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POSITIVE SOLUTIONS FOR SINGULAR SEMI-POSITONE NEUMANN BOUNDARY-VALUE PROBLEMS

YONG-PING SUN, YAN SUN

Abstract. In this paper, we study the singular semi-positone Neumann boundary-value problem

$$
\begin{gathered}
-u'' + m^2 u = \lambda f(t,u) + g(t,u), \quad 0 < t < 1, \\
u'(0) = u'(1) = 0,
\end{gathered}
$$

where m is a positive constant. Under some suitable assumptions on the functions f and g, for sufficiently small λ, we prove the existence of a positive solution. Our approach is based on the Krasnasel'skii fixed point theorem in cones.

1. Introduction

In this paper, we shall study the following singular semi-positone Neumann boundary-value problem (NBVP)

$$
\begin{gathered}
-u'' + m^2 u = \lambda f(t,u) + g(t,u), \quad 0 < t < 1, \\
u'(0) = u'(1) = 0,
\end{gathered}
\tag{1.1}
$$

where $m > 0$ is a constant, $\lambda > 0$ is a parameter, $f : (0,1) \times [0,+\infty) \to [0,+\infty)$ and $g : [0,1] \times [0,+\infty) \to (-\infty,+\infty)$ are continuous.

We say problem (1.1) is singular because f may be singular at $t = 0$ and/or $t = 1$. When $g(t,u) \not\equiv 0$, problem (1.1) is a semi-positone problem, this situation arises naturally in chemical reaction theory [8]. In recent years, attention has been paid to (1.1) in the case of $g(t,u) \equiv 0$; see, for example, [11, 12, 13] and the references therein. Attention has been paid also to the semi-positone boundary-value problem; see, for example, [6, 7, 9] and the references therein. As far as the authors know, there are no existence results for the singular semi-positone NBVP. Recently, Xu [9] studied the existence of positive solutions for the singular semi-positone boundary-value problem

$$
\begin{gathered}
u'' + f(t,u) + q(t) = 0, \quad 0 < t < 1, \\
u(0) = u(1) = 0,
\end{gathered}
$$

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where $f : (0,1) \times [0,+\infty) \to [0,+\infty)$ and $q : (0,1) \to (-\infty,+\infty)$ are continuous.

Motivated by the papers mentioned above, we study the existence of positive solutions for the singular semi-positone NBVP (1.1), and give an explicit interval for λ. Our results can be regarded as an extension and improvement of the corresponding results of [12, 13]. The paper is organized as follows. In Section 2, we present some lemmas that will be used to prove the main result and Krasnasel'skii fixed point theorem in cones. In Section 3, we prove the main result of this paper.

2. Preliminaries

We consider problem in the Banach space $E = C[0,1]$ equipped with the norm $\|u\| = \sup_{t\in[0,1]} |u(t)|$. Let $G(t,s)$ be the Green's function for the Boundary-value problem

$$\begin{gathered} -u'' + m^2 u = 0, \quad 0 < t < 1, \\ u'(0) = u'(1) = 0\,. \end{gathered} \tag{2.1}$$

Then

$$G(t,s) = \frac{1}{\rho}\begin{cases} \varphi(s)\varphi(1-t), & 0 \le s \le t \le 1, \\ \varphi(t)\varphi(1-s), & 0 \le t \le s \le 1, \end{cases}$$

where $\rho = \frac{1}{2}m(e^m - e^{-m})$, $\varphi(t) = \frac{1}{2}(e^{mt} + e^{-mt})$. It is obvious that $\varphi(t)$ is increasing on $[0,1]$, and

$$G(t,s) \le G(s,s), \quad 0 \le t, s \le 1.$$

Lemma 2.1. *Let $G(t,s)$ be the Green's function for the NBVP* (2.1).

(1) *Assume that $0 < \theta < \frac{1}{2}$, then*

$$G(t,s) \ge M_\theta G(s,s), \quad \theta \le t \le 1-\theta,\ 0 \le s \le 1.$$

where

$$M_\theta = \frac{e^{m\theta} + e^{-m\theta}}{e^m + e^{-m}}.$$

(2)

$$G(t,s) \ge C\varphi(t)\varphi(1-t)G(t_0,s), \quad t, t_0, s \in [0,1],$$

where $C = 1/\varphi^2(1)$.

