Electronic Journal of Differential Equations, Vol. 2003(2003), No. 49, pp. 1–16.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

EXISTENCE OF POSITIVE SOLUTIONS FOR DIRICHLET PROBLEMS OF SOME SINGULAR ELLIPTIC EQUATIONS

ZHIREN JIN

Abstract. When an unbounded domain is inside a slab, existence of a positive solution is proved for the Dirichlet problem of a class of semilinear elliptic equations similar to the singular Emden-Fowler equation. The proof is based on a super and sub-solution method. A super solution is constructed by Perron's method together with a family of auxiliary functions.

1. Introduction and Main Results

Let Ω be an unbounded domain in $\mathbb{R}^n$ $(n \geq 3)$ with $C^{2,\alpha}$ $(0 < \alpha < 1)$ boundary. We assume that Ω is inside a slab of width $2M$:

$$\Omega \subset S_M = \{(\mathbf{x}, y) \in \mathbb{R}^n : |y| < M\}$$

where $\mathbf{x} = (x_1, x_2, \dots, x_{n-1})$ and throughout the paper, y will be identified with x_n.

We consider the existence of positive solutions for the Dirichlet problem

$$-\sum_{i,j=1}^{n} a_{ij}(\mathbf{x}, y) D_{ij} u = p(\mathbf{x}, y) u^{-\gamma} \quad \text{on } \Omega; \quad u = 0 \quad \text{on } \partial\Omega; \tag{1.1}$$

where (a_{ij}) is a positive definite matrix in which each entry is a local Hölder continuous function on $\overline{\Omega}$, $p(\mathbf{x}, y)$ is a also local Hölder continuous on $\overline{\Omega}$, $\gamma > 0$ is a constant.

The main result of the paper is as follows.

Theorem 1.1. *Assume*

1. $p(\mathbf{x}_0, y_0) > 0$ *for some* $(\mathbf{x}_0, y_0) \in \Omega$;
2. *there is a positive constant* C *such that*
$$0 \leq p(\mathbf{x}, y) \leq C(|\mathbf{x}| + 1)^{\gamma} \quad \text{for} \quad (\mathbf{x}, \mathrm{y}) \in \Omega; \tag{1.2}$$
3. $\mathrm{Trace}(a_{ij}) = 1$ *and there is a constant* $c_1 > 0$, *such that*
$$a_{nn}(\mathbf{x}, y) \geq c_1 \quad \text{on } \overline{\Omega}. \tag{1.3}$$

Then (1.1) has a positive solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$.

2000 *Mathematics Subject Classification.* 35J25, 35J60, 35J65.
Key words and phrases. Elliptic boundary value problems, positive solutions, singular semilinear equations, unbounded domains, Perron's method, super solutions.

Submitted December 11, 2002. Published April 29, 2003.

When the principal part in (1.1) is the Laplace operator, (1.1) becomes a boundary value problem for the singular Emden-Fowler equation

$$-\Delta u = p(\mathbf{x}, y)u^{-\gamma} \quad \text{on } \Omega; \quad u = 0 \quad \text{on } \partial\Omega. \tag{1.4}$$

The singular Emden-Fowler is related to the theory of heat conduction in electrical conduction materials and in the studies of boundary layer phenomena for viscous fluids [2, 16]. The existence of positive solutions of the equation on exterior domains (including $\mathbb{R}^n$) has been considered by quite a number of authors (for example, see [4, 5, 8, 11, 12, 15], and references therein). The main approach used to prove existence is to construct super and sub- solutions. To construct super solutions, one needs to assume that $p(\mathbf{x}, y)$ decays near infinity in an appropriate rate. A super solution is usually found in the class of radial symmetric functions. If Ω is an exterior domain (not inside a slab), $\gamma > 0$ and there is C such that $p(\mathbf{x}, y) \geq \frac{C}{(1+|\mathbf{x}|^2+y^2)}$ for $|\mathbf{x}|^2 + y^2$ large, then (1.4) has no positive solutions ([11]). On the other hand, if there are constants $\sigma > 1$ and C, such that $0 \leq p(\mathbf{x}, y) \leq \frac{C}{(1+|\mathbf{x}|^2+y^2)^\sigma}$ for $|\mathbf{x}|^2 + y^2$ large, (1.4) has a positive solution ([8]). When Ω is an unbounded domain inside a slab, the situation is quite different. The traditional way to construct a super solution by finding an appropriate radial symmetric function is no longer valid since the domain now is inside a slab (the generality of the coefficient matrix (a_{ij}) also makes finding a radial symmetric super solution impossible). In this paper, we combine an idea from [13] and a family of auxiliary functions constructed in [10] to construct a super solution which is then used to prove the existence of a positive solution of (1.1).

Actually the procedure in the paper can be applied to prove the existence of a positive solution for the Dirichlet problem of more general elliptic equations. A statement for the general case will be given in the last section of the paper. Here we just state a special case of the general result.

Theorem 1.2. *Assume*

(1) $p(\mathbf{x}_0, y_0) > 0$ *for some* $(\mathbf{x}_0, y_0) \in \Omega$*;*
(2) *there is a positive constant* C *such that*

$$0 \leq p(\mathbf{x}, y) \leq Ce^{|\mathbf{x}|} \quad \text{for} \quad (\mathbf{x}, \mathrm{y}) \in \Omega, \tag{1.5}$$

(3) $\mathrm{Trace}(a_{ij}) = 1$*, and there is a constant* $c_1 > 0$*, such that*

$$a_{nn}(\mathbf{x}, y) \geq c_1 \quad \text{on} \quad \overline{\Omega}. \tag{1.6}$$

Then the problem

$$-\sum_{i,j=1}^{n} a_{ij}(\mathbf{x}, y)D_{ij}u = p(\mathbf{x}, y)e^{-u} \quad \text{on } \Omega; \quad u = 0 \quad \text{on } \partial\Omega \tag{1.7}$$

has a positive solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$*.*

This paper is organized as follows. In Section 2, we construct a family of auxiliary functions that are defined on a family of subdomains of Ω. In Section 3, we combine the family of auxiliary functions constructed in Section 2 and an idea from [13] to prove that (1.1) has a positive supper solution. In Section 4, we prove that (1.1) has a positive solution by the procedure used in [8]. In Section 5, we discuss the general case.

2. A Family of Auxiliary Functions

In this section, we will construct families of sub-domains $\Omega_{\mathbf{x}_0}$ of Ω and functions $T_{\mathbf{x}_0} + z$ (see definitions below) so that

$$-\sum_{i,j=1}^{n} a_{ij}(\mathbf{x}, y) D_{ij}(T_{\mathbf{x}_0} + z) \geq p(\mathbf{x}, y)(T_{\mathbf{x}_0} + z)^{-\gamma} \quad \text{on } \Omega_{\mathbf{x}_0} \tag{2.1}$$

and the graphs of the functions $T_{\mathbf{x}_0} + z$ have special relative positions (see below).

Our construction is based on the construction of a family of auxiliary functions used in [10] (the construction in [10] was adapted from [9] which in turn was inspired from [6] and [14]). We consider the operator

$$Qu = \sum_{i,j=1}^{n} a_{ij}(\mathbf{x}, y) D_{ij} u.$$

We first extend a_{ij} $(1 \leq i, j \leq n)$ to be continuous functions on $\overline{S_M}$ in such a way that we still have $\operatorname{Trace}(a_{ij}) = 1$ and

$$a_{nn}(\mathbf{x}, y) \geq c_1 \quad \text{on } S_M. \tag{2.2}$$

In the rest of the paper, we will use c_m (for some integer $m \geq 2$) to denote a constant depending only on c_1 and M. Once a constant c_m is used in a formula, it will represent the same constant if the same notation appears again in the paper.

