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Existence of positive solutions for two nonlinear eigenvalue problems *

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Abstract

We study the existence of positive solutions for the following two nonlinear eigenvalue problems

$$\Delta u - g(.,u)u + \lambda f(.,u)u = 0,$$
$$\Delta u - g(.,u)u + \lambda f(.,u) = 0,$$

in a bounded regular domain in $\mathbb{R}^2$ with $u = 0$ on the boundary. We assume that f and g are in Kato class of functions.

1 Introduction

In this paper, we shall study the existence of positive solutions for the following nonlinear eigenvalue problems: (P_λ):

$$\begin{gathered} \Delta u - g(.,u)u + \lambda f(.,u)u = 0, \quad \text{in } D, \\ u > 0, \quad \text{in} D, \\ u = 0, \quad \text{on } \partial D, \end{gathered} \tag{1.1}$$

and (Q_λ):

$$\begin{gathered} \Delta u - g(.,u)u + \lambda f(.,u) = 0, \quad \text{in } D, \\ u > 0, \quad \text{in } D, \\ u = 0, \quad \text{on } \partial D. \end{gathered} \tag{1.2}$$

In this paper, D is a regular bounded domain in $\mathbb{R}^2$, Δ is the Laplacian and the functions f and g are in a new Kato class K introduced in [11]. Solutions to these problems are understood as distributional solutions in D. For the reader's convenience, we recall the definition of class K, some of its properties, and some examples below and in section 2.

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Definition 1.1 A Borel measurable function φ on D belongs to the class K if φ satisfies the condition

$$\lim_{\alpha\to 0}\sup_{x\in D}\int_{(|x-y|\leq\alpha)\cap D}\frac{\rho(y)}{\rho(x)}\log(1+\frac{\rho(x)\rho(y)}{|x-y|^2})|\varphi(y)|dy=0, \tag{1.3}$$

where $\rho(x)$ is the distance from x to ∂D.

Hansen and Hueber in [9, 10] studied the existence of eigenvalues for the linear problem

$$\begin{gathered}\Delta u-\mu u+\lambda\nu u=0,\ \text{ in }\Omega,\\ u>0,\quad\text{in }\Omega,\\ u=0,\quad\text{on }\partial\Omega,\end{gathered} \tag{1.4}$$

in the general framework of harmonic spaces where Ω is a regular bounded domain in $\mathbb{R}^n, n\geq 1$ and the measures μ and ν generate continuous potentials. They showed that (1.4) has a principal eigenvalue with a corresponding positive eigenfunction. These results were generalized later in [2]. Namely, the authors proved when f and g are locally in the Kato class K_n and under some assumptions, the existence of eigenvalues λ for which problems (1.1) and (1.2) have nonnegative eigenfunctions.

Recall that a function φ in D belongs to the Kato class K_2 [1, 4] if

$$\lim_{\alpha\to 0}\sup_{x\in D}\int_{(|x-y|\leq\alpha)\cap D}\log(\frac{1}{|x-y|})|\varphi(y)|dy=0. \tag{1.5}$$

In [1] Aizenman and Simon identified the class K_2 as the natural class of functions so that the week solutions of the equation $\Delta u+\varphi u=0$ are continuous. We point out that the class K properly contains the Kato class K_2 (see [11]).

Now we present concrete examples of functions in the class K [11].

1. Let φ be a radial function in $B(0,1)$. Then, the function φ is in the class K if and only if
$$\int_0^1 r\log(\frac{1}{r})|\varphi(r)|dr<\infty.$$
2. Let $\lambda<2$, then the function defined in D by $\rho_\lambda(y)=\frac{1}{(\rho(y))^\lambda}$ is in the class K. Note that if $1\leq\lambda<2$ then $1/(\rho(y))^\lambda\notin L^1(D)$.
3. Let $p>1$. Then $L^p(D)\subset K_2\subset L^1(D)\cap K\subset K\subset L^1_{\rm loc}(D)$. In case that φ is radial and D is a ball, we prove that φ is in K_2 if nd only if φ is in $K\cap L^1(D)$.

In the sequel, for $f:D\times\mathbb{R}\to\mathbb{R}$ a Borel function and $a>0$, we denote

$$f^-=\max(-f,0),\quad f^a(x)=\sup_{t\in[0,a]}|f(x,t)|,$$
$$\text{and}\quad f_a(x)=\sup_{t\in[0,a]}|f(x,t)|.$$

For the remaining of this paper, we assume the following two conditions:

(H1) f is a nonnegative measurable function on $D \times \mathbb{R}$, continuous with respect to the second variable such that for all $c \geq 0$, $f(.,c) \in K$.

(H2) g is a measurable functions on $D \times \mathbb{R}$, continuous with respect to the second variable such that for all $c \geq 0$, $g(.,c) \in K$.

Our main results are stated as follows.

Theorem 1.2 *Assume (H1)–(H2) and that there exists a constant $a > 0$ such that*

$$Vf_a > 0, \quad \textit{and} \quad \|V(g^-)^a\|_\infty < 1, \tag{1.6}$$

where $V = (-\Delta)^{-1}$ denotes the potential kernel associated to Δ. Then there exists $\lambda > 0$ such that (1.1) has a continuous solution u_λ satisfying $\|u_\lambda\|_\infty = a$.

