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OPTIMAL CONTROL FOR A NONLINEAR AGE-STRUCTURED POPULATION DYNAMICS MODEL

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Abstract. We investigate the optimal harvesting problem for a nonlinear age-dependent and spatially structured population dynamics model where the birth process is described by a nonlocal and nonlinear boundary condition. We establish an existence and uniqueness result and prove the existence of an optimal control. We also establish necessary optimality conditions.

1. Introduction and setting of the problem

We consider a general mathematical model describing the dynamics of a single species population with age dependence and spatial structure. Let $u(x, t, a)$ be the distribution of individuals of age $a \geq 0$ at time $t \geq 0$ and location x in $\overline{\Omega}$. Here Ω is a bounded open subset of $\mathbb{R}^N$, $N \in \{1, 2, 3\}$, with a suitably smooth boundary $\partial\Omega$. Thus

$$P(x,t) = \int_0^{A_\dagger} u(x,t,a)\, da \tag{1.1}$$

is the total population at time t and location x, where $A_\dagger$ is the maximal age of an individual. Let $\beta(x, t, a, P(x, t)) \geq 0$ be the natural fertility-rate, and let $\mu(x, t, a, P(x, t)) \geq 0$ be the natural death-rate of individuals of age a at time t and location x. We also assume that the flux of population takes the form $k\nabla u(x, t, a)$ with $k > 0$, where ∇ is the gradient vector with respect to the spatial variable x.

In this paper we are concerned with the optimal harvesting problem on the time interval $(0, T)$, $T > 0$, subject to an external supply of individuals $f(x, t, a) \geq 0$ and to a specific harvesting effort $v(x, t, a)$, where $(x, t, a) \in Q = \Omega \times (0, T) \times (0, A_\dagger)$.

So, we deal with the problem of finding the harvesting effort v in order to obtain the best harvest; i.e.,

Maximize, over all $v \in \mathcal{V}$, the value of

$$\int_Q v(x,t,a)g(x,t,a)u^v(x,t,a)\, dx\, dt\; da\,, \tag{1.2}$$

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where g is a given bounded function, and u^v is the solution of

$$\begin{aligned}
&\partial_t u + \partial_a u - k\Delta_x u + \mu(x,t,a,P(x,t))u = f - vu, && \text{in } Q\\
&\frac{\partial u}{\partial \eta}(x,t,a) = 0, && \text{on } \Sigma\\
&u(x,t,0) = \int_0^{A_\dagger} \beta(x,t,a,P(x,t))u(x,t,a)\,da, && \text{in } \Omega\times(0,T)\\
&u(x,0,a) = u_0(x,a), && \text{in } \Omega\times(0,A_\dagger),
\end{aligned} \tag{1.3}$$

where $\Sigma = \partial\Omega\times(0,T)\times(0,A_\dagger)$. From a biological point of view $g(x,t,a)\geq 0$ is a weight (the price of an individual of age a at time t and location x) and $u_0(x,a)\geq 0$ is the initial distribution of population.

The set of controllers is

$$\mathcal{V} = \Big\{ v\in L^2(Q) : \zeta_1(x,t,a)\leq v(x,t,a)\leq \zeta_2(x,t,a) \ \text{ a.e. } (x,t,a)\in Q \Big\}$$

for some $\zeta_1,\zeta_2\in L^\infty(Q)$, $0\leq \zeta_1(x,t,a)\leq \zeta_2(x,t,a)$ a.e. in Q. The harvesting problem for linear initial value age-structured population has been previously studied in Brokate [2, 3], Gurtin et al [4, 5], Murphy et al [8] and the periodic case in Aniţa et al [1].