It was proved in [10] (also see Appendix I) that there are positive decreasing functions $\chi(t)$, $h_a(t)$ and a positive increasing function $A(t)$ ($\chi(t)$ depending on c_1 only, $h_a(t)$ and $A(t)$ depending on c_1 and M only), such that for any number K, there is a number H_0, depending only on K, M and c_1, such that for $H \geq H_0$, we have (for $0 < t < 2M$)

$$A(H) \leq h_a^{-1}(t) \leq A(H)e^{\chi(H)}, \quad 22MH \leq c_1 A(H) e^{\chi(H)} \leq 66MH, \tag{2.3}$$

$$8K \leq A(H)e^{\chi(H)}, \quad 0 < \chi(H) < 1, \tag{2.4}$$

and the non-negative function

$$z = z_{\mathbf{x}_0} = A(H)e^{\chi(H)} - \{(h_a^{-1}(y + M))^2 - |\mathbf{x} - \mathbf{x}_0|^2\}^{1/2} \tag{2.5}$$

satisfies

$$Qz \leq \frac{-3c_1}{22eMH} \quad \text{in } \Omega_{\mathbf{x}_0,H,K}, \tag{2.6}$$

$$z \geq K \quad \text{on } \partial\Omega_{\mathbf{x}_0,H,K} \cap \{|y| < M\}, \quad z(\mathbf{x}_0, y) \leq \frac{2M}{H} \quad \text{for } |y| \leq M, \tag{2.7}$$

where

$$\Omega_{\mathbf{x}_0,H,K} = \{(\mathbf{x}, y) : |y| < M, |\mathbf{x} - \mathbf{x}_0| < \sqrt{\frac{2K}{A(H)e^{\chi(H)}}} h_a^{-1}(y + M)\}. \tag{2.8}$$

(For verifications of (2.3)-(2.4) and (2.6)-(2.7), see Appendix I.)

Now we set

$$K = 100, \quad H = H_0 + 4M, \quad \Omega_{\mathbf{x}_0} = \Omega_{\mathbf{x}_0,H,K}. \tag{2.9}$$

Then (2.6)-(2.7) becomes

$$Qz \leq -c_2 \quad \text{in } \Omega_{\mathbf{x}_0}, \tag{2.10}$$

$$z \geq 100 \quad \text{on } \partial\Omega_{\mathbf{x}_0} \cap \{|y| < M\}, \quad z(\mathbf{x}_0, y) \leq 1 \quad \text{for } |y| \leq M. \tag{2.11}$$

Now we construct a family of auxiliary functions as follows.

If $(\mathbf{x}, y) \in \Omega_{\mathbf{x}_0}$, from (2.3) and (2.8), we have

$$|\mathbf{x} - \mathbf{x}_0| < \sqrt{200A(H)e^{\chi(H)}} \le \sqrt{13200MH/c_1} = c_4.$$

For C defined in (1.2), we set

$$T_{\mathbf{x}_0} = (\frac{C}{c_2})^{1/\gamma}(|\mathbf{x}_0| + c_4 + 1). \tag{2.12}$$

Then we have that on $\Omega_{\mathbf{x}_0}$,

$$p(\mathbf{x}, y)(T_{\mathbf{x}_0} + z)^{-\gamma} \le C(|\mathbf{x}| + 1)^\gamma T_{\mathbf{x}_0}^{-\gamma} \le \frac{C(|\mathbf{x}_0| + c_4 + 1)^\gamma}{T_{\mathbf{x}_0}^\gamma} = c_2.$$

Thus

$$-Q(T_{\mathbf{x}_0} + z) \ge c_2 \ge p(\mathbf{x}, y)(T_{\mathbf{x}_0} + z)^{-\gamma} \quad \text{on } \Omega_{\mathbf{x}_0}. \tag{2.13}$$

When $\mathbf{x}_0$ changes, we obtain families of auxiliary functions $T_{\mathbf{x}_0} + z$ and domains $\Omega_{\mathbf{x}_0}$ satisfying (2.1).

To be able to use the family of auxiliary functions, we need to investigate relative positions of the graphs of these auxiliary functions.

For two points $\mathbf{x}_0$ and $\mathbf{x}_1$ in R^{n-1}, when $\Omega_{\mathbf{x}_1}$ either covers the whole segment of the set $\{(\mathbf{x}_0, y) | |y| \le M\}$ or does not intersect with the set, from (2.3) and (2.8), we have either

$$|\mathbf{x}_1 - \mathbf{x}_0| \le \sqrt{200A(H)e^{-\chi(H)}} \quad \text{or} \quad |\mathbf{x}_1 - \mathbf{x}_0| \ge \sqrt{200A(H)e^{\chi(H)}}. \tag{2.14}$$

Then when $\Omega_{\mathbf{x}_1}$ covers part of some neighborhood of $\{(\mathbf{x}_0, y) : |y| \le M\}$, we have

$$\sqrt{195A(H)e^{-\chi(H)}} \le |\mathbf{x}_1 - \mathbf{x}_0| \le \sqrt{205A(H)e^{\chi(H)}}. \tag{2.15}$$

Let $\mathbf{x}_1$ and $\mathbf{x}_0$ satisfy (2.15) and δ_0 be a small positive number such that $2\delta_0 < \sqrt{195A(H)e^{-\chi(H)}}$. If $(\mathbf{x}, y) \in \Omega_{\mathbf{x}_1}$ for some y and $|\mathbf{x} - \mathbf{x}_0| \le \delta_0$, by (2.3), (2.5) and (2.15), we have

$$\begin{aligned}
&T_{\mathbf{x}_1} + z_{\mathbf{x}_1}(\mathbf{x}, y) \\
&\ge T_{\mathbf{x}_1} + A(H)e^{\chi(H)} - \{A(H)^2 e^{2\chi(H)} - |\mathbf{x} - \mathbf{x}_1|\}^{1/2} \\
&\ge T_{\mathbf{x}_1} + A(H)e^{\chi(H)} - \big\{A(H)^2 e^{2\chi(H)} - (\sqrt{195A(H)e^{-\chi(H)}} - \delta_0)^2\big\}^{1/2} \\
&\ge T_{\mathbf{x}_1} + A(H)e^{\chi(H)} \\
&\quad - \big\{A(H)^2 e^{2\chi(H)} - 195A(H)e^{-\chi(H)} + 2\delta_0\sqrt{195A(H)e^{-\chi(H)}}\big\}^{1/2} \\
&\ge T_{\mathbf{x}_1} + A(H)e^{\chi(H)}\Big(1 - \Big(1 - \frac{195}{A(H)e^{3\chi(H)}} + \frac{2\delta_0\sqrt{195A(H)e^{-\chi(H)}}}{A(H)^2 e^{2\chi(H)}}\Big)^{1/2}\Big) \\
&\text{(by the inequality } \sqrt{1-t} \le 1 - \tfrac{1}{2}t \text{ for } 0 < t < 1 \text{ and (2.4))} \\
&\ge T_{\mathbf{x}_1} + A(H)e^{\chi(H)}(\frac{195}{2A(H)e^{3\chi(H)}} - \frac{2\delta_0\sqrt{195A(H)e^{-\chi(H)}}}{2A(H)^2 e^{2\chi(H)}}) \\
&= T_{\mathbf{x}_1} + \frac{195}{2e^{2\chi(H)}} - \frac{\delta_0\sqrt{195A(H)e^{-\chi(H)}}}{A(H)e^{\chi(H)}} > T_{\mathbf{x}_1} + 10 - \frac{\delta_0\sqrt{195A(H)e^{-\chi(H)}}}{A(H)e^{\chi(H)}}.
\end{aligned}$$

Thus there is a δ_0 small such that for all $|\mathbf{x}-\mathbf{x}_0| \le \delta_0$ with $(\mathbf{x},y)\in\Omega_{\mathbf{x}_1}$, if $\mathbf{x}_1$ and $\mathbf{x}_0$ satisfy (2.15), we have

$$T_{\mathbf{x}_1} + z_{\mathbf{x}_1}(\mathbf{x},y) \ge T_{\mathbf{x}_1} + 8. \tag{2.16}$$

Further for all $\mathbf{x}_0$ and $\mathbf{x}_1$ satisfying (2.15),

$$\begin{aligned} T_{\mathbf{x}_0} + 2 &\le T_{\mathbf{x}_1} + T_{\mathbf{x}_0} - T_{\mathbf{x}_1} + 2 \\ &\le T_{\mathbf{x}_1} + (\frac{C}{c_2})^{\frac{1}{\gamma}}(|\mathbf{x}_0| - |\mathbf{x}_1|) + 2 \\ &\le T_{\mathbf{x}_1} + (\frac{C}{c_2})^{\frac{1}{\gamma}}|\mathbf{x}_1 - \mathbf{x}_0| + 2 \\ &\le T_{\mathbf{x}_1} + (\frac{C}{c_2})^{\frac{1}{\gamma}}\sqrt{205A(H)e^{\chi(H)}} + 2 \\ &\le T_{\mathbf{x}_1} + (\frac{C}{c_2})^{\frac{1}{\gamma}}c_5 + 2 \end{aligned}$$

where $c_5 = \sqrt{205A(H)e^{\chi(H)}}$. Thus if we assume that C in (1.2) satisfies

$$C \le 6^{\gamma} c_5^{-\gamma} c_2, \tag{2.17}$$

we have that for all $\mathbf{x}_0$ and $\mathbf{x}_1$ satisfying (2.15),

$$T_{\mathbf{x}_0} + 2 \le T_{\mathbf{x}_1} + 8\,. \tag{2.18}$$

From (2.8) and (2.11), we can choose a number $\delta_2(\mathbf{x}_0) > 0$ such that for all $\mathbf{x}\in R^{n-1}$ with $|\mathbf{x}_0 - \mathbf{x}| \le \delta_2(\mathbf{x}_0)$, we have $(\mathbf{x},y)\in\Omega_{\mathbf{x}_0}$ for all $|y| < M$, and

$$T_{\mathbf{x}_0} + z_{\mathbf{x}_0}(\mathbf{x},y) \le T_{\mathbf{x}_0} + 2\,. \tag{2.19}$$