Example Let g be a measurable function defined on $B(0,1) \times \mathbb{R}_+$. Suppose that there exists a nonnegative function q in $(0,1)$ such that

$$|g(x,t)| \leq q(|x|), \text{ for all } (x,t) \in B(0,1) \times \mathbb{R}_+,$$

and

$$\int_0^1 r \log \frac{1}{r} q(r) dr < 1.$$

Since $\|V(g^-)^a\| \leq \int_0^1 r \log \frac{1}{r} q(r) dr < 1$ then for any $a > 0$, there exists $\lambda \geq 0$ such that the problem

$$\begin{gathered} \Delta u(x) - g(x, u(x))u(x) + \lambda u(x) \exp(u(x)) = 0, x \in B(0,1), \\ u(x) > 0, x \in B(0,1), \\ u(x) = 0, x \in \partial B, \end{gathered}$$

has a continuous solution u such that $\|u\|_\infty = a$.

Now, we introduce the following definition which will be needed below.

Definition 1.3 We say that a measurable function f in $D \times \mathbb{R}$ is locally K-Lipschitz with respect to the second variable if for every $c > 0$, there exists a nonnegative function φ in K such that for all x in D and t in $[-c, c]$

$$|f(x,t) - f(x,t')| \leq \varphi(x)|t - t'|.$$

Theorem 1.4 *Assume that f and g are nonnegative, locally K-Lipschitz with respect to the second variable and satisfying (H1)–(H2). Also assume that there exist two nonnegative functions ϕ and ψ in K such that*

$$V\phi > 0 \textit{ and } f(x,t) \geq t\phi(x), \textit{ for all } (x,t) \in D \times [0,\infty), \tag{1.7}$$

$$Vf_a > 0, \textit{ for all } a > 0, \tag{1.8}$$

$$g(x,t) \leq \psi(x), \textit{ for all } (x,t) \in D \times [0,\infty). \tag{1.9}$$

Then there exists $\lambda^* \in (0,\infty)$ such that

(i) *For any* $0 < \lambda < \lambda^*$, *the problem* (1.2) *has a positive minimal solution* $u_\lambda \in C(D)$ *and for any* $\lambda > \lambda^*$, *there is no positive solution for* (1.2).

(ii) *The function* $\lambda \to u_\lambda$ *is nondecreasing.*

By a minimal solution, we mean a solution u of (1.2) such that if w is any solution of (1.2), then $u \leq w$.

Corollary 1.5 *Assume that* f *and* g *are nonnegative, locally Lipschitz with respect to the second variable and satisfy the same conditions as in Theorem 1.4, then the function* $\lambda \mapsto u_\lambda$ *is increasing.*

Remark 1.6 If $g(x,u) = 0$ and $f(.,u) = f(u)$, then (1.2) becomes the corresponding problem of semilinear equation

$$\begin{gathered} \Delta u + \lambda f(u) = 0, \quad \text{in } D, \\ u > 0, \quad \text{in } D, \\ u = 0, \quad \text{on } \partial D \end{gathered} \tag{1.10}$$

which has been widely studied [5, 6, 7]. It is shown that if f satisfies the condition

(H) f is a C^1 positive nondecreasing convex function on $[0,\infty)$ such that

$$\lim_{t\to\infty} \frac{f(t)}{t} = \infty,$$

then there exists an extremal positive value $\lambda^* < \infty$ for the parameter λ such that

(i) For any $0 < \lambda < \lambda^*$ there exists a positive minimal classical solution $u_\lambda \in C^2(\overline{\Omega})$ while there is no such solution for (1.10) if $\lambda > \lambda^*$.

(ii) The function $\lambda \to u_\lambda$ is increasing.

In those papers, the existence of solutions was obtained by applying the variational methods in critical point theory or by using the general theory of bifurcation of Rabinowitz to get a curve of solutions of (1.10).

It is worth mentioning that we have the minimum requirements on the smoothness of f and g. Indeed, there is no assumptions on the monotony neither on the convexity of the function f as we will see in the examples given below and the condition (1.7) is less restrictive than the condition $\lim_{t\to\infty} \frac{f(t)}{t} = \infty$.

Example Let $p > 0$. Let Ψ and Φ be nonnegative functions in K such that $V\Phi > 0$. Then, the results of Theorem 1.4 hold for the problem

$$\begin{gathered} \Delta u(x) - \frac{\Psi(x)u(x)}{1+u^p(x)} + \lambda\Phi(x)(1+u^2(x)|\log u(x)|) = 0, \quad \text{in } D, \\ u > 0, \quad \text{in } D, \\ u = 0, \quad \text{on } \partial D. \end{gathered}$$

Example Let $p > 0$. Let Ψ and Φ be nonnegative functions in K such that $V\Phi > 0$. Then, the results of Theorem 1.4 hold for the problem

$$\begin{gathered}\Delta u(x) - \frac{\Psi(x)u(x)}{1+u^p(x)} + \lambda\Phi(x)(1+u(x)) = 0, \quad \text{in } B(0,1), \\ u > 0, \quad \text{in } B(0,1), \\ u = 0, \quad \text{on } \partial B(0,1).\end{gathered}$$