Proof. (1) Let $t \in [\theta, 1-\theta]$. For $s \le t$,

$$\frac{G(t,s)}{G(s,s)} = \frac{\varphi(1-t)}{\varphi(1-s)} \ge \frac{\varphi(\theta)}{\varphi(1)} = \frac{e^{m\theta} + e^{-m\theta}}{e^m + e^{-m}} = M_\theta.$$

If $t \le s$, then

$$\frac{G(t,s)}{G(s,s)} = \frac{\varphi(t)}{\varphi(s)} \ge \frac{\varphi(\theta)}{\varphi(1)} = \frac{e^{m\theta} + e^{-m\theta}}{e^m + e^{-m}} = M_\theta.$$

Thus

$$G(t,s) \ge M_\theta G(s,s), \quad \theta \le t \le 1-\theta,\ 0 \le s \le 1.$$

(2) When $t, t_0 \le s$,

$$\begin{aligned} \frac{G(t,s)}{G(t_0,s)} &= \frac{\varphi(t)\varphi(1-s)}{\varphi(t_0)\varphi(1-s)} = \frac{\varphi(t)\varphi(1-t)}{\varphi(t_0)\varphi(1-t)} \\ &\ge \frac{1}{\varphi^2(1)}\varphi(t)\varphi(1-t) = C\varphi(t)\varphi(1-t). \end{aligned}$$

If $t \le s \le t_0$,

$$\begin{aligned}\frac{G(t,s)}{G(t_0,s)} &= \frac{\varphi(t)\varphi(1-s)}{\varphi(s)\varphi(1-t_0)} = \frac{\varphi(t)\varphi(1-t)}{\varphi(s)\varphi(1-t)} \cdot \frac{\varphi(1-s)}{\varphi(1-t_0)} \\ &\ge \frac{1}{\varphi^2(1)}\varphi(t)\varphi(1-t) = C\varphi(t)\varphi(1-t).\end{aligned}$$

If $t_0 \le s \le t$,

$$\begin{aligned}\frac{G(t,s)}{G(t_0,s)} &= \frac{\varphi(s)\varphi(1-t)}{\varphi(t_0)\varphi(1-s)} = \frac{\varphi(t)\varphi(1-t)}{\varphi(t)\varphi(1-s)} \cdot \frac{\varphi(s)}{\varphi(t_0)} \\ &\ge \frac{1}{\varphi^2(1)}\varphi(t)\varphi(1-t) = C\varphi(t)\varphi(1-t).\end{aligned}$$

For $s \le t, t_0$,

$$\begin{aligned}\frac{G(t,s)}{G(t_0,s)} &= \frac{\varphi(s)\varphi(1-t)}{\varphi(s)\varphi(1-t_0)} = \frac{\varphi(t)\varphi(1-t)}{\varphi(t)\varphi(1-t_0)} \\ &\ge \frac{1}{\varphi^2(1)}\varphi(t)\varphi(1-t) = C\varphi(t)\varphi(1-t).\end{aligned}$$

Therefore,

$$G(t,s) \ge C\varphi(t)\varphi(1-t)G(t_0,s), \quad t,\ t_0,\ s \in [0,1].$$

This completes the proof. □

Lemma 2.2. *Let $y \in C((0,1),[0,\infty)), 0 < \int_0^1 y(s)ds < \infty$. Then the NBVP*

$$\begin{gathered}-w'' + m^2 w = y(t), \quad 0 < t < 1, \\ w'(0) = w'(1) = 0,\end{gathered} \tag{2.2}$$

has a unique solution w and there exists a constant C_y such that

$$C\|w\|\varphi(t)\varphi(1-t) \le w(t) \le C_y\varphi(t)\varphi(1-t), \quad 0 \le t \le 1. \tag{2.3}$$