Now if we set $\delta_{\mathbf{x}_0} = \min\{\delta_0, \delta_2(\mathbf{x}_0)\}$, from (2.16), (2.18) and (2.19), we have

$$T_{\mathbf{x}_0} + z_{\mathbf{x}_0}(\mathbf{x},y) \le T_{\mathbf{x}_1} + z_{\mathbf{x}_1}(\mathbf{x},y) \tag{2.20}$$

for all $\mathbf{x}_0$ and $\mathbf{x}_1$ satisfying (2.15), $|\mathbf{x}_0 - \mathbf{x}| \le \delta_{\mathbf{x}_0}$ and $(\mathbf{x},y)\in\Omega_{\mathbf{x}_1}$.

Finally we define a family of open subsets of Ω that will be needed in next section.

For each point $(\mathbf{x}_0,y_0)\in\overline{\Omega}$, we define an open set $O(\mathbf{x}_0,y_0)$ as follows:

(1) If $(\mathbf{x}_0,y_0)\in\Omega$, we choose a ball B with center $(\mathbf{x}_0,y_0)$ and a radius less than $\delta_{\mathbf{x}_0}$ so that $B\subset\Omega$. We then set $O(\mathbf{x}_0,y_0)=B$;

(2) If $(\mathbf{x}_0,y_0)\in\partial\Omega$, since Ω has $C^{2,\alpha}$ boundary, there is a ball B with center $(\mathbf{x}_0,y_0)$ and a radius less than $\delta_{\mathbf{x}_0}$, such that there is a $C^{2,\alpha}$ diffeomorphism Φ satisfying

$$\Phi(B\cap\Omega)\subset\mathbb{R}^n_+, \quad \Phi(B\cap\partial\Omega)\subset\partial\mathbb{R}^n_+; \quad \Phi(\mathbf{x}_0,y_0)=\mathbf{0}.$$

Now we choose a domain J with C^3 boundary with following properties: (a) $J\subset\Phi(B\cap\Omega)$; (b) $\partial J\cap\partial\mathbb{R}^n_+$ is a neighborhood of $\mathbf{0}$ in $\partial\mathbb{R}^n_+$. Certainly there are many different J's having those properties. One example is given in the Appendix II at the end of paper to illustrate how to construct such a domain J.

Now we set $O(\mathbf{x}_0,y_0)=\Phi^{-1}(J)$. It is easy to see that $O(\mathbf{x}_0,y_0)\subset B\cap\Omega$, $O(\mathbf{x}_0,y_0)$ has a $C^{2,\alpha}$ boundary and $\partial O(\mathbf{x}_0,y_0)\cap\partial\Omega$ is a neighborhood of $(\mathbf{x}_0,y_0)$ in $\partial\Omega$. Let Π be the collection of all such open sets $O(\mathbf{x}_0,y_0)$ defined in (1) and (2).

3. A Super Solution of (1.1)

In this section, using the family of auxiliary functions $T_{\mathbf{x}_0} + z$ constructed in Section 2 and an idea from [13] (that basically says that the Perron's method still works if we can find a family of appropriate auxiliary functions that works like a super solution), we will show that there is a positive function $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, satisfies

$$-\sum_{i,j=1}^{n} a_{ij}(\mathbf{x}, y) D_{ij} u = p(\mathbf{x}, y) u^{-\gamma} \quad \text{on } \Omega, \quad u = \tau \quad \text{on } \partial\Omega.$$

for some constant $\tau > 0$. Then u will be a super solution of (1.1).

If $u = c_0 v$ for some constant c_0, v will satisfy

$$-\sum_{i,j=1}^{n} a_{ij}(\mathbf{x}, y) D_{ij} v = c_0^{-\gamma-1} p(\mathbf{x}, y) v^{-\gamma} \quad \text{on } \Omega, \quad v = \tau/c_0 \quad \text{on } \partial\Omega\,.$$

Thus without loss of generality, we may assume C in (1.2) satisfying (2.17). Then all constructions in Section 2 are valid.

Let $v > 0$ be a function on $\overline{\Omega}$, for a point $(\mathbf{x}_0, y_0) \in \overline{\Omega}$, we define a new function $M_{(\mathbf{x}_0,y_0)}(v)$, called the lift of v over $O(\mathbf{x}_0, y_0)$ as follows:

$$M_{(\mathbf{x}_0,y_0)}(v)(\mathbf{x}, y) = v(\mathbf{x}, y) \quad \text{if} \quad (\mathbf{x}, y) \in \Omega \setminus O(\mathbf{x}_0, y_0)$$
$$M_{(\mathbf{x}_0,y_0)}(v)(\mathbf{x}, y) = w(\mathbf{x}, y) \quad \text{if} \quad (\mathbf{x}, y) \in O(\mathbf{x}_0, y_0)$$

where $w(\mathbf{x}, y)$ is the positive solution of the boundary-value problem

$$-\sum_{i,j=1}^{n} a_{ij}(\mathbf{x}, y) D_{ij} w = p(\mathbf{x}, y) w^{-\gamma} \quad \text{in } O(\mathbf{x}_0, y_0), \quad w = v \quad \text{on } \partial O(\mathbf{x}_0, y_0)\,. \tag{3.1}$$

It is easy to see (3.1) has a unique positive solution in $C^2(O(\mathbf{x}_0, y_0)) \cap C^0(\overline{O(\mathbf{x}_0, y_0)})$. Indeed $m_1 = \min\{v(\mathbf{x}, y) : (\mathbf{x}, y) \in \partial O(\mathbf{x}_0, y_0)\}$ is a sub-solution since $p(\mathbf{x}, y)$ is non-negative, $m_2 + T_{\mathbf{x}_0} + z_{\mathbf{x}_0}$ is a super solution by (2.1), where $m_2 = \max\{v(\mathbf{x}, y) : (\mathbf{x}, y) \in \partial O(\mathbf{x}_0, y_0)\}$. Then we can conclude the existence of a desired solution (for example, see [1] or [3]). Uniqueness of positive solutions of (3.1) follows from a standard argument.

Set $\tau = (C/c_2)^{1/\gamma} c_4$ (see (2.12) for the source of the constants).

We define a class Ξ of functions as follows: a function v is in Ξ if

(1) $v \in C^0(\overline{\Omega})$, $v > 0$ on $\overline{\Omega}$ and $v \le \tau$ on $\partial\Omega$;
(2) For any $(\mathbf{x}_0, y_0) \in \overline{\Omega}$, $v \le M_{(\mathbf{x}_0,y_0)}(v)$;
(3) $v \le T_{\mathbf{x}_0} + z_{\mathbf{x}_0}$ on $\Omega_{\mathbf{x}_0} \cap \Omega$ for any $(\mathbf{x}_0, y_0) \in \overline{\Omega}$.

By the following well-known lemma, it is easy to check the function $v = \tau$ is in Ξ. Thus Ξ is not empty.

Lemma 3.1. *Let D be a bounded domain, $f(\mathbf{x}, y, t)$ be a C^1 function that is decreasing in t. If w_1, w_2 are in $C^2(D) \cap C^0(\overline{D})$, $w_1 \le w_2$ on ∂D, and*

$$-\sum_{i,j=1}^{n} a_{ij}(\mathbf{x}, y) D_{ij} w_1 \le f(\mathbf{x}, y, w_1) \quad \text{in } D,$$
$$-\sum_{i,j=1}^{n} a_{ij}(\mathbf{x}, y) D_{ij} w_2 \ge f(\mathbf{x}, y, w_2) \quad \text{in } D$$

then $w_1 \le w_2$ *on* D.