Theorem 1.7 *Assume that f and g are nonnegative, locally K-Lipschitz with respect to the second variable and satisfying (H1)–(H2) and* (1.7)*. Moreover suppose that there exists a function θ in K such that*

$$f(x,t) \leq \theta(x), \text{ for all } (x,t) \in D \times [0,+\infty[.$$

Then for any $\lambda > 0$, the problem (1.2) *has at least a positive continuous solution in D.*

Example Let $0 < a < b$ and $\beta < 2$. Let $D = \{x \in D, a < |x| < b\}$. Consider the problem

$$\begin{gathered}\Delta u(x) + \lambda \frac{2+\sin u(x)}{(|x|-a)^\beta (b-|x|)^\beta} = 0, x \in D, \\ u(x) > 0, x \in D, \\ u(x) = 0, x \in \partial D.\end{gathered}$$

Then for any $\lambda > 0$, this problem has at least a positive continuous solution on D.

We shall prove Theorem 1.2 in section 3, and Theorem 1.4 and Theorem 1.7 in section 4. To prove the Theorems, we shall convert the problems into suitable integral equations and use Shauder fixed point theorem and the iteration method to establish existence.

As usual, we denote by $B(D)$ the set of Borel measurable functions in D and $B_b(D)$ the set of bounded ones. $C(D)$ will denote the set of continuous functions in D and

$$C_0(D) = \{v \in C(D) : \lim_{x \to \partial D} v(x) = 0\}.$$

Throughout this paper, the letter C will denote a generic positive constant which may vary from line to line.

2 Preliminaries

First, we give some properties of functions belonging to the class K which will be used later and are proved in [11]. Let $G(x,y)$ be the Green's function for

D corresponding to the Laplacian Δ. Then by [4] and [13], there exists $C > 0$ such that for $x, y \in D$,

$$\frac{1}{C}\log(1+\frac{\rho(x)\rho(y)}{|x-y|^2}) \le G(x,y) \le C\log(1+\frac{\rho(x)\rho(y)}{|x-y|^2}), \tag{2.1}$$

$$\frac{\rho(y)}{\rho(x)}G(x,y) \le C(1+G(x,y)). \tag{2.2}$$

Furthermore, G_D satisfies the **3G-Theorem** [13], which states that there exists a constant C_0 depending only on D such that for all x, y and z in D, we have

$$\frac{G(x,z)G(z,y)}{G(x,y)} \le C_0\Big[\frac{\rho(z)}{\rho(x)}G(x,z) + \frac{\rho(z)}{\rho(y)}G(z,y)\Big]. \tag{2.3}$$

Proposition 2.1 *Let φ be a function in K. Then the function $y \mapsto \rho^2(y)\varphi(y)$ is in $L^1(D)$.*

In the sequel, we use the notation

$$\|\varphi\|_D = \sup_{x\in D}\int_D \frac{\rho(y)}{\rho(x)}\log(1+\frac{\rho(x)\rho(y)}{|x-y|^2})|\varphi(y)|dy. \tag{2.4}$$

Proposition 2.2 *If $\varphi \in K$ then $\|\varphi\|_D < \infty$.*

Proposition 2.3 *For any function φ belonging to K, any nonnegative superharmonic function h in D and all $x \in D$*

$$\int_D G(x,y)h(y)|\varphi(y)|dy \le 2C_0\|\varphi\|_D h(x), \tag{2.5}$$

where the constant C_0 is given in (2.3).

Corollary 2.4 *Let φ be a function in K. Then*

$$\sup_{x\in D}\int_D G(x,y)|\varphi(y)|dy < \infty. \tag{2.6}$$

Corollary 2.5 *Let φ be a function in K. Then the function $y \mapsto \rho(y)\varphi(y)$ is in $L^1(D)$.*

3 Proof of Theorem 1.2

For this section, we need some preliminary results. Recall that the potential kernel V is defined on $B^+(D)$ by

$$V\varphi(x) = \int_D G(x,y)\varphi(y)dy,\ x \in D.$$

Hence, for $\varphi \in B^+(D)$ such that $\varphi \in L^1_{\rm loc}(D)$ and $V\varphi \in L^1_{\rm loc}(D)$, we have in the distributional sense that $\Delta(V\varphi) = -\varphi$, in D. We point out if $V\varphi \neq \infty$, we have $V\varphi \in L^1_{\rm loc}(D)$, (see [4, p. 51]. Recall that V satisfies the complete maximum principle, i.e., for each $\phi \in B^+(D)$ and v a nonnegative superharmonic function on D such that $V\phi \leq v$ in $\{\phi > 0\}$ we have $V\phi \leq v$ in D [12, Theorem 3.6]. In the sequel, for $\varphi \in K$, we define the kernel V^φ on $B_b(D)$ by

$$V^\varphi w = V\varphi w,\ \forall w \in B_b(D).$$

Lemma 3.1 ([11]) *Let $x_0 \in \overline{D}$. Then for any function φ belonging to K and any positive superharmonic function h in D, we have*

$$\lim_{\delta\to 0}\sup_{x\in D}\frac{1}{h(x)}\int_{B(x_0,\delta)\cap D} G(x,y)h(y)|\varphi(y)|\,dy = 0. \tag{3.1}$$

Consequently, we obtain the following result.

