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EXPONENTIAL STABILITY OF TRAVELING WAVES FOR NON-MONOTONE DELAYED REACTION-DIFFUSION EQUATIONS

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Abstract. This article concerns the exponential stability of non-critical traveling waves (the wave speed is greater than the minimum speed) for non-monotone time-delayed reaction-diffusion equations. With the help of the weighted energy method, we prove that the non-critical travelling waves are exponentially stable when the initial perturbation around the wave is small.

1. Introduction

In this article, we study the stability of traveling waves for the non-monotone delayed reaction diffusion equation

$$\frac{\partial v(t,x)}{\partial t} = D\frac{\partial^2 v(t,x)}{\partial x^2} - d(v(t,x)) + f(v(t-r,x)), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R} \tag{1.1}$$

with the initial condition

$$v(s,x) = v_0(s,x), \quad s \in [-r,0],\ x \in \mathbb{R}. \tag{1.2}$$

where $D > 0$, $r \geq 0$ are constants. The nonlinear functions $d(u)$ and $f(u)$ satisfy the following hypotheses:

- (H1) $d \in C^2([0,\infty],\mathbb{R})$, $f \in C^2([0,\infty],\mathbb{R})$; there exist only two constant equilibria 0 and $K > 0$ such that $f(0) = d(0)$, $f(K) = d(K)$, $d'(0) - f'(0) < 0$ and $d'(K) - f'(K) > 0$.
- (H2) There exists $K^* \geq K$ such that $d(K^*) \geq \max\{f(v) | 0 \leq v \leq K^*\}$ and $d(v) < d(K^*)$ for all $v \in [0,K^*)$, $f'(0)v \geq f(v) > 0$, $d(v) \geq d'(0)v$ and $f'(0)v > d(v)$ for all $v \in (0,K^*]$.
- (H3) $d(v)$ is strictly increasing on $[0,K^*]$ and $d(v) < f(v) < 2d(K) - d(v)$ for $v \in [0,K)$, $d(v) > f(v) > 2d(K) - d(v)$ for $v \in (K,K^*]$.
- (H4) $d'(v) \geq d'(0)$ and $|f'(v)| \leq f'(0)$ for all $v \in (0,K^*]$.

Equation (1.1) includes several practical models. Letting $d(v(t,x)) = \delta v(t,x)$, (1.1) reduces to the time-delayed reaction-diffusion equation

$$\frac{\partial v(t,x)}{\partial t} = D\frac{\partial^2 v(t,x)}{\partial x^2} - \delta v(t,x) + f(v(t-r,x)). \tag{1.3}$$

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This model represents the single species population distribution such as the Australian blowfly [11, 12, 27]. Here $v(t,x)$ denotes the mature population of the blowflies at location x and time t, $D > 0$ and $\delta > 0$ are the diffusion coefficient and death rate of the mature population, the time delay $r > 0$ is the time taken from birth to maturity, and $f(v(t-r,x))$ is the birth function. Especially, taking $f(v) = pve^{-av}$, $p > 0$, $a > 0$, (1.3) is a typical Nicholson's blowflies model; i.e.,

$$\frac{\partial v(t,x)}{\partial t} = D\frac{\partial^2 v(t,x)}{\partial x^2} - \delta v(t,x) + pv(t-r)e^{-av(t-r)}. \tag{1.4}$$

When the birth function f is monotone, authors in [9, 13, 22, 23, 28, 29] investigated the existence of monotone traveling waves by using the monotone iteration and fixed-points theorem with help of the upper-lower solutions. Schaaf [26] first studied linear stability for the delayed reaction diffusion with the quasi-monotone nonlinear terms, which includes (1.1), by using a spectral method. The authors in [21] investigated the nonlinear stability of traveling waves by using the (technical) weighted energy method. Then authors in [25] further employed its global stability by using the weighted energy technique and the comparison principle. These results were then extended to more general delayed reaction diffusion equations with the quasi-monotone nonlinearity in [15, 16, 30]. By using the Fourier transform, Green's function and the weighted energy method, the authors in [24, 25] showed the global stability of critical traveling waves, which depends on the monotonicity of both the equation and traveling waves.

However, because of the lack of monotonicity, the simple but useful methods have failed. For this case [6, 7, 8, 17, 30, 10] show the existence of traveling waves by developing different methods. Especially, the study on the stability of traveling waves is quite limited. Wu, Zhao and Liu [34] first showed the stability of traveling waves with the large wave speed for (1.3) by using weighted energy method. Recently, Lin et al [14] established the stability of traveling waves (including oscillating traveling waves) for (1.4) by using the weighted energy method and the nonlinear Halanay inequality. Then Chern et al [3] followed the recent study [14] and further answered all critical traveling waves for (1.4) are time-asymptotically stable with the help of some new development.

To the best of our knowledge, the stability of traveling waves for the more general non-monotone delayed reaction diffusion equation (1.1) is still not investigated. The methods in [34, 3] can still be used owing to the boundedness of the solution with the initial condition for the non-monotone delayed reaction diffusion equation (1.1), which was proved in [32].