Now we set

$$u(\mathbf{x}, y) = \sup_{v \in \Xi} v(\mathbf{x}, y), \quad (\mathbf{x}, y) \in \overline{\Omega}\,.$$

We will show that u is in $C^2(\Omega) \cap C^0(\overline{\Omega})$ and satisfies

$$-\sum_{i,j=1}^{n} a_{ij}(\mathbf{x}, y) D_{ij} u = p(\mathbf{x}, y) u^{-\gamma} \quad \text{on } \Omega; \quad u = \tau \quad \text{on } \partial\Omega.$$

First we need some lemmas.

Lemma 3.2. *If* $0 < v_1 \le v_2$*, then* $M_{(\mathbf{x}_0,y_0)}(v_1) \le M_{(\mathbf{x}_0,y_0)}(v_2)$ *for any* $(\mathbf{x}_0, y_0) \in \overline{\Omega}$.

Proof. Let w_1, w_2 be the positive solutions for the following problems

$$\begin{gathered} -\sum_{i,j=1}^{n} a_{ij}(\mathbf{x}, y) D_{ij} w_k = p(\mathbf{x}, y) w_k^{-\gamma} \quad \text{in } O(\mathbf{x}_0, y_0), \\ w_k = v_k \quad \text{on } \partial O(\mathbf{x}_0, y_0), \quad k = 1, 2. \end{gathered}$$

Since $w_1 = v_1 \le v_2 = w_2$ on $\partial O(\mathbf{x}_0, y_0)$, $p(\mathbf{x}, y)t^{-\gamma}$ is decreasing on t, from lemma 1, we see $w_1 \le w_2$ on $O(\mathbf{x}_0, y_0)$. On $\Omega \backslash O(\mathbf{x}_0, y_0)$, $M_{(\mathbf{x}_0,y_0)}(v_1) = v_1$, $M_{(\mathbf{x}_0,y_0)}(v_2) = v_2$. Thus $M_{(\mathbf{x}_0,y_0)}(v_1) \le M_{(\mathbf{x}_0,y_0)}(v_2)$. □

Lemma 3.3. *If* $v_1 \in \Xi$*,* $v_2 \in \Xi$*, then* $\max\{v_1, v_2\} \in \Xi$.

Proof. If $v_1 \in \Xi$, $v_2 \in \Xi$, it is clear that $\max\{v_1, v_2\} \in C^0(\overline{\Omega})$, $\max\{v_1, v_2\} > 0$ on $\overline{\Omega}$ and $\max\{v_1, v_2\} \le \tau$ on $\partial\Omega$. It is also clear that $\max\{v_1, v_2\} \le T_{\mathbf{x}_0} + z_{\mathbf{x}_0}$ on $\Omega_{\mathbf{x}_0} \cap \Omega$ for any $(\mathbf{x}_0, y_0) \in \overline{\Omega}$. Since

$$v_1 \le \max\{v_1, v_2\}, \quad v_2 \le \max\{v_1, v_2\}$$

we have (by lemma 2) that for any $(\mathbf{x}_0, y_0) \in \overline{\Omega}$,

$$M_{(\mathbf{x}_0,y_0)}(v_1) \le M_{(\mathbf{x}_0,y_0)}(\max\{v_1, v_2\}), \quad M_{(\mathbf{x}_0,y_0)}(v_2) \le M_{(\mathbf{x}_0,y_0)}(\max\{v_1, v_2\}).$$

Since $v_1 \in \Xi$ and $v_2 \in \Xi$ imply

$$v_1 \le M_{(\mathbf{x}_0,y_0)}(v_1), \quad v_2 \le M_{(\mathbf{x}_0,y_0)}(v_2),$$

we have

$$\max\{v_1, v_2\} \le M_{(\mathbf{x}_0,y_0)}(\max\{v_1, v_2\}).$$

Thus $\max\{v_1, v_2\} \in \Xi$. □

Lemma 3.4. *If* $v \in \Xi$*, then* $M_{(\mathbf{x}_0,y_0)}(v) \in \Xi$ *for any* $(\mathbf{x}_0, y_0) \in \overline{\Omega}$.

Proof. By the definition of $M_{(\mathbf{x}_0,y_0)}(v)$, it is clear that $M_{(\mathbf{x}_0,y_0)}(v) > 0$ on $\overline{\Omega}$, $M_{(\mathbf{x}_0,y_0)}(v) \in C^0(\overline{\Omega})$ and $M_{(\mathbf{x}_0,y_0)}(v) \le \tau$ on $\partial\Omega$.

For any $(\mathbf{x}^*, y^*) \in \overline{\Omega}$, we first show that

$$M_{(\mathbf{x}_0,y_0)}(v)(\mathbf{x}, y) \le M_{(\mathbf{x}^*,y^*)}(M_{(\mathbf{x}_0,y_0)}(v))(\mathbf{x}, y). \tag{3.2}$$

We only need to prove that (3.2) is true for $(\mathbf{x}, y) \in O(\mathbf{x}^*, y^*)$. Since

$$v \le M_{(\mathbf{x}_0,y_0)}(v),$$

we have (by lemma 2)

$$M_{(\mathbf{x}^*,y^*)}(v) \le M_{(\mathbf{x}^*,y^*)}(M_{(\mathbf{x}_0,y_0)}(v)).$$

Then from $v \le M_{(\mathbf{x}^*,y^*)}(v)$ (by lemma 2 again), we have

$$v \le M_{(\mathbf{x}^*,y^*)}(M_{(\mathbf{x}_0,y_0)}(v)).$$

Thus for $(\mathbf{x},y) \in O(\mathbf{x}^*,y^*) \setminus O(\mathbf{x}_0,y_0)$,

$$M_{(\mathbf{x}_0,y_0)}(v)(\mathbf{x},y) = v(\mathbf{x},y) \le M_{(\mathbf{x}^*,y^*)}(M_{(\mathbf{x}_0,y_0)}(v))(\mathbf{x},y). \tag{3.3}$$

That is, (3.2) is true on $O(\mathbf{x}^*,y^*) \setminus O(\mathbf{x}_0,y_0)$, Now for $\Omega_1 = O(\mathbf{x}^*,y^*) \cap O(\mathbf{x}_0,y_0)$, if we set

$$M_{(\mathbf{x}_0,y_0)}(v) = w_1, \quad M_{(\mathbf{x}^*,y^*)}(M_{(\mathbf{x}_0,y_0)}(v)) = w_2$$

we have

$$-\sum_{i,j=1}^n a_{ij}(\mathbf{x},y)D_{ij}w_1 = p(\mathbf{x},y)w_1^{-\gamma} \quad \text{on } \Omega_1,$$
$$-\sum_{i,j=1}^n a_{ij}(\mathbf{x},y)D_{ij}w_2 = p(\mathbf{x},y)w_2^{-\gamma} \quad \text{on } \Omega_1.$$

On $\partial\Omega_1$, $w_1 \le w_2$ on $O(\mathbf{x}^*,y^*) \cap \partial O(\mathbf{x}_0,y_0)$ by (3.3) and $w_1 \le w_2$ on $\partial O(\mathbf{x}^*,y^*) \cap O(\mathbf{x}_0,y_0)$ since (3.2) is true on $\Omega \setminus O(\mathbf{x}^*,y^*)$. Then lemma 1 implies $w_1 \le w_2$ on Ω_1. Thus (3.2) is true on $O(\mathbf{x}^*,y^*) \cap O(\mathbf{x}_0,y_0)$ and on $O(\mathbf{x}^*,y^*)$. □

Now we prove that $M_{(\mathbf{x}_0,y_0)}(v) \le T_{\mathbf{x}_1} + z_{\mathbf{x}_1}$ on $\Omega_{\mathbf{x}_1} \cap \Omega$ for all $(\mathbf{x}_1,y_1) \in \overline{\Omega}$.

