{\mathbb{R}^2} n^k \left|\nabla\left(\log n^k\right) - \chi\nabla c^k\right|^2\,dx\right)ds \le F[n^k(\cdot,t_0)]\,.$$

The compactness in $L^2([t_0,t]\times\mathbb{R}^2)$ follows from Lemma 2.9. Passing to the limit as $k\to\infty$, we get

$$F[n(\cdot,t)] + \int_{t_0}^t \left(\int_{\mathbb{R}^2} n\left|\nabla\left(\log n\right) - \chi\nabla c\right|^2\,dx\right)ds \le F[n(\cdot,t_0)]\,.$$

Since, as a function of s, $\int_{\mathbb{R}^2} n(\cdot,s)\left|\nabla\left(\log n(\cdot,s)\right) - \chi\nabla c(\cdot,s)\right|^2\,dx$ is integrable on $(0,t)$, we can pass to the limit $t_0\to 0$. By convexity of $n\mapsto n\log n$, it is easy to check that $\lim_{t_0\to 0_+} F[n(\cdot,t_0)] \le F[n_0]$. □

Apply Corollary 3.6 with $n_0^k = \min\{n_0, \varepsilon_k^{-1}\}$ as in the regularization procedure of Section 2.5–2.7. This completes the proof of Theorem 1.1.

4. Intermediate asymptotics and self-similar solutions

In this section, we investigate the behavior of the solutions as time t goes to infinity and prove Theorem 1.2. The key tool is the free energy written in rescaled variables, F^R, which is defined below. The main difficulty comes from the fact that the uniqueness of the solutions to (1.3) for a given $M\in(0,8\pi/\chi)$ is not known. This is not crucial for the proof of Theorem 1.2 because, in the self-similar variables, the decay of the entropy selects a unique solution to (1.3). In this section, we will anyway prove several additional properties (radial symmetry, regularity, decay at infinity) of the solution of (1.3) and comment on related issues.

4.1. **Self-similar variables.** Assume that $\chi M < 8\pi$, consider a solution of (1.1) and define the rescaled functions u and v by:

$$n(x,t) = \frac{1}{R^2(t)} u\big(\frac{x}{R(t)}, \tau(t)\big) \quad \text{and} \quad c(x,t) = v\big(\frac{x}{R(t)}, \tau(t)\big) \tag{4.1}$$

with

$$R(t) = \sqrt{1+2t} \quad \text{and} \quad \tau(t) = \log R(t)\,.$$

The rescaled system is

$$\begin{gathered} \frac{\partial u}{\partial t} = \Delta u - \nabla\cdot(u(x+\chi\nabla v)) \quad x\in\mathbb{R}^2\,,\ t>0\,,\\ v = -\frac{1}{2\pi}\log|\cdot| * u \quad x\in\mathbb{R}^2\,,\ t>0\,,\\ u(\cdot,t=0) = n_0 \geq 0 \quad x\in\mathbb{R}^2\,. \end{gathered} \tag{4.2}$$

The free energy now takes the form

$$F^R[u] := \int_{\mathbb{R}^2} u\log u\,dx - \frac{\chi}{2}\int_{\mathbb{R}^2} uv\,dx + \frac{1}{2}\int_{\mathbb{R}^2}|x|^2 u\,dx\,.$$

If (u,v) is a smooth solution of (4.2) which decays sufficiently at infinity, then

$$\frac{d}{dt}F^R[u(\cdot,t)] = -\int_{\mathbb{R}^2} u\,|\nabla\log u - \chi\nabla v + x|^2\,dx\,.$$

Because of the hypercontractivity, the above inequality holds as an inequality for the solution of Theorem 1.1 after rescaling:

$$\frac{d}{dt}F^R[u(\cdot,t)] \leq -\int_{\mathbb{R}^2} u\,|\nabla\log u - \chi\nabla v + x|^2\,dx\,.$$

For a rigorous proof, one has to redo the argument of Section 3.4. Since there is no additional difficulty this is left to the reader.

4.2. **The self-similar solution.** System (4.2) has the interesting property that for $\chi M < 8\pi$, it has a stationary solution which minimizes the free energy.