Proposition 3.2 *Let $\varphi \in K$. Then the function $V\varphi$ defined on D by*

$$V\varphi(x) = \int_D G(x,y)\varphi(y)dy$$

is in $C_0(D)$.

Proof Let $x_0 \in D$ and $r > 0$. Let $x, x' \in B(x_0, \frac{r}{2}) \cap D$. Then

$$\begin{aligned}|V\varphi(x) - V\varphi(x')| &\leq \int_D |G(x,y) - G(x',y)||\varphi(y)|dy\\ &\leq 2\sup_{\xi\in D}\int_{B(x_0,r)\cap D} G(\xi,y)|\varphi(y)|dy\\ &\quad + \int_{D\cap(|y-x_0|\geq r)} |G(x,y) - G(x',y)||\varphi(y)|dy.\end{aligned}$$

Since D is bounded, by (2.1), there exists $C > 0$ such that for all $x \in B(x_0, \frac{r}{2})\cap D$ and $y \in (D\backslash B(x_0,r))$,

$$G(x,y) \leq C\rho(y).$$

Moreover, $G(x,y)$ is continuous on $(x,y) \in (B(x_0,\frac{r}{2})\cap D)\times(D\backslash B(x_0,r))$. Then by Corollary 2.5 and Lebesgue's theorem,

$$\int_{D\cap(|y-x_0|\geq r)} |G(x,y) - G(x',y)||\varphi(y)|dy \to 0 \quad \text{as } |x - x'| \to 0.$$

Hence, by (3.1) with $h = 1$ we obtain that $V\varphi$ is continuous in D.

Now, we show that

$$\lim_{x\to\partial D} V\varphi(x) = 0.$$

Let $x_0 \in \partial D$, $r > 0$, and $x \in B(x_0, \frac{r}{2}) \cap D$. Then

$$\begin{aligned}|V\varphi(x)| &\leq \int_D G(x,y)\,|\varphi(y)|\,dy \\ &\leq \sup_{\xi \in D} \int_{B(x_0,r)\cap D} G(\xi,y)|\varphi(y)|dy + \int_{D\cap(|y-x_0|\geq r)} G(x,y)|\varphi(y)|dy.\end{aligned}$$

Since for all $y \in D$, $\lim_{x\to\partial D} G(x,y) = 0$, it follows, as in the above argument, that

$$\lim_{x\to\partial D} V\varphi(x) = 0.$$

□

Proposition 3.3 *Let φ in K. Then, the operator V^{φ} is compact on $B_b(D)$.*

Proof Let $M > 0$ and

$$S = \{w \in B_b(D) : \|w\|_\infty \leq M\}.$$

For $w \in S$, we have

$$\big|V^{\varphi}w(x)\big| = \big|\int_D G(x,y)\varphi(y)w(y)dy\big| \leq M \sup_{x\in D}\int_D G(x,y)|\varphi(y)|dy.$$

Since $\varphi \in K$, from Corollary 2.4, $V^{\varphi}(S)$ is uniformly bounded.

Next, we prove the equicontinuity of $V^{\varphi}(S)$ in $B_b(D)$. Let $x_0 \in \overline{D}$, $r > 0$, $x, x' \in B(x_0, \frac{r}{2}) \cap D$. Then for $w \in S$,

$$|V^{\varphi}w(x) - V^{\varphi}w(x')| \leq M\int_D |G(x,y) - G(x',y)||\varphi(y)|dy.$$

Since $\varphi \in K$ then by Proposition 3.2, we get

$$|V^{\varphi}w(x) - V^{\varphi}w(x')| \to 0 \quad \text{as } |x - x'| \to 0,$$

uniformly for all $w \in S$. Finally, by Ascoli's Theorem the family $V^{\varphi}(S)$ is relatively compact in $B_b(D)$. □

Proposition 3.4 ([3]) *Let φ be in K such that $\|V^{\varphi^-}\|_\infty < 1$. Then the operator $(I + V^{\varphi})$ is invertible on $B_b(D)$. Moreover, for every nonnegative superharmonic function s in D, we have*

$$\begin{gathered}(I + V^{\varphi})^{-1}s \geq 0,\\ \{(I + V^{\varphi})^{-1}s > 0\} = \{s > 0\}.\end{gathered}$$

Let $a > 0$ be such that (1.6) holds, and set

$$F_a = \{u \in C(D) : 0 \leq u \leq a\}.$$

Theorem 3.5 *Assume (H1)–(H2) and* (1.6). *Then for $u \in F_a$, the problem*

$$\begin{gathered} \Delta v - g(.,u)v + \lambda f(.,u)v = 0, \quad \text{in } D, \\ v > 0, \quad \text{in } D, \\ v = 0, \quad \text{on } \partial D \end{gathered} \tag{3.2}$$

has a principal eigenvalue $\lambda^u > 0$ and a corresponding eigenfunction v^u continuous on D and satisfying

$$\|v^u\|_\infty = a.$$

Moreover, the set $\{\lambda^u, u \in F_a\}$ is bounded.