2. Preliminaries and main result

We first introduce some notation. Throughout this paper, $C > 0$ denotes a generic constant, while $C_i > 0$ $(i = 0, 1, 2\dots)$ represents a special constant. Letting I be an interval, especially $I = \mathbb{R}$, $L^2(I)$ is the space of the square integrable function on I, and $H^k(I)(k \geq 0)$ is the Sobolev space of the L^2-function $f(x)$ defined on I whose derivatives $\frac{d^i}{dx^i}f$, $i = 1, \dots, k$, also belong to $L^2(I)$. $L^2_\omega(I)$ represents the weighted L^2-space with the weight $\omega(x) > 0$ and its norm is defined by

$$\|f\|_{L^2_\omega} = \Big(\int_I \omega(x)f^2(x)dx\Big)^{1/2}.$$

$H^k_\omega(I)$ is the weighted Sobolev space with the norm given by

$$\|f\|_{H^k_\omega} = \Big(\sum_{i=0}^{k}\int_I \omega(x)|\frac{d^i}{dx^i}f(x)|^2dx\Big)^{1/2}.$$

Letting $T > 0$ and B space, we denote by $C^0([0,T];B)$ the space of the B-valued continuous functions on $[0,T]$, and $L^2([0,T];B)$ as the space of B-valued L^2-function on $[0,T]$. The corresponding spaces of the B-valued function on $[0,\infty)$ are defined similarly.

The traveling waves for (1.1) connecting 0 and K are the special solution to (1.1) in the form of $v(t,x) = \phi(x+ct)$, namely, ϕ satisfies

$$c\phi'(\xi) - D\phi''(\xi) + d(\phi(\xi)) - f(\phi(\xi - cr)) = 0, \tag{2.1}$$

$$\phi(-\infty) = 0, \quad \phi(+\infty) = K. \tag{2.2}$$

If $f'(0) > d'(0)$, there exists a unique number $c_* > 0$ such that for $c > c_*$, the characteristic equation $\Delta(c,\lambda) = 0$ of linearized equation at 0 for (2.1) has two positive roots $\lambda_1 = \lambda_1(c) > 0$ and $\lambda_2 = \lambda_2(c) > 0$, where

$$\Delta(c,\lambda) := c\lambda - D\lambda^2 + d'(0) - e^{-\lambda cr}f'(0). \tag{2.3}$$

Moreover,

$$c\lambda - D\lambda^2 + d'(0) > e^{-\lambda cr}f'(0), \quad \text{for } \lambda_1 < \lambda < \lambda_2. \tag{2.4}$$

Let us recall the existence and uniqueness of traveling waves for (1.1) with the non-monotone nonlinearity, (see [17]) and some related results can also be found in [5, 33] and the boundedness of the solution with the initial condition for the non-monotone delayed reaction diffusion equation (1.1), see [32].

Proposition 2.1. *Assume that* (H1)–(H3) *hold, there exists a unique number $c_* > 0$ such for every $c > c_*$, Equation* (1.1) *has a unique (up to translation) traveling wave solution $\phi(\xi)$ satisfying $\phi(-\infty) = 0$, $\phi(+\infty) = K$ and $0 \le \phi(\xi) \le K^*$ for all $\xi \in \mathbb{R}$.*

Proposition 2.2. *Assume that* (H1)–(H4) *hold and $0 \le v_0(s,x) \le K^*$ for all $(s,x) \in [-r,0]\times\mathbb{R}$. Then the solution of Cauchy problem* (1.1) *and* (1.2) *satisfies*

$$0 \le v(t,x) \le K^* \quad \text{for all} \quad (t,x) \in [0,+\infty)\times\mathbb{R}.$$

For $\lambda_1 < \lambda < \lambda_2$ and some number ξ_*, define the weight function

$$\omega(\xi) = \begin{cases} e^{-2\lambda(\xi-\xi_*)}, & \text{for } \xi < \xi_*,\\ 1, & \text{for } \xi \ge \xi_*. \end{cases} \tag{2.5}$$

For a given weight function $\omega(\xi)$ and letting $T \ge 0$, we define the solution spaces as

$$X(-r,T) = \{u | u(t,\xi) \in C([-r,T]; C(\mathbb{R}) \cap H^1_\omega(\mathbb{R}))\}$$

and

$$M(T)^2 = \sup_{t\in[-r,T]}\Big(\|u(t)\|^2_C + \|u(t)\|^2_{H^1_\omega}\Big).$$

In particular, when $T = \infty$, we can also define the solution space as $X(-r,\infty)$ and the norm of the solution space as $M(\infty)$. Now, we state the stability result for (1.1).