By the definition of $M_{(\mathbf{x}_0,y_0)}(v)$, we only need to consider the graph of the function $M_{(\mathbf{x}_0,y_0)}(v)$ over $O(\mathbf{x}_0,y_0)$. If $O(\mathbf{x}_0,y_0)$ is covered completely by $\Omega_{\mathbf{x}_1}$, since $v \le T_{\mathbf{x}_1} + z_{\mathbf{x}_1}$ and $T_{\mathbf{x}_1} + z_{\mathbf{x}_1}$ satisfies (2.1), $T_{\mathbf{x}_1} + z_{\mathbf{x}_1}$ is a super solution of (3.1) on $O(\mathbf{x}_0,y_0)$. Then Lemma 3.1 implies $M_{(\mathbf{x}_0,y_0)}(v) \le T_{\mathbf{x}_1} + z_{\mathbf{x}_1}$ on $O(\mathbf{x}_0,y_0)$. In the case that $O(\mathbf{x}_0,y_0)$ does not intersect with $\Omega_{\mathbf{x}_1}$, the conclusion is trivial. Now we consider the case that $O(\mathbf{x}_0,y_0)$ is partially covered by $\Omega_{\mathbf{x}_1}$. Since $O(\mathbf{x}_0,y_0)$ is covered by $\Omega_{\mathbf{x}_0}$, we always have

$$M_{(\mathbf{x}_0,y_0)}(v) \le T_{\mathbf{x}_0} + z_{\mathbf{x}_0} \quad \text{on } O(\mathbf{x}_0,y_0). \tag{3.4}$$

Then by the choice of $\delta_{\mathbf{x}_0}$, $O(\mathbf{x}_0,y_0)$, and the fact that $O(\mathbf{x}_0,y_0) \cap T_{\mathbf{x}_1}$ is not empty, we have that $\mathbf{x}_0$ and $\mathbf{x}_1$ satisfy (2.15), and for all $(\mathbf{x},y) \in O(\mathbf{x}_0,y_0) \cap \Omega_{\mathbf{x}_1}$, $|\mathbf{x}_0 - \mathbf{x}| \le \delta_{\mathbf{x}_0}$. Then by (2.20), the graph of $T_{\mathbf{x}_0} + z_{\mathbf{x}_0}$ over $O(\mathbf{x}_0,y_0) \cap \Omega_{\mathbf{x}_1}$ is under the graph of $T_{\mathbf{x}_1} + z_{\mathbf{x}_1}$. Thus the conclusion follows from (3.4).

Now we are ready to prove that u has the desired properties.

Let $(\mathbf{x}_0,y_0) \in \overline{\Omega}$. By the definition of $u(\mathbf{x}_0,y_0)$, there is a sequence of functions v_k in Ξ such that

$$u(\mathbf{x}_0,y_0) = \lim_{k\to\infty} v_k(\mathbf{x}_0,y_0).$$

By lemma 3 and the fact that $v = \tau$ is in Ξ, replacing v_k by $\max\{v_k,\tau\}$ if it is necessary, we may assume that $v_k \ge \tau$ on Ω. We replace v_k by $M_{(\mathbf{x}_0,y_0)}(v_k)$. Then we have a sequence of functions w_k satisfying

$$u(\mathbf{x}_0,y_0) = \lim_{k\to\infty} w_k(\mathbf{x}_0,y_0),$$
$$-\sum_{i,j=1}^n a_{ij}(\mathbf{x},y)D_{ij}w_k = p(\mathbf{x},y)w_k^{-\gamma} \quad \textit{on} \quad O(\mathbf{x}_0,y_0),$$
$$w_k = v_k \quad \textit{on} \quad \partial O(\mathbf{x}_0,y_0).$$

Since for all k,

$$\tau \le v_k \le w_k \le T_{\mathbf{x}_0} + z_{\mathbf{x}_0} \quad \text{on } O(\mathbf{x}_0,y_0).$$

By [7, Theorem 9.11] and an approximation of the boundary value by smooth functions, we see that there is a subsequence of w_k, for convenience still denoted by w_k, converges to a $C^2(O(\mathbf{x}_0, y_0)) \cap C^0(\overline{O(\mathbf{x}_0, y_0)})$ function $w(x)$ in $C^2(O(\mathbf{x}_0, y_0)) \cap C^0(\overline{O(\mathbf{x}_0, y_0)})$. Thus $w(x)$ satisfies

$$-\sum_{i,j=1}^{n} a_{ij}(\mathbf{x}, y) D_{ij} w = p(\mathbf{x}, y) w^{-\gamma} \quad on \quad O(\mathbf{x}_0, y_0)$$

and $u(\mathbf{x}_0, y_0) = w(\mathbf{x}_0, y_0)$. We claim that $u = w$ on $O(\mathbf{x}_0, y_0)$. Indeed, if there is another point $(\mathbf{x}_2, y_2) \in O(\mathbf{x}_0, y_0)$ such that $u(\mathbf{x}_2, y_2)$ is not equal to $w(\mathbf{x}_2, y_2)$, then $u(\mathbf{x}_2, y_2) > w(\mathbf{x}_2, y_2)$. Then there is a function $u_0 \in \Xi$, such that

$$w(\mathbf{x}_2, y_2) < u_0(\mathbf{x}_2, y_2) \le u(\mathbf{x}_2, y_2).$$

Now the sequence $\max\{u_0, M_{(\mathbf{x}_0, y_0)}(v_k)\}$ satisfying

$$v_k \le \max\{u_0, M_{(\mathbf{x}_0, y_0)}(v_k)\} \le u.$$

Then similar to the way we obtain w, $M_{(\mathbf{x}_0, y_0)}(\max\{u_0, M_{(\mathbf{x}_0, y_0)}(v_k)\})$ will produce a function w_1 satisfying

$$-\sum_{i,j=1}^{n} a_{ij}(\mathbf{x}, y) D_{ij} w_1 = p(\mathbf{x}, y) w_1^{-\gamma} \quad \text{on } O(\mathbf{x}_0, y_0),$$
$$w \le w_1 \quad on \quad O(\mathbf{x}_0, y_0), \quad w(\mathbf{x}_2, y_2) < u_0(\mathbf{x}_2, y_2) \le w_1(\mathbf{x}_2, y_2),$$
$$w(\mathbf{x}_0, y_0) = w_1(\mathbf{x}_0, y_0) = u(\mathbf{x}_0, y_0).$$

That is, $w_1(\mathbf{x}, y) - w(\mathbf{x}, y)$ is non-negative, not identically zero on $O(\mathbf{x}_0, y_0)$ and achieves its minimum value zero inside $O(\mathbf{x}_0, y_0)$. However, from the equations satisfied by w and w_1, we have that on $O(\mathbf{x}_0, y_0)$,

$$-\sum_{i,j=1}^{n} a_{i,j}(\mathbf{x}, y) D_{ij}(w_1 - w) + \gamma p(\mathbf{x}, y)(w + \theta(w_1 - w))^{-\gamma-1}(w_1 - w) = 0$$

for some continuous function θ. Then by the standard maximum principle (for example, see [7, Theorem 3.5]), we get a contradiction. Thus $u = w$ on $O(\mathbf{x}_0, y_0)$. Therefore $u \in C^2(\Omega)$ and

$$-\sum_{i,j=1}^{n} a_{i,j}(\mathbf{x}, y) D_{ij} u = p(\mathbf{x}, y) u^{-\gamma} \quad \text{on } \Omega.$$

When $(\mathbf{x}_0, y_0) \in \partial\Omega$, $\partial O(\mathbf{x}_0, y_0) \cap \partial\Omega$ is a neighborhood of $(\mathbf{x}_0, y_0)$ in $\partial\Omega$. Since $\max\{\tau, v_k\} = \tau$ on $\partial\Omega$, $u = \tau$ on $\partial\Omega$ and $w = \tau$ on $\partial O(\mathbf{x}_0, y_0) \cap \partial\Omega$. Since w is continuous up to the boundary of $O(\mathbf{x}_0, y_0)$, u is continuous on $\partial O(\mathbf{x}_0, y_0) \cap \partial\Omega$ from inside $O(\mathbf{x}_0, y_0)$. Thus $u \in C^0(\overline{\Omega})$ and $u = \tau$ on $\partial\Omega$.

4. Proof of Existence

Using the super solution u constructed in Section 3, we can prove the existence of a positive solution of (1.1) exactly in the same way as that in [8] (the generality of the principal term of the elliptic operator will not cause any extra difficulty). We just sketch the proof here.

Since Ω is an unbounded domain with $C^{2,\alpha}$ boundary, we can choose a sequence of subdomains of Ω, denoted by Ω_m, $m = 1, 2, 3, \dots$, such that

(1) $\Omega_m \subset \Omega_{m+1} \subset \Omega$ for all m;

(2) $\cup\Omega_m = \Omega$;
(3) each Ω_m is a bounded domain with $C^{2,\alpha}$ boundary;
(4) $dist(0, \partial\Omega \setminus \partial\Omega_m) \to \infty$ as $m \to \infty$.

