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SOLITARY WAVES FOR MAXWELL-SCHRÖDINGER EQUATIONS

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Abstract. In this paper we study solitary waves for the coupled system of Schrödinger- Maxwell equations in the three-dimensional space. We prove the existence of a sequence of radial solitary waves for these equations with a fixed L^2 norm. We study the asymptotic behavior and the smoothness of these solutions. We show also that the eigenvalues are negative and the first one is isolated.

1. Introduction

The classical correspondence rules in quantum mechanics are

$$E \to i\hbar\partial_t, \quad p \to -i\hbar\nabla, \quad \nabla = (\nabla_1, \nabla_2, \nabla_3), \quad \nabla_j = \partial x_j, \quad j = 1, 2, 3, \tag{1.1}$$

where E is the energy and $p = (p_1, p_2, p_3)$ is the momentum (see for example [9, Section 4, Chapter V]). Using these rules, we can derive some basic wave equations in quantum mechanics from the corresponding laws of classical mechanics. For example, the classical relation

$$E = \frac{p^2}{2m} + V(x), \quad p^2 = p_1^2 + p_2^2 + p_3^2, \tag{1.2}$$

represents the energy as a sum of kinetic energy $p^2/2m$ and a potential energy term $V(x)$. The well - known Schrödinger equation for the wave function $\psi(t, x)$ can be written as

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\Delta\psi + V(x)\psi \tag{1.3}$$

and this equation is a consequence of (1.1) and the relation (1.2). Here $\hbar$ is the Plank constant, m is the mass of the field ψ and $V(x)$ is a given external potential. For the case of potential created by the nucleus of the some atoms (see Section 4, Chapter V in [9] for example) we have a Coulomb potential

$$V(x) = -\frac{e^2 Z}{|x|}, \tag{1.4}$$

where e is the electron charge, while Z is the number of protons in the nucleus.

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The interaction between the electromagnetic field and the wave function related to a quantistic non-relativistic charged particle (considered as classical fields) is described by the Maxwell - Schrödinger system. More precisely, let $\psi = \psi(x,t)$ be the wave function and let $\mathcal{A} = (A_0, A_1, A_2, A_3)$ be the electromagnetic potentials of a charged non- relativistic particle. Then the corresponding Maxwell - Schrödinger system (in Lorentz gauge) has the form (see the next section for the derivation of this system)

$$\begin{gathered} \frac{1}{c^2}\partial_{tt}\mathcal{A} - \Delta\mathcal{A} = \mathcal{J}, \\ i\hbar\partial_{t,\mathcal{A}}\psi + \frac{\hbar^2}{2m}\Delta_{\mathcal{A}}\psi - V(x)\psi = 0, \\ \frac{1}{c}\partial_t A_0 + \sum_{k=1}^{3}\partial_{x_k}A_k = 0, \end{gathered} \tag{1.5}$$

where c is the light velocity (in vacuum),

$$\begin{gathered} \partial_{t,\mathcal{A}} = \partial_t + i\frac{e}{\hbar}A_0, \quad \Delta_{\mathcal{A}} = \sum_{k=1}^{3}\partial_{k,\mathcal{A}}^2, \\ \partial_{k,\mathcal{A}} = \partial_{x_k} + i\frac{e}{\hbar c}A_k, \quad \mathcal{J} = (J_0, J_1, J_2, J_3), \\ J_0 = 4\pi e|\psi|^2, \quad J_k = 4\pi\frac{\hbar e}{mc}\operatorname{Im}(\bar{\psi}\partial_{k,\mathcal{A}}\psi). \end{gathered} \tag{1.6}$$

We choose units in which

$$\hbar = c = 1, \quad \alpha = \frac{e^2}{4\pi} \approx \frac{1}{137}.$$

Also for simplicity we take $m = 1$.

We consider special solitary type solutions to the system (1.5) of the form

$$\psi(x,t) = u(x)e^{-i\omega t/\hbar}, \quad x \in \mathbb{R}^3, t \in \mathbb{R},$$

and

$$A_0 = \varphi(x), \quad A_j(x) = 0, \quad j = 1,2,3, \quad x \in \mathbb{R}^3,$$

where $\omega \in \mathbb{R}$ and u is real valued. Then the system (1.5) takes the simpler form

$$\begin{gathered} -\frac{1}{2}\Delta u + e\varphi u + V(x)u = \omega u, \quad x \in \mathbb{R}^3, \\ -\Delta\varphi = 4\pi e u^2, \quad x \in \mathbb{R}^3, \\ \int_{\mathbb{R}^3} u^2 = N, \end{gathered} \tag{1.7}$$

where the last equation is due to the probabilistic interpretation of the wave function. In this work we shall assume the following relation between N and Z is satisfied

$$N \le Z. \tag{1.8}$$

The equations in (1.7) have a variational structure, in fact they are the Euler - Lagrange equations related to the functional:

$$F(u,\varphi) = \frac{1}{4}\int_{\mathbb{R}^3}|\nabla u|^2\,dx + \frac{e}{2}\int_{\mathbb{R}^3}\varphi u^2\,dx + \frac{1}{2}\int_{\mathbb{R}^3}V(x)u^2\,dx - \frac{1}{16\pi}\int_{\mathbb{R}^3}|\nabla\varphi|^2\,dx. \tag{1.9}$$

It is easy to see that this functional is well - defined, when

$$u \in H^1(\mathbb{R}^3), \quad \int_{\mathbb{R}^3} |\nabla\varphi|^2\,dx < \infty.$$

This functional is strongly indefinite, which means that F is neither bounded from below nor from above and this indefiniteness cannot be removed by a compact perturbation. Moreover, F is not even. Later on (see (3.5)) we shall introduce a functional $J(u)$ that is bounded from below and such that the critical points of J can be associated with the critical points of F.

The first natural question is connected with the simplest case $V \equiv 0$ (that is $Z = 0$), namely

$$\begin{gathered} -\frac{1}{2}\Delta u + \varphi e u = \omega u, x \in \mathbb{R}^3, \\ -\Delta\varphi = 4\pi e u^2,\ x \in \mathbb{R}^3. \end{gathered} \tag{1.10}$$

It is well-known that the similar physical model of Maxwell - Dirac system with zero external field admits solitary solutions (see [14]), i.e. nontrivial solutions in the Schwartz class $S(\mathbb{R}^3)$.

Our first result is as follows.

Theorem 1.1. *Let (u, φ, ω) be a solution of (1.10) such that u, φ are radial and*

$$u \in H^1(\mathbb{R}^3), \quad \int_{\mathbb{R}^3} |\nabla\varphi|^2\,dx < \infty.$$

Then $u \equiv \varphi \equiv 0$.

The above result shows that the Schrödinger - Maxwell equations with zero potential have only trivial solution. This fact justifies the study of the Schrödinger - Maxwell equations with nonzero external potential.

We shall look for soliton type solutions u, i.e. very regular solutions decaying rapidly at infinity. First, we establish the existence of H^1 radially symmetric solutions.

Theorem 1.2. *Under the assumptions (1.4) and (1.8), there exists a sequence of real negative numbers $\{\omega_k\}_{k\in\mathbb{N}}$ so that $\omega_k \to 0$ and for any ω_k there exists a couple (u_k, φ_k) of solutions of (1.7) such that*

$$u_k \in H^1(\mathbb{R}^3), \quad \int_{\mathbb{R}^3} |\nabla\varphi_k|^2\,dx < \infty.$$

Moreover u_k, φ_k are radially symmetric functions.

A more precise information about the localization of the eigenvalues ω is given in the following.

Theorem 1.3. *Assume (1.4) and $N < Z$. Let (u, φ, ω) be a nontrivial solution of the equations in (1.7) such that u, φ radial and*

$$u \in H^1(\mathbb{R}^3), \quad \int_{\mathbb{R}^3} |\nabla\varphi|^2\,dx < \infty.$$

Then we have

$$\omega < 0. \tag{1.11}$$

On the other hand, the solutions constructed in Theorem 1.2 are only radial ones. Therefore, it remains as an open problem the existence of non-radial solutions.

Some qualitative properties of the solutions for the case $N \le Z$ are described in the following.

Theorem 1.4. *Under the assumptions (1.4), if (u, φ, ω) is a solution of (1.7) with u and φ radially symmetric maps and such that*

$$u \in H^1(\mathbb{R}^3), \quad \int_{\mathbb{R}^3} |\nabla \varphi|^2 \, dx < \infty,$$

then

(a) $u(r) \in C^\infty([0,1]), \varphi(r) \in C^\infty([0,1])$
(b) *If $N = Z$ then $u \in S(|x| > 1)$, with $S(|x| > 1)$ being the Schwartz class of rapidly decreasing functions.*

Remark 1.5. Property (b) in the above theorem shows that the soliton behavior of the solutions can be established, when the neutrality condition $N = Z$ is satisfied. The physical importance of the neutrality condition is discussed in [17] (see (5.2) page 24 in [17]).

Finally the topological properties of the set of the solutions are stated as follows.

Theorem 1.6. *Under the assumptions (1.4) and (1.8), let (u, φ, ω) be a solution of (1.7) such that $\omega < 0$ is the first eigenvalue, u and φ are radially symmetric maps and such that*

$$u \in H^1(\mathbb{R}^3), \quad \int_{\mathbb{R}^3} |\nabla \varphi|^2 \, dx < \infty.$$

Then the solution (u, φ, ω) is isolated, i.e. there exists a neighborhood $U \subset H^1(\mathbb{R}^3)$ of u, one W of φ such that

$$\int_{\mathbb{R}^3} |\nabla \phi|^2 \, dx < \infty, \quad \phi \in W,$$

and one $\Omega \subset \mathbb{R}$ of ω such that each $(v, \phi, \lambda) \in U \times W \times \Omega$ with $(v, \phi, \lambda) \neq (u, \varphi, \omega)$, v and ϕ radially symmetric maps satisfying the following

$$\int_{\mathbb{R}^3} |v|^2 \, dx = N,$$

is not a solution of (1.7).

For the sake of completeness we want to recall that the existence of solitary waves has been studied by Benci and Fortunato (see [6]) in the case in which the charged particle “lives” in a bounded space region Ω. Moreover, the Maxwell equations coupled with nonlinear Klein-Gordon equation, with Dirac equation, with nonlinear Schrödinger equation and with the Schrödinger equation under the action of some external potential have been studied respectively in [7, 14, 10, 11, 12]. Also, we recall the classical papers [4, 5, 13].

The plan of the work is the following. In Section 2 we prove some preliminary variational results, that permit to reduce (1.7) to a single equation. Moreover we show the variational structure of the problem. In Section 3 we prove some topological properties of the energy functional associated to (3.4). In Section 4 we prove Theorem 1.1 and 1.3. In Section 5, 6 and 7 we prove Theorem 1.2, 1.4 and 1.6, respectively.

2. Derivation of the equations

The relations (1.1) have to be modified as follows (see Section 2, part I in [22]

$$
\begin{aligned}
&E \to i\hbar\partial_{t,\varphi}, \quad p \to -i\hbar\nabla_{\mathbb{A}},\\
&\partial_{t,\varphi} = \partial_t + \frac{ie}{\hbar}\varphi, \quad \nabla_{\mathbb{A}} = \nabla - \frac{ie}{\hbar c}\mathbb{A},
\end{aligned}
\tag{2.1}
$$

when an external electromagnetic potential $(\varphi, \mathbb{A})$, $\mathbb{A} = (\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3)$ is presented. Here $c > 0$ is the light speed. Then the relation (1.2) leads to the following Schrödinger equation with electromagnetic potential and external Coulomb potential

$$
i\hbar\partial_{t,\varphi}\psi = -\frac{\hbar^2}{2m}\nabla_{\mathbb{A}}^2\psi + V(x)\psi. \tag{2.2}
$$

The corresponding Lagrangian density (see (6.7), Section 6.2 in [16]) is

$$
\mathcal{L}_{\varphi,\mathbb{A}}(\psi) = \frac{i\hbar}{4}\left(\overline{\psi}\ \partial_{t,\varphi}\psi - \psi\ \overline{\partial_{t,\varphi}\psi}\right) - \frac{\hbar^2}{4m}|\nabla_{\mathbb{A}}\psi|^2 - \frac{V}{2}|\psi|^2. \tag{2.3}
$$

Equation (2.2) is then the Euler - Lagrange equation for the functional

$$
\int_{\mathbb{R}^{1+3}} \mathcal{L}_{\varphi,\mathbb{A}}(\psi).
$$

We have the following charge conservation law for any solution to (2.2)

$$
\int_{\mathbb{R}^3} |\psi(t,x)|^2\, dx = N, \tag{2.4}
$$

where N has the interpretation as number of electrons.