We assume the following hypotheses:

(H1) The fertility rate satisfies $\beta\in L^\infty(Q\times\mathbb{R})$, $\beta(x,t,a,P)\geq 0$ a.e. $(x,t,a,P)\in Q\times\mathbb{R}$ and is decreasing and locally Lipschitz continuous with respect to the variable P

(H2) The mortality rate satisfies $\mu\in L^\infty_{\rm loc}(\overline{\Omega}\times[0,T]\times[0,A_\dagger)\times\mathbb{R})$, and μ is increasing and locally Lipschitz continuous with respect to the variable P, $\mu(x,t,a,P)\geq \mu_0(a,t)\geq 0$ a.e. $(x,t,a,P)\in Q\times\mathbb{R}$, where $\mu_0\in L^\infty_{\rm loc}([0,T]\times[0,A_\dagger))$ and

$$\int^{A_\dagger} \mu_0(t+a-A_\dagger,a)\,da = +\infty, \quad \text{a.e. } t\in(0,T)\,.$$

The last condition in (H2) implies that each individual in the population dies before age $A_\dagger$. In addition, we assume the following on u_0, f, g:

(H3) $u_0\in L^\infty(\Omega\times(0,A_\dagger))$, $u_0(x,a)\geq 0$ a.e. $(x,a)\in\Omega\times(0,A_\dagger)$.

(H4) $f,\ g\in L^\infty(Q)$, $f(x,t,a),\ g(x,t,a)\geq 0$ a.e. $(x,t,a)\in Q$.

This paper is organized as follows. In Section 2 we prove that under the assumptions listed above and for any $v\in\mathcal{V}$, (1.3) admits a unique and nonnegative solution. A compactness result for the same system is also proved. In Section 3 we treat the existence of an optimal control for problem (1.2). Section 4 is devoted to the deduction of the necessary optimality conditions for the optimal harvesting problem.

2. Existence, uniqueness and compactness of solutions

The first part of this section is devoted to the existence and uniqueness of solutions to system (1.3), under assumptions (H1)–(H4) and with $v\in\mathcal{V}$ fixed. By a solution to (1.3), we mean a function $u\in L^2(Q)$ which belongs to $C(\overline{S};L^2(\Omega))\cap AC(S;L^2(\Omega))\cap L^2(S;H^1(\Omega))\cap L^2_{loc}(S;L^2(\Omega))$, for almost any characteristic line S

of equation $a - t = const.$, $(t, a) \in (0, T) \times (0, A_\dagger)$ and satisfies

$$\begin{aligned}
&Du(x,t,a) - k\Delta_x u(x,t,a) + \mu(x,t,a,P(x,t))u(x,t,a) \\
&= f(x,t,a) - v(x,t,a)u(x,t,a), && \text{a.e. in } Q \\
&\frac{\partial u}{\partial \eta}(x,t,a) = 0, && \text{a.e. in } \Sigma \\
&\lim_{h\to 0^+} u(x,t+h,h) = \int_0^{A_\dagger} \beta(x,t,a,P(x,t))u(x,t,a)\,da, && \text{a.e. in } \Omega \times (0,T) \\
&\lim_{h\to 0^+} u(x,h,a+h) = u_0(x,a), && \text{a.e. in } \Omega \times (0,A_\dagger),
\end{aligned}$$

where P is given by (1.1) and Du denotes the directional derivative

$$Du(x,t,a) = \lim_{h\to 0} \frac{1}{h}\big[u(x,t+h,a+h) - u(x,t,a)\big].$$

Theorem 2.1. *For any $v \in \mathcal{V}$, (1.3) admits a unique and nonnegative solution u^v which belongs to $L^\infty(Q)$.*

Proof. Denote by Λ the mapping $\Lambda : \widetilde{u} \mapsto u^{\widetilde{u},v}$, where $u^{\widetilde{u},v}$ is the solution of

$$\begin{aligned}
&Du - k\Delta_x u + \mu(x,t,a,\widetilde{P}(x,t))u = f - v(x,t,a)u, && (x,t,a) \in Q \\
&\frac{\partial u}{\partial \eta}(x,t,a) = 0, && (x,t,a) \in \Sigma \\
&u(x,t,0) = \int_0^{A_\dagger} \beta(x,t,a,\widetilde{P}(x,t))u(x,t,a)\,da, && (x,t) \in \Omega \times (0,T) \\
&u(x,0,a) = u_0(x,a), && (x,a) \in \Omega \times (0,A_\dagger),
\end{aligned}$$