Lemma 4.1. *The functional F^R is bounded from below on the set*

$$\big\{u\in L^1_+(\mathbb{R}^2) : |x|^2 u\in L^1(\mathbb{R}^2)\,,\ \int_{\mathbb{R}^2} u\log u\,dx < \infty\big\}$$

if and only if $\chi\|u\|_{L^1(\mathbb{R}^2)} \leq 8\pi$.

Proof. If $\chi\|u\|_{L^1(\mathbb{R}^2)} \leq 8\pi$, the result is a straightforward consequence of Lemma 2.4. Notice that by Lemma 2.11, (iii), $u\log u$ is then bounded in $L^1(\mathbb{R}^2)$.

The functional $F^R[u]$ has an interesting scaling property. For a given u, let $u_\lambda(x) = \lambda^{-2}u(\lambda^{-1}x)$. It is straightforward to check that $\|u_\lambda\|_{L^1(\mathbb{R}^2)} =: M$ does not depend on $\lambda > 0$ and

$$F^R[u_\lambda] = F^R[u] - 2M\big(1-\frac{\chi M}{8\pi}\big)\log\lambda + \frac{\lambda-1}{2}\int_{\mathbb{R}^2}|x|^2 u\,dx\,.$$

As a function of λ, $F^R[u_\lambda]$ is clearly bounded from below if $\chi M < 8\pi$, and not bounded from below if $\chi M > 8\pi$, which completes the proof. $\square$

The free energy has a minimum which is a radial stationary solution of (4.2), see [13]. Such a solution is of course a natural candidate for the large time asymptotics of any solution of (4.2). In [13], there are also indications that (1.3) should have a unique solution for any given M, and there are strong numerical evidences supporting this fact. However, we are not able to discard the possibility that more than one solution to (1.3) exists for any given M.

Lemma 4.2. *Let $\chi M < 8\pi$. If u is a solution of (4.2), with initial data u_0 satisfying Assumptions (2.1), corresponding to a solution of (1.1) as given in Theorem 1.1, then as $t \to \infty$, $(s,x) \mapsto u(x,t+s)$ converges in $L^\infty(0,T;L^1(\mathbb{R}^2))$ for any positive T to a solution of (1.3) which is a stationary solution of (4.2) and moreover satisfies:*

$$\lim_{t\to\infty} \int_{\mathbb{R}^2} |x|^2 u(x,t)\,dx = \int_{\mathbb{R}^2} |x|^2 u_\infty\,dx = 2M\big(1-\frac{\chi M}{8\pi}\big)\,. \tag{4.3}$$

Proof. We use the free energy production term

$$F^R[u_0] - \liminf_{t\to\infty} F^R[u(\cdot,t)] = \lim_{t\to\infty} \int_0^t \Big(\int_{\mathbb{R}^2} u\,|\nabla \log u - \chi\nabla v + x|^2\,dx\Big)\,ds\,.$$

As a consequence,

$$\lim_{t\to\infty} \int_t^\infty \Big(\int_{\mathbb{R}^2} u\,|\nabla \log u - \chi\nabla v + x|^2\,dx\Big)\,ds = 0\,, \tag{4.4}$$

which shows that, up to the extraction of subsequences, the limit u_∞ of $u(\cdot,t+\cdot)$, which exists for the same reasons as in the proof of Theorem 1.1, satisfies

$$\nabla \log u_\infty - \chi\nabla v_\infty + x = 0\,, \quad v_\infty = -\frac{1}{2\pi}\log|\cdot| * u_\infty\,,$$

where the first equation holds at least a.e. in the support of u_∞. This is equivalent to write that (u_∞, v_∞) solves (1.3). Notice that the limit is unique because of (4.4) even if the uniqueness of the solutions of (1.3) is not established. Because of (4.4), we also know that u_∞ does not depend on the choice of the subsequence.