Equation (2.2) is linear in ψ and the electromagnetic potential is assumed to be a known real - valued function. The description of the interaction between electromagnetic and Schrödinger fields involves quantum fields equations for an electrodynamic non - relativistic many body system. A classical approximation of these quantum fields equations gives a simplified nonlinear model for the following Lagrangian density

$$
\mathcal{L}_{M-S}(\psi,\varphi,\mathbb{A}) = \mathcal{L}_{\varphi,\ \mathbb{A}}(\psi) + D\ \mathcal{L}_M(A), \tag{2.5}
$$

where $D > 0$ is a suitable constant and

$$
\mathcal{L}_M(A) = -\frac{1}{4}\sum_{\mu,\nu=0}^{3} F_{\mu\nu}F^{\mu\nu} \tag{2.6}
$$

is the Lagrangian density for the free Maxwell equation, i.e. $F_{\mu\nu}$ is the electromagnetic antisymmetric tensor, such that

$$
F_{\mu\nu} = -F_{\nu\mu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad \nu,\mu = 0,1,2,3. \tag{2.7}
$$

Here $\partial_0 = c^{-1}\partial_t$, $\partial_j = \partial_{x_j}$, $j = 1,2,3$. The four potential A_μ is defined as follows

$$
A_0 = \varphi, \quad A_j = -\mathbb{A}_j, \quad j = 1,2,3. \tag{2.8}
$$

It is easy to compute all components of $F_{\mu\nu}$:

$$
F_{0j} = -c^{-1}\partial_t\mathbb{A}_j - \partial_j\varphi, \quad F_{jk} = \partial_k\mathbb{A}_j - \partial_j\mathbb{A}_k, \quad j,k = 1,2,3. \tag{2.9}
$$

Since the Minkowski metric with respect to coordinates

$$
x^0 = ct, \quad x^j = x_j, \quad j = 1,2,3
$$

is $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, we find

$$F^{0j} = -F_{0j}, \quad F^{jk} = F_{jk}, \quad j, k = 1, 2, 3, \tag{2.10}$$

so

$$\begin{aligned}\sum_{\mu,\nu=0}^{3} F_{\mu\nu}F^{\mu\nu} &= -2\sum_{j=1}^{3}(F_{0j})^2 + 2\sum_{1\le j<k\le 3}(F_{jk})^2 = \\ &= -2|c^{-1}\partial_t \mathbb{A} + \nabla\varphi|^2 + 2|\nabla\times\mathbb{A}|^2,\end{aligned}$$

where $a \times b$ denotes the vector product in $\mathbb{R}^3$. The Lagrangian density in (2.5) for the Maxwell - Schrödinger system becomes now

$$\begin{aligned}\mathcal{L}_{M-S}(\psi, \varphi, \mathbb{A}) = \frac{i\hbar}{4}\left(\overline{\psi}\ \partial_{t,\varphi}\psi - \psi\ \overline{\partial_{t,\varphi}\ \psi}\right) - \frac{\hbar^2}{4m}|\nabla_{\mathbb{A}}\psi|^2 - \frac{V}{2}|\psi|^2 \\ + \frac{D}{2}|c^{-1}\partial_t\mathbb{A} + \nabla\varphi|^2 - \frac{D}{2}|\nabla\times\mathbb{A}|^2,\end{aligned} \tag{2.11}$$

where $D > 0$ is a dimensionless constant. Taking the variation of the functional

$$\int_{\mathbb{R}^{1+3}} \mathcal{L}_{M-S}(\psi, \varphi, \mathbb{A})$$

with respect to $\bar{\psi}$, we obtain the Scrödinger equation (2.2) and this is the second equation in (1.5). The variation with respect to φ gives the equation

$$-\frac{e}{2}|\psi|^2 - D\Delta\varphi - \frac{D}{c}\partial_t(\nabla\cdot\mathbb{A}) = 0, \tag{2.12}$$

while the variation with respect to $\mathbb{A}$ implies

$$i\frac{\hbar e}{4mc}(\nabla\overline{\psi}\psi - \nabla\psi\overline{\psi}) - \frac{e^2}{2mc^2}\mathbb{A}|\psi|^2 - \frac{D}{c^2}\partial_t^2\mathbb{A} + D\Delta\mathbb{A} - D\nabla(\nabla\cdot\mathbb{A}) - \frac{D}{c}\partial_t\nabla\varphi = 0. \tag{2.13}$$

We shall take (for simplicity)

$$D = \frac{1}{8\pi} \tag{2.14}$$

and shall assume that the electromagnetic potential satisfies the following Lorentz gauge condition

$$\frac{1}{c}\partial_t A^0 + \sum_{k=1}^{3}\partial_{x_k}A^k = 0. \tag{2.15}$$

Then a combination between (2.12) and this Lorentz gauge condition implies

$$-\Delta\varphi + \frac{1}{c^2}\partial_t^2\varphi = 4\pi e|\psi|^2, \tag{2.16}$$

In a similar way from (2.13) we get (using the gauge condition)

$$\frac{\hbar e}{2mc}\,\mathrm{Im}\left(\nabla_{k,\mathcal{A}}\psi\ \overline{\psi}\right) - \frac{1}{c^2}\partial_t^2\mathbb{A}_k + \Delta\mathbb{A}_k = 0, \quad k = 1, 2, 3. \tag{2.17}$$

Equations (2.16) and (2.17) can be rewritten as

$$\frac{1}{c^2}\partial_{tt}\mathcal{A} - \Delta\mathcal{A} = \mathcal{J}, \tag{2.18}$$

where

$$\mathcal{J} = (J_0, J_1, J_2, J_3), \quad J_0 = 4\pi e|\psi|^2, \quad J_k = 4\pi\frac{\hbar e}{mc}\,\mathrm{Im}(\bar{\psi}\partial_{k,\mathcal{A}}\psi) \tag{2.19}$$

and this coincides with the first equation in (1.5).

3. The Variational Setting

In this section we shall prove a variational principle that permits to reduce (1.7) to the study of the critical points of an even functional, which is not strongly indefinite. To this end we need some technical preliminaries.

We define the space $\mathcal{D}^{1,2}(\mathbb{R}^3)$ as the closure of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_{\mathcal{D}^{1,2}} := \Big(\int_{\mathbb{R}^3} |\nabla u|^2\,dx\Big)^{1/2}.$$

The Sobolev - Hardy inequality (see [21]) implies the following lemma.

Lemma 3.1. *For all $\rho \in L^{6/5}(\mathbb{R}^3)$ there exists only one $\varphi \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ such that $\Delta\varphi = \rho$. Moreover there results*

$$\|\varphi\|^2_{\mathcal{D}^{1,2}} \le c\|\rho\|^2_{L^{6/5}} \tag{3.1}$$

and the map

$$\rho \in L^{6/5}(\mathbb{R}^3) \mapsto \varphi = \Delta^{-1}(\rho) \in \mathcal{D}^{1,2}(\mathbb{R}^3)$$

is continuous.

Moreover, the classical Sobolev embedding and a duality argument guarantee the properties

$$\begin{gathered} H^1(\mathbb{R}^3) \subseteq L^q(\mathbb{R}^3) \quad \text{for} 2 \le q \le 6 \\ L^{q'}(\mathbb{R}^3) \subseteq \big(H^1(\mathbb{R}^3)\big)' \quad \text{for} \frac{6}{5} \le q' \le 2. \end{gathered} \tag{3.2}$$

Denoting by $H^1_r(\mathbb{R}^3)$ the set of all H^1 radial functions. Then the classical Strauss Lemma shows that (see [23] or [8, Theorem A.I'])

$$H^1_r(\mathbb{R}^3) \text{ is compactly embedded into } L^q(\mathbb{R}^3), 2 < q < 6. \tag{3.3}$$

By Lemma 3.1 and by using the Sobolev inequalities, for any given $u \in H^1(\mathbb{R}^3)$ the second equation of (1.7) has the unique solution

$$\varphi = -4\pi e \Delta^{-1} u^2 \big(\in \mathcal{D}^{1,2}(\mathbb{R}^3)\big).$$

For this reason we can reduce the system (1.7) to the equations

$$-\frac{1}{2}\Delta u - 4\pi e^2(\Delta^{-1}u^2)u + V(x)u = \omega u, \quad \int_{\mathbb{R}^3} |u|^2\,dx = N. \tag{3.4}$$

Observe that (3.4) is the Euler-Lagrange equation of the functional

$$J(u) = \frac{1}{4}\int_{\mathbb{R}^3} |\nabla u|^2\,dx + \pi\ e^2 \int_{\mathbb{R}^3} |\nabla\Delta^{-1}u^2|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|u|^2\,dx, \tag{3.5}$$

constrained on the manifold

$$B := \big\{u \in H^1(\mathbb{R}^3) \big| \|u\|^2_{L^2} = N\big\}.$$

Note that the functional $J(u)$ can be defined for complex valued u, while its critical points are only real-valued.

Given any integer $k \ge 1$ we set

$$H^k_r(\mathbb{R}^3) := \{u \in H^k(\mathbb{R}^3) : u(x) = u(|x|),\ x \in \mathbb{R}^3\}.$$

Lemma 3.2. *There results:*

(i) *J is even*
(ii) *J is C^1 on $H^1(\mathbb{R}^3)$ and its critical points constrained on B are the solutions of (3.4)*

(iii) *any critical point of* $J\big|_{H^1_r(\mathbb{R}^3)\cap B}$ *is also a critical point of* $J\big|_B$.

Proof. The proof of (i) is trivial. Since

$$\frac{d}{d\lambda}\Big(\int_{\mathbb{R}^3}|\nabla\Delta^{-1}|u+\lambda v|^2|^2\,dx\Big)\Big|_{\lambda=0}=-4\int_{\mathbb{R}^3}(\Delta^{-1}u|v)\,dx,$$

(ii) holds true. Now we prove (iii). Consider the $O(3)$ group action T_g on $H^1(\mathbb{R}^3)$ defined by

$$T_gu(x)=u(g(x)),$$

where $g\in O(3)$ and $u\in H^1(\mathbb{R}^3)$. Then the conclusion follows by well known arguments (see for example [23]) because J is invariant under the T_g action, namely

$$J(T_gu)=J(u),$$

where $g\in O(3)$ and $u\in H^1(\mathbb{R}^3)$. So, by [19] or [25, Theorem 1.28], iii) is proved. □

4. Topological Results

In this section we shall prove some topological properties of the functional J.

Lemma 4.1. *The functional J is weakly lower semicontinuous on $H^1_r(\mathbb{R}^3)$. In particular, the operator*

$$T:u\in H^1_r(\mathbb{R}^3)\mapsto(\Delta^{-1}u^2)u\in\big(H^1_r(\mathbb{R}^3)\big)'$$

is compact and the functionals

$$J_1:u\in H^1_r(\mathbb{R}^3)\mapsto\int_{\mathbb{R}^3}|\nabla\Delta^{-1}u^2|^2\,dx,$$
$$J_2:u\in H^1(\mathbb{R}^3)\mapsto\int_{\mathbb{R}^3}V(x)u^2\,dx$$

are weakly continuous.

Proof. We prove that T is compact. Let $\{u_k\}\subset H^1_r(\mathbb{R}^3)$ be bounded. Passing to a subsequence, there exists $u\in H^1_r(\mathbb{R}^3)$ such that $u_k\rightharpoonup u$ weakly in $H^1_r(\mathbb{R}^3)$. By (3.1) and Sobolev inequalities (3.2) we see that $\{\Delta^{-1}u_k^2\}$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^3)$. Passing to a subsequence, there exists $h\in\mathcal{D}^{1,2}(\mathbb{R}^3)$ such that

$$\Delta^{-1}u_k^2\rightharpoonup h\quad\text{weakly in }\mathcal{D}^{1,2}(\mathbb{R}^3).\tag{4.1}$$

We have to prove that

$$(\Delta^{-1}u_k^2)u_k\to hu\quad\text{in }(H^1_r(\mathbb{R}^3))'.\tag{4.2}$$

Denote

$$q=\frac{12}{5},\quad r=\frac{12}{7},\quad \alpha=\frac{q}{r}=\frac{7}{5},\quad \alpha'=\frac{\alpha}{\alpha-1}=\frac{7}{2},$$

clearly $2<q<6$, $\frac{6}{5}<r<2$, $\alpha'=\frac{6}{r}$. We have

$$\|(\Delta^{-1}u_k^2)u_k-hu\|_{L^r}\le\|(\Delta^{-1}u_k^2)u_k-(\Delta^{-1}u_k^2)u\|_{L^r}+\|(\Delta^{-1}u_k^2)u-hu\|_{L^r},\tag{4.3}$$

by Hölder inequality (note that $1/r=1/6+1/q$)

$$\|(\Delta^{-1}u_k^2)u_k-(\Delta^{-1}u_k^2)u\|_{L^r}\le\|\Delta^{-1}u_k^2\|_{L^6}\|u_k-u\|_{L^q},$$

then, using the compactness of the embedding (3.3), we see that $u_k\to u$ in $L^q(\mathbb{R}^3)$ and $\{\Delta^{-1}u_k^2\}$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^3)(\hookrightarrow L^6(\mathbb{R}^3))$, we have:

$$\|(\Delta^{-1}u_k^2)u_k-(\Delta^{-1}u_k^2)u\|_{L^r}\to0.\tag{4.4}$$

From the fact that $u_k \to u$ in $L^q(\mathbb{R}^3)$, we see that

$$u_k^2 \to u^2 \quad \text{in } L^{6/5}(\mathbb{R}^3).$$

Applying Lemma 3.1, we find

$$\Delta^{-1}u_k^2 \to \Delta^{-1}u^2 \quad \text{in } \mathcal{D}^{1,2}(\mathbb{R}^3).$$

Now the Sobolev embedding (3.2) guarantees that

$$\Delta^{-1}u_k^2 \to \Delta^{-1}u^2 \quad \text{in } L^6(\mathbb{R}^3).$$

Comparing this result with (4.1), we conclude that $h = \Delta^{-1}u^2$ and via

$$\|(\Delta^{-1}u_k^2)u - hu\|_{L^r} \le \|\Delta^{-1}u_k^2 - h\|_{L^{\alpha' r}}\|u\|_{L^q},$$

we get

$$\|(\Delta^{-1}u_k^2)u - hu\|_{L^r} \to 0. \tag{4.5}$$

So we have, by (4.3), (4.4) and (4.5), that

$$(\Delta^{-1}u_k^2)u_k \to hu \quad \text{in } L^r(\mathbb{R}^3).$$

From the properties (3.2) we arrive at (4.2).