Theorem 2.3 (Stability)**.** *Assume that* (H1)–(H4) *hold and* $|f'(K)|$ *is sufficiently small. For any given traveling wave* $\phi(\xi)$ *of* (1.4) *with speed* $c > c_*$*, if the initial perturbation is small; i.e.,*

$$\max_{s\in[-r,0]} \|(v_0-\phi)(s)\|_C^2 + \|(v_0-\phi)(0)\|_{H_\omega^1}^2 + \int_{-r}^0 \|(v_0-\phi)(s)\|_{H_\omega^1}^2 ds \le \delta_0^2,$$

then the unique solution $v(t,x)$ *of* (1.1) *and* (1.2) *exists globally and satisfies*

$$v(t,x)-\phi(x+ct) \in C([-r,\infty); C(\mathbb{R})\cap H_\omega^1(\mathbb{R})), \tag{2.6}$$

$$\sup_{x\in\mathbb{R}} |v(t,x)-\phi(x+ct)| \le Ce^{-\mu t}, \quad t>0 \tag{2.7}$$

for the constant $C>0$ *and* $\mu>0$.

3. Proof of stability

To obtain the stability of non-monotone delayed reaction diffusion equations, we need to give some results.

Lemma 3.1. *Assume that* (H1)–(H4) *and* ϕ *is a traveling wave for* (1.4)*. Then there exists* $A>0$ *such that*

$$|\phi'(\xi)| \le A \quad \text{and} \quad \lim_{\xi\to\pm\infty} \phi'(\xi) = 0$$

Proof. Letting

$$\rho_1 = \frac{c-\sqrt{c^2+4Dd'(0)}}{2D} \quad \text{and} \quad \rho_2 = \frac{c+\sqrt{c^2+4Dd'(0)}}{2D},$$

it follows from (1.4) that

$$\phi(\xi) = \frac{1}{D(\rho_2-\rho_1)}\Big[\int_{-\infty}^{\xi} e^{\rho_1(\xi-s)}H(\phi)(s)ds + \int_{\xi}^{+\infty} e^{\rho_2(\xi-s)}H(\phi)(s)ds\Big],$$

where $H(\phi)(s) = f(\phi(s-cr)) + d'(0)\phi(\xi) - d(\phi(\xi))$. Differentiating the above equation with respect to ξ, we obtain

$$\phi'(\xi) = \frac{1}{D(\rho_2-\rho_1)}\Big[\int_{-\infty}^{\xi} \rho_1 e^{\rho_1(\xi-s)}H(\phi)(s)ds + \int_{\xi}^{+\infty} \rho_2 e^{\rho_2(\xi-s)}H(\phi)(s)ds\Big]. \tag{3.1}$$

Since $\rho_2-\rho_1 \ge 2\sqrt{\frac{d'(0)}{D}}$, we obtain

$$|\phi'(\xi)| \le \frac{1}{\sqrt{Dd'(0)}} \max_{s\in\mathbb{R}} |H(\phi)(s)| := A \quad \text{for all } \xi\in\mathbb{R}.$$

Finally, (3.1) and the L.Hopital's rule imply that $\lim_{\xi\to+\infty}\phi'(\xi)=0$. the proof is complete. □

Lemma 3.2. *Assume that* $f'(0) > d'(0)$ *and* $|f'(K)|$ *is sufficiently small. Then, for every* $c>c_*$*, there exist* $\xi_0, \xi_* \in \mathbb{R}$ *with* $\xi_* > \xi_0$ *such that*

$$\max\{|f'(\phi(\xi_0-cr))|, |f'(\phi(\xi_0))|\} \le \frac{\min\{\Delta(c,\lambda)+e^{-\lambda cr}f'(0), d'(0)\}}{\cosh(\lambda cr)},$$

and for $\xi \ge \xi_* - cr > \xi_0$,

$$\max\{|f'(\phi(\xi-cr))|, |f'(\phi(\xi))|\} \le \max\{|f'(\phi(\xi_0-cr))|, |f'(\phi(\xi_0))|\}.$$

Since $\lim_{\xi\to\pm\infty}\phi(\xi)=K$ and $f'(K)$ is sufficiently small, the conclusion of the above lemma obviously holds.

Letting $u(t,\xi):=v(t,x)-\phi(\xi), \xi=x+ct$, where $\phi(x+ct)$ is a given traveling wave solution of (1.1), the Cauchy problem (1.1) and (1.2) can be reformulated as

$$\begin{gathered} u_t(t,\xi)+cu_\xi(t,\xi)-Du_{\xi\xi}(t,\xi)=g(u(t-r,\xi-cr))-p(u(t,\xi)),\\ (t,\xi)\in(0,+\infty)\times\mathbb{R},\\ u(s,\xi)=v_0(s,\xi-cs)-\phi(\xi)=:u_0(s,\xi),\quad (s,\xi)\in[-r,0]\times\mathbb{R}, \end{gathered}\tag{3.2}$$

where

$$g(u)=f(\phi+u)-f(\phi),\quad p(u)=d(\phi+u)-d(\phi).$$

By the iteration technique and the energy method (see [14, 18, 19]), we can obtain the existence of local solutions for (3.2).