We can find a number μ, such that for each large m, the eigenvalue problem

$$-\sum_{i,j=1}^{n} a_{ij}(\mathbf{x}, y) D_{ij} w = \lambda(\mu p(\mathbf{x}, y)) w \quad \text{on } \Omega_m, \quad w = 0 \quad \text{on } \partial\Omega_m$$

has a first eigenvalue $\lambda_1 < 1$ with its first eigenfunction ϕ_m. We can assume $\max \phi_m = 1$. Choose δ_m such that $\delta_m \le \frac{1}{2}\tau$ and

$$\mu p(\mathbf{x}, y) t \le p(\mathbf{x}, y) t^{-\gamma} \quad \text{for } (\mathbf{x}, y) \in \Omega_m, \quad 0 < t < \delta_m.$$

Then

$$-\sum_{i,j=1}^{n} a_{ij}(\mathbf{x}, y) D_{ij} w = p(\mathbf{x}, y) w^{-\gamma} \quad \text{on } \Omega_m, \quad w = 0 \quad \text{on } \partial\Omega_m \tag{4.1}$$

has a pair of super and sub solutions $u(\mathbf{x}, y)$, $\delta_m \phi_m$. Thus (4.1) has a solution w_m that can be proved to satisfy

$$0 < w_m < u \quad \text{on } \Omega_m,$$
$$\frac{1}{2}\delta_s \phi_s \le w_m \quad \text{on } \Omega_m$$

for all $m > s$. Finally we take limit of w_m to get a desired solution.

5. The General Case

Now we consider the boundary-value problem

$$-\sum_{i,j=1}^{n} a_{ij}(\mathbf{x}, y) D_{ij} u = g(\mathbf{x}, y, u) \quad \text{on } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \tag{5.1}$$

In addition to the assumptions on (a_{ij}) and Ω given at the beginning of the paper, we assume the following conditions.

(1) Trace $a_{ij}) = 1$;
(2) There is a constant $c_1 > 0$ such that $a_{nn} \ge c_1$ on $\overline{\Omega}$;
(3) There is a family of increasing positive functions $T = T(t)$ satisfying (with $T_{\mathbf{x}} = T(|\mathbf{x}|)$)
   (a) $|T_{\mathbf{x}_0} - T_{\mathbf{x}}| \le |\mathbf{x}_0 - \mathbf{x}|/c_5$;
   (b) $g(\mathbf{x}, y, T_{\mathbf{x}_0} + z_{\mathbf{x}_0}) \le c_2$ on $\Omega_{\mathbf{x}_0}$ ($\Omega_{\mathbf{x}_0}$, $z_{\mathbf{x}_0}$ and c_2 are defined in Section 2);
(4) $g(\mathbf{x}, y, t)$ is non-negative, in $C^1(\overline{\Omega} \times \mathbb{R}^n_+)$ and decreasing on t.
(5) $\lim_{t \longrightarrow 0^+} \frac{g(\mathbf{x}, y, t)}{t} \ge v_0(\mathbf{x}, y)$ uniformly for $(\mathbf{x}, y)$ in any bounded subset on $\overline{\Omega}$, where $v_0(\mathbf{x}, y)$ is a non-negative function satisfying that when m is large, the eigenvalue problem

$$-\sum_{i,j=1}^{n} a_{ij}(\mathbf{x}, y) D_{ij} w = \lambda v_0(\mathbf{x}, y) w \quad \textit{on} \quad \Omega_m, \quad u = 0 \quad \textit{on} \quad \partial\Omega_m.$$

has a first eigenvalue $\lambda_1 < 1$.

Then we have the following conclusion.

Theorem 5.1. *Under the assumptions (1)-(5), (5.1) has a positive solution.*

Proof. We just sketch the proof here. Assumptions (1)–(3) assure that $T_{\mathbf{x}_0} + z_{\mathbf{x}_0}$ is a family of auxiliary functions satisfying (2.1) on $\Omega_{\mathbf{x}_0}$ and the graphs of these function have the desired relative positions as discussed in Section 2.

Assumption (4) assures that lemma 1 can be applied and the boundary value problem

$$-\sum_{i,j=1}^{n} a_{ij}(\mathbf{x}, y)D_{ij}w = g(\mathbf{x}, y, w) \quad \text{in } O(\mathbf{x}_0, y_0), \quad w = v \text{on } \partial O(\mathbf{x}_0, y_0) \tag{5.2}$$

has a unique positive solution for each positive function v on $\overline{\Omega}$. Thus the lift $M_{(\mathbf{x}_0,y_0)}$ and the class Ξ of functions are well defined. The proofs of lemmas 2-4 and the existence of the super solution u are the same.

Finally the assumption (5) assures that the proof in Section 4 still works out like that in [8]. □

Now we apply theorem 3 to the case that $g(\mathbf{x}, y, u) = p(\mathbf{x}, y)e^{-u}$. We consider a modified problem:

$$-\sum_{i,j=1}^{n} a_{ij}(\mathbf{x}, y)D_{ij}u = \frac{p(\mathbf{x}, y)e^{-c_5 u}}{c_5} \quad \text{on } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \tag{5.3}$$

If we can find a positive solution u of (5.3), then $c_5 u$ is a positive solution of (1.7).

For (5.3), we set

$$T(t) = \frac{1}{c_5}(t + c_4) + \frac{1}{c_5} \ln \frac{C}{c_2 c_5} + A$$

where A is a positive constant such that $\frac{1}{c_5} \ln \frac{C}{c_5} + A > 1$, C is defined in (1.5) and c_2, c_4, c_5 are defined in Section 2. Then $T(t)$ is increasing and the assumption (3)(a) is obviously satisfied for $T_{\mathbf{x}} = T(|\mathbf{x}|)$. For (3)(b), on $\Omega_{\mathbf{x}_0}$,

$$\begin{aligned}
\frac{1}{c_5} p(\mathbf{x}, y)e^{-c_5(T_{\mathbf{x}_0} + z_{\mathbf{x}_0})} &\le \frac{C}{c_5} e^{|\mathbf{x}|} e^{-c_5 T_{\mathbf{x}_0}} \\
&\le \frac{C}{c_5} e^{|\mathbf{x}_0| + c_4} e^{-c_5 T_{\mathbf{x}_0}} \\
&= \frac{C}{c_5} e^{|\mathbf{x}_0| + c_4} e^{-|\mathbf{x}_0| - c_4 - \ln \frac{C}{c_2 c_5} - c_5 A} \\
&= c_2 e^{-c_5 A} < c_2 .
\end{aligned}$$

Assumption (4) is obvious. For assumption (5), let λ_1 be the first eigenvalue of the eigenvalue problem (Ω_1 is defined in Section 4)

$$-\sum_{i,j=1}^{n} a_{ij}(\mathbf{x}, y)D_{ij}w = \lambda p(\mathbf{x}, y)w \quad \text{on } \Omega_1, \quad w = 0 \quad \text{on } \partial\Omega_1.$$

Set $v_0 = 2\lambda_1 p(\mathbf{x}, y)$, then it is easy to see that

$$\lim_{t \to 0^+} \frac{p(\mathbf{x}, y)e^{-t}}{t} \ge v_0(\mathbf{x}, y) \quad \text{uniformly on } \overline{\Omega}.$$

It is also easy to see that v_0 has the desired property. Thus assumption (5) is satisfied. Therefore we can conclude that Theorem 1.22 is true.

6. Appendix I: Verifications of (2.3), (2.4), (2.6),(2.7)

In this appendix, we verify (2.3)-(2.4) and (2.6)-(2.7) used in Section 2. All the computations here are copied from [10].