We prove that J_1 is weakly continuous. Here it suffices to observe that the operator

$$Q : u \in H_r^1(\mathbb{R}^3) \mapsto u^2 \in L^{6/5}(\mathbb{R}^3)$$

is compact, indeed, by the compact embeddings of $H_r^1(\mathbb{R}^3)$, the operator:

$$H_r^1(\mathbb{R}^3) \hookrightarrow\hookrightarrow L^{12/5}(\mathbb{R}^3) \xrightarrow{Q} L^{6/5}(\mathbb{R}^3)$$

is compact and, by Lemma 3.1, the following one $\Delta^{-1} : L^{6/5}(\mathbb{R}^3) \to \mathcal{D}^{1,2}(\mathbb{R}^3)$ is continuous.

We prove that J_2 is weakly continuous. Let $\{u_k\} \subset H^1(\mathbb{R}^3)$ and $u \in H^1(\mathbb{R}^3)$ such that

$$u_k \rightharpoonup u \quad \text{weakly in } H^1(\mathbb{R}^3).$$

Since $u_k \rightharpoonup u$ weakly in $L^2(\mathbb{R}^3)$, there exists $C > 0$ such that

$$\|u_k\|_{L^2} \le C, \quad \|u\|_{L^2} \le C, \quad k \in \mathbb{N}.$$

Fix $\varepsilon > 0$, there results

$$\int_{\{|x|>z/\varepsilon\}} V(x)u_k^2\,dx \le C\varepsilon, \quad \int_{\{|x|>z/\varepsilon\}} V(x)u^2\,dx \le C\varepsilon, \quad k \in \mathbb{N}. \tag{4.6}$$

By the Sobolev inequality, $u_k^2 \rightharpoonup u^2$ weakly in $L^3(\mathbb{R}^3)$, since $V \in L^{3/2}(\{|x| \le z/\varepsilon\})$, there results

$$\int_{\{|x|\le z/\varepsilon\}} V(x)u_k^2\,dx \to \int_{\{|x|\le z/\varepsilon\}} V(x)u^2\,dx.$$

Then, by the previous one and (4.6), we can conclude

$$\int_{\mathbb{R}^3} V(x)u_k^2\,dx \to \int_{\mathbb{R}^3} V(x)u^2\,dx.$$

Since, by well known arguments, the functional

$$u \in H^1(\mathbb{R}^3) \mapsto \int_{\mathbb{R}^3} |\nabla u|^2\,dx$$

is weakly lower semicontinuous. The proof is complete. □

Lemma 4.2. *The functional J is coercive in $H^1_r(\mathbb{R}^3)$, i. e. for all sequence $\{u_k\} \subset H^1_r(\mathbb{R}^3)$ such that $\|u_k\|_{H^1} \to +\infty$ there results $\lim_k J(u_k) = +\infty$.*

Proof. Denote

$$B_{H^1_r} = \{u \in H^1_r(\mathbb{R}^3) \big| \|u\|_{H^1} = 1\}.$$

Let $\{u_k\} \subset H^1_r(\mathbb{R}^3)$ be a sequence, such that $\|u_k\|_{H^1} \to +\infty$. Take

$$\lambda_k = \|u_k\|_{H^1} \quad \text{and} \quad \tilde{u}_k = \frac{u_k}{\lambda_k}.$$

Then obviously, $u_k = \lambda_k \tilde{u}_k$ with $\lambda_k \in \mathbb{R}$ tending to $+\infty$ and $\tilde{u}_k \in B_{H^1_r}$. We have

$$J(u_k) = a_k \lambda_k^2 + b_k \lambda_k^4 - c_k \lambda_k^2,$$

with

$$\begin{aligned} a_k &= \frac{1}{4} \int_{\mathbb{R}^3} |\nabla \tilde{u}_k|^2 \, dx \in \Big[0, \frac{1}{4}\Big], \\ b_k &= \pi e^2 \int_{\mathbb{R}^3} |\nabla \Delta^{-1} \tilde{u}_k^2|^2 \, dx \geq 0, \\ c_k &= \frac{1}{2} \int_{\mathbb{R}^3} V(x) \tilde{u}_k^2 \, dx \geq 0. \end{aligned}$$

Observe that by Sobolev inequality there results

$$\begin{aligned} 2c_k &= \int_{\{|x| \leq 1\}} V(x) \tilde{u}_k^2 \, dx + \int_{\{|x|>1\}} V(x) \tilde{u}_k^2 \, dx \\ &\leq \|V\|_{L^{\frac{3}{2}}(\{|x|\leq 1\})} \|\tilde{u}_k\|^2_{L^6} + \sup_{|x|\geq 1} V(x) \|\tilde{u}_k\|^2_{L^2} \\ &\leq \Big(C\|V\|_{L^{\frac{3}{2}}(\{|x|\leq 1\})} + \sup_{|x|\geq 1} V(x) \Big) \|\tilde{u}_k\|^2_{H^1} \\ &= \Big(C\|V\|_{L^{\frac{3}{2}}(\{|x|\leq 1\})} + \sup_{|x|\geq 1} V(x) \Big), \end{aligned}$$

where $C > 0$ is the Sobolev embedding constant. Since, by Lemma 4.1, $u \in H^1_r(\mathbb{R}^3) \mapsto \int_{\mathbb{R}^3} |\nabla \Delta^{-1} u^2|^2 \, dx$ is weakly continuous and $B_{H^1_r}$ is bounded in $H^1_r(\mathbb{R}^3)$, there exists $\alpha > 0$ such that $b_k \geq \alpha > 0$. Then we can conclude that

$$\lim_k J(u_k) = +\infty,$$

and so the proof is complete. □

The two previous lemma guarantee that J is bounded from below. Alternatively, we can give a direct proof of this fact.

Lemma 4.3. *The functional J is bounded from below on B.*

Proof. For each $u \in B$ there results

$$J(u) \geq \frac{1}{4} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^3} V(x) |u|^2 \, dx. \tag{4.7}$$

By Kato's Inequality (see (7.13) page 35 in [17]) and since $u \in B$

$$\int_{\mathbb{R}^3} V(x)|u|^2 \, dx = Z \int_{\mathbb{R}^3} \frac{|u|^2}{|x|} \, dx \leq CZ \|u\|_{L^2} \|\nabla u\|_{L^2} = CNZ \|\nabla u\|_{L^2},$$

for some constant $C > 0$. So, by (4.7),

$$J(u) \geq \frac{1}{4}\|\nabla u\|_{L^2}^2 - \frac{CNZ}{2}\|\nabla u\|_{L^2}. \tag{4.8}$$

Since the map

$$x \in \mathbb{R} \mapsto \frac{1}{4}x^2 - \frac{NCZ}{2}x$$

is bounded from below, by (4.8), the claim is done. □

5. Spectral Results

The main result of this section is as follows.

Proposition 5.1. *Let $(u,\omega) \in H_r^1(\mathbb{R}^3) \times \mathbb{R}$ be solution of the equation in (3.4). If*

$$0 < \int_{\mathbb{R}^3} u^2\,dx \leq N, \tag{5.1}$$

$$V(x) = -\frac{Ze^2}{|x|}, \tag{5.2}$$

then

$$\omega < 0 \quad \text{provided } Z > N, \tag{5.3}$$

$$\omega \leq 0 \quad \text{provided } Z = N. \tag{5.4}$$

This proposition implies that Theorem 1.3 is valid. However, to prove the above proposition some Lemmas are needed.

Lemma 5.2. *Let $u \in C^2(\{|x| \geq R\})$ be a solution of*

$$\Delta u + p(x)u = 0, \quad |x| \geq R, \tag{5.5}$$

for some $R > 0$, if $p \in C(\mathbb{R}^3)$ and there exist $\alpha, R_0 > R$ such that

$$p(|x|) \geq 0, \quad |x| \geq R_0, \tag{5.6}$$

$$\frac{\partial p}{\partial r} + \frac{2(1-\alpha)}{|x|}p \geq 0, \quad |x| \geq R_0, \tag{5.7}$$

then

$$\liminf_{R\to+\infty} \frac{1}{R^\alpha}\int_{\{R_0\leq|x|\leq R\}} p(x)u^2(x)\,dx > 0. \tag{5.8}$$

The proof of the above lemma is a direct consequence of [2, Theorem 3].

Lemma 5.3. *Let $u \in H_r^1(\mathbb{R}^3), v \in L^1(\mathbb{R}^3) \cap L^{6/5}(\mathbb{R}^3)$ radial, $\omega > 0$ or $\omega = 0$ and*

$$v \geq 0, \quad \int_{\mathbb{R}^3} v\,dx < Z. \tag{5.9}$$

If u, v satisfy the equation

$$-\frac{1}{2}\Delta u - 4\pi e^2(\Delta^{-1}v)u + V(x)u = \omega u, \tag{5.10}$$

then $u \equiv 0$.

Proof. Assume, by absurd, that there exist $u \not\equiv 0$ and $\omega \geq 0$ satisfying (5.9) and (5.10). Denote

$$p(x) := 8\pi e^2 (\Delta^{-1} v)(x) - 2V(x) + 2\omega, \quad x \in \mathbb{R}^3,$$

clearly u is solution of (5.5). We shall apply Lemma 5.2 for this take α, $0 < \alpha < \frac{1}{2}$. By [18] or Lemma 9.2 in the Appendix,

$$4\pi(\Delta^{-1} v)(x) = -\int_{\mathbb{R}^3} \frac{v(y)}{\max\{|x|, |y|\}} \, dy, \quad x \in \mathbb{R}^3, \tag{5.11}$$

so

$$p(|x|) = 2e^2 \int_{|y| \geq |x|} \Big(\frac{1}{|x|} - \frac{1}{|y|}\Big) v(y) \, dy + 2\omega + 2\frac{Z - N}{|x|} e^2 \geq 0. \tag{5.12}$$

For $r = |x|$, there results

$$\begin{aligned} \frac{\partial p}{\partial r}(x) + \frac{2(1-\alpha)}{|x|} p(x) &= 8\pi e^2 \Big(\frac{\partial(\Delta^{-1} v)}{\partial r}(x) + \frac{2(1-\alpha)}{|x|} (\Delta^{-1} v)(x) \Big) \\ &\quad - 2\Big(\frac{\partial V}{\partial r}(x) + \frac{2(1-\alpha)}{|x|} V(x) \Big) + \frac{4(1-\alpha)\omega}{|x|}. \end{aligned} \tag{5.13}$$

Moreover, by (1.4),

$$-\frac{\partial V}{\partial r}(x) - \frac{2(1-\alpha)}{|x|} V(x) = -\frac{Z}{r^2} + \frac{2(1-\alpha)Z}{r^2} = \frac{(1-2\alpha)Z}{r^2}. \tag{5.14}$$