Theorem 3.3. *Assume that* (H1)–(H4) *hold. For any given traveling wave $\phi(\xi)$ with $c>c_*$, suppose $u_0(s,\xi)\in X(-r,0)$, and $M(0)\le\delta_1$, where δ_1 is a given positive constant. Then there exists a small $t_0=t_0(\delta_1)>0$ such that the local solution $u(t,\xi)$ of* (3.2) *uniquely exists for $t\in[-r,t_0]$ and satisfies $u\in X(-r,t_0)$ and $M(t_0)\le aM(0)$ for some constant a.*

Proof. Let $u^{(0)}(t,\xi):=u_0(t,\xi)\in X(-r,0)\subseteq X(-r,t_0)$. Then define the iteration $u^{(n+1)}=\mathcal{T}(u^{(n)})$ for $n\ge 0$ by

$$\begin{gathered} \frac{\partial u^{(n+1)}}{\partial t}+c\frac{\partial u^{(n+1)}}{\partial\xi}-D\frac{\partial^2u^{(n+1)}}{\partial\xi^2}=g(u^{(n)}(t-r,\xi-cr))-p(u^{(n)}(t,\xi)),\\ u^{(n+1)}(s,\xi)=u_0(s,\xi),\quad s\in[-r,0],\ \xi\in\mathbb{R}. \end{gathered}\tag{3.3}$$

Using Fourier transform, (3.3) can be written as

$$\begin{aligned} u^{(n+1)}(t,\xi)&=\int_{\mathbb{R}}\Gamma(\eta,t)u_0(0,\xi-\eta)d\eta+\int_0^t\int_{\mathbb{R}}\Gamma(\eta,t-s)\\ &\quad\times\big[g(u^{(n)}(s-r,\xi-\eta+cr))-p(u^{(n)}(s,\xi-\eta))\big]\,d\eta\,d\xi, \end{aligned}\tag{3.4}$$

where $\Gamma(\eta,t)$ is the heat kernel

$$\Gamma(\eta,t)=\frac{1}{\sqrt{4\pi Dt}}e^{-\frac{(\eta+ct)^2}{4Dt}}.$$

By applying regular energy estimates to both sides of (3.3), indicated as

$$\int_0^t\int_{\mathbb{R}}\Big(\sum_{k=0}^{1}\partial_\xi^k((3.3))\times\omega(\xi)\partial_\xi^k u^{(n+1)}\Big)\,d\xi\,ds,$$

we can estimate

$$\begin{aligned} &\|u^{(n+1)}(t)\|_{H^1_\omega}^2+\int_0^t\|u^{(n+1)}(s)\|_{H^1_\omega}^2ds\\ &\le C\Big(\|u_0(0)\|_{H^1_\omega}^2+\int_{-r}^0\|u_0(s)\|_{H^1_\omega}^2ds+\int_0^t\|u^{(n)}(s)\|_{H^1_\omega}^2ds\Big),\quad t\in[0,t_0] \end{aligned}\tag{3.5}$$

for some positive constant $C>0$. From (3.4) it follows that

$$\|u^{(n+1)}(t)\|_C\le C\|u_0(0)\|_C+Ct_0\sup_{t\in[-r,t_0]}\|u^{(n)}(t)\|_C,\quad t\in[0,t_0].\tag{3.6}$$

According to (3.5) and (3.6), it holds that

$$M_{u^{(n+1)}}(t_0) \le C\Big(\max_{s\in[-r,0]}\|u_0(s)\|_C^2+\|u_0(0)\|_{H^1_\omega}^2+\int_{-r}^0\|u_0(s)\|_{H^1_\omega}^2\,ds\Big)+Ct_0M_{u^{(n)}}(t_0).$$

Thus, when $\max_{s\in[-r,0]}\|u_0(s)\|_C^2+\|u_0(0)\|_{H^1_\omega}^2+\int_{-r}^0\|u_0(s)\|_{H^1_\omega}^2\,ds \ll 1$ with $0 < t_0 \ll 1$, $u^{(n+1)} = \mathcal{T}(u^{(n)})$ defined in (3.3) is a contraction mapping from $X(-r,t_0)$ to $X(-r,t_0)$. Hence, by using Banach fixed point theorem, (3.2) admits a unique local solution in $X(-r,t_0)$. This completes the proof. □