As in the proof of Lemma 2.1, consider a smooth function $\varphi_\varepsilon(|x|)$ with compact support that grows nicely to $|x|^2$ as $\varepsilon \to 0$. If (u,v) is a solution to (4.2), we compute

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2} \varphi_\varepsilon u\,dx \\
&= \int_{\mathbb{R}^2} \Delta\varphi_\varepsilon u\,dx - \frac{\chi}{4\pi}\int_{\mathbb{R}^2} \frac{(\nabla\varphi_\varepsilon(x) - \nabla\varphi_\varepsilon(y))\cdot(x-y)}{|x-y|^2} u(x,t)u(y,t)\,dx\,dy \\
&\quad - 2\int_{\mathbb{R}^2} |x|^2 u\,dx\,.
\end{aligned}$$

Since ε vanishes we may pass to the limit and obtain

$$\frac{d}{dt}\int_{\mathbb{R}^2} |x|^2 u\,dx = 4M\big(1-\frac{\chi M}{8\pi}\big) - 2\int_{\mathbb{R}^2} |x|^2 u\,dx\,.$$

This proves that for any $t>0$,

$$\int_{\mathbb{R}^2} |x|^2 u(x,t)\,dx = \int_{\mathbb{R}^2} |x|^2 n_0\,dx\; e^{-2t} + 2M\big(1-\frac{\chi M}{8\pi}\big)\,(1-e^{-2t})\,.$$

Passing to the limit $t \to \infty$, we get

$$\int_{\mathbb{R}^2} |x|^2 u_\infty \, dx \le 2M\big(1 - \frac{\chi M}{8\pi}\big)\,.$$

However, u_∞ is a solution of Equation (4.2), which satisfies the same assumptions as n_0. Since it is a stationary solution with finite second moments, we have

$$\int_{\mathbb{R}^2} |x|^2 u_\infty \, dx = 2M\big(1 - \frac{\chi M}{8\pi}\big)\,.$$

$\square$

Notice that under the constraint $\|u_\infty\|_{L^1(\mathbb{R}^2)} = M$, u_∞ is a critical point of the free energy. If we knew that (1.3) has at most one solution for a given $M > 0$, u_∞ would automatically be the unique minimizer of the free energy. This result is not known although one can establish that u_∞ is radially symmetric. This is done using the two following results, Lemmata 4.3 and 4.4.

Lemma 4.3. *Let $u \in L^1_+(\mathbb{R}^2, (1+|x|^2)\, dx)$ with $M := \int_{\mathbb{R}^2} u\, dx$ and $\int_{\mathbb{R}^2} u \log u \, dx < \infty$. Define*

$$v(x) := -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log|x-y| u(y)\, dy\,.$$

Then there exists a positive constant C such that, for any $x \in \mathbb{R}^2$ with $|x| > 1$,

$$\big|\ v(x) + \frac{M}{2\pi} \log|x|\big| \le C\,.$$

Note that as a straightforward consequence, v is non-positive outside of a ball.

Proof. We estimate

$$\big|v(x) + \frac{M}{2\pi}\log|x|\big| = \big|\frac{1}{2\pi}\int_{\mathbb{R}^2} \log\big(\frac{|x-y|}{|x|}\big) u(y)\, dy\big| \le \text{(I)} + \text{(II)} + \text{(III)}$$

where

$$\text{(I)} := -\frac{1}{2\pi}\int_{\Omega_{\rm I}} \log\big(\frac{|x-y|}{|x|}\big) u(y)\, dy \quad \text{with } \Omega_{\rm I} := \{(x,y) \in \mathbb{R}^2 : \frac{|x-y|}{|x|} \le \frac{1}{2}\}$$

$$\text{(II)} := \big|\frac{1}{2\pi}\int_{\Omega_{\rm II}} \log\big(\frac{|x-y|}{|x|}\big) u(y)\, dy\big| \quad \text{with } \Omega_{\rm II} := \{(x,y) \in \mathbb{R}^2 : \frac{1}{2} < \frac{|x-y|}{|x|} \le 2\}$$

$$\text{(III)} := \frac{1}{2\pi}\int_{\Omega_{\rm III}} \log\big(\frac{|x-y|}{|x|}\big) u(y)\, dy \quad \text{with } \Omega_{\rm III} := \{(x,y) \in \mathbb{R}^2 : \frac{|x-y|}{|x|} > 2\}\,.$$