Proof. It is obvious that $w(t) = \int_0^1 G(t,s)y(s)ds$ is the unique solution of (2.2). First, let $t_0 \in (0,1)$ such that $\|w\| = w(t_0) = \int_0^1 G(t_0,s)y(s)ds$. By Lemma 2.1, we have

$$\begin{aligned}w(t) &= \int_0^1 G(t,s)y(s)ds \\ &\ge \int_0^1 C\varphi(t)\varphi(1-t)G(t_0,s)y(s)ds \\ &= C\varphi(t)\varphi(1-t)\int_0^1 G(t_0,s)y(s)ds \\ &= C\varphi(t)\varphi(1-t)\|w\|,\end{aligned}$$

which is the first inequality of (2.3). On the other hand,

$$\begin{aligned} w(t) &= \int_0^1 G(t,s)y(s)ds \\ &= \frac{1}{\rho}\int_0^t \varphi(s)\varphi(1-t)y(s)ds + \frac{1}{\rho}\int_t^1 \varphi(t)\varphi(1-s)y(s)ds \\ &\le \frac{1}{\rho}\varphi(1-t)\varphi(t)\int_0^t y(s)ds + \frac{1}{\rho}\varphi(t)\varphi(1-t)\int_t^1 y(s)ds \\ &= \frac{1}{\rho}\varphi(1-t)\varphi(t)\int_0^1 y(s)ds. \end{aligned}$$

By setting

$$C_y = \frac{1}{\rho}\int_0^1 y(s)ds,$$

then the second inequality of (2.3) is proved. □

Remark 2.3. From Lemma 2.2 we know, if $y(t) \equiv M$, then $C_y = C_M = \frac{M}{\rho}$.

We make the following assumptions

- (H1) $f(t,u) \le p(t)q(u)$, where $p : (0,1) \to [0,+\infty)$ and $q : [0,+\infty) \to [0,+\infty)$ are continuous.
- (H2) $|g(t,u)| \le M$, where $M > 0$ is a constant.
- (H3) $0 < \int_0^1 G(s,s)p(s)ds < +\infty$.
- (H4) $\lim_{u\to+\infty} \frac{f(t,u)}{u} = +\infty$ uniformly on any compact subinterval of $(0,1)$.

Let

$$\begin{gathered} C^+[0,1] = \{u \in C[0,1] : u(t) \ge 0,\ 0 \le t \le 1\}, \\ K = \{u : u \in C^+[0,1], \min_{\theta\le t\le 1-\theta} u(t) \ge M_\theta \|u\|\}. \end{gathered}$$

It is obvious that $C^+[0,1]$ and K are cones of E. Let $v(t)$ be the solution of the boundrary-value problem

$$\begin{gathered} -v'' + m^2 v = M,\ 0 < t < 1, \\ v'(0) = v'(1) = 0. \end{gathered}$$

By Lemma 2.2, $v(t) \le C_M\varphi(t)\varphi(1-t) = \frac{M}{\rho}\varphi(t)\varphi(1-t)$. Set

$$\begin{gathered} [y(t)]^* = \begin{cases} y(t), & y(t) \ge 0, \\ 0, & y(t) < 0, \end{cases} \quad 0 < t < 1, \\ F(t,u) = \lambda f(t,[u-v]^*) + g(t,[u-v]^*) + M,\ 0 \le t \le 1. \end{gathered}$$

Consider the boundary-value problem

$$\begin{gathered} -u'' + m^2 u = F(t,u), \quad 0 < t < 1, \\ u'(0) = u'(1) = 0. \end{gathered} \tag{2.4}$$

It is no difficulty to prove that $u = u_0 - v$ is a positive solution of (1.1) if and only if u_0 is a positive solution of (2.4) and $u_0(t) > v(t)$, $0 < t < 1$.

Define an operator $T_\lambda : C^+[0,1] \to C^+[0,1]$ by

$$(T_\lambda u)(t) = \lambda\int_0^1 G(t,s)F(s,u(s))ds, \quad u \in K.$$

Lemma 2.4. *Let (H1)–(H3) hold. Then $T_\lambda : K \to K$ is a completely continuous operator.*

Proof. For any $u \in K$, $t \in [0,1]$, we have

$$(T_\lambda u)(t) = \lambda \int_0^1 G(t,s)F(s,u(s))ds \le \lambda \int_0^1 G(s,s)F(s,u(s))ds.$$

thus

$$\|T_\lambda u\| \le \lambda \int_0^1 G(s,s)F(s,u(s))ds.$$

On the other hand, by Lemma 2.1,

$$\begin{aligned}\min_{\theta\le t\le 1-\theta}(T_\lambda u)(t) &= \min_{\theta\le t\le 1-\theta} \lambda \int_0^1 G(t,s)F(s,u(s))ds\\ &\ge M_\theta \lambda \int_0^1 G(s,s)F(s,u(s))ds\\ &\ge M_\theta \|T_\lambda u\|.\end{aligned}$$