A priori estimate. We rewrite (3.2) as

$$\begin{gathered} u_t(t,\xi)+cu_\xi(t,\xi)-Du_{\xi\xi}(t,\xi)+d'(\phi(\xi))u(t,\xi)-f'(\phi(\xi-cr))u(t-r,\xi-cr)\\ =G(u)(t,\xi)-E(u)(t,\xi),\quad (t,\xi)\in(0,+\infty)\times\mathbb{R},\\ u(s,\xi)=u_0(s,\xi),\quad (s,\xi)\in[-r,0]\times\mathbb{R}, \end{gathered} \tag{3.7}$$

where

$$\begin{aligned} G(u)(t,\xi)&=f(u(t-r,\xi-cr)+\phi(\xi-cr))-f(\phi(\xi-cr))\\ &\quad -f'(\phi(\xi-cr))u(t-r,\xi-cr), \end{aligned} \tag{3.8}$$

$$E(u)(t,\xi)=d(u(t,\xi)+\phi(\xi))-d(\phi(\xi))-d'(\phi(\xi))u(t,\xi). \tag{3.9}$$

Lemma 3.4. *Let $u(t,\xi)\in X(-r,T)$. Then*

$$\begin{aligned} &\|u(t)\|_{L^2_\omega}^2+\int_0^t e^{-2\mu(t-s)}\int_{\mathbb{R}}[B_{\eta,\mu,\omega}(\xi)-CM(t)]\omega(\xi)u^2(s,\xi)\,d\xi\,ds\\ &\le Ce^{-2\mu t}\Big(\|u_0(0)\|_{L^2_\omega}^2+\int_{-r}^0\|u_0(s)\|_{L^2_\omega}^2\,ds\Big), \end{aligned} \tag{3.10}$$

where

$$B_{\eta,\mu,\omega}(\xi):=A_{\eta,\omega}(\xi)-2\mu-\frac{1}{\eta}(e^{2\mu r}-1)|f'(\phi(\xi))|\frac{\omega(\xi+cr)}{\omega(\xi)}, \tag{3.11}$$

$$\begin{aligned} A_{\eta,\omega}(\xi):=&-c\frac{\omega'(\xi)}{\omega(\xi)}+2d'(\phi(\xi))-\frac{D}{2}\Big(\frac{\omega'(\xi)}{\omega(\xi)}\Big)^2-\eta|f'(\phi(\xi-cr))|\\ &-\frac{1}{\eta}|f'(\phi(\xi))|\frac{\omega(\xi+cr)}{\omega(\xi)}, \end{aligned} \tag{3.12}$$

and μ,η are positive constants.

Proof. Multiplying (3.7) by $e^{2\mu t}\omega(\xi)u(t,\xi)$ with $\xi\in\mathbb{R}$ and $0\le t\le T$, we have

$$\begin{aligned} &\Big\{\frac12 e^{2\mu t}\omega u^2\Big\}_t+e^{2\mu t}\Big\{\frac12 c\omega u^2-D\omega uu_\xi\Big\}_\xi+De^{2\mu t}\omega u_\xi^2+De^{2\mu t}\omega' u_\xi u\\ &+\Big\{-\frac{c}{2}\frac{\omega'}{\omega}+d'(\phi(\xi))-\mu\Big\}e^{2\mu t}\omega u^2-e^{2\mu t}\omega(\xi)u(t,\xi)f'(\phi(\xi-cr))u(t-r,\xi-cr)\\ &=e^{2\mu t}\omega(\xi)u(t,\xi)[G(u)-E(u)]. \end{aligned} \tag{3.13}$$

By the Cauchy-Schwarz inequality,

$$|De^{2\mu t}\omega' u_\xi u|\le De^{2\mu t}\omega u_\xi^2+\frac{D}{4}e^{2\mu t}\Big(\frac{\omega'}{\omega}\Big)^2\omega u^2,$$

we reduce (3.13) to

$$
\begin{aligned}
&\Big\{\frac{1}{2}e^{2\mu t}\omega u^2\Big\}_t + e^{2\mu t}\Big\{\frac{1}{2}c\omega u^2 - D\omega u u_\xi\Big\}_\xi \\
&+ \Big\{-\frac{c}{2}\frac{\omega'}{\omega} + d'(\phi(\xi)) - \mu - \frac{D}{4}\Big(\frac{\omega'}{\omega}\Big)^2\Big\}e^{2\mu t}\omega u^2 \\
&- e^{2\mu t}\omega(\xi)u(t,\xi)f'(\phi(\xi - cr))u(t-r,\xi-cr) \\
&\le e^{2\mu t}\omega(\xi)u(t,\xi)[G(u) - E(u)].
\end{aligned} \tag{3.14}
$$

Integrating (3.14) over $\mathbb{R}\times[0,t]$ with respect to ξ and t, we have

$$
\begin{aligned}
&e^{2\mu t}\|u(t)\|^2_{L^2_\omega} + \int_0^t\int_{\mathbb{R}} e^{2\mu s}\Big\{-c\frac{\omega'(\xi)}{\omega(\xi)} + 2d'(\phi(\xi)) - 2\mu - \frac{D}{2}\Big(\frac{\omega'(\xi)}{\omega(\xi)}\Big)^2\Big\} \\
&\times \omega(\xi)u^2(s,\xi)\,d\xi\,ds \\
&- 2\int_0^t\int_{\mathbb{R}} e^{2\mu s}\omega(\xi)f'(\phi(\xi-cr))u(s,\xi)u(s-r,\xi-cr)\,d\xi\,ds \\
&\le \|u_0(0)\|^2_{L^2_\omega} + 2\int_0^t\int_{\mathbb{R}} e^{2\mu s}\omega(\xi)u(s,\xi)[G(u)(s,\xi) - E(u)(s,\xi)]\,d\xi\,ds.
\end{aligned} \tag{3.15}
$$