Using $|x-y|^2 \le 2(|x|^2 + |y|^2)$ and $\log(1+t) \le t$, we get

$$\begin{aligned} 4\pi\text{(III)} &= \int_{\Omega_{\rm III}} \log\big(\frac{|x-y|^2}{|x|^2}\big) u(y)\, dy \\ &\le \int_{\Omega_{\rm III}} \log\big(2 + 2\frac{|y|^2}{|x|^2}\big) u(y)\, dy \le M + \frac{2}{|x|^2}\int_{\Omega_{\rm III}} |y|^2 u(y)\, dy\,. \end{aligned}$$

On $\Omega_{\rm II}$, $|\log(|x-y|/|x|)\,|$ is bounded by $\log 2$: (II) $\le M\frac{\log 2}{2\pi}$. For the last term, denote $z_x(y) = \frac{|x|}{|x-y|}$:

$$\text{(I)} = \frac{1}{2\pi}\int_{\Omega_{\rm I}} \log\big(z_x(y)\big)\, u(y)\, dy\,.$$

By Jensen's inequality

$$\int_{\Omega_{\rm I}} u(y)\log\Big(\frac{u(y)}{z_x(y)}\Big)\,dy \geq \int_{\Omega_{\rm I}} u(y)\log\Big(\frac{\int_{\Omega_{\rm I}} u(y)\,dy}{\int_{\Omega_{\rm I}} z_x(y)\,dy}\Big)\,dy\,,$$

we get

$$2\pi({\rm I}) \leq \int_{\Omega_{\rm I}} u(y)\log(u(y))\,dy - \int_{\Omega_{\rm I}} u(y)\log\Big(\frac{\int_{\Omega_{\rm I}} u(y)\,dy}{\int_{\Omega_{\rm I}} z_x(y)\,dy}\Big)\,dy\,.$$

The right hand side is bounded since $u\log u$ is bounded in $L^1(\mathbb{R}^2)$ by Lemma 2.6,

$$\int_{\Omega_{\rm I}} z_x(y)\,dy = \int_{\Omega_{\rm I}} \frac{|x|}{|x-y|}\,dy = \pi|x|^2\,,$$
$$\int_{\Omega_{\rm I}} u(y)\,dy \leq \frac{4}{|x|^2}\int_{\Omega_{\rm I}} |y|^2 u(y)\,dy\,.$$

Hence we can control (I) because $\int_{\Omega_{\rm I}} u(y)\,dy \log\Big(\int_{\Omega_{\rm I}} z_x(y)\,dy\Big) \leq \frac{4}{|x|^2}\log\big(\pi|x|^2\big)$. $\square$

This suffices to prove that the solution is radially symmetric, see [46].

Lemma 4.4 ([46])**.** *Assume that V is a non-negative non-trivial radial function on $\mathbb{R}^2$ such that $\lim_{|x|\to\infty}|x|^\alpha V(x)<\infty$ for some $\alpha\geq 0$. If u is a solution of*

$$\Delta u + V(x)e^u = 0 \quad x\in\mathbb{R}^2$$

such that $u_+\in L^\infty(\mathbb{R}^2)$, then u is radially symmetric about the origin and $x\cdot\nabla u(x)<0$ for any $x\in\mathbb{R}^2$.

Note here that because of the asymptotic logarithmic behavior of v_∞, the result of Gidas, Ni and Nirenberg, [24], does not directly apply. The boundedness from above is essential, otherwise non-radial solutions can be found, even with no singularity. Consider for instance the perturbation $\Theta(x)=\frac12\theta(x_1^2-x_2^2)$ for any $x=(x_1,x_2)$, for some fixed $\theta\in(0,1)$, and define the potential $\phi(x)=\frac12|x|^2-\Theta(x)$. By a fixed-point method we can find a solution of

$$w(x) = -\frac{1}{2\pi}\log|\cdot| * M\frac{e^{\chi w-\phi(x)}}{\int_{\mathbb{R}^2} e^{\chi w(y)-\phi(y)}\,dy}$$