Therefore, $T_\lambda(K) \subset K$. For a natural number $n \ge 2$, define

$$F_n(t,x) = \begin{cases} \min\{F(t,x),\ F(\frac{1}{n},x)\}, & 0 < t \le \frac{1}{n},\\ F(t,x), & \frac{1}{n} < t < 1-\frac{1}{n},\\ \min\{F(t,x),\ F(\frac{1}{n},x)\}, & 1-\frac{1}{n} \le t < 1,\end{cases}$$

and

$$(T_n u)(t) = \lambda \int_0^1 G(t,s)F_n(t,u(s))ds, \quad \forall u \in E.$$

It is easy to prove that T_n is completely continuous. Let $D \subset E$ be a bounded set, then there is a constant $L > 0$ such that $\|u\| \le L$ for all $u \in D$, hence $[u(s) - x(s)]^* \le u(s) \le \|u\| \le L$. We have

$$\begin{aligned}&\Big|(T_\lambda u)(t) - (T_n u)(t)\Big|\\ &\le \lambda \int_0^{1/n} G(t,s)\Big|F(s,u(s)) - F(\frac{1}{n},u(s))\Big|ds\\ &\quad + \lambda \int_{1-\frac{1}{n}}^1 G(t,s)\Big|F(s,u(s)) - F(1-\frac{1}{n},u(s))\Big|ds\\ &\le 2\lambda\Big(\int_0^{1/n} G(t,s)[p(s)q(u(s)) + M]ds + \int_{1-\frac{1}{n}}^1 G(t,s)[p(s)q(u(s)) + M]ds\Big)\\ &\le 2\lambda \max_{0\le x\le L} q(x)\Big(\int_0^{1/n} G(s,s)p(s)ds + \int_{1-\frac{1}{n}}^1 G(s,s)p(s)ds\Big)\\ &\quad + 2M\Big(\int_0^{1/n} G(s,s)ds + \int_{1-\frac{1}{n}}^1 G(s,s)ds\Big)\\ &\to 0 (n \to \infty).\end{aligned}$$

Therefore, T_n converge uniformly to T_λ on any bounded subset of E. This implies that T_λ is a completely continuous operator. □

The following Krasnosel'skii fixed point theorem in a cone plays an important role in proving the main result [10].

Theorem 2.5. *Let E be Banach space and $K \subset E$ be a cone in E. Suppose Ω_1 and Ω_2 are open subset of E with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that*

(A) $\|Tu\| \le \|u\|$ *for all* $u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \ge \|u\|$ *for all* $u \in K \cap \partial\Omega_2$ *or*
(B) $\|Tu\| \le \|u\|$ *for all* $u \in K \cap \partial\Omega_2$ *and* $\|Tu\| \ge \|u\|$ *for all* $u \in K \cap \partial\Omega_1$.

Then T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Main Result

In this section, we present and prove our main result.

Theorem 3.1. *Suppose (H1)–(H4)hold, then* (1.1) *has at least one positive solution $u \in C(0,1) \cap C^2[0,1]$ if*

$$0 < \lambda \le \Big[\max_{0\le t\le r} q(\tau) \int_0^1 G(s,s)p(s)ds)\Big]^{-1},$$

where $r = \max\big\{1 + 2M\int_0^1 G(s,s)ds,\ \frac{M}{\rho C}\big\}$.