Since

$$
\begin{aligned}
&2\Big|\int_0^t\int_{\mathbb{R}} e^{2\mu s}\omega(\xi)f'(\phi(\xi-cr))u(s,\xi)u(s-r,\xi-cr)\,d\xi\,ds\Big| \\
&\le \eta\int_0^t\int_{\mathbb{R}} e^{2\mu s}\omega(\xi)|f'(\phi(\xi-cr))|u^2(s,\xi)\,d\xi\,ds \\
&\quad + \frac{1}{\eta}\int_0^t\int_{\mathbb{R}} e^{2\mu s}\omega(\xi)|f'(\phi(\xi-cr))|u^2(s-r,\xi-cr)\,d\xi\,ds \\
&= \eta\int_0^t\int_{\mathbb{R}} e^{2\mu s}\omega(\xi)|f'(\phi(\xi-cr))|u^2(s,\xi)\,d\xi\,ds \\
&\quad + \frac{1}{\eta}e^{2\mu r}\int_{-r}^{t-r}\int_{\mathbb{R}} e^{2\mu s}\omega(\xi+cr)|f'(\phi(\xi))|u^2(s,\xi)\,d\xi\,ds \\
&\le \eta\int_0^t\int_{\mathbb{R}} e^{2\mu s}\omega(\xi)|f'(\phi(\xi-cr))|u^2(s,\xi)\,d\xi\,ds \\
&\quad + \frac{1}{\eta}e^{2\mu r}\int_0^t\int_{\mathbb{R}} e^{2\mu s}\omega(\xi+cr)|f'(\phi(\xi))|u^2(s,\xi)\,d\xi\,ds \\
&\quad + \frac{1}{\eta}e^{2\mu r}\int_{-r}^0\int_{\mathbb{R}} e^{2\mu s}\omega(\xi+cr)|f'(\phi(\xi))|u_0^2(s,\xi)\,d\xi\,ds,
\end{aligned} \tag{3.16}
$$

Substituting (3.16) in (3.15), we have

$$
\begin{aligned}
&e^{2\mu t}\|u(t)\|^2_{L^2_\omega} + \int_0^t\int_{\mathbb{R}} e^{2\mu s}B_{\eta,\mu,\omega}(\xi)\omega(\xi)u^2(s,\xi)\,d\xi\,ds \\
&\le \|u_0(0)\|^2_{L^2_\omega} + \frac{e^{2\mu r}}{\eta}\int_{-r}^0\int_{\mathbb{R}} e^{2\mu s}\omega(\xi+cr)|f'(\phi(\xi)|u_0^2(s,\xi)\,d\xi\,ds \\
&\quad + 2\int_0^t\int_{\mathbb{R}} e^{2\mu s}\omega(\xi)u(s,\xi)[G(u)(s,\xi) - E(u)(s,\xi)]\,d\xi\,ds,
\end{aligned} \tag{3.17}
$$

where $B_{\eta,\mu,\omega}(\xi)$ is given by (3.11).

By standard Sobolev's embedding inequality $H^1(\mathbb{R}) \hookrightarrow C^0(\mathbb{R})$ and the embedding inequality $H^1_\omega(\mathbb{R}) \hookrightarrow H^1(\mathbb{R})$ (since $\omega(\xi) \ge 1$, for all $\xi \in \mathbb{R}$), we have, for all $\xi \in \mathbb{R}$ and $-r \le t \le T$,

$$|u(t,\xi)| \le \sup_{\xi\in\mathbb{R}} |u(t,\xi)| \le \sigma_0 \|u(t,\cdot)\|_{H^1} \le \sigma_0 \|u(t,\cdot)\|_{H^1_\omega} \le \sigma_0 M(t), \tag{3.18}$$

where $\sigma_0 > 0$ is the embedding constant. Since

$$\begin{aligned}|G(u)(t,\xi)| &= |f(u(t-r,\xi-cr)+\phi(\xi-cr)) - f(\phi(\xi-cr)) \\ &\quad - f'(\phi(\xi-cr))u(t-r,\xi-cr)| \le C|u(t-r,\xi-cr)|^2,\end{aligned}$$