since, as $|x|\to\infty$, $\phi(x)\sim\frac12\big[(1-\theta)x_1^2+(1+\theta)x_2^2\big]\to+\infty$. This solution is such that $w(x)\sim-\frac{M}{2\pi}\log|x|$ for reasons similar to the ones of Lemma 4.3. Hence $v(x):=w(x)+\Theta(x)/\chi$ is a non-radial solution of the above equation with $\log V(x)=-\frac12|x|^2$, which behaves like $\Theta(x)/\chi$ as $|x|\to\infty$ with $|x_1|\neq|x_2|$. This gives a non radial solution of Equation (1.3).

Lemma 4.5. *If $\chi M>8\pi$, Equation* (4.2) *has no stationary solution (u_∞,v_∞) such that $\|u_\infty\|_{L^1(\mathbb{R}^2)}=M$ and $\int_{\mathbb{R}^2}|x|^2u_\infty\,dx<\infty$. If $\chi M<8\pi$, Equation* (4.2) *has at least one radial stationary solution given by* (1.3)*. This solution is C^∞ and u_∞ is dominated as $|x|\to\infty$ by $e^{-(1-\varepsilon)|x|^2/2}$ for any $\varepsilon\in(0,1)$.*

Proof. The existence of a stationary solution if $\chi M<8\pi$ is easy. It follows from Lemma 4.2 but can also be achieved by minimizing the free energy, see [13]. If the initial condition is radial or if the minimization is done among radial solutions, then the stationary solution is also radial. Direct approaches (fixed-point methods, ODE shooting methods) can also be used.

If $\chi M > 8\pi$ and if there was a stationary solution with finite second moment, we could write

$$0 = \frac{d}{dt} \int_{\mathbb{R}^2} |x|^2 u_\infty \, dx = 4M\big(1 - \frac{\chi M}{8\pi}\big) - 2 \int_{\mathbb{R}^2} |x|^2 u_\infty \, dx \, .$$

Since the right hand side is negative, this is simply impossible. □

In the rescaled variables, the solution of (4.2) converges to a radial stationary solution u_∞ of (1.3). It is not difficult to check that $\bar{n}(x,t) := \frac{1}{2t} u_\infty \left(\frac{1}{2}\log(2t), x/\sqrt{2t}\right)$ and $\bar{c}(x,t) := v_\infty \left(\frac{1}{2}\log(2t), x/\sqrt{2t}\right)$ gives a self-similar solution of (1.1), which is supposed to describe the large time asymptotics of (1.1), and this is what we are going to clarify in the last section.

4.3. **Intermediate asymptotics.**

Lemma 4.6. *Under the assumptions of Lemma 4.2,*

$$\lim_{t\to\infty} F^R[u(\cdot, \cdot + t)] = F^R[u_\infty] \, .$$

Proof. By (4.3), we already know that $\lim_{t\to\infty} \int_{\mathbb{R}^2} |x|^2 u(x,t)\, dx = \int_{\mathbb{R}^2} |x|^2 u_\infty \, dx$. Using the estimates of Sections 2.5–2.7 and Lemma 2.9, we know that $u(\cdot, \cdot + t)$ converges to u_∞ in $L^2((0,1)\times\mathbb{R}^2)$ and that $\int_{\mathbb{R}^2} u(\cdot, \cdot + t) v(\cdot, \cdot + t)\, dx$ converges to $\int_{\mathbb{R}^2} u_\infty v_\infty \, dx$. Concerning the entropy, it is sufficient to prove that $u(\cdot, \cdot + t)\log u(\cdot, \cdot + t)$ weakly converges in $L^1((0,1)\times\mathbb{R}^2)$ to $u_\infty \log u_\infty$. By Lemma 2.6, there is a uniform L^1 bound. Concentration is prohibited by the convergence in $L^2((0,1)\times\mathbb{R}^2)$. Vanishing or dichotomy cannot occur either: Take indeed $R > 0$, large, and compute $\int_{|x|>R} u|\log u| = (\mathrm{I}) + (\mathrm{II})$, with

$$(\mathrm{I}) = \int_{|x|>R,\ u\geq 1} u \log u \, dx \leq \frac{1}{2} \int_{|x|>R,\ u\geq 1} |u|^2 \, dx \, ,$$

$$(\mathrm{II}) = -\int_{|x|>R,\ u<1} u \log u \, dx \leq \frac{1}{2} \int_{|x|>R,\ u<1} |x|^2 u \, dx - m \log\left(\frac{m}{2\pi}\right) \, .$$