with $\widetilde{P}(x,t) = \int_0^{A_\dagger} \widetilde{u}(x,t,a)\,da$. Let $L^p_+(Q) = \{u \in L^p(Q) : u(x,t,a) \geq 0 \text{ a.e. in } Q\}$. Then the mapping Λ is well defined from $L^2_+(Q)$ to $L^2_+(Q)$; see Garroni et al [6]. The comparison result in Garroni et al [6] and in Langlais [7] implies

$$0 \leq u^{\widetilde{u},v}(x,t,a) \leq \overline{u}(x,t,a) \quad \text{a.e. in } Q,$$

where $\overline{u} \in L^\infty_+(Q)$ is the solution of (1.3) corresponding to a null mortality rate and to a maximal fertility rate equal to $\|\beta\|_{L^\infty(Q\times\mathbb{R})}$.

For any $\widetilde{u}_1,\ \widetilde{u}_2 \in L^2(Q)$ we denote $\widetilde{P}_i(x,t) = \int_0^{A_\dagger} \widetilde{u}_i(x,t,a)\,da$, with $(x,t) \in \Omega \times (0,T)$, and $i \in \{1,2\}$. Using now the definition of Λ we obtain

$$\begin{aligned}
&\int_{Q_t} [D(\Lambda\widetilde{u}_1 - \Lambda\widetilde{u}_2) - k\Delta_x(\Lambda\widetilde{u}_1 - \Lambda\widetilde{u}_2) + \mu(x,s,a,\widetilde{P}_1(x,t))(\Lambda\widetilde{u}_1 - \Lambda\widetilde{u}_2) \\
&+(\mu(x,s,a,\widetilde{P}_1) - \mu(x,s,a,\widetilde{P}_2))\widetilde{u}_2 + v(\Lambda\widetilde{u}_1 - \Lambda\widetilde{u}_2)](\Lambda\widetilde{u}_1 - \Lambda\widetilde{u}_2)\,dx\,ds\,da = 0,
\end{aligned}$$

where $Q_t = \Omega \times (0,t) \times (0,A_\dagger)$, $t \in (0,T)$.

Using Gauss-Ostrogradski's formula and the Lipschitz continuity of μ and β with respect to P, we get after some calculations that

$$\|(\Lambda\widetilde{u}_1 - \Lambda\widetilde{u}_2)(t)\|^2_{L^2(\Omega\times(0,A_\dagger))} \leq C \int_0^t \|(\widetilde{u}_1 - \widetilde{u}_2)(s)\|^2_{L^2(\Omega\times(0,A_\dagger))}\,ds,$$

where C is a positive constant. Banach's fixed point theorem allows us to conclude the existence of a unique fixed point for Λ. Since the solution u^v satisfies

$$0 \leq u^v(x,t,a) \leq \overline{u}(x,t,a) \quad \text{a.e. in } Q$$

and $\overline{u} \in L^\infty_+(Q)$, we complete the proof. ♢

For $v \in \mathcal{V}$, let

$$P^v(x,t) = \int_0^{A_\dagger} u^v(x,t,a)\,da \quad (x,t) \in \Omega \times (0,T)\,.$$

We shall prove now a compactness result which is one of the main ingredients in the next section.

Lemma 2.2. *The set* $\{P^v;\ v \in \mathcal{V}\}$ *is relatively compact in* $L^2(\Omega \times (0,T))$.

Proof. Because u^v is a solution of (1.3), for any $\varepsilon > 0$ small enough we have that

$$P^{v,\varepsilon}(x,t) = \int_0^{A_\dagger - \varepsilon} u^v(x,t,a)\,da, \quad (x,t) \in \Omega \times (0,T)$$

is a solution of

$$\begin{aligned} P_t^{v,\varepsilon} - k\Delta_x P^{v,\varepsilon} &= \int_0^{A_\dagger-\varepsilon} (f - (\mu(x,t,a,P^v(x,t)) + v)u^v)da - u^v(x,t,A_\dagger - \varepsilon)\\ &\quad + \int_0^{A_\dagger} \beta(x,t,a,P^v(x,t))u^v(x,t,a)\,da, \quad \text{a.e. in } \Omega \times (0,T)\\ &\frac{\partial P^{v,\varepsilon}}{\partial \eta}(x,t) = 0, \quad \text{a.e. } \partial\Omega \times (0,T)\\ P^{v,\varepsilon}(x,0) &= \int_0^{A_\dagger-\varepsilon} u_0(x,a)\,da, \quad \text{a.e. in } \Omega\,. \end{aligned}$$