Proof. By (H1) and (H2), f^a and g^a are in K. Let $u \in F_a$. Since

$$\|V^{g^-(.,u)}\|_\infty \leq \|V(g^-)^a\|_\infty < 1,$$

we have by Proposition 3.4 that the operator $(I + V^{g(.,u)})$ is invertible on $B_b(D)$. Let Γ be the operator defined on $B_b(D)$ as

$$\Gamma = (I + V^{g(.,u)})^{-1} V^{f(.,u)}.$$

Since $f(.,u) \leq f^a$, we deduce from Proposition 3.3 that Γ is compact on $B_b(D)$. Therefore, from the general Fredholm theory for compact operators we get the existence of a principal eigenvalue $\mu^u > 0$ with a corresponding positive eigenfunction v^u such that $\|v^u\|_\infty = a$. By setting $\lambda^u = \frac{1}{\mu^u}$, we get the desired result. On the other hand, v^u satisfies

$$\Delta(v^u + V(g^a v^u) - \lambda^u V(f_a v^u)) = (g(.,u) - g^a)v^u + \lambda^u(-f(.,u) + f_a)v^u \leq 0.$$

It follows that v^u is a supersolution of the problem

$$\begin{gathered} \Delta v - g^a v + \lambda^u f_a v = 0, \text{in } D, \\ v > 0, \quad \text{in } D, \\ v = 0, \quad \text{on } \partial D. \end{gathered} \tag{3.3}$$

Hence, by a result in [10], we get $\lambda^u \leq \widetilde{\lambda}$, where $\widetilde{\lambda}$ is the principal eigenvalue of (3.3). □

Proof of Theorem 1.2 Let T be the operator defined on F_a by

$$\begin{aligned} Tu(x) &= v^u(x) \\ &= -\int_D G(x,y)g(y,u(y))v^u(y)\,dy + \lambda^u \int_D G(x,y)f(y,u(y))v^u(y)\,dy. \end{aligned}$$

We will show that T has a fixed point in F_a. To this end, we need to check that T is a compact mapping from F_a to itself. First, we will show that the family of functions $T(F_a) = \{T(u), u \in F_a\}$ is relatively compact in $C(\overline{D})$. Let $u \in F_a$ and $x \in D$, then by Theorem 3.5,

$$\begin{aligned}|Tu(x)| &= \Big|\int_D G(x,y)v^u(y)[\lambda^u f(y,u(y)) - g(y,u(y))]\Big| \\ &\le a(1+C)\int_D G(x,y)[f^a(y)+g^a(y)]dy.\end{aligned}$$

Since f^a, $g^a \in K$, from Proposition 2.3 with $h = 1$, we deduce that

$$\|Tu\|_\infty \le C(\|f^a\|_D + \|g^a\|_D).$$

Thus the family $T(F_a)$ is uniformly bounded.

Now, we prove the equicontinuity of $T(F_a)$ in $C(\overline{D})$. Let $x_0 \in \overline{D}$, $\delta > 0$, $x, x' \in B(x_0, \frac{\delta}{2}) \cap D$, and $u \in F_a$. Then

$$\begin{aligned}|Tu(x) - Tu(x')| \le& 2a(1+C)\sup_{x\in D}\int_{B(x_0,\delta)\cap D} G(x,y)(f^a(y)+g^a(y))\,dy \\ &+ a\int_{B^c(x_0,\delta)\cap D} |G(x,y) - G(x',y)|(g^a(y) + Cf^a(y))\,dy.\end{aligned}$$

Since D is bounded, for $|x-y| \ge \frac{\delta}{2}$, $G(x,y) \le C\rho(y)$. Since $f^a + g^a$ is in K and $G(x,y)$ is continuous for $(x,y) \in (B(x_0,\frac{\delta}{2}) \cap \overline{D}) \times B^c(x_0,\delta) \cap D$, it follows, by Corollary 2.5 and Lebesgue's theorem, that

$$\int_{B^c(x_0,\delta)\cap D} |G(x,y) - G(x',y)|(g^a(y) + Cf^a(y))dy \to 0$$

as $|x - x'| \to 0$. Then it follows from Lemma 3.1 that

$$|Tu(x) - Tu(x')| \to 0 \quad \text{as } |x-x'| \to 0$$

uniformly for all $u \in F_a$. Then by Ascoli's theorem, the family $T(F_a)$ is relatively compact in $C(\overline{D})$.