Set $\Phi_1(\rho) = \rho^{-2}$ if $0 < \rho < 1$ and $\Phi_1(\rho) = \frac{11}{c_1}$ if $\rho \geq 1$, and define a function χ by

$$\chi(\alpha) = \int_\alpha^\infty \frac{d\rho}{\rho^3\Phi_1(\rho)} \quad \text{for } \alpha > 0.$$

It is clear that $\chi(\alpha)$ is a decreasing function with range $(0, \infty)$. Let η be the inverse of χ. Then η is a positive, decreasing function with range $(0, \infty)$. Let $c^* = 11/c_1$. For $\alpha > 1$, we have

$$\chi(\alpha) = \int_\alpha^\infty \frac{d\rho}{\rho^3\Phi_1(\rho)} = \int_\alpha^\infty \frac{d\rho}{c^*\rho^3} = \frac{1}{2c^*}\alpha^{-2}. \tag{6.1}$$

Thus

$$\eta(\beta) = (2c^*\beta)^{-\frac{1}{2}} \quad for \quad 0 < \beta < (2c^*)^{-1}. \tag{6.2}$$

Let $H \geq 2$. Since $\eta(\chi(H)) = H$ and η is decreasing, we have $\eta(\beta) > H$ for $0 < \beta < \chi(H)$. We define a function $A(H)$ by

$$A(H) = 2M(\int_1^{e^{\chi(H)}} \eta(\ln t)dt)^{-1}. \tag{6.3}$$

For the rest of this article, we set $a = A(H)$ and define

$$h_a(r) = \int_r^{ae^{\chi(H)}} \eta(\ln\frac{t}{a})\, dt \quad \text{for } a \leq r \leq ae^{\chi(H)}. \tag{6.4}$$

Then

$$h_a(ae^{\chi(H)}) = 0, \quad h_a(a) = h_{A(H)}(A(H)) = 2M. \tag{6.5}$$

For $a < r \leq ae^{\chi(H)}$,

$$h_a'(r) = -\eta(\ln\frac{r}{a}) < 0,\ |h_a'(r)| > H, \quad h_a''(r) = \frac{1}{r}(\eta(\ln\frac{r}{a}))^3\Phi_1(\eta(\ln\frac{r}{a})). \tag{6.6}$$

Thus for $a < r \leq ae^{\chi(H)}$,

$$\frac{h_a''(r)}{(h_a'(r))^2} = -\frac{h_a'(r)}{r}\Phi_1(-h_a'(r)). \tag{6.7}$$

Let h_a^{-1} be the inverse of h_a. Then h_a^{-1} is decreasing and

$$h_a^{-1}(0) = A(H)e^{\chi(H)}, \quad h_a^{-1}(2M) = A(H). \tag{6.8}$$

Thus we have the first half of (2.3). Further for $-M \leq y \leq M$,

$$(h_a^{-1})'(y+M) = \frac{1}{h_a'(h_a^{-1}(y+M))}$$

$$\begin{aligned}(h_a^{-1})''(y+M) &= (\frac{1}{h_a'(h_a^{-1}(y+M))})' \\ &= -\frac{h_a''(h_a^{-1}(y+M))(h_a^{-1})'(y+M)}{(h_a'(h_a^{-1}(y+M)))^2} \\ &= -\frac{h_a''(h_a^{-1}(y+M))}{(h_a'(h_a^{-1}(y+M)))^3} \\ &= \frac{1}{h_a^{-1}(y+M)}\Phi_1(-h_a'(h_a^{-1}(y+M))).\end{aligned}$$

Thus

$$(h_a^{-1})''(y+M)h_a^{-1}(y+M) = \Phi_1(-h_a'(h_a^{-1}(y+M))). \tag{6.9}$$

Now we choose an $H_0 > 2$ such that for $H \geq H_0$,

$$H_0 > \frac{1}{\sqrt{2c^*}} + 3M + 4 + \frac{24nc_1K}{M}, \quad \sqrt{\frac{4K}{A(H)e^{\chi(H)}}} \leq \frac{1}{\sqrt{2}}. \tag{6.10}$$

Then we have (2.4). For $H > H_0$, by (6.1), (6.2), we have

$$\begin{aligned}A(H)^{-1} &= (2M)^{-1}\int_1^{e^{\chi(H)}} \eta(\ln t)dt \\ &= (2M)^{-1}\int_0^{\chi(H)} \eta(m)e^m dm \\ &= (2M)^{-1}\int_0^{\chi(H)} \frac{e^m}{\sqrt{2c^*m}}dm\,.\end{aligned}$$

From

$$\frac{1}{\sqrt{2c^*}}\int_0^{\chi(H)} \frac{1}{\sqrt{m}}dm \leq \int_0^{\chi(H)} \frac{e^m}{\sqrt{2c^*m}}dm \leq \frac{e^{\chi(H)}}{\sqrt{2c^*}}\int_0^{\chi(H)} \frac{1}{\sqrt{m}}dm\,,$$

we have

$$\frac{1}{c^*H} = \frac{2\sqrt{\chi(H)}}{\sqrt{2c^*}} \leq \int_0^{\chi(H)} \frac{e^m}{\sqrt{2c^*m}}dm \leq \frac{2e^{\chi(H)}\sqrt{\chi(H)}}{\sqrt{2c^*}} = \frac{e^{\frac{1}{2c^*H^2}}}{c^*H}.$$

Thus

$$2Mc^*H \geq A(H) \geq 2Mc^*He^{-\chi(H)} = 2Mc^*He^{-\frac{1}{2c^*H^2}}. \tag{6.11}$$

Thus we have the second half of (2.3) since $c^* = 11/c_1$.

For $\mathbf{x}_0 \in \mathbb{R}^{n-1}$, and a fixed constant K, we define a domain $\Omega_{\mathbf{x}_0,H,K}$ in $(\mathbf{x}, y)$ space by (2.8) and define a function $z = z(\mathbf{x}, y)$ by (2.5). Since $h_a^{-1}(y+M) \geq 0$ for $|y| \leq M$, $(\mathbf{x}_0, y) \in \Omega_{\mathbf{x}_0,H,K}$ for $|y| < M$. Further it is clear that the function $z = z(\mathbf{x}, y)$ is well defined on $\Omega_{\mathbf{x}_0,H,K}$.

Now we verify the first half of (2.7), on $\partial\Omega_{\mathbf{x}_0,H,K} \cap \{(\mathbf{x}, y) : |y| < M\}$,

$$|\mathbf{x} - \mathbf{x}_0| = \sqrt{\frac{2K}{A(H)e^{\chi(H)}}}h^{-1}(y+M);$$

then from (6.8), we have

$$\begin{aligned} z &= A(H)e^{\chi(H)} - \{(h_a^{-1}(y+M))^2 - |\mathbf{x}-\mathbf{x}_0|^2\}^{1/2} \\ &= A(H)e^{\chi(H)} - h_a^{-1}(y+M)(1-\frac{2K}{A(H)e^{\chi(H)}})^{1/2} \\ &\geq A(H)e^{\chi(H)} - A(H)e^{\chi(H)}(1-\frac{2K}{A(H)e^{\chi(H)}})^{1/2} \\ &\geq A(H)e^{\chi(H)}(1-(1-\frac{2K}{2A(H)e^{\chi(H)}})) = K. \end{aligned}$$

Here we have used (6.10) and the fact that $\sqrt{1-t} \leq 1 - \frac{1}{2}t$ for $0 < t < 1$. For the second half of (2.7), since $h_a^{-1}(r)$ and η are decreasing functions, we have

$$\begin{aligned} \frac{-1}{h_a'(h_a^{-1}(y+M))} &= \frac{1}{\eta(\ln(\frac{1}{a}h_a^{-1}(y+M)))} \\ &\leq \frac{1}{\eta(\ln e^{\chi(H)})} \\ &= \frac{1}{\eta(\chi(H))} = \frac{1}{H}, \quad \text{for } |y| \leq -M. \end{aligned} \tag{6.12}$$

Then by (2.5), we have

$$\frac{\partial z}{\partial y}(\mathbf{x}_0, y) = \frac{-1}{h_a'(h_a^{-1}(y+M))} \leq \frac{1}{H}, \quad \text{for } |y| \leq -M.$$

Now the second half of (2.7) follows from this and

$$z(\mathbf{x}_0, -M) = A(H)e^{\chi(H)} - h_a^{-1}(0) = A(H)e^{\chi(H)} - A(H)e^{\chi(H)} = 0.$$

For (2.6), we set $S = \{(h_a^{-1}(y+M))^2 - |\mathbf{x}-\mathbf{x}_0|^2\}^{1/2}$. Then we have that for $1 \leq i \leq n-1$,

$$\frac{\partial z}{\partial x_i} = \frac{1}{S}(x_i - x_{0i}), \quad \frac{\partial z}{\partial y} = -\frac{1}{S}h_a^{-1}(h_a^{-1})'.$$

By (6.10) and (6.11), on $\Omega_{\mathbf{x}_0,H,K}$, we have

$$\frac{1}{2}h_a^{-1}(y+M) \leq S \leq h_a^{-1}(y+M),$$

and

$$\frac{|\mathbf{x}-\mathbf{x}_0|}{S} \leq 2(\frac{2K}{A(H)e^{\chi(H)}})^{1/2} \leq 2(\frac{2K}{2Mc^*H})^{1/2}.$$