Since $\{vu^v\}$ and $\{\mu(\cdot,\cdot,\cdot,P^v)u^v\}$ are bounded in $L^\infty(\Omega \times (0,T) \times (0,A_\dagger - \varepsilon))$, $\{\beta(\cdot,\cdot,\cdot,P^v)u^v\}$ is bounded in $L^\infty(\Omega \times (0,T) \times (0,A_\dagger))$ and $\{u^v(\cdot,\cdot,A_\dagger - \varepsilon)\}$ is bounded in $L^\infty(\Omega \times (0,T))$ - with respect to $v \in \mathcal{V}$ (as a consequence of the proof of Theorem 2.1), we conclude that $\{P_t^{v,\varepsilon} - k\Delta_x P^{v,\varepsilon}\}$ is bounded in $L^\infty(\Omega \times (0,T))$. This implies via Aubin's compactness theorem that for any $\varepsilon > 0$ small enough, the set $\{P^{v,\varepsilon}; v \in \mathcal{V}\}$ is relatively compact in $L^2(\Omega \times (0,T))$. On the other hand

$$|P^{v,\varepsilon}(x,t) - P^v(x,t)| \le \int_{A_\dagger-\varepsilon}^{A_\dagger} |u^v(x,t,a)|\,da \le \varepsilon \|\overline{u}\|_{L^\infty(Q)}\,,$$

for all $\varepsilon > 0$, and all $v \in \mathcal{V}$, a.e. (x,t) in $\Omega \times (0,T)$. Combining these two results we conclude the relative compactness of $\{P^v; v \in \mathcal{V}\}$ in $L^2(Q)$. ♢

3. Existence of an optimal control

In this section, we prove the existence of an optimal pair (an optimal control v^* and the corresponding solution u^{v^*} for problem (1.2)). Indeed we have the following theorem.

Theorem 3.1. *Problem* (1.2) *admits at least one optimal pair.*

Proof. Let $\varphi : \mathcal{V} \to \mathbb{R}^+$, be defined by

$$\varphi(v) = \int_Q v(x,t,a)g(x,t,a)u^v(x,t,a)\,dx\,dt\,da$$

and let $d = \sup_{v\in\mathcal{V}} \varphi(v)$. Since by the proof of Theorem 2.1

$$0 \le \varphi(v) \le \int_Q \zeta_2(x,t,a)g(x,t,a)\overline{u}(x,t,a)\,dx\,dt\,da\,,$$

we have $d \in [0, +\infty)$. Now let $\{v_n\}_{n\in\mathbb{N}^*} \subset \mathcal{V}$ be a sequence such that

$$d - \frac{1}{n} < \varphi(v_n) \le d \,.$$

Since $0 \le u^{v_n}(x,t,a) \le \overline{u}(x,t,a)$ a.e. in Q, we conclude that there exists a subsequence, also denoted by $\{v_n\}_{n\in N^*}$, such that

$$u^{v_n} \to u^* \text{ weakly in } L^2(Q) \,.$$

Using Mazur's theorem we obtain the existence of a sequence $\{\widetilde{u}_n\}_{n\in N^*}$ such that

$$\widetilde{u}_n(x,t,a) = \sum_{i=n+1}^{k_n} u^{v_i}, \quad \lambda_i^n \ge 0, \quad \sum_{i=n+1}^{k_n} \lambda_i^n = 1$$

and $\widetilde{u}_n \to u^*$ in $L^2(Q)$.