we obtain

$$\begin{aligned}
&2\Big|\int_0^t \int_{\mathbb{R}} e^{2\mu s}\omega(\xi)u(s,\xi)G(u)(s,\xi)\,d\xi\,ds\Big| \\
&\le C\sigma_0 M(t)\int_0^t \int_{\mathbb{R}} e^{2\mu s}\omega(\xi)u^2(s-r,\xi-cr)\,d\xi\,ds \\
&= C\sigma_0 M(t)\int_{-r}^{t-r} \int_{\mathbb{R}} e^{2\mu (s+r)}\omega(\xi+cr)u^2(s,\xi)\,d\xi\,ds \\
&\le CM(t)e^{2\mu r}\Big\{\int_0^t \int_{\mathbb{R}} e^{2\mu s}\omega(\xi)u^2(s,\xi)\,d\xi\,ds + \int_{-r}^0 \int_{\mathbb{R}} e^{2\mu s}\omega(\xi+cr)u_0^2(s,\xi)\,d\xi\,ds\Big\} \\
&\le CM(t)\Big\{\int_0^t \int_{\mathbb{R}} e^{2\mu s}\omega(\xi)u^2(s,\xi)\,d\xi\,ds + \int_{-r}^0 \int_{\mathbb{R}} e^{2\mu s}\omega(\xi+cr)u_0^2(s,\xi)\,d\xi\,ds\Big\}.
\end{aligned} \tag{3.19}$$

On the other hand,

$$|E(u)(t,\xi)| = |d(u(t,\xi)+\phi(\xi)) - d(\phi(\xi)) - d'(\phi(\xi))u(t,\xi)| \le C|u(t,\xi)|^2,$$

we can also obtain

$$2\Big|\int_0^t \int_{\mathbb{R}} e^{2\mu s}\omega(\xi)u(s,\xi)E(u)(s,\xi)\,d\xi\,ds\Big| \le CM(t)\int_0^t \int_{\mathbb{R}} e^{2\mu s}\omega(\xi)u^2(s,\xi)\,d\xi\,ds. \tag{3.20}$$

From (3.19), (3.20) and (3.17), we have

$$\begin{aligned}
&e^{2\mu t}\|u(t)\|^2_{L^2_\omega} + \int_0^t \int_{\mathbb{R}} e^{2\mu s}[B_{\eta,\mu,\omega}(\xi) - CM(t)]\omega(\xi)u^2(s,\xi)\,d\xi\,ds \\
&\le \|u_0(0)\|^2_{L^2_\omega} + \int_{-r}^0 \int_{\mathbb{R}} e^{2\mu s}\omega(\xi+cr)[CM(t) + \frac{e^{2\mu r}}{\eta}|f'(\phi(\xi)|]u_0^2(s,\xi)\,d\xi\,ds \quad (3.21)\\
&\le C\Big(\|u_0(0)\|^2_{L^2_\omega} + \int_{-r}^0 \|u_0(s)\|^2_{L^2_\omega}\,ds\Big),
\end{aligned}$$

which immediately implies (3.10). This completes the proof. □

Next we prove a key inequality.

Lemma 3.5. *Letting $\eta = e^{-\lambda cr}$, there exists a unique number $c_* > 0$, such for every $c > c_*$, there exists a constant $C_1 > 0$ such that*

$$A_{\eta,\omega}(\xi) \ge C_1 > 0 \quad \text{for } \xi \in \mathbb{R}. \tag{3.22}$$

Proof. We distinguish three cases:

Case 1: For $\xi < \xi_* - cr$, $\omega(\xi) = e^{-2\lambda(\xi-\xi_*)}$ and $\omega(\xi+cr) = e^{-2\lambda(\xi-\xi_*+cr)}$

$$\frac{\omega'(\xi)}{\omega(\xi)} = -2\lambda, \quad \frac{\omega(\xi+cr)}{\omega(\xi)} = e^{-2\lambda cr}.$$

By (H4) and (2.4), we can have

$$\begin{aligned} A_{\eta,\omega}(\xi) &:= -c\frac{\omega'(\xi)}{\omega(\xi)} + 2d'(\phi(\xi)) - \frac{D}{2}\Big(\frac{\omega'(\xi)}{\omega(\xi)}\Big)^2 - \eta|f'(\phi(\xi-cr))| \\ &\quad - \frac{1}{\eta}|f'(\phi(\xi))|\frac{\omega(\xi+cr)}{\omega(\xi)} \\ &= 2c\lambda - 2D\lambda^2 + 2d'(\phi(\xi)) - e^{-\lambda cr}|f'(\phi(\xi-cr))| - e^{-\lambda cr}|f'(\phi(\xi))| \\ &\ge 2\Big(c\lambda - D\lambda^2 + d'(0) - e^{-\lambda cr}f'(0)\Big) =: C_{11} > 0. \end{aligned} \tag{3.23}$$