In the first case, we have used the inequality $u \log u \leq u^2/2$ for any $u \geq 1$, while the second estimate is based on Jensen's inequality in the spirit of the proof of Lemma 2.6:

$$m := \int_{|x|>R,\ u<1} u \, dx \leq \frac{1}{R^2} \int_{|x|>R,\ u<1} |x|^2 u \, dx \, .$$

Because of the convergence of the two quantities $\int_{|x|>R,\ u<1} |u|^2 \, dx$ and $\int_{|x|>R,\ u<1} |x|^2 u \, dx$ to 0 as $R \to \infty$, we have the uniform estimate

$$\lim_{R\to\infty} \int_{|x|>R} u|\log u| = 0 \, ,$$

which completes the proof. □

The result we have shown above is actually slightly better, since it proves that all terms in the free energy, namely the entropy, the energy corresponding to the potential $\frac{1}{2}|x|^2$ and the self-consistent potential energy, converge to the corresponding values for the limiting stationary solution.

As noted above, u_∞ is a critical point of F^R under the constraint $\|u\|_{L^1(\mathbb{R}^2)} = M$. We can therefore rewrite $F^R[u] - F^R[u_\infty]$ as

$$F^R[u] - F^R[u_\infty] = \int_{\mathbb{R}^2} u \log\big(\frac{u}{u_\infty}\big)\,dx - \frac{\chi}{2}\int_{\mathbb{R}^2} |\nabla v - \nabla v_\infty|^2\,dx\,,$$

and both terms in the above expression converge to 0 as $t \to \infty$, if u is a solution of (1.1). Since for any nonnegative functions f, $g \in L^1(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2} f\,dx = \int_{\mathbb{R}^2} g\,dx = M$,

$$\|f-g\|^2_{L^1(\mathbb{R}^2)} \le \frac{1}{4M}\int_{\mathbb{R}^2} f \log\big(\frac{f}{g}\big)\,dx$$

by the Csiszár-Kullback inequality, [21, 36], this proves the following statement.

Corollary 4.7. *Under the assumptions of Lemma 4.2,*

$$\lim_{t\to\infty} \|u(\cdot, \cdot + t) - u_\infty\|_{L^1(\mathbb{R}^2)} = 0, \quad \lim_{t\to\infty} \|\nabla v(\cdot, \cdot + t) - \nabla v_\infty\|_{L^2(\mathbb{R}^2)} = 0\,.$$

Undoing the change of variables (4.1), this proves Theorem 1.2.

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(Adrien Blanchet) CEREMADE (UMR CNRS no. 7534), Université Paris Dauphine, Place de Lattre de Tassigny, 75775 Paris Cédex 16, France; and CERMICS, ENPC, 6–8 avenue Blaise Pascal, Champs-sur-Marne, 77455 Marne-la-Vallée Cédex 2, France
E-mail address: `blanchet@ceremade.dauphine.fr`
URL: `http://www.ceremade.dauphine.fr/~blanchet/`

(Jean Dolbeault) CEREMADE (UMR CNRS no. 7534), Université Paris Dauphine, Place de Lattre de Tassigny, 75775 Paris Cédex 16, France
Phone: (33) 1 44 05 46 78, Fax: (33) 1 44 05 45 99
E-mail address: `dolbeaul@ceremade.dauphine.fr`
URL: `http://www.ceremade.dauphine.fr/~dolbeaul/`

(Benoît Perthame) DMA (UMR CNRS no. 8553), Ecole Normale Supérieure, 45 rue d'Ulm, 75005 Paris Cédex 05, France

E-mail address: Benoit.Perthame@ens.fr

URL: http://www.dma.ens.fr/users/perthame