Proof. By Lemma 2.4, we know that T_λ is a completely continuous operator. Let

$$\Omega_1 = \{u \in C[0,1] : \|u\| < r\}.$$

For all $u \in K \cap \partial\Omega_1,\ t \in [0,1]$, we have

$$\begin{aligned}
(T_\lambda u)(t) &= \lambda \int_0^1 G(t,s)F(t,u(s))ds \\
&= \int_0^1 G(t,s)(\lambda f(s,[u(s)-x(s)]^*) + g(s,[u(s)-x(s)]^*) + M)ds \\
&\le \int_0^1 G(t,s)\lambda p(s)q([u(s)-x(s)]^*)ds + 2M\int_0^1 G(t,s)ds \\
&\le \lambda \max_{0\le\tau\le r} q(\tau) \int_0^1 G(s,s)p(s)ds + 2M\int_0^1 G(s,s)ds \\
&\le 1 + 2M\int_0^1 G(s,s)ds \le r = \|u\|.
\end{aligned}$$

This implies

$$\|T_\lambda u\| \le \|u\|, \quad \text{for } u \in K \cap \partial\Omega_1. \tag{3.1}$$

On the other hand, choose N large enough such that

$$\frac{1}{2}\lambda M_\theta N C\varphi^2(\theta)\int_\theta^{1-\theta} G(s,s)ds \ge 1.$$

By (H4), there exists a constant $B > 0$ such that

$$\frac{f(s,u)}{u} > N, \quad \text{for } (s,u) \in [\theta, 1-\theta] \times [B, \infty).$$

Set

$$\Omega_2 = \{u \in C[0,1] : \|u\| < R\}, \quad R = \max\Big\{2r, \frac{2M}{\rho C}, \frac{2B}{C\varphi^2(\theta)}\Big\}.$$

For any $u \in K \cap \partial\Omega_2$, $s \in [0,1]$,

$$\begin{aligned} u(s) - v(s) &\geq u(s) - C_M \varphi(s)\varphi(1-s) = u(s) - \frac{M}{\rho}\varphi(s)\varphi(1-s) \\ &= u(s) - \frac{M}{\rho C R} C\|u\|\varphi(s)\varphi(1-s) \geq u(s) - \frac{1}{2}u(s) = \frac{1}{2}u(s) \geq 0. \end{aligned}$$

Thus

$$\begin{aligned} \min_{\theta \leq s \leq 1-\theta}(u(s) - v(s)) &\geq \min_{\theta \leq s \leq 1-\theta} \frac{1}{2}u(s) \\ &\geq \min_{\theta \leq s \leq 1-\theta} \frac{C}{2}\|u\|\varphi(s)\varphi(1-s) \\ &\geq \frac{CR}{2}\varphi^2(\theta) \geq B. \end{aligned}$$

Therefore, for $t \in [\theta, 1-\theta]$,

$$\begin{aligned} (T_\lambda u)(t) &= \lambda \int_0^1 G(t,s)F(s,u(s)) \\ &= \int_0^1 G(t,s)(\lambda f(s,[u(s)-v(s)]^*) + g(s,[u(s)-v(s)]^*) + M)ds \\ &\geq \int_\theta^{1-\theta} G(t,s)\lambda f(s,[u(s)-v(s)]^*)ds \\ &\geq \lambda M_\theta \int_\theta^{1-\theta} G(s,s)N(u(s)-v(s))ds \\ &\geq \lambda M_\theta \int_\theta^{1-\theta} G(s,s)N\frac{u(s)}{2}ds \\ &\geq \lambda M_\theta \int_\theta^{1-\theta} G(s,s)N\frac{C}{2}\|u\|\varphi(s)\varphi(1-s)ds \\ &\geq \frac{1}{2}\lambda M_\theta N C\varphi^2(\theta)\int_\theta^{1-\theta} G(s,s)ds\|u\| \geq \|u\|. \end{aligned}$$

Thus

$$\|T_\lambda u\| \geq \|u\|, \quad \text{for } u \in K \cap \partial\Omega_2. \tag{3.2}$$

Applying (B) of Theorem 2.5 to (3.1) and (3.2) yields that T_λ has a fixed point u_0 with $r \leq \|u_0\| \leq R$. By Lemma 2.1 it follows that

$$\begin{aligned} u_0(t) &\geq C\|u_0\|\varphi(t)\varphi(1-t) \\ &= Cr\varphi(t)\varphi(1-t) \\ &= \frac{r\rho C}{M} \cdot \frac{M}{\rho}\varphi(t)\varphi(1-t) \\ &\geq \frac{M}{\rho}\varphi(t)\varphi(1-t) = v(t). \end{aligned}$$

Set $u(t) = u_0(t) - v(t)$, then $u(t)$ is a $C[0,1] \cap C^2(0,1)$ positive solution to (1.1). This completes the proof. $\square$

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