So using this relation and Lemma 9.3 from the Appendix, we find

$$\begin{aligned} &4\pi\Big(\frac{\partial \Delta^{-1} v}{\partial r}(x) + \frac{2(1-\alpha)}{|x|} \Delta^{-1} v(x) \Big) \\ &= \int_{|y|<r} \frac{v(y)}{|x|^2} \, dy - \frac{2(1-\alpha)}{r} \int_{\mathbb{R}^3} \frac{v(y)}{\max\{|x|, |y|\}} \, dy \\ &= \int_{|y|<r} \frac{v(y)}{|x|^2} \, dy - \frac{2(1-\alpha)}{r} \int_{\{|y| \leq r\}} \frac{v(y)}{\max\{|x|, |y|\}} \, dy \\ &\quad - \frac{2(1-\alpha)}{r} \int_{\{|y| \geq r\}} \frac{v(y)}{\max\{|x|, |y|\}} \, dy \\ &= \int_{\{|y| \leq r\}} \frac{v(y)}{\max\{|x|^2, |y|^2\}} \, dy - \frac{2(1-\alpha)}{r^2} \int_{\{|y| \leq r\}} v(y) \, dy \\ &\quad - \frac{2(1-\alpha)}{r} \int_{\{|y| \geq r\}} \frac{v(y)}{|y|} \, dy \\ &\geq \int_{\{|y| \leq r\}} \frac{v(y)}{r^2} \, dy - \frac{2(1-\alpha)}{r^2} \int_{\{|y| \leq r\}} v(y) \, dy - \frac{2(1-\alpha)}{r^2} \int_{\{|y| \geq r\}} v(y) \, dy \\ &\geq -\frac{(1-2\alpha)}{r^2} \int_{\mathbb{R}^3} v(y) \, dy - \frac{2(1-\alpha)}{r^2} \int_{\{|y| \geq r\}} v(y) \, dy. \end{aligned} \tag{5.15}$$

By (5.13), (5.14) and (5.15),

$$\begin{aligned} &\frac{\partial p}{\partial r}(x) + \frac{2(1-\alpha)}{|x|} p(x) \\ &\geq 2\frac{(1-2\alpha)}{r^2} \Big(Z - \int_{\mathbb{R}^3} v(y) \, dy \Big) + 4\frac{(1-\alpha)}{r} \Big(\omega - \frac{1}{r} \int_{\{|y| \geq r\}} v(y) \, dy \Big). \end{aligned} \tag{5.16}$$

If $\omega > 0$, then there exists $R_0 > 0$ such that

$$\frac{1}{|x|}\int_{\{|y|\geq|x|\}} v(y)\,dy \leq \frac{\omega}{2}, \quad |x| \geq R_0.$$

If $\omega = 0$ and $Z > N$, then for any $\varepsilon > 0$ one can find $R_0 > 0$ such that

$$\int_{\{|y|\geq|x|\}} v(y)\,dy \leq \varepsilon, \quad |x| \geq R_0.$$

In both cases, by (1.8), (5.9) and (5.16), since $0 < \alpha < \frac{1}{2}$, we have

$$\frac{\partial p}{\partial r}(x) + \frac{2(1-\alpha)}{|x|}p(x) \geq 0, \quad |x| \geq R_0. \tag{5.17}$$

By (5.5) and Lemma 5.2, the formula (5.8) holds true. On the other hand, we have

$$\int_{\mathbb{R}^3} u^2(\Delta^{-1}v)\,dx \leq \|u\|^2_{L^{12/5}}\|\Delta^{-1}v\|_{L^6} \tag{5.18}$$

and, as in Lemma 4.2,

$$\int_{\mathbb{R}^3} u^2|V|\,dx \leq \|V\|_{L^{3/2}(\{|x|\leq 1\})}\|u\|^2_{L^6} + Z\|u\|^2_{L^2}, \tag{5.19}$$

so, by (5.18) and (5.19),

$$\begin{aligned}
&\int_{\{R_0\leq|x|\leq R\}} pu^2\,dx \\
&\leq \int_{\mathbb{R}^3} pu^2\,dx \\
&= 2\Big(4\pi e^2\int_{\mathbb{R}^3}(\Delta^{-1}v)u^2\,dx + \int_{\mathbb{R}^3} Vu^2\,dx + \int_{\mathbb{R}^3}\omega u^2\,dx\Big) \\
&\leq 8\pi e^2\|u\|^2_{L^{12/5}}\|\Delta^{-1}v\|_{L^6} + 2\|V\|_{L^{3/2}(\{|x|\leq 1\})}\|u\|^2_{L^6} + 2Z\|u\|^2_{L^2} + 2\omega\|u\|^2_{L^2}.
\end{aligned} \tag{5.20}$$

Then

$$\lim_{R\to+\infty}\frac{1}{R^\alpha}\int_{\{R_0\leq|x|\leq R\}} p(x)u^2(x)\,dx = 0, \tag{5.21}$$

and this is absurd, since (5.21) contradicts (5.8), this concludes the proof. □

Corollary 5.4. *If $V \equiv 0$ and the assumptions of Lemma 5.3 are satisfied, then $u \equiv 0$.*

Proof. Suppose, by absurd, that there is $u \not\equiv 0$ solution of (5.10), multiplying by u and integrating on $\mathbb{R}^3$, we get $\omega > 0$. We are going to apply the Agmon's result of Lemma 5.2. For this we have to verify the condition (5.7) for $|x|$ large enough, $0 < \alpha < \frac{1}{2}$ and

$$p(x) := 2(\Delta^{-1}v)(x) + 2\omega, \quad x \in \mathbb{R}^3.$$

The argument of the previous lemma (with $Z = 0$) gives

$$\begin{aligned}
&\frac{\partial p}{\partial r}(x) + \frac{2(1-\alpha)}{|x|}p(x) \\
&\geq 4\frac{(1-\alpha)\omega}{r} - 2\frac{(1-2\alpha)}{r^2}\int_{\mathbb{R}^3} v(y)\,dy - 4\frac{(1-\alpha)}{r^2}\int_{\{|y|\geq r\}} v(y)\,dy.
\end{aligned}$$

So, for $R_0 > 0$ sufficiently large

$$\frac{\partial p}{\partial r}(x) + \frac{2(1-\alpha)}{|x|} p(x) \geq 0, \quad |x| \geq R_0.$$

By (5.20), with $Z = 0$ we have

$$\frac{1}{R^\alpha} \int_{\{R_0 \leq |x| \leq R\}} p u^2 \, dx \leq \frac{2}{R^\alpha} \left(\|u\|^2_{L^{12/5}} \|\Delta^{-1} v\|_{L^6} + \omega \|u\|^2_{L^2} \right).$$

This is absurd, because it contradicts (5.8), then $u \equiv 0$. □

Proof of Theorem 1.1. Denote $v(x) := u^2(x)$, $x \in \mathbb{R}^3$. By the Sobolev inequalities

$$v \in L^1(\mathbb{R}^3) \cap L^{6/5}(\mathbb{R}^3),$$

it is radial and, by the constraint in (3.4), it satisfies also (5.9). Since $\omega \geq 0$, the claim is direct consequence of the previous corollary and of the equivalence between (1.7) and (3.4). □

Lemma 5.5. *If $u \in H^1_r(\mathbb{R}^3)$ is a solution of (3.4), such that*

$$\int_{\mathbb{R}^3} u^2 \, dx < Z, \tag{5.22}$$

$$\omega \geq 0, \tag{5.23}$$

or

$$\int_{\mathbb{R}^3} u^2 \, dx = Z, \tag{5.24}$$

$$\omega > 0, \tag{5.25}$$

then $u \equiv 0$.

Proof. Denote $v(x) := u^2(x)$, for $x \in \mathbb{R}^3$. By the Sobolev inequalities

$$v \in L^1(\mathbb{R}^3) \cap L^r(\mathbb{R}^3), \quad \frac{6}{5} < r \leq 2,$$

it is radial and, by (5.22) or (5.24), it satisfies also (5.9). Applying Lemma 5.3, we complete the proof. □

Then the proof of Proposition 5.1 is a direct consequence of the lemma above.

The proof of Theorem 1.3 is a direct consequence of Lemma 5.5 and of the equivalence between (1.7) and (3.4).

6. Proof of Theorem 1.2

In this section we shall prove Theorem 1.2. We begin proving some lemmas.

Lemma 6.1. *The functional $J\big|_{H^1_r(\mathbb{R}^3) \cap B}$ satisfies the Palais-Smale condition in each level $]-\infty, -\varepsilon], \varepsilon > 0$, i.e. any sequence $\{u_k\} \subset H^1_r(\mathbb{R}^3) \cap B$ such that $\{J(u_k)\}$ is bounded and*

$$J(u_k) \leq -\varepsilon, \quad J\big|'_{H^1_r(\mathbb{R}^3) \cap B}(u_k) \to 0, \tag{6.1}$$

contains a converging subsequence.

Proof. Fix $\varepsilon > 0$. Let $\{u_k\} \subset H_r^1(\mathbb{R}^3) \cap B$ be such that $\{J(u_k)\}$ is bounded and satisfies (6.1). First of all observe that, by (iii) of Lemma 3.2, there results

$$J\big|'_{H_r^1(\mathbb{R}^3)\cap B}(u) = 0 \Longleftrightarrow J\big|'_B(u) = 0,$$

then we can assume

$$J\big|'_B(u_k) \to 0.$$

Since $J(u_k) \leq -\varepsilon$, by Lemma 4.2, $\{u_k\}$ is bounded in $H_r^1(\mathbb{R}^3)$, passing to a subsequence, there exists $u \in H_r^1(\mathbb{R}^3)$ such that

$$u_k \rightharpoonup u \qquad \text{weakly in } H_r^1(\mathbb{R}^3). \tag{6.2}$$

We shall prove that

$$u_k \to u \qquad \text{in } H_r^1(\mathbb{R}^3). \tag{6.3}$$

By definition, there exists $\{\omega_k\} \subset \mathbb{R}$ such that

$$J\big|'_B(u_k) = J'(u_k) - \omega_k u_k, \quad k \in \mathbb{N}.$$

Observe that, since $\{u_k\} \subset B$, we have

$$\begin{aligned}
&N\omega_k \\
&= \langle J\big|'_B(u_k), u_k\rangle - \langle J'(u_k), u_k\rangle \\
&= \langle J\big|'_B(u_k), u_k\rangle - \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_k|^2\,dx - 4\pi e^2 \int_{\mathbb{R}^3} |\nabla\Delta^{-1}u_k^2|^2\,dx + \int_{\mathbb{R}^3} V(x)|u_k|^2\,dx \\
&= \langle J\big|'_B(u_k), u_k\rangle - 2J(u_k) - 2\pi e^2 \int_{\mathbb{R}^3} |\nabla\Delta^{-1}u_k^2|^2\,dx,
\end{aligned}$$

by Lemma 4.1 and (6.1), $\{\omega_k\}$ is bounded in $\mathbb{R}$ and so passing to a subsequence there results

$$\omega_k \to \omega\,, \tag{6.4}$$

$$-\frac{1}{2}\Delta u - 4\pi e^2(\Delta^{-1}u^2)u - V(x)u = \omega u\,. \tag{6.5}$$

If $\omega < 0$, by Lemma 4.1, (6.2), (6.4) and (6.5),

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_k|^2\,dx - \omega\int_{\mathbb{R}^3} u_k^2\,dx \\
&= \big\langle J\big|'_B(u_k),\, u_k\big\rangle - 4\pi e^2 \int_{\mathbb{R}^3} |\nabla\Delta^{-1}u_k^2|^2\,dx + \int_{\mathbb{R}^3} V(x)u_k^2\,dx + (\omega_k - \omega)\int_{\mathbb{R}^3} u_k^2\,dx \\
&\to -4\pi e^2 \int_{\mathbb{R}^3} |\nabla\Delta^{-1}u^2|^2\,dx + \int_{\mathbb{R}^3} V(x)u^2\,dx \\
&= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2\,dx - \omega\int_{\mathbb{R}^3} u^2\,dx,
\end{aligned} \tag{6.6}$$

and then (6.3) follows.