Consider now the sequence of controls

$$\widetilde{v}_n(x,t,a) = \begin{cases} \frac{\sum_{i=n+1}^{k_n} \lambda_i^n v_i(x,t,a) u^{v_i}(x,t,a)}{\sum_{i=n+1}^{k_n} \lambda_i^n u^{v_i}(x,t,a)} & \text{if } \sum_{i=n+1}^{k_n} \lambda_i^n u^{v_i}(x,t,a) \ne 0 \\ \zeta_1(x,t,a), & \text{if } \sum_{i=n+1}^{k_n} \lambda_i^n u^{v_i}(x,t,a) = 0\,. \end{cases}$$

For these controls we have $\widetilde{v}_n \in \mathcal{V}$. Lemma 2.2 implies the existence of a subsequence, also denoted by $\{v_n\}_{n\in N^*}$ such that

$$P^{v_n} \to P^* \quad \text{in } L^2(\Omega\times(0,T)) \tag{3.1}$$

and since $u^{v_n} \to u^*$ weakly in $L^2(Q)$, then we obtain that

$$\int_0^{A_\dagger} u^{v_n}(\cdot,\cdot,a)\,da \to \int_0^{A_\dagger} u^*(\cdot,\cdot,a)\,da \quad \text{weakly in } L^2(\Omega\times(0,T)).$$

Consequently we get that

$$P^*(x,t) = \int_0^{A_\dagger} u^*(x,t,a)\,da \quad \text{a.e. in } \Omega\times(0,T)\,.$$

We can take a subsequence, also denoted by $\{\widetilde{v}_n\}_{n\in N^*}$, such that

$$\widetilde{v}_n \to v^* \quad \text{weakly in } L^2(Q),$$

with $v^* \in \mathcal{V}$. It is obvious now that $\widetilde{u}_n$ is a solution of

$$\begin{aligned} &Du - k\Delta_x u + \sum_{i=n+1}^{k_n} \lambda_i^n \mu(x,t,a,P^{v_i}(x,t))u^{v_i} \\ &= f - \widetilde{v}_n u, && \text{in } Q \\ &\frac{\partial u}{\partial \eta}(x,t,a) = 0, && \text{on } \Sigma \\ &u(x,t,0) = \int_0^{A_\dagger} \sum_{i=n+1}^{k_n} \lambda_i^n \beta(x,t,a,P^{v_i}(x,t))u^{v_i}\,da, && \text{in } \Omega\times(0,T) \\ &u(x,0,a) = u_0(x,a), && \text{in } \Omega\times(0,A_\dagger)\,. \end{aligned} \tag{3.2}$$

By (3.1) we deduce the existence of a subsequence (also denoted by $\{v_n\}$) such that

$$\begin{gathered} \mu(\cdot,\cdot,\cdot,P^{v_n}) \to \mu(\cdot,\cdot,\cdot,P^*) \quad \text{a.e. in } Q\,, \\ \beta(\cdot,\cdot,\cdot,P^{v_n}) \to \beta(\cdot,\cdot,\cdot,P^*) \quad \text{a.e. in } Q\,. \end{gathered}$$

Since $\widetilde{u}_n \to u^*$ in $L^2(Q)$, we have

$$\sum_{i=n+1}^{k_n} \lambda_i^n \mu(x,t,a,P^{v_i}(x,t))u^{v_i}(x,t,a) \to \mu(x,t,a,P^*(x,t))u^*(x,t,a)$$

a.e. in Q, and

$$\sum_{i=n+1}^{k_n} \lambda_i^n \beta(x,t,a,P^{v_i}(x,t))u^{v_i}(x,t,a) \to \beta(x,t,a,P^*(x,t))u^*(x,t,a)$$

a.e. in Q. Passing to the limit in (3.2) we obtain that u^* is the solution of (1.3) corresponding to v^*. Moreover we have

$$\begin{aligned}
&\sum_{i=n+1}^{k_n} \lambda_i^n \int_Q v_i(x,t,a)g(x,t,a)u^{v_i}(x,t,a)\,dx\,dt\,da \\
&= \int_Q \widetilde{v}_n(x,t,a)g(x,t,a)\widetilde{u}_n(x,t,a)\,dx\,dt\,da \\
&= \sum_{i=n+1}^{k_n} \lambda_i^n \varphi(v_i) \to \varphi(v^*)
\end{aligned}$$

(as $n \to +\infty$). We may infer now that $d = \varphi(v^*)$. ♢

4. Necessary optimality conditions

Concerning the necessary optimality conditions the following result holds under the assumptions (H1)–(H4).