Thus, by (6.12), we have

$$|\frac{\partial z}{\partial x_i}| \leq 2(\frac{c_1K}{MH})^{1/2}, \quad |\frac{\partial z}{\partial y}| \leq \frac{h_a^{-1}(y+M)}{S|h_a'(h_a^{-1}(y+M))|} \leq \frac{2}{H}. \tag{6.13}$$

Hence from (6.10), and the assumption that $\operatorname{Trace} a_{ij}) = 1$ (hence all eigenvalues of (a_{ij}) are less than or equal to 1), we have

$$|\sum_{i,j=1}^{n} a_{ij}\frac{\partial z}{\partial x_i}\frac{\partial z}{\partial x_j}| \leq |Dz|^2 \leq 1. \tag{6.14}$$

Now we have

$$
\begin{aligned}
Qz &= \sum_{i,j=1}^{n} a_{ij}(\mathbf{x},y)D_{ij}z \\
&= \frac{1}{S}\sum_{i=1}^{n-1} a_{ii} + \frac{1}{S^3}\sum_{i,j=1}^{n-1} a_{ij}(x_i-x_i^0)(x_j-x_j^0) - \frac{1}{S^3}\sum_{i=1}^{n-1} a_{in}(x_i-x_i^0)h_a^{-1}(h_a^{-1})' \\
&\quad - \frac{1}{S}a_{nn}((h_a^{-1})^2 + h_a^{-1}(h_a^{-1})'') + \frac{1}{S^3}a_{nn}(h_a^{-1})^2((h_a^{-1})')^2 \\
&= \frac{1}{S}\big\{1 - a_{nn} + \sum_{i,j=1}^{n} a_{ij}\frac{\partial z}{\partial x_i}\frac{\partial z}{\partial x_j} - a_{nn}((h_a^{-1})^2 + h_a^{-1}(h_a^{-1})'')\big\} \quad (\text{since } a_{nn} > 0) \\
&\le \frac{1}{S}\big\{1 + \sum_{i,j=1}^{n} a_{ij}\frac{\partial z}{\partial x_i}\frac{\partial z}{\partial x_j} - a_{nn}h_a^{-1}(h_a^{-1})''\big\}\,.
\end{aligned}
$$

By (2.2), (6.9)), (6.11) and (6.14)) the above expression is bounded by

$$
\frac{-9}{S} \le \frac{-9}{h_a^{-1}(y+M)} \le \frac{-9}{A(H)e^{\chi(H)}} \le \frac{-9}{2Mc^*He^{\frac{1}{2c^*H^2}}} \le \frac{-3c_1}{22eMH}.
$$

This shows (2.6).

7. Appendix II: A Construction of the Domain J

In this part, we give a construction of the domain J used at the end of Section 2 in the definition of Π. Let

$$
\mathbb{R}^n_+ = \{(y_1,y_2,\dots,y_n)|y_n > 0\}\,,
$$

$$
J_1 = \big\{(y_1,y_n) : y_1 = \pm 1,\ |y_n| \le 1 \text{ or } y_n = \pm 1,\ |y_1| \le 1\big\}
$$

That is, J_1 is a square with side length 2 and center $(0,0)$ in (y_1,y_n) plane. In polar coordinate we can write ∂J_1 as

$$
(y_1,y_n) = (k(\theta)\cos\theta, k(\theta)\sin\theta), \quad 0 \le \theta \le 2\pi,
$$

where $k(\theta)$ is a positive, continuous, periodic function of period 2π, $k(\theta)$ is C^∞ except at $\theta = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$. Then we can smooth out $k(\theta)$ near those points to get a function $k_1(\theta)$ such that $k_1(\theta)$ is a positive, C^∞, periodic function of period 2π, $k_1(\theta) = k(\theta)$ except in some small neighborhoods of $\theta = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$, and $k_1(\theta) \le k(\theta)$ for all θ. Indeed we can modify $k(\theta)$ as follows:

Let $s(t)$ be a C^∞ function satisfying

1. $s(t) = 0$ if $t \le 1$;
2. $0 < s(t) \le \frac{1}{8}$ if $1 < t \le 2$;
3. $s(t) \ge 0$ for all t;
4. $s(t) = 1$ if $t \ge 4$.

Fixed a positive constant $\epsilon < \frac{\pi}{100}$. Near $\theta = \frac{\pi}{4}$, we define

$$
k_1(\theta) = k(\theta)s\big(\frac{1}{\epsilon}|\theta - \frac{\pi}{4}|\big) + \frac{1}{8}\big(1 - s(\frac{2}{\epsilon}|\theta - \frac{\pi}{4}|)\big).
$$

Then using the fact that $\max k(\theta) = \sqrt{2}$, $\min k(\theta) = 1$, we can verify that $k_1(\theta)$ is positive, smooth and

$$
k_1(\theta) = k(\theta) \ \text{ if } \ |\theta - \frac{\pi}{4}| \ge 4\epsilon; \quad 0 < k_1(\theta) \le k(\theta).
$$

In a similar way, we can modify $k(\theta)$ near other points $-\pi/4$ and $\pm 3\pi/4$. Now let J_2 be the domain in (y_1, y_n) plane bounded by the curve

$$(y_1, y_n) = (k_1(\theta)\cos\theta, k_1(\theta)\sin\theta), \quad 0 \le \theta \le 2\pi.$$

We then rotate the set

$$\{(y_1, 0, \cdot, \cdot, \cdot, 0, y_n) : (y_1, y_n) \in J_2\}$$

with respect to y_n axis to get a domain J_3. Finally, J is obtained from J_3 by appropriate translation and scaling.

Acknowledgement. The author thanks the anonymous referee for his/her comments and suggestions that make the paper more readable.

References

[1] Amann, H., *On the number of solutions of nonlinear equations in ordered Banach spaces*, J. Functional Analysis, **11** (1972), 346–284.

[2] Callegari, A. & Nachman, A., *A nonlinear singular boundary value problem in the theory of pseudoplastic fluids*, SIAM J. Appl. Math. **38** (1980), 275–281.

[3] Crandall, M. G., Rabinowitz, P. & Tata, L., *On a Dirichlet problem with singular nonlinearity*, Comm. PDE, **2** (1977), 193–222

[4] Dalmasso, R., *Solutions d'equations elliptiques semi-linearires singulieres*, Ann. Mat. Pura. Appl, **153**(1988),191–201.

[5] Edelson, A., *Entire solutions of singular elliptic equations*, J. Math. Anal. Appl. **139** (1989), 523–532.

[6] Finn, R., *New Estimates for equations of minimum surface type*, Arch. Rational Mech. Anal., **7** (1963), 337–375.

[7] Gilbarg, D. & Trudinger, N., *Second order elliptic partial differential equations*, second edition, Springer 1983.

[8] Jin, Z., *Solutions for a class of singular semilinear elliptic equations*, Nonlinear analysis, **31** (1998), 475–492

[9] Jin, Z.& Lancaster, K., *Phragmén-Lindelöf Theorems and the Asymptotic Behavior of solutions of quasilinear elliptic equations in slabs*, Proc. Roy. Soc. Edinburgh **130A** (2000), 335–373.

[10] Jin, Z. & K. Lancaster, *The convergent rate of solutions of Dirichlet problems for quasilinear equations* (2002), submitted for publication.

[11] Kusano, T. & Swanson, C. A., *Asymptotic properties of semilinear elliptic equations*, Funkcial Ekvac. **26** (1983), 115–129.

[12] Lair, Alan V. & Shaker, Aihua W.,*Classical and weak solutions of a singular semilinear elliptic problem*, J. Math. Anal. Appl. **211** (1997), 371–385.

[13] Lopez, R., *Constant mean curvature graphs in a strip of R^2*, Pacific J. Math. **206** (2002), 359–373.

[14] Serrin, J., *The problem of Dirichlet for quasilinear equations with many independent variables*, Phil. Trans. Royal Soc. Lond. A **264** (1969), 413–496.

[15] Shaker, Aihua W., *On singular semilinear elliptic equations*, J. Math. Anal. Appl. **173**(1993), 222–228.

[16] Wong, J. S. W., *On the generalized Emden-Fowler equation*, SIAM Review, **17** (1975), 339–360.

Zhiren Jin
Department of Mathematics and Statistics, Wichita State University, Wichita, Kansas, 67260-0033, USA
E-mail address: zhiren@math.twsu.edu