Now we consider the case $\omega \geq 0$. If $\|u\|_{L^2} = 0$, by Lemma 4.1, we have

$$0 = J(u) \leq \liminf_k J(u_k) \leq -\varepsilon,$$

that is absurd. If $0 < \|u\|_{L^2}^2 < N$ then u is solution of the equation in (3.4), (5.22) and (5.23) hold. So, by Lemma 5.5 , we have $u \equiv 0$ and also this is absurd. Finally, if $\|u\|_{L^2}^2 = N$, we have, from (6.2)

$$u_k \to u \qquad \text{in } L^2(\mathbb{R}^3), \tag{6.7}$$

then (6.3) is direct consequence of (6.4), (6.6) and (6.7). This concludes the proof. □

Remark 6.2. Let $\rho \in L^1(\mathbb{R}^3) \cap L^r(\mathbb{R}^3)$, with $\frac{6}{5} < r \le 2, \varphi \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ radially symmetric maps such that $\Delta\varphi = \rho$. Denote

$$\rho_\nu(x) := \rho(\nu x), \quad x \in \mathbb{R}^3, \nu \ge 0$$

we claim that the unique solution φ_ν of the equation

$$\Delta\varphi_\nu = \rho_\nu$$

is $\varphi_\nu(x) := \nu^{-2}\varphi(\nu x)$, $x \in \mathbb{R}^3$. Indeed, denoting $r = |x|$, there results

$$\begin{aligned} \Delta\varphi_\nu(x) = \Delta\varphi_\nu(r) &= \partial_{rr}^2\varphi_\nu(r) + \frac{2}{r}\partial_r\varphi_\nu(r) \\ &= \frac{1}{\nu^2}\Big(\nu^2\partial_{rr}^2\varphi(\nu r) + \frac{2\nu}{r}\partial_r\varphi(\nu r)\Big) \\ &= \partial_{rr}^2\varphi(\nu r) + \frac{2}{r\nu}\partial_r\varphi(\nu r) \\ &= \Delta\varphi(\nu x) = \rho(\nu x) = \rho_\nu(x). \end{aligned} \tag{6.8}$$

Moreover

$$\begin{aligned} \int_{\mathbb{R}^3} |\nabla\Delta^{-1}\rho_\nu(x)|^2\,dx &= \int_{\mathbb{R}^3} |\nabla\varphi_\nu(x)|^2\,dx = \frac{1}{\nu^2}\int_{\mathbb{R}^3} |\nabla\varphi(\nu x)|^2\,dx \\ &= \frac{1}{\nu^5}\int_{\mathbb{R}^3} |\nabla\varphi(x)|^2\,dx = \frac{1}{\nu^5}\int_{\mathbb{R}^3} |\nabla\Delta^{-1}\rho(x)|^2\,dx. \end{aligned} \tag{6.9}$$

For the last part of this section we need more notation. Define

$$\begin{gathered} c_k := \inf\big\{\sup J(A)\big|A \in \mathcal{A}, \gamma(A) \ge k\big\}, \quad k \in \mathbb{N}\backslash\{0\}, \\ \tilde{c}_k := \inf\big\{\sup J\big(h\big(S^{k-1}\big)\big)\big|h \in \Omega_k\big\}, \quad k \in \mathbb{N}\backslash\{0\}, \\ \tilde{c}_{k,\lambda} := \inf\big\{\sup J\big(h\big(S^{k-1}\big)\big)\big|h \in \Omega_{k,\lambda}\big\}, \quad k \in \mathbb{N}\backslash\{0\}, \lambda > 0, \end{gathered}$$

where

$$\begin{gathered} \mathcal{A} := \big\{A \subset H_r^1(\mathbb{R}^3) \cap B : A \text{ is closed and symmetric}\big\}, \\ \Omega_k := \big\{h : S^{k-1} \to H_r^1(\mathbb{R}^3) \cap B : h \text{is continuous and odd}\big\}, \quad k \in \mathbb{N}\backslash\{0\}, \\ \Omega_{k,\lambda} := \big\{h : S^{k-1} \to H_r^1(\mathbb{R}^3) \cap B_\lambda : h \text{is continuous and odd}\big\}, \quad k \in \mathbb{N}\backslash\{0\}, \lambda > 0, \\ B_\lambda := \big\{u \in H^1(\mathbb{R}^3) : \|u\|_{L^2} = \lambda\big\}, \quad \lambda > 0 \end{gathered}$$

and γ is the Krasnoselskii Genus (see e. g. [3, Definition 1.1]).

Lemma 6.3. *There results*

$$c_k \le \tilde{c}_k \le \tilde{c}_{k,\lambda}, \tag{6.10}$$

for each $k \in \mathbb{N}\backslash\{0\}$ and $0 < \lambda \le \sqrt{N}$.

Proof. Fix $k \in \mathbb{N}\backslash\{0\}$. We prove that

$$c_k \le \tilde{c}_k. \tag{6.11}$$

Let $h \in \Omega_k$, since h is continuous and odd the set $J(h(S^{k-1}))$ is closed and symmetric. Moreover $h(S^{k-1}) \subset B$ and, by the invariance property of the Genus, there results

$$\gamma\big(h\big(S^{k-1}\big)\big) \geq \gamma\big(S^{k-1}\big) = k.$$

So we have $c_k \leq \sup J\big(h\big(S^{k-1}\big)\big)$ and then (6.11) is proved.

Now, we prove that

$$\tilde{c}_k \leq \tilde{c}_{k,\lambda}, \quad 0 < \lambda \leq \sqrt{N}. \tag{6.12}$$

Fix $0 < \lambda \leq \sqrt{N}$ and define

$$h_\lambda(\xi)(x) = \frac{1}{\lambda^5} h(\xi)\Big(\frac{x}{\lambda^4}\Big), \quad h \in \Omega_k, \xi \in S^{k-1}.$$

Let $h \in \Omega_k$ and $\xi \in S^{k-1}$ such that

$$\frac{3}{2N^2} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx - \int_{\mathbb{R}^3} V(x)|u|^2 \, dx \geq 0, \tag{6.13}$$

where $u := h(\xi)$. Set

$$\nu := \frac{1}{\lambda^4}, \quad u_\nu(x) := h_\lambda(x) = \nu^{5/4}(x)u(\nu x)$$

and observe that, by (6.9), there results

$$\int_{\mathbb{R}^3} |u_\nu|^2 \, dx = \frac{1}{\nu^{1/2}} \int_{\mathbb{R}^3} |u|^2 \, dx = \lambda^2 N,$$
$$\int_{\mathbb{R}^3} |\nabla u_\nu|^2 \, dx = \nu^{3/2} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx,$$
$$\int_{\mathbb{R}^3} |\nabla \Delta^{-1} u_\nu^2(x)|^2 \, dx = \int_{\mathbb{R}^3} |\nabla \Delta^{-1} u^2(x)|^2 \, dx,$$
$$\int_{\mathbb{R}^3} V(x)|u_\nu|^2 \, dx = \nu^{1/2} \int_{\mathbb{R}^3} V(x)|u|^2 \, dx.$$

Consider the map

$$f(\nu) := J(u_\nu) = \frac{\nu^{3/2}}{4} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx + \pi e^2 \int_{\mathbb{R}^3} |\nabla \Delta^{-1} u^2|^2 \, dx - \frac{\nu^{1/2}}{2} \int_{\mathbb{R}^3} V(x)|u|^2 \, dx,$$

there results

$$\frac{df}{d\nu}(\nu) = \frac{3\nu^{1/2}}{8} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx - \frac{1}{4\nu^{1/2}} \int_{\mathbb{R}^3} V(x)|u|^2 \, dx.$$

Clearly

$$\frac{df}{d\nu}(\nu) \geq 0 \Longleftrightarrow \frac{3\nu}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx - \int_{\mathbb{R}^3} V(x)|u|^2 \, dx \geq 0$$

and then, by (6.13), f is increasing for $\nu \geq 1/N^2$, namely

$$J(h(\xi)) = J(u) \leq J(u_\nu) = J(h_\lambda(\xi)).$$

Since, if there exists $\xi' \in S^{k-1}, \xi \neq \xi'$ such that $h(\xi')$ does not satisfy (6.13), we have $J(h(\xi')) \leq J(h(\xi))$, then

$$\sup J\big(h\big(S^{k-1}\big)\big) \leq \sup J\big(h_\lambda\big(S^{k-1}\big)\big).$$

This concludes the proof of (6.12). $\square$

Lemma 6.4. *For all $k \in \mathbb{N}\backslash\{0\}$, there exist a subspace $V_k \subset H^1_r(\mathbb{R}^3)$ of dimension k and $\nu > 0$ such that*

$$\int_{\mathbb{R}^3} \Big(\frac{1}{2}|\nabla u|^2 - V(x)u^2\Big)\,dx \le -\nu,$$

for all $u \in V_k \cap B$.

Proof. Let u be a smooth map with compact support such that

$$\int_{\mathbb{R}^3} |u|^2\,dx = N, \quad \operatorname{supp}(u) \subset B_2(0)\backslash B_1(0),$$

where

$$B_\rho(x) := \big\{y \in \mathbb{R}^3 \big| |x-y| < \rho\big\}, \quad x \in \mathbb{R}^3, \rho > 0.$$

Denote

$$u_\lambda(x) := \lambda^{3/2}u(\lambda x), \quad \lambda > 0, x \in \mathbb{R}^3,$$

there results

$$\int_{\mathbb{R}^3} |u|^2\,dx = \int_{\mathbb{R}^3} |u_\lambda|^2\,dx = N, \quad \operatorname{supp}(u_\lambda) \subset B_{2/\lambda}(0)\backslash B_{1/\lambda}(0).$$

We have

$$\begin{aligned}\int_{\mathbb{R}^3} \Big(\frac{1}{2}|\nabla u_\lambda|^2 - V(x)u_\lambda^2\Big)\,dx &= \int_{\mathbb{R}^3} \Big(\lambda^2\frac{1}{2}|\nabla u|^2 - V\big(x/\lambda\big)u^2\Big)\,dx \\ &\le \lambda^2 \int_{\mathbb{R}^3} \frac{1}{2}|\nabla u|^2\,dx - N \inf_{\operatorname{supp}(u)/\lambda} V \\ &\le \lambda^2 \int_{\mathbb{R}^3} \frac{1}{2}|\nabla u|^2\,dx - \frac{z\lambda}{2}N.\end{aligned}$$

There exists $\lambda_0 > 0$ such that

$$\int_{\mathbb{R}^3} \Big(\frac{1}{2}|\nabla u_{\lambda_0}|^2 - V(x)u_{\lambda_0}^2\Big)\,dx < 0.$$

Let $k \in \mathbb{N}\backslash\{0\}$ and $u_1, u_2, \dots, u_k$ be smooth maps with compact supports such that

$$\int_{\mathbb{R}^3} |u_i|^2\,dx = 1, \quad \operatorname{supp}(u_i) \subset B_{2i}(0)\backslash B_i(0), \quad i = 1, 2, \dots, k.$$

Using an analogous argument we are able to find $\lambda_1, \lambda_2, \dots, \lambda_k > 0$ such that

$$\int_{\mathbb{R}^3} \Big(\frac{1}{2}|\nabla u_{i_{\lambda_i}}|^2 - V(x)u_{i_{\lambda_i}}^2\Big)\,dx < 0, \quad i = 1, 2, \dots, k\,.$$

Let $0 < \bar\lambda < \min\{\lambda_1, \lambda_2, \dots, \lambda_k\}$ and let V_k be the subspace spanned by $u_{1_{\bar\lambda}}$, $u_{2_{\bar\lambda}}$, $\dots$, $u_{k_{\bar\lambda}}$. Since the supports of this maps are pairwise disjoint, V_k has dimension k. Since for all $i = 1, 2, \dots, k$ and $\lambda \le \lambda_i$, there results

$$\int_{\mathbb{R}^3} \Big(\frac{1}{2}|\nabla u_{i_\lambda}|^2 - V(x)u_{i_\lambda}^2\Big) < 0$$

and $V_k \cap B$ is compact, the claim is proved. □

Lemma 6.5. *There results*

$$c_k < 0, \tag{6.14}$$

for each $k \in \mathbb{N}\backslash\{0\}$.

Proof. Let $k \in \mathbb{N}\backslash\{0\}$, by Lemma 6.4, there exist $V_k \subset H_r^1(\mathbb{R}^3)$ subspace of dimension k and $\nu > 0$ such that, for all $u \in V_k \cap B$,

$$\int_{\mathbb{R}^3} \Big(\frac{1}{2}|\nabla u|^2 - V(x)u^2\Big)\,dx \le -\nu.$$

Let $\lambda > 0$ and define

$$h_\lambda : V_k \cap B \to H_r^1(\mathbb{R}^3), \quad h_\lambda(u) = \lambda^{1/2}u.$$

Fixed $u \in V_k \cap B$ and $0 < \lambda < \sqrt{N}$, there results

$$J(h_\lambda(u)) \le -\lambda/2\nu + c\lambda^2 \le -\lambda/2\nu + c\lambda^2, \tag{6.15}$$

where c is a positive constant. Then there exists $0 < \bar\lambda < \sqrt{N}$ such that for all $u \in V_k \cap B$ there results $J(h_{\bar\lambda}(u)) < 0$. Since $h_{\bar\lambda} \in \Omega_{\bar\lambda}$ and $V_k \cap B \simeq S^{k-1}$, by Lemma 6.3 and the compactness of S^{k-1}, we have

$$c_k \le \tilde c_k \le \tilde c_{k,\bar\lambda} \le \sup J(h_{\bar\lambda}(V_k \cap B)) < 0\,.$$

The proof is complete. □

Corollary 6.6. *There results*

$$\inf_{u \in H_r^1(\mathbb{R}^3)\cap B} J(u) < 0. \tag{6.16}$$

The proof of this corollary is a direct consequence of the previous Lemma.