Next, we shall prove the continuity of T in the supremum norm. Let $(u_n)_{n\ge 0}$ be a sequence in F_a which converges uniformly to $u \in F_a$. Since $T(F_a)$ is a relatively compact family in $C(\overline{D})$ then without loss of generality, we may suppose that there exists w in F_a such that $(T(u_n))_n$ converges uniformly to w. Similarly, since $(\lambda^{u_n})_n$ is bounded, we may suppose that $(\lambda^{u_n})_n$ converges to a nonnegative real λ. Let $x \in D$. Then we have

$$\begin{aligned}&\lambda^{u_n}\int_D G(x,y)v^{u_n}(y)f(y,u_n(y))dy - \lambda\int_D G(x,y)w(y)f(y,u(y))\,dy \\ &= \lambda^{u_n}\int_D G(x,y)[v^{u_n}(y)f(y,u_n(y)) - w(y)f(y,u(y))]\,dy \\ &\quad + (\lambda^{u_n} - \lambda)\int_D G(x,y)w(y)f(y,u(y))\,dy.\end{aligned}$$

Since

$$|\lambda^{u_n} G(x,y)[v^{u_n}(y)f(y,u_n(y)) - w(y)f(y,u(y))]| \leq CG(x,y)f^a(y),$$

by (H1), (2.5) and Lebegue's theorem, we have

$$\Big|\lambda^{u_n}\int_D G(x,y)v^{u_n}(y)f(y,u_n(y))dy - \lambda\int_D G(x,y)w(y)f(y,u(y))dy\Big| \to 0$$

uniformly in D as $n\to\infty$. Similarly, we have

$$\Big|\int_D G(x,y)v^{u_n}(y)g(y,u_n(y))dy - \int_D G(x,y)w(y)g(y,u(y))dy\Big| \to 0$$

uniformly in D as $n\to\infty$. Using the relationship

$$v^{u_n}(x) + \int_D G(x,y)[g(y,u_n(y)) - \lambda^{u_n} f(y,u_n(y))]v^{u_n}(y)dy = 0$$

and letting $n\to\infty$, we get

$$w(x) + \int_D G(x,y)g(y,u(y))w(y)dy - \lambda\int_D G(x,y)f(y,u(y))w(y)dy = 0.$$

Hence w is a solution of (3.2) with $\|w\|_\infty = a$. Then λ is the principal eigenvalue λ^u of (3.2) and

$$w = v^u = T(u).$$

Now, the Shauder fixed point theorem implies the existence of $u\in F_a$ such that $T(u)=u$. ☐

4 Proof of Theorems 1.4 and 1.7

To establish the existence results, we shall use the method of sub-solution and super solution. By definition, we will say that $\underline{u}$ is a subsolution to (1.2) if

$$\begin{gathered}\Delta\underline{u} - g(.,\underline{u})\underline{u} + \lambda f(.,\underline{u}) \geq 0, \quad \text{in } D,\\ \underline{u}\leq 0, \quad \text{on } \partial D\end{gathered}$$

in the sense of distributions. Similarly, $\overline{u}$ is a supersolution to (1.2) if in the above expressions the reverse inequalities hold.

Proposition 4.1 *Assume that there exist $\underline{u}$ and $\overline{u}$ in $B_b^+(D)$ such that $\overline{u}$ is a supersolution of (1.2) and $\underline{u}$ is a subsolution of (1.2) satisfying $\underline{u}\leq\overline{u}$.*
If (H1)–(H2) hold and f and g are nonnegative locally K-Lipschitz such that $Vf(.,\underline{u})>0$, then there exists a solution w of (1.2) satisfying

$$\underline{u}\leq w\leq\overline{u}, \text{ in } D.$$

Proof Let $a = \|\overline{u}\|_\infty$. Since f and g are K-Lipchitz with respect to the second variable, then there exist two nonnegative functions f_1 and g_1 belonging to K such that the maps

$$\begin{gathered} t \mapsto \lambda f(x,t) + f_1(x)t, \\ t \mapsto -g(x,t)t + g_1(x)t \end{gathered}$$

are nondecreasing on $[0, a]$. Set

$$F_a = \{u \in C(D) : 0 \le u \le a\}.$$

For $u \in F_a$, let v^u be the unique solution in D of the linear problem

$$\begin{gathered} \Delta v^u - (f_1 + g_1)v^u = (g(.,u) - g_1 u)u - \lambda f(.,u) - f_1 u, \\ v^u = 0, \quad \text{on } \partial D. \end{gathered} \tag{4.1}$$

Let T be the operator on F_a defined by

$$Tu = v^u.$$

We claim that T is nondecreasing on F_a. Indeed, let u_1 and u_2 in F_a such that $u_1 \le u_2$. It follows that

$$\begin{aligned} &\Delta(v^{u_1} - v^{u_2}) - (f_1 + g_1)(v^{u_1} - v^{u_2}) \\ &= (g(.,u_1) - g_1 u_1)u_1 - (g(.,u_2) - g_1 u_2)u_2 \\ &\quad - \lambda f(.,u_1) - f_1 u_1 + \lambda f(.,u_2) + f_1 u_2 \ge 0. \end{aligned}$$

Since $v^{u_1} - v^{u_2} = 0$ on ∂D, we get by the complete maximum principle that

$$v^{u_1} - v^{u_2} \le 0 \quad \text{in } D$$

and therefore T is nondecreasing on F_a. Let $\underline{u}$ be a subsolution of (1.2), then by (4.1), we have

$$\begin{gathered} \Delta(T\underline{u} - \underline{u}) - (f_1 + g_1)(T\underline{u} - \underline{u}) \le 0, \text{ in D}, \\ T\underline{u} - \underline{u} \ge 0, \text{ on } \partial D. \end{gathered}$$