Theorem 4.1. *Assume β and μ are C^1 with respect to P. If (u^*,v^*) is an optimal pair for (1.2) and if q is the solution of*

$$\begin{aligned}
&-Dq(x,t,a) - k\Delta_x q(x,t,a) + \mu(x,t,a,P^{v^*}(x,t))q(x,t,a) \\
&+ \int_0^{A_\dagger} \mu'_P(x,t,s,P^*(x,t))u^*(x,t,s)q(x,t,s)\ ds \\
&-\Big(\beta(x,t,a,P^*(x,t)) + \int_0^{A_\dagger} \beta'_P(x,t,s,P^*(x,t))u^*(x,t,s)\ ds\Big)q(x,t,0) \\
&\qquad = -v^*(g+q)(x,t,a), \quad (x,t,a)\in Q \qquad (4.1) \\
&\qquad \frac{\partial q}{\partial \eta}(x,t,a) = 0, \qquad (x,t,a)\in\Sigma \\
&\qquad q(x,t,A_\dagger) = 0, \qquad (x,t)\in\Omega\times(0,T) \\
&\qquad q(x,T,a) = 0, \qquad (x,a)\in\Omega\times(0,A_\dagger)\,,
\end{aligned}$$

then we have

$$v^*(x,t,a) = \begin{cases} \zeta_1(x,t,a) & \text{if } (g+q)(x,t,a) < 0 \\ \zeta_2(x,t,a) & \text{if } (g+q)(x,t,a) > 0\,. \end{cases}$$

Here μ'_P and β'_p are the derivatives of μ and β with respect to P.

Proof. Existence and uniqueness of q, a solution of (4.1) follows in the same way as the existence and uniqueness of the solution of (1.3). Since (v^*, u^*) is an optimal pair for (1.2) we get

$$\begin{aligned}&\int_Q v^*(x,t,a)g(x,t,a)u^{v^*}(x,t,a)\,dx\,dt\,da\\&\geq \int_Q (v^*(x,t,a)+\delta v(x,t,a))g(x,t,a)u^{v^*+\delta v}(x,t,a)\,dx\,dt\,da\end{aligned}$$

for all δ positive and small enough, for all $v \in L^\infty(Q)$ such that

$$\begin{aligned}v(x,t,a)\leq 0 \quad &\text{if } v^*(x,t,a)=\zeta_2(x,t,a)\\ v(x,t,a)\geq 0 \quad &\text{if } v^*(x,t,a)=\zeta_1(x,t,a)\,.\end{aligned}$$

This implies

$$\begin{aligned}&\int_Q v^*(x,t,a)g(x,t,a)\frac{u^{v^*+\delta v}(x,t,a)-u^{v^*}(x,t,a)}{\delta}\,dx\,dt\,da\\&+\int_Q v(x,t,a)g(x,t,a)u^{v^*+\delta v}(x,t,a)\,dx\,dt\,da\leq 0\,.\end{aligned}\tag{4.2}$$

Using the definition of solution to (1.3) and the comparison result in Garroni et al [6], we can prove that for any $v \in L^\infty(Q)$ as above, the following convergence holds

$$u^{v^*+\delta v}(x,t,a)\longrightarrow u^{v^*}(x,t,a) \text{ in } L^\infty(0,T;L^2((0,A_\dagger)\times\Omega))$$

as $\delta \longrightarrow 0^+$. Let

$$z^\delta(x,t,a)=\frac{u^{v^*+\delta v}(x,t,a)-u^{v^*}(x,t,a)}{\delta},\quad (x,t,a)\in Q\,.$$

Then the function z^δ is a solution of

$$\begin{gathered}Dz^\delta-k\Delta_x z^\delta+\frac{1}{\delta}\big(\mu(x,t,a,P^{v^*+\delta v}(x,t))u^{v^*+\delta v}-\mu(x,t,a,P^{v^*}(x,t))u^{v^*}\big)\\ =-v^*z^\delta-v(x,t,a)u^{v^*+\delta v},\quad (x,t,a)\in Q\\ \frac{\partial z^\delta}{\partial\eta}(x,t,a)=0,\quad (x,t,a)\in\Sigma\\ z^\delta(x,t,0)=\int_0^{A_\dagger}\frac{\beta(x,t,a,P^{v^*+\delta v}(x,t))u^{v^*+\delta v}-\beta(x,t,a,P^{v^*}(x,t))u^{v^*}}{\delta}\,da,\\ (x,t)\in\Omega\times(0,T)\\ z^\delta(x,0,a)=0,\quad (x,a)\in\Omega\times(0,A_\dagger)\end{gathered}$$