Lemma 6.7. *Let $k \in \mathbb{N}, E \subset H^1(\mathbb{R}^3)$ be a subspace of dimension k and $A \in \mathcal{A}$, if*

$$\gamma(A) \ge k+1 \tag{6.17}$$

then

$$A \cap E^\perp \ne \emptyset. \tag{6.18}$$

Proof. Assume, by absurd that (6.18) is false, there results

$$P(A) \subset E\backslash\{0\}, \tag{6.19}$$

where $P : H^1(\mathbb{R}^3) \to E$ is the orthogonal projection on E. So we have

$$\gamma(P(A)) \le k. \tag{6.20}$$

On the other side, since P is continuous and odd, by the invariance property of the Genus there results

$$k+1 \le \gamma(A) \le \gamma(P(A)).$$

Since this is in contradiction with (6.20), the proof is complete. □

Lemma 6.8. *The functional J has a sequence $\{u_k\}_{k\in\mathbb{N}} \subset H_r^1(\mathbb{R}^3) \cap B$ of critical points such that $\omega_k < 0$ and $\omega_k \to 0$, where $\{\omega_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ is the sequence of the Lagrange multipliers associated to the critical points.*

Proof. By Lemmas 6.1 and 6.5 (see [20, Theorem 9.1]) there exists a sequence $\{u_k\}_{k\in\mathbb{N}} \subset H_r^1(\mathbb{R}^3) \cap B$ of critical points of the functional J. Call $\{\omega_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ the sequence of the Lagrange multipliers associated to this critical points, namely

$$J'(u_k) - \omega_k u_k = 0, \quad k \in \mathbb{N}\backslash\{0\}.$$

By Lemma 5.3, there results $\omega_k < 0$ for $k \in \mathbb{N}\backslash\{0\}$. We have to prove that

$$\omega_k \to 0. \tag{6.21}$$

Let $\{V_k\}$ be a sequence of subspaces of $H^1_r(\mathbb{R}^3)$, such that

$$\dim(V_k) = k, \quad \bigcup_{k\in\mathbb{N}\setminus\{0\}} V_k \text{ is dense in } H^1_r(\mathbb{R}^3).$$

Moreover, let $\{A_k\} \subset \mathcal{A}$ such that

$$\gamma(A_k) \geq k, \quad c_k \leq \sup J(A_k) \leq \frac{c_k}{2}, \quad k \in \mathbb{N}\setminus\{0\}. \tag{6.22}$$

Call

$$W_k := V_{k-1}^{\perp}, \quad k \in \mathbb{N}\setminus\{0\},$$

by Lemma 6.7, there results $W_k \cap A_k \neq \emptyset$, $k \in \mathbb{N}\setminus\{0\}$. Let $\{v_k\} \subset H^1_r(\mathbb{R}^3) \cap B$ such that

$$v_k \in W_k \cap A_k, \quad k \in \mathbb{N}\setminus\{0\},$$

clearly

$$v_k \rightharpoonup 0 \qquad \text{weakly in} \quad H^1_r(\mathbb{R}^3) \tag{6.23}$$

and, by (6.22),

$$\sup J(V_k) \leq \frac{c_k}{2}, \quad k \in \mathbb{N}\setminus\{0\}. \tag{6.24}$$

By (6.23) and Lemma 4.1 we have

$$0 \leq \liminf_k J(v_k) \tag{6.25}$$

and, by (6.24),

$$\limsup_k J(v_k) \leq \lim_k \frac{c_k}{2} \leq 0. \tag{6.26}$$

By (6.25) and (6.26), we deduce $c_k \to 0$. Since $2c_k \leq \omega_k < 0$, (6.21) is done. □

Proof of Theorem 1.2. Since $F(u, 4\pi\Delta^{-1}u^2) = J(u)$ for all $u \in H^1(\mathbb{R}^3)$, by Lemma 3.2 and the previous one the claim is proved. □

7. Proof of Theorem 1.4

Our next step is to show that the radially symmetric solutions

$$u \in H^1(\mathbb{R}^3), \quad \nabla\varphi \in L^2(\mathbb{R}^3), \tag{7.1}$$

to the equation

$$\begin{gathered} -\frac{1}{2}\Delta u - e\varphi u - \frac{Z}{|x|}u = \omega u, x \in \mathbb{R}^3, \\ \Delta\varphi = 4\pi e u^2, \ x \in \mathbb{R}^3, \\ \int_{\mathbb{R}^3} u^2\, dx = N, \end{gathered} \tag{7.2}$$

constructed in the previous section, are more regular. More precisely, we shall derive the higher regularity

$$\nabla u \in H^{k-1}(|x| > \varepsilon), \quad \nabla\varphi \in H^{k-1}(|x| > \varepsilon), \tag{7.3}$$

where k is arbitrary integer and $\varepsilon > 0$.

Lemma 7.1. *If the assumption* (7.1) *is satisfied, then*

$$\nabla u \in H^1(\mathbb{R}^3), \quad \nabla\varphi \in H^1(\mathbb{R}^3), \quad u \in L^\infty(\mathbb{R}^3), \quad \varphi \in L^\infty(\mathbb{R}^3). \tag{7.4}$$

Proof. The assumption (7.1) and the Sobolev embedding in $\mathbb{R}^3$ guarantee that

$$\varphi \in L^6, \quad u \in L^p, \quad 2 \le p \le 6. \tag{7.5}$$

This property and the Hölder inequality imply that the nonlinear term φu in the first equation in (7.2) is in L^2. The fact that $|x|^{-1}u \in L^2$ follows from the Hardy inequality and the fact that $\nabla u \in L^2$. Therefore this equation shows that $\Delta u \in L^2$, so $u \in H^2$. Using the second equation and the fact that $u^2 \in L^2$ we conclude that $\nabla\varphi \in H^1$. Finally the property $u \in L^\infty$ follows from the estimate

$$\|u\|_{L^\infty} \le C\|\nabla u\|_{H^1(\mathbb{R}^3)}.$$

This estimate follows from the Fourier representation

$$u(x) = (2\pi)^{-3}\int_{\mathbb{R}^3} \mathrm{e}^{-ix\xi}\hat{u}(\xi)d\xi,$$

the Cauchy inequality and the fact that

$$|\xi|^{-1}(1+|\xi|)^{-1} \in L^2(\mathbb{R}^3).$$

The Lemma is established. □

In the same way, proceeding inductively, we obtain the following result.

Lemma 7.2. *Under assumption* (7.1)*, for any integer $k \ge 2$ and for any positive number $\varepsilon > 0$ we have*

$$u \in H^k(|x| > \varepsilon), \quad \nabla\varphi \in H^{k-1}(|x| > \varepsilon). \tag{7.6}$$

To study more precisely the behavior of the solution $u(x) = u(|x|)$ we introduce polar coordinates $r = |x|$ and set

$$\mathbb{U}(r) = ru(r), \quad \mathbb{V}(r) = -r\varphi(r). \tag{7.7}$$

Using the identities

$$\Delta\big(\frac{\mathbb{U}(r)}{r}\big) = \frac{\mathbb{U}''(r)}{r}, \quad \Delta\big(\frac{\mathbb{V}(r)}{r}\big) = \frac{\mathbb{V}''(r)}{r},$$

where $\mathbb{U}'(r) = \partial_r\mathbb{U}(r)$, we can rewrite (7.2) in the form

$$\begin{gathered} -\frac{\mathbb{U}''}{2} + e\frac{\mathbb{V}}{r}\mathbb{U} - \frac{Z}{r}\mathbb{U} = \omega\mathbb{U}, \quad r > 0, \\ -\mathbb{V}'' = 4\pi e\frac{\mathbb{U}^2}{r}, \quad r > 0. \end{gathered} \tag{7.8}$$

We shall need the following result.

Lemma 7.3. *Let $k \ge 1$ be an integer and $\varepsilon > 0$ be a real number. We have the following properties:*

(a) *if $u(x) = u(|x|) \in H^k(\mathbb{R}^3)$, then $\mathbb{U}(r) \in H^k(0,\infty)$*
(b) *$u(x) = u(|x|) \in H^k(|x| > \varepsilon)$, if and only if $\mathbb{U}(r) \in H^k(\varepsilon,\infty)$.*

Proof. The proof of (a) follows from the relation

$$\partial_r^k\mathbb{U}(r) = r\partial_r^k u(r) + k\partial_r^{k-1}u(r)$$

valid for any integer $k \ge 1$. Note that the Hardy inequality implies

$$\int_0^\infty |\partial_r^{k-1}u(r)|^2 dr \le C\|u\|^2_{H^k(\mathbb{R}^3)}.$$

For property (b), we can use the relation

$$\partial_r^k u(r) = \sum_{j=1}^{k} \frac{c_{k,j}}{r^j} \partial_r^{k-j} \mathbb{U}(r)$$

and the fact that r^{-j} is bounded for $r \geq \varepsilon > 0$. □

Lemma 7.4. *The functions* $\mathbb{U}(r), \mathbb{V}(r)$ *are smooth near* $r = 0$.

Proof. From $u \in H^2$ (see Lemma 7.1) it follows $u \in L^\infty$, so

$$|\mathbb{U}(r)| = r|u(r)| \leq Cr$$

near $r = 0$. In the same way $\varphi \in L^\infty$ (Lemma 7.1) implies that

$$|\mathbb{V}(r)| = r|\varphi(r)| \leq Cr$$

near $r = 0$. The system (7.2) shows that

$$|\mathbb{U}''(r)| + |\mathbb{V}''(r)| \leq C,$$

so $\mathbb{U}(r), \mathbb{V}(r) \in C^1([0,1])$. Setting $a_1 = \mathbb{U}'(0)$, $b_1 = \mathbb{V}'(0)$, we can make the representation

$$\mathbb{U}(r) = a_1 r + \mathbb{U}_1(r), \quad \mathbb{V}(r) = b_1 r + \mathbb{V}_1(r),$$

where $\mathbb{U}_1, \mathbb{V}_1 \in o(r)$ satisfy

$$\begin{gathered} -\frac{\mathbb{U}_1''}{2} + \frac{\mathbb{V}_1}{r}\mathbb{U}_1 - \frac{Z}{r}\mathbb{U}_1 - \omega \mathbb{U}_1 = c_1 + O(r), \quad r > 0, \\ -\mathbb{V}_1'' - 4\pi \frac{\mathbb{U}_1^2}{r} = O(r), \quad r > 0, \end{gathered} \tag{7.9}$$

where $c_1 = \omega a_1$. These equations imply

$$\mathbb{U}_1''(r) = c_1 + O(r), \quad \mathbb{V}_1''(r) = O(r),$$

so

$$\mathbb{U}_1(r) = \frac{c_1 r^2}{2} + O(r^3), \quad \mathbb{V}_1(r) = O(r^3)$$

near $r = 0$ and these relations imply $\mathbb{U}_1(r), \mathbb{V}_1(r) \in C^2([0,1])$. Continuing further we obtain inductively

$$\mathbb{U}(r) = a_1 r + a_2 r^2 + \cdots + a_k r^k + \mathbb{U}_k(r), \quad \mathbb{V}(r) = b_1 r + b_2 r^2 + \cdots + b_k r^k + \mathbb{V}_k(r).$$

Here $\mathbb{U}_k, \mathbb{V}_k \in o(r^k)$ satisfy

$$\begin{gathered} -\frac{\mathbb{U}_k''}{2} + \frac{\mathbb{V}_k}{r}\mathbb{U}_k - \frac{z}{r}\mathbb{U}_k - \omega \mathbb{U}_k = c_k r^{k-1} + O(r^k), \quad r > 0, \\ -\mathbb{V}_k'' - 4\pi \frac{\mathbb{U}_k^2}{r} = \tilde{c}_k r^{k-1} + O(r^k), \quad r > 0. \end{gathered} \tag{7.10}$$

These relations imply

$$\mathbb{U}_k(r) = \frac{c_k r^{k+1}}{k(k+1)} + O(r^{k+2}), \quad \mathbb{V}_k(r) = \frac{\tilde{c}_k r^{k+1}}{k(k+1)} + O(r^{k+2})$$

near $r = 0$ and these relations imply $\mathbb{U}_k(r), \mathbb{V}_k(r) \in C^{k+1}([0,1])$. □

Our next step is to obtain the decay of the solution. We look for soliton type solutions u to (1.7), i.e. very regular solutions decaying rapidly at infinity. Our next step is to obtain a very rapid decay of the radial field $u(|x|)$ at infinity.

Lemma 7.5. *If the assumption* (7.1) *is satisfied, then*

$$\mathbb{U} \in H^k((1,+\infty)), \quad \mathbb{V}' \in H^{k-1}((1,+\infty)), \tag{7.11}$$

and

$$|\mathbb{U}'(r)|^2 + |\mathbb{U}(r)|^2 \le \frac{C}{r^k}, \quad 0 \le \mathbb{V}'(r) \le \frac{C}{r^k} \tag{7.12}$$

for each integer $k \ge 2, r \ge 1$.