Using the complete maximum principle, we obtain

$$T\underline{u} \ge \underline{u}, \text{ in D}.$$

Similarly, we show that $T\overline{u} \le \overline{u}$. Since T is nondecreasing, then the sequences defined inductively by

$$\begin{gathered} u_0 = \underline{u}, \quad u_n = Tu_{n-1}; \\ v_0 = \overline{u}, \quad v_n = Tv_{n-1} \end{gathered}$$

are monotonic and satisfy

$$\underline{u} \le u_n \le v_n \le \overline{u}.$$

Let
$$u = \lim_{n\to\infty} u_n \quad \text{and} \quad v = \lim_{n\to\infty} v_n.$$
Since $T(F_a)$ is compact in $C_b(D)$ then the pointwise convergence implies the uniform convergence.
$$u = Tu \quad \text{and} \quad v = Tv.$$
Hence, it follows from (4.1) that u is a solution of
$$\begin{gathered}\Delta u - g(.,u)u = -\lambda f(.,u), \quad \text{in } D,\\ u = 0, \quad \text{on } \partial D.\end{gathered} \tag{4.2}$$
i.e., $u = \lambda(I + V^{g(.,u)})^{-1}[V(f(.,u)]$. Thus we deduce from Proposition 3.4, that
$$u = \lambda(I + V^{g(.,u)})^{-1}[V(f(.,u)] \geq \lambda(I + V^{g(.,u)})^{-1}[V(f(.,\underline{u})] > 0 \quad \text{in } D$$
which implies that u and v are solutions of (1.2) satisfying
$$\underline{u} \leq u \leq v \leq \overline{u}.$$
Moreover, u and v are extremal solutions. □

Lemma 4.2 *Assume that f and g are nonnegative and satisfying (H1)–(H2) and* (1.8)*. Then, for any $a > 0$, there exists a real $\lambda > 0$ such that the problem* (1.2) *has a continuous solution u_λ satisfying $\|u_\lambda\|_\infty = a$.*

Proof. Let $a > 0$. For each $u \in F_a$, let λ_u be such that
$$\lambda_u \|(I + V^{g(.,u)})^{-1}(Vf(.,u))\|_\infty = a$$
and let T be the operator defined on F_a by
$$Tu = (I + V^{g(.,u)})^{-1}(\lambda_u Vf(.,u)).$$
Then
$$\lambda_u = \frac{a}{\|(I + V^{g(.,u)})^{-1}(Vf(.,u))\|_\infty} \leq \frac{a}{\|(I + V^{g^a})^{-1}Vf_a\|_\infty}\,.$$
Hence, $\{\lambda^u, u \in F_a\}$ is bounded. As in the proof of Theorem 1.1 we prove that T has a fixed point $u \in F_a$, $Tu = u$. Moreover, by (1.8) we have
$$Vf(.,u) > 0, \quad \text{in } D.$$
Using Proposition 3.4 and the fact that g is nonnegative, we obtain
$$Tu = (I + V^{g(.,u)})^{-1}(\lambda_u Vf(.,u)) > 0, \quad \text{in } D$$
which completes the proof. □

Proposition 4.3 *Let $a > 0$ and $F_a = \{u \in C(D) : 0 \leq u \leq a\}$. Let S_a be the set of all $\lambda \geq 0$ such that the problem* (1.2) *has a continuous solution $u_\lambda \in F_a$. Then, there exists $\lambda(a) \in]0,\infty[$ such that $S_a = [0, \lambda(a)]$.*

Proof. By Lemma 4.2, S_a is nonempty. Assume that we can solve (Q_{λ_0}) for some $\lambda_0 > 0$ and let u_0 be a solution of (Q_{λ_0}). Then one can solve (1.2) for all $0 \leq \lambda \leq \lambda_0$ since u_0 is clearly a supersolution to (1.2) and $\underline{u} = 0$ is a subsolution of (1.2). Thus, if $\lambda(a)$ denotes the supermum of all λ in S_a, we claim that

$$\lambda(a) < \infty.$$

Indeed, let $\lambda \in S_a$ and u_λ be a solution in F_a of (Q_λ). Then we have

$$\Delta u_\lambda - g^a u_\lambda + \lambda \frac{f_a}{a} u_\lambda = \lambda(\frac{f_a}{a} - \frac{f(.,u_\lambda)}{u_\lambda})u_\lambda + (g(.,u_\lambda) - g^a)u_\lambda \leq 0.$$

Consequently, u_λ is a supersolution of

$$\begin{gathered} \Delta u - g^a u + \lambda \frac{f_a}{a} u = 0 \quad \text{in } D, \\ u > 0 \quad \text{in } D, \\ u = 0 \quad \text{on } \partial D. \end{gathered} \tag{4.3}$$

Hence, $\lambda \leq \widetilde{\lambda}$, where $\widetilde{\lambda}$ is the principal eigenvalue of (4.3).