and using again the definition of solution to (1.3) and the comparison result in Garroni et al [6], we can prove that $z^\delta \to z$ in $L^\infty(Q)$ as $\delta \to 0$, where z is the solution of

$$\begin{gathered}Dz-k\Delta_x z+\mu(x,t,a,P^{v^*}(x,t))z(x,t,a)\\ +\mu'_P(x,t,a,P^{v^*}(x,t))u^{v^*}(x,t,a)\int_0^{A_\dagger}z(x,t,s)ds\\ =-v^*z-v(x,t,a)u^{v^*},\quad (x,t,a)\in Q\\ \frac{\partial z}{\partial\eta}(x,t,a)=0,\quad (x,t,a)\in\Sigma\\ z(x,t,0)=\int_0^{A_\dagger}\beta(x,t,a,P^{v^*}(x,t))z(x,t,a)\,da\end{gathered}$$

$$+\int_0^{A_\dagger}\Big(\beta'_P(x,t,a,P^{v^*}(x,t))u^{v^*}\int_0^{A_\dagger} z(x,t,s)ds\Big)\,da, \quad (x,t)\in\Omega\times(0,T)$$
$$z(x,0,a)=0,\quad (x,a)\in\Omega\times(0,A_\dagger)\,.$$

Passing to the limit in (4.2), $\delta\to 0^+$, we conclude that

$$\int_Q v^*(x,t,a)g(x,t,a)z(x,t,a)\,dx\,dt\,da$$
$$+\int_Q v(x,t,a)g(x,t,a)u^{v^*}(x,t,a)\,dx\,dt\,da\le 0\,,$$

for all $v\in L^\infty(Q)$ such that

$$\begin{aligned} v(x,t,a)\le 0 &\quad \text{if } v^*(x,t,a)=\zeta_2(x,t,a)\\ v(x,t,a)\ge 0 &\quad \text{if } v^*(x,t,a)=\zeta_1(x,t,a)\,. \end{aligned}$$

Multiplying (4.1) by z and integrating over Q we get after some calculation that

$$\int_Q (v^*gz)(x,t,a)\,dx\,dt\,da=\int_Q (vu^{v^*}q)(x,t,a)\,dx\,dt\,da$$

and consequently

$$\int_Q v(x,t,a)u^{v^*}(x,t,a)(g+q)(x,t,a)\,dx\,dt\,da\le 0\,,$$

for all $v\in L^\infty(Q)$ such that

$$\begin{aligned} v(x,t,a)\le 0 &\quad \text{if } v^*(x,t,a)=\zeta_2(x,t,a)\\ v(x,t,a)\ge 0 &\quad \text{if } v^*(x,t,a)=\zeta_1(x,t,a)\,. \end{aligned}$$

This implies $u^{v^*}(g+q)\in N_{\mathcal V}(v^*)$, where $N_{\mathcal V}(v^*)$ is the normal cone at $\mathcal V$ in v^* (in $L^2(Q)$).

For any $(x,t,a)\in Q$ such that $u^{v^*}(x,t,a)\ne 0$, we conclude

$$v^*(x,t,a)=\begin{cases}\zeta_1(x,t,a) & \text{if } (g+q)(x,t,a)<0\\ \zeta_2(x,t,a) & \text{if } (g+q)(x,t,a)>0\,.\end{cases}$$

On the other hand, for any $(x,t,a)\in Q$ such that $u^{v^*}(x,t,a)=0$, it is obvious that we can change the value of the optimal control v^* in (x,t,a) with any arbitrary value belonging to $[\zeta_1(x,t,a),\zeta_2(x,t,a)]$ and the state corresponding to this new control is the same and the value of the cost functional also remains the same. The conclusion of Theorem 4.1 is now obvious. ♢

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