Proof. The Sobolev embedding and Lemma 7.2 imply that

$$\begin{gathered}\int_0^{+\infty} |\mathbb{U}(r)|^2 dr + \int_0^{+\infty} |\mathbb{U}'(r)|^2 dr \le C\|u\|^2_{H^1(\mathbb{R}^3)},\\ \int_0^{+\infty} |\mathbb{V}'(r)|^2 dr \le C\|\varphi\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^3)}.\end{gathered} \tag{7.13}$$

Note that we have used the Hardy inequality

$$\int_0^{+\infty} |f(r)|^2 dr \le C \int_0^{+\infty} |f'(r)|^2 \, r^2 dr \tag{7.14}$$

in the above estimates (see [15, Theorem 330] or [24, Remark 1, Section 3.2.6]). Hence

$$\mathbb{U} \in H^1((0,+\infty)), \quad \mathbb{V}' \in L^2((0,+\infty)).$$

Proceeding further inductively we find (7.11).

The above properties and the Sobolev embedding imply

$$\lim_{r\to+\infty} |\mathbb{U}(r)| = 0, \quad \lim_{r\to+\infty} |\mathbb{U}'(r)| = 0, \tag{7.15}$$

In a similar way we get

$$\lim_{r\to+\infty} |\mathbb{V}'(r)| = 0. \tag{7.16}$$

We can improve the last property. Indeed, integrating the second equality in (7.8) we find

$$\mathbb{V}'(r) = \int_r^\infty \frac{\mathbb{U}^2(\tau)}{\tau} d\tau. \tag{7.17}$$

Since

$$\int_r^\infty \mathbb{U}^2(\tau) d\tau \le C, \tag{7.18}$$

we get

$$0 \le \mathbb{V}'(r) \le \frac{C}{r}. \tag{7.19}$$

Our next step is to obtain weighted Sobolev estimates. From the first equation in (7.8) we have

$$\begin{gathered}\frac{\mathbb{U}''}{2}(r) + \omega\mathbb{U}(r) = \mathbb{F}(r),\\ \mathbb{F}(r) = \frac{\mathbb{V}}{r}\mathbb{U} - \frac{Z}{r}\mathbb{U}.\end{gathered} \tag{7.20}$$

Since the initial data for $\mathbb{U}$ are

$$\mathbb{U}(0) = 0, \quad \mathbb{U}'(0) = a_1, \tag{7.21}$$

we have the following integral equation satisfied by $\mathbb{U}$

$$\mathbb{U}(r) = \sinh(\sqrt{-2\omega} r) a_1 + \int_0^r \sinh(\sqrt{-2\omega}(r-\rho))\mathbb{F}(\rho) d\rho. \tag{7.22}$$

It is easy to see that the function $\mathbb{F}$ satisfies the estimate

$$\mathbb{F}(r) = O(r^{-1}), \quad r \geq 1. \tag{7.23}$$

Then the condition (7.15) and simple qualitative study of the integral equation in (7.22) guarantees that

$$a_1 + \int_0^\infty \mathrm{e}^{\sqrt{-2\omega}\rho}\mathbb{F}(\rho)d\rho = 0.$$

This fact enables one to represent $\mathbb{U}$ as follows

$$\mathbb{U}(r) = \mathrm{e}^{-\sqrt{-2\omega}r}a_1 - \tag{7.24}$$

$$- \int_r^\infty \mathrm{e}^{\sqrt{-2\omega}(r-\rho)}\mathbb{F}(\rho)d\rho - \int_0^r \mathrm{e}^{-\sqrt{-2\omega}(r-\rho)}\mathbb{F}(\rho)d\rho. \tag{7.25}$$

The first term in the right side of (7.24) is exponentially decaying. The second term we can represent as the following sum

$$\int_r^{2r} \mathrm{e}^{\sqrt{-2\omega}(r-\rho)}\mathbb{F}(\rho)d\rho + \int_{2r}^\infty \mathrm{e}^{\sqrt{-2\omega}(r-\rho)}\mathbb{F}(\rho)d\rho.$$

It is clear that

$$\int_{2r}^\infty \mathrm{e}^{\sqrt{-2\omega}(r-\rho)}\mathbb{F}(\rho)d\rho$$

is decaying exponentially, while

$$\int_r^{2r} \mathrm{e}^{\sqrt{-2\omega}(r-\rho)}\mathbb{F}(\rho)d\rho \leq \frac{C}{r}\int_r^{2r} \mathrm{e}^{\sqrt{-2\omega}(r-\rho)}d\rho = \frac{C_1}{r}$$

due to (7.23). In a similar way we can treat the last term

$$\int_0^r \mathrm{e}^{-\sqrt{-2\omega}(r-\rho)}\mathbb{F}(\rho)d\rho$$

in (7.24). This term now is a sum of type

$$\int_0^{r/2} \mathrm{e}^{-\sqrt{-2\omega}(r-\rho)}\mathbb{F}(\rho)d\rho + \int_{r/2}^r \mathrm{e}^{-\sqrt{-2\omega}(r-\rho)}\mathbb{F}(\rho)d\rho.$$

The term

$$\int_0^{r/2} \mathrm{e}^{-\sqrt{-2\omega}(r-\rho)}\mathbb{F}(\rho)d\rho$$

decays exponentially in r and the property (7.24) implies that

$$\int_{r/2}^r \mathrm{e}^{-\sqrt{-2\omega}(r-\rho)}\mathbb{F}(\rho)d\rho = O(r^{-1}).$$

The above observation and (7.24) implies

$$\mathbb{U} = O(r^{-1}),$$

$$\mathbb{F}(r) = \frac{\mathbb{V}}{r}\mathbb{U} - \frac{Z}{r}\mathbb{U} = O(r^{-2}).$$

This estimate implies a stronger version of (7.18)

$$\int_r^\infty \mathbb{U}^2(\tau)d\tau \leq \frac{C}{r}, \tag{7.26}$$

and from (7.17) we improve (7.19) as follows

$$0 \le \mathbb{V}'(r) \le \frac{C}{r^2}. \tag{7.27}$$

This argument shows that combining (7.19) and (7.20) we can obtain inductively

$$\sum_{j=0}^{k} |\mathbb{U}^{(j)}(r)|^2 \le \frac{C}{r^n}\,, \tag{7.28}$$

$$\sum_{j=1}^{k} |\mathbb{V}^{(j)}(r)|^2 \le \frac{C}{r^n} \tag{7.29}$$

for any integers $k \ge 1$ and $n \ge 2$. $\square$

The proof of Theorem 1.4 is an immediate consequence of Lemmas 7.5 and (7.2), with the change of variables (7.7).

8. Proof of Theorem 1.6

Define the functional

$$I(u,\omega) := \frac{1}{4}\int_{\mathbb{R}^3} |\nabla u|^2\,dx + \pi e^2 \int_{\mathbb{R}^3} |\nabla \Delta^{-1} u^2|^2\,dx - \frac{1}{2}\int_{\mathbb{R}^3} V(x)|u|^2\,dx - \frac{\omega}{2}\int_{\mathbb{R}^3} |u|^2\,dx, \tag{8.1}$$

for each $(u,\omega) \in H^1_r(\mathbb{R}^3) \times \mathbb{R}$. There results

$$\begin{gathered}
\frac{\partial I}{\partial u}(u,\omega) = -\frac{1}{2}\Delta u - 4\pi e^2(\Delta^{-1}u^2)u - V(x)u - \omega u,\\
\frac{\partial I}{\partial \omega}(u,\omega) = -\frac{1}{2}\int_{\mathbb{R}^3} |u|^2\,dx,\\
\frac{\partial^2 I}{\partial u^2}(u,\omega)h = -\frac{1}{2}\Delta h - 4\pi(\Delta^{-1}u^2)h - 8\pi e^2 \Delta^{-1}(hu)u - V(x)h - \omega h, h \in H^1_r(\mathbb{R}^3),\\
\frac{\partial^2 I}{\partial u \partial \omega}(u,\omega) = -u,\\
\frac{\partial^2 I}{\partial \omega^2}(u,\omega) = 0.
\end{gathered}$$

Let $\nabla I : H^1_r(\mathbb{R}^3) \times \mathbb{R} \to \big(H^1_r(\mathbb{R}^3)\big)' \times \mathbb{R}$,

$$\nabla I(u,\omega) = \begin{pmatrix} \frac{\partial I}{\partial u}(u,\omega) \\ \frac{\partial I}{\partial \omega}(u,\omega) \end{pmatrix}$$

be the Jacobian matrix of I and $HI(u,\omega) : H^1_r(\mathbb{R}^3) \times \mathbb{R} \to \big(H^1_r(\mathbb{R}^3)\big)' \times \mathbb{R}$,

$$HI(u,\omega) = \begin{pmatrix} \frac{\partial^2 I}{\partial u^2}(u,\omega) & \frac{\partial^2 I}{\partial u \partial \omega}(u,\omega) \\ \frac{\partial^2 I}{\partial u \partial \omega}(u,\omega) & \frac{\partial^2 I}{\partial \omega^2}(u,\omega) \end{pmatrix}$$

be the Hessian matrix of I in (u,ω). More precisely

$$\begin{aligned} &HI(u,\omega)(h,k)\\ &=\begin{pmatrix} \frac{\partial^2 I}{\partial u^2}(u,\omega)h+\frac{\partial^2 I}{\partial u\partial\omega}(u,\omega)k\\ \frac{\partial^2 I}{\partial u\partial\omega}(u,\omega)h+\frac{\partial^2 I}{\partial\omega^2}(u,\omega)k \end{pmatrix}\\ &=\begin{pmatrix} -\frac{1}{2}\Delta h-4\pi(\Delta^{-1}u^2)h-8\pi e^2(\Delta^{-1}(hu))u-V(x)h-\omega h-ku\\ -\int_{\mathbb{R}^3}uh\,dx \end{pmatrix}, \end{aligned} \tag{8.2}$$

for each $u,h\in H^1_r(\mathbb{R}^3)$ and $k,\omega\in\mathbb{R}$. Finally denote

$$B':=B\cap H^1_r(\mathbb{R}^3). \tag{8.3}$$

Lemma 8.1. *Let $u_0\in B'$ (see (8.3)) be a critical point of $J\big|_{B'}$ that corresponds to the minimum*

$$\omega_0=\inf_{u\in H^1\setminus\{0\},\|u\|^2_{L^2}=N}J(u), \tag{8.4}$$

namely

$$0=J\big|'_{B'}(u_0)=J\big|'_{B}(u_0)=J'(u_0)-\omega_0u_0.$$

The operator

$$h\in\Big\{h\in H^1_r(\mathbb{R}^3);\int h(x)u_0(x)\,dx=0\Big\}\longmapsto\frac{\partial^2 I}{\partial u^2}(u_0,\omega_0)h\in\big(H^1_r(\mathbb{R}^3)\big)'$$

has a trivial kernel and

$$\big\langle\frac{\partial^2 I}{\partial u^2}(u_0,\omega_0)h\big|h\big\rangle=0,\quad \int h(x)u_0(x)\,dx=0\Longrightarrow h\equiv 0. \tag{8.5}$$

Proof. Repeating the qualitative argument in the proof of Lemma 7.5, we see that any solution of

$$\frac{\partial^2 I}{\partial u^2}(u_0,\omega_0)h=0$$

decays rapidly at infinity and it is smooth as a function of $r\geq 0$. Another interpretation of the first eigenvalue $\omega_0=\omega(N)<0$ is the following one

$$\omega_0=N\inf_{u\in H^1\setminus\{0\}}\frac{J(u)}{\|u\|^2_{L^2}}. \tag{8.6}$$

Let

$$\Big\langle\frac{\partial^2 I}{\partial u^2}(u_0,\omega_0)h\big|h\Big\rangle=0$$

for some $h\in H^1$ orthogonal (in L^2) to u_0. Take $u_0+i\varepsilon h$ with $\varepsilon>0$ small enough (will be chosen later on). Then a simple calculation implies

$$\frac{J(u_0+i\varepsilon h)}{\|u_0+i\varepsilon h\|^2_{L^2}}=\frac{J(u_0)+o(\varepsilon^2)}{\|u_0\|^2-\varepsilon^2\|h\|^2}=\frac{J(u_0)}{N}+\varepsilon^2\frac{\|h\|^2}{N}J(u_0)+o(\varepsilon^2).$$

Hence, the assumption $\|h\|\neq 0$ will contradict the fact that ω_0 is defined as the minimum in (8.6). This completes the proof. $\square$

Lemma 8.2. *Let $u_0 \in B'$ (see (8.3)) be a critical point of $J\big|_{B'}$ that corresponds to the minimum as in the previous Lemma. The operator*
$$(h,k) \in H^1_r(\mathbb{R}^3) \times \mathbb{R} \mapsto HI(u_0,\omega_0)(h,k) \in \big(H^1_r(\mathbb{R}^3)\big)' \times \mathbb{R}$$
is invertible.