Case 2: For $\xi_* - cr \le \xi \le \xi_*$, then $\omega(\xi) = e^{-2\lambda(\xi-\xi_*)}$ and $\omega(\xi+cr) = 1$, and by Lemma 3.2, we have

$$\begin{aligned} &A_{\eta,\omega}(\xi) \\ &:= -c\frac{\omega'(\xi)}{\omega(\xi)} + 2d'(\phi(\xi)) - \frac{D}{2}\Big(\frac{\omega'(\xi)}{\omega(\xi)}\Big)^2 - \eta|f'(\phi(\xi-cr))| \\ &\quad - \frac{1}{\eta}|f'(\phi(\xi))|\frac{\omega(\xi+cr)}{\omega(\xi)} \\ &= 2c\lambda - 2D\lambda^2 + 2d'(\phi(\xi)) - \eta|f'(\phi(\xi-cr))| \\ &\quad - \frac{1}{\eta}|f'(\phi(\xi))|e^{2\lambda(\xi-\xi_*)} \\ &\ge 2(c\lambda - D\lambda^2 + d'(0) - e^{-\lambda cr}f'(0)) + 2e^{-\lambda cr}f'(0) - \eta|f'(\phi(\xi-cr))| \\ &\quad - \frac{1}{\eta}|f'(\phi(\xi))| \\ &\ge 2\triangle(c,\lambda) + 2e^{-\lambda cr}f'(0) - (\eta + \frac{1}{\eta})\max\{|f'(\phi(\xi-cr))|, |f'(\phi(\xi))|\} \\ &\ge 2\triangle(c,\lambda) + 2e^{-\lambda cr}f'(0) - 2\max\{|f'(\phi(\xi-cr))|, |f'(\phi(\xi))|\}\cosh(\lambda cr) \\ &\ge 2\Big(\triangle(c,\lambda) + e^{-\lambda cr}f'(0) - \max\{|f'(\phi(\xi_0-cr))|, |f'(\phi(\xi_0))|\}\cosh(\lambda cr)\Big) \\ &=: C_{12} > 0. \end{aligned} \tag{3.24}$$

Case 3: For $\xi \ge \xi_*$, $\omega(\xi) = \omega(\xi+cr) = 1$, and by Lemma 3.2, we obtain

$$\begin{aligned} A_{\eta,\omega}(\xi) :&= 2d'(\phi(\xi)) - \eta|f'(\phi(\xi-cr))| - \frac{1}{\eta}|f'(\phi(\xi))| \\ &\ge 2\Big(d'(0) - (\eta + \frac{1}{\eta})\max\{|f'(\phi(\xi-cr))|, |f'(\phi(\xi))|\}\Big) \\ &\ge 2\Big(d'(0) - \max\{|f'(\phi(\xi_0-cr))|, |f'(\phi(\xi_0))|\}\cosh(\lambda cr)\Big) \\ &=: C_{13} > 0. \end{aligned} \tag{3.25}$$

Combining (3.23)–(3.25), we obtain $A_{\eta,\omega}(\xi) \geq C_1$, where $C_1 = \min_{i=1,2,3}\{C_{1i}\} > 0$. This completes the proof. □

Lemma 3.6. *Let $u(t,\xi) \in X(-r,T)$. Then there exists a constant $\mu^* > 0$ such that for $0 < \mu < \mu^*$, it holds*

$$\begin{aligned} &\|u(t)\|^2_{L^2_\omega} + \int_0^t \int_{\mathbb{R}} e^{-2\mu(t-s)} \|u(s)\|^2_{L^2_\omega}\,ds \\ &\leq Ce^{-2\mu t}\Big(\|u_0(0)\|^2_{L^2_\omega} + \int_{-r}^0 \|u_0(s)\|^2_{L^2_\omega}\,ds\Big), \end{aligned} \tag{3.26}$$

provided $M(t) \ll 1$.

Proof. We distinguish three cases:

Case 1: For $\xi < \xi_* - cr$, $\omega(\xi) = e^{-2\lambda(\xi-\xi_*)}$, $\omega(\xi+cr) = e^{-2\lambda(\xi-\xi_*+cr)}$, and according to Lemma 3.5,

$$\begin{aligned} B_{\eta,\mu,\omega}(\xi) &= A_{\eta,\omega}(\xi) - 2\mu - \frac{1}{\eta}(e^{2\mu r}-1)|f'(\phi(\xi))|\frac{\omega(\xi+cr)}{\omega(\xi)} \\ &\geq C_1 - 2\mu - f'(0)e^{-\lambda cr}(e^{2\mu r}-1) \\ &=: C_{21} > 0 \text{ for } \ 0 < \mu < \mu_1, \end{aligned} \tag{3.27}$$

where μ_1 is the unique root of the equation

$$C_1 - 2\mu - f'(0)e^{-\lambda cr}(e^{2\mu r}-1) = 0.$$

Case 2: For $\xi_* - cr \leq \xi \leq \xi_*$, then $\omega(\xi) = e^{-2\lambda(\xi-\xi_*)}$ and $\omega(\xi+cr) = 1$,

$$\begin{aligned} B_{\eta,\mu,\omega}(\xi) &= A_{\eta,\omega}(\xi) - 2\mu - \frac{1}{\eta}(e^{2\mu r}-1)|f'(\phi(\xi))|\frac{\omega(\xi+cr)}