Finally, we shall prove that $\lambda(a) \in S_a$. Let $\lambda_n \in S_a$ such that $\lambda_n \to \lambda(a)$ and $u_n \in F_a$ be a solution of (Q_{λ_n}). Then

$$u_n(x) = \int_D G(x,y)[-g(y,u_n(y))u_n(y) + \lambda_n f(y,u_n(y))]dy, \quad \forall x \in D.$$

Since the family

$$\{x \mapsto \int_D G(x,y)[-g(y,u_n(y))u_n(y) + \lambda_n f(y,u_n(y))dy, n \in \mathbb{N}\}$$

is equicontinuous, we may suppose that there exists a continuous function $u \in F_a$ such that u_n converges uniformly to u. Thus

$$u(x) = \int_D G(x,y)[-g(y,u(y))u + \lambda(a) f(y,u(y))dy, \quad \forall x \in D.$$

It follows that u is a solution of $(Q_{\lambda(a)})$ and consequently $\lambda(a) \in S_a$.

Proof of Theorem 1.4 (i) Let $S = \cup_{a \geq 0} S_a$. Since $S_{a_1} \subset S_{a_2}$ if $a_1 \leq a_2$, it follows that S is an interval. Let λ^* be the supermum of all λ in S. We claim that $\lambda^* < \infty$. Indeed, let $\lambda \in S$ and u_λ be a solution of (Q_λ). Then u_λ satisfies

$$\Delta u - \psi u + \lambda \phi u = \lambda(\phi - \frac{f(u)}{u})u + (g(.,u) - \psi)u \leq 0.$$

So, $\lambda \leq \widetilde{\lambda}$, where $\widetilde{\lambda}$ is the principal eigenvalue of the linear equation

$$\begin{gathered} \Delta u - \psi u + \lambda \phi u = 0 \quad \text{in } D, \\ u = 0 \quad \text{on } \partial D. \end{gathered}$$

Next, we will show the existence of the minimal solution of (1.2) for $\lambda \leq \lambda^*$. Indeed, let $0 < \lambda < \lambda^*$ and let u_λ be a solution of (1.2). Using the proof of Proposition 4.1, we set

$$u_0 = 0, \quad u_n = T(u_{n-1}) \quad \text{for } n \geq 1.$$

Since T is increasing and $u_\lambda = Tu_\lambda$, the function $\widetilde{u}_\lambda = \lim_{n\to\infty} u_n$ is a solutions of (1.2) satisfying

$$0 \leq \widetilde{u}_\lambda \leq u_\lambda.$$

It follows that $\widetilde{u}_\lambda$ is the minimal solution of (Q_λ).

(ii) Let $0 < \lambda < \lambda^*$ and $\mu < \lambda$. Since $\widetilde{u}_\lambda$ is a supersolution of (Q_μ), then by Proposition 4.1, there exists a positive solution u_μ of (Q_μ) such that

$$0 \leq u_\mu \leq \widetilde{u}_\lambda.$$

Hence $\widetilde{u}_\mu \leq \widetilde{u}_\lambda$. □

Proof of Corollary 1.5 There exists $\gamma > 0$ such that the function

$$t \mapsto \mu f(x,t) - g(x,t)t + \gamma t$$

is nondecreasing on $[0, \|\widetilde{u}_\lambda\|_\infty]$ for every $x \in D$. Since $\widetilde{u}_\mu \leq \widetilde{u}_\lambda$, it follows that

$$\begin{aligned}
&\Delta(\widetilde{u}_\mu - \widetilde{u}_\lambda) - \gamma(\widetilde{u}_\mu - \widetilde{u}_\lambda) \\
&\geq \mu f(\widetilde{u}_\lambda) - g(\widetilde{u}_\lambda)\widetilde{u}_\lambda + \gamma\widetilde{u}_\lambda - [\mu f(\widetilde{u}_\mu) - g(\widetilde{u}_\mu)\widetilde{u}_\mu + \gamma\widetilde{u}_\mu] \geq 0.
\end{aligned}$$

Thus, by Hopf theorem [8, Theorem 3.5], we obtain that $\widetilde{u}_\mu < \widetilde{u}_\lambda$ in D.

Proof of Theorem 1.7 By (1.8) for every $n \in \mathbb{N}$, $Vf_n > 0$. Hence, by Lemma 4.2, there exist $\lambda_n > 0$ and a solution u_{λ_n} of (Q_{λ_n}) such that $\|u_{\lambda_n}\|_n = n$. Since

$$u_{\lambda_n} + Vg(., u_{\lambda_n})u_{\lambda_n} = \lambda_n Vf(., u_{\lambda_n}),$$

then

$$\lambda_n \geq \frac{u_{\lambda_n}(x)}{\|Vf(., u_{\lambda_n})\|_\infty}, \forall x \in D.$$

Hence

$$\lambda_n \geq \frac{u_{\lambda_n}(x)}{\|V\theta\|_\infty}, \forall x \in D.$$

Thus, we obtain $\lambda_n \geq n/\|V\theta\|_\infty$. Therefore, $\lim_{n\to\infty} \lambda_n = \infty$. Since the mapping $a \to \lambda(a)$ is increasing, we get

$$\cup_{a>0} S_a = [0, \infty).$$

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