Proof. Let $u_0 \in B'$ be a critical point of $J\big|_{B'}$ with multiplier ω_0 as in the previous lemma, call
$$A := \frac{\partial^2 I}{\partial u^2}(u_0,\omega_0).$$
We begin proving that $HI(u_0,\omega_0)$ is injective. Let $h \in H^1_r(\mathbb{R}^3)$ and $k \in \mathbb{R}$ such that
$$HI(u_0,\omega_0)(h,k) = 0, \tag{8.7}$$
we have to prove that
$$h = k = 0. \tag{8.8}$$
By (8.2) and (8.7), we have
$$Ah - ku_0 = 0, \quad -\int_{\mathbb{R}^3} u_0 h\,dx = 0. \tag{8.9}$$
Multiplying the first of (8.9) by h and integrating on $\mathbb{R}^3$, we have
$$\int_{\mathbb{R}^3} \big(Ah\big)h\,dx = -k\int_{\mathbb{R}^3} u_0 h\,dx = 0,$$
and by (8.5) and the definition of A
$$h \equiv 0. \tag{8.10}$$
On the other hand, multiplying the first of (8.9) by u_0 and integrating on $\mathbb{R}^3$, since $u_0 \in B'$, we have
$$kN = k\int_{\mathbb{R}^3} u_0^2\,dx = \int_{\mathbb{R}^3} \big(Ah\big)u_0\,dx = 0. \tag{8.11}$$
Since (8.8) is direct consequence of (8.10) and (8.11), $HI(u_0,\omega_0)$ is injective.

We prove that $HI(u_0,\omega_0)$ is surjective. Observe that the operator A is selfadjoint, indeed
$$\begin{aligned}(Ah,f)_{L^2} = \frac{1}{2}\int_{\mathbb{R}^3}(\nabla h,\nabla f)\,dx - 4\pi e^2\int_{\mathbb{R}^3}(\Delta^{-1}u^2)hf\;dx \\ + 8\pi\int_{\mathbb{R}^3}\big(\nabla\Delta^{-1}(hu),\nabla\Delta^{-1}(fu)\big)\,dx - \int_{\mathbb{R}^3}V(x)hf\;dx - \omega\int_{\mathbb{R}^3}hf\;dx,\end{aligned}$$
for each h, f in $H^1_r(\mathbb{R}^3)$. Moreover, also the operator $HI(u_0,\omega_0)$ is selfadjont, indeed
$$\begin{aligned}\big(HI(u_0,\omega_0)(h,k),(f,\alpha)\big)_{L^2\times\mathbb{R}} &= \Big(\big(Ah-ku_0,-(u_0,h)_{L^2}\big),\big(f,\alpha\big)\Big)_{L^2\times\mathbb{R}} \\ &= \big(Ah-ku_0,f\big)_{L^2} - \alpha(u_0,h)_{L^2} \\ &= \big(Ah,f\big)_{L^2} - k\big(u_0,f\big)_{L^2} - \alpha(u_0,h)_{L^2}\end{aligned}$$
and
$$\begin{aligned}\big(HI(u_0,\omega_0)(f,\alpha),(h,k)\big)_{L^2\times\mathbb{R}} &= \Big(\big(Af-\alpha u_0,-(u_0,f)_{L^2}\big),\big(h,k\big)\Big)_{L^2\times\mathbb{R}} \\ &= \big(Af-\alpha u_0,h\big)_{L^2} - k(u_0,f)_{L^2} \\ &= \big(Af,h\big)_{L^2} - \alpha(u_0,h)_{L^2} - k\big(u_0,f\big)_{L^2},\end{aligned}$$

since A is selfadjoint

$$\big(HI(u_0,\omega_0)(h,k),(f,\alpha)\big)_{L^2\times\mathbb{R}} = \big(HI(u_0,\omega_0)(f,\alpha),(h,k)\big)_{L^2\times\mathbb{R}}$$

for each h,f in $H^1_r(\mathbb{R}^3)$ and k,α in $\mathbb{R}$. Since $HI(u_0,\omega_0)$ is injective and selfadjoint, there results

$$\begin{aligned}\operatorname{Im}\big(HI(u_0,\omega_0)\big) &= \Big(\ker\big(HI(u_0,\omega_0)^*\big)\Big)^\perp\\ &= \Big(\ker\big(HI(u_0,\omega_0)\big)\Big)^\perp\\ &= H^1_r(\mathbb{R}^3)\times\mathbb{R},\end{aligned} \tag{8.12}$$

then $HI(u_0,\omega_0)$ is surjective. The claim is direct consequence of the Closed Graph Theorem. □

Lemma 8.3. *The critical points of the functional $J\big|_{B'}$ that correspond to the minimum are isolated, i.e. for each $u\in B'$ critical point of $J\big|_{B'}$, with the Lagrange multiplier satisfying (8.4), there exists a neighborhood $U\subset H^1(\mathbb{R}^3)$ of u such that any element of $B'\cap U$ is not a critical point of it.*

Proof. Let $u_0\in B'$ be a critical point of $J\big|_{B'}$ corresponding to the minimum as in the previous lemmas, then

$$0 = J\big|_{B'}'(u_0) = J\big|_{B}'(u_0) = J'(u_0)-\omega_0u_0 = \frac{\partial I}{\partial u}(u_0,\omega_0)$$

and since $u_0\in B'$,

$$\frac{\partial I}{\partial\omega}(u_0,\omega_0) = -\frac12\int_{\mathbb{R}^3}u_0^2\,dx = -\frac N2,$$

we have

$$\nabla I(u_0,\omega_0) = \begin{pmatrix}0\\ -N/2\end{pmatrix}.$$

By Lemma 8.2 and the Implicit Function Theorem there exist $U\subset H^1_r(\mathbb{R}^3)$ neighborhood of u_0, $\Omega\subset\mathbb{R}$ neighborhood of ω_0, $W\subset\big(H^1_r(\mathbb{R}^3)\big)'\times\mathbb{R}$ neighborhood of $\big(0,-\frac N2\big)$ and $G:W\to U\times\Omega$ such that

$$\begin{gathered}G\big(\nabla I(u,\omega)\big) = (u,\omega),\quad (u,\omega)\in U\times\Omega,\\ \nabla I\big(G(f,\alpha)\big) = (f,\alpha),\quad (f,\alpha)\in W.\end{gathered} \tag{8.13}$$

Assume, by absurd, that u_0 is not isolate, namely there exists a sequence $\{u_k\}\subset B'$ of critical points of $J\big|_{B'}$, such that

$$u_k\neq u_0,\quad u_k\to u_0\quad \text{in} H^1(\mathbb{R}^3). \tag{8.14}$$

Moreover, there exists a sequence $\{\omega_k\}\subset\mathbb{R}$ such that

$$0 = J\big|_{B'}'(u_k) = J'(u_k)-\omega_ku_k = \frac{\partial I}{\partial u}(u_k,\omega_k).$$

Since $u_k\in B'$ and by (8.14), we have

$$\omega_k = \big\langle J'(u_k)\big|u_k\big\rangle \to \big\langle J'(u_0)\big|u_0\big\rangle = \omega_0. \tag{8.15}$$

By (8.14) and (8.15), there exists $k_0\in\mathbb{N}$ such that $(u_k,\omega_k)\in U\times\Omega$ for $k\geq k_0$. Finally, fixed $k\geq k_0$, since

$$\nabla I(u_k,\omega_k) = \begin{pmatrix}0\\ -N/2\end{pmatrix},$$

by (8.13), we have

$$(u_k,\omega_k) = G\big(\nabla I(u_k,\omega_k)\big) = G\begin{pmatrix} 0 \\ -N/2 \end{pmatrix} = G\big(\nabla I(u_0,\omega_0)\big) = (u_0,\omega_0).$$

Since this contradicts (8.14), the claim is done. $\square$

Lemma 8.4. *The first eigenvalue of the operator $J\big|'_{B'}$ (see (8.4)) is isolated, i.e. there exists a neighborhood $\Omega \subset \mathbb{R}$ of ω_0 such that any element of Ω is not an eigenvalue of the previous operator.*

Proof. Assume, by absurd, that the first eigenvalue ω_0 is not isolated, namely there exists a sequence $\{\omega_k\} \subset \mathbb{R}$ of eigenvalues such that

$$\omega_k \to \omega_0. \tag{8.16}$$

By definition, there exists $\{u_k\} \subset B'$ such that

$$0 = J\big|'_{B'}(u_k) = J'(u_k) - \omega_k u_k, \quad k \in \mathbb{N}. \tag{8.17}$$

Observe that, by Lemma 5.5, $\omega_k, \omega_0 < 0$, then there exists $\varepsilon > 0$ such that

$$\omega_k, \omega_0 \le -\varepsilon, \quad k \in \mathbb{N}. \tag{8.18}$$

Moreover, by Lemma 4.3 and since $\{u_k\} \subset B'$

$$-\infty < \min_{u \in H^1_r(\mathbb{R}^3)} J(u) \le J(u_k) \le \sup_k \frac{\omega_k}{2} \le -\frac{\varepsilon}{2}, \tag{8.19}$$

then $\{J(u_k)\}$ is bounded and, by (8.17),

$$J\big|'_{B'}(u_k) \to 0. \tag{8.20}$$

By the Palais-Smale Condition (see Lemma 6.1) there exists $u_0 \in B'$ such that, passing to a subsequence,

$$u_k \to u_0, \quad \text{in} H^1(\mathbb{R}^3).$$

By (8.16) and (8.17),

$$0 = J\big|'_{B'}(u_0) = J'(u_0) - \omega_0 u_0,$$

namely u_0 is a not isolated critical point of the functional $J\big|_{B'}$. Since this contradicts Lemma 8.3, the proof is done. $\square$

Proof of Theorem 1.6. Since $F(u, 4\pi\Delta^{-1}u^2) = J(u)$ for all $u \in H^1(\mathbb{R}^3)$, by Lemmas 8.3 and 8.4 the claim is complete. $\square$

9. Appendix

Here we shall prove for completeness the relation (5.11). First, for the partial case of space dimensions $n = 3$ we need the following relation (a generalization of this relation for space dimensions $n \ge 3$ can be found in [1]).

Lemma 9.1 (see [1])**.** *If $f(x) = f(|x|)$ is an $L^\infty(\mathbb{R}^3)$ function, then for any $r > 0$ and $x \neq 0$ we have the relation*

$$\int_{\mathbb{S}^2} f(|x + r\omega|)d\omega = \frac{2\pi}{|x|r} \int_{||x|-r|}^{|x|+r} f(\lambda)\lambda d\lambda. \tag{9.1}$$

Proof. It is sufficient to consider only the case $x = (0, 0, |x|)$ and to pass to polar coordinates

$$\omega_1 = \sin\theta\cos\varphi\,, \quad \omega_2 = \sin\theta\sin\varphi\,, \quad \omega_3 = \cos\theta\,.$$

Then $d\omega = \sin\theta\, d\theta\, d\varphi$ and

$$\int_{\mathbb{S}^2} f(|x + r\omega|) d\omega = 2\pi \int_0^\pi f\big(\sqrt{|x|^2 + r^2 + 2|x| r \cos\theta}\big) \sin\theta\, d\theta.$$

Making the change of variable

$$\theta \to \lambda = \sqrt{|x|^2 + r^2 + 2|x| r \cos\theta},$$

the proof is complete. □

Now we are ready to verify (5.11).

Lemma 9.2. *If $v(x) = v(|x|)$ is a radial $C_0^\infty(\mathbb{R}^3)$ function, then the solution of the equation $\Delta u = v$ can be represented as follows*

$$4\pi u(x) = -\int_{\mathbb{R}^3} v(|y|) \frac{dy}{\max(|x|, |y|)}, \quad x \in \mathbb{R}^3. \tag{9.2}$$

Proof. Starting with the classical representation

$$4\pi u(x) = \int_{\mathbb{R}^3} |x - y|^{-1} v(|y|)\, dy,$$

we introduce polar coordinates $r = |y|$, $\omega = y/|y|$ apply Lemma 9.1 and find

$$u(x) = -\frac{1}{2|x|} \int_0^\infty \Big(\int_{||x|-r|}^{|x|+r} d\lambda \Big) v(r) r\, dr\,.$$

Note that the right side of (9.2) becomes

$$-4\pi \int_0^\infty v(r) \frac{r^2 dr}{\max(|x|, r)}.$$

Using the fact that

$$\frac{1}{|x| r} \int_{||x|-r|}^{|x|+r} d\lambda = \frac{2}{\max(|x|, r)},$$

we obtain (9.2) and this completes the proof. □

Using the relation

$$4\pi u(x) = -\int_0^r v(\rho) \frac{\rho^2 d\rho}{r} - \int_r^\infty v(\rho) \rho d\rho, \quad r = |x|$$

and differentiating with respect to $r = |x|$, we arrive at the following lemma.

Lemma 9.3. *If $v(x) = v(|x|)$ is a radial $C_0^\infty(\mathbb{R}^3)$ function, then the solution of the equation $\Delta u = v$ satisfies the relation*

$$4\pi \frac{\partial \Delta^{-1} v}{\partial r}(x) = \int_{|y|<r} \frac{v(y)}{|x|^2}\, dy, \tag{9.3}$$

for each $x \in \mathbb{R}^3, x \neq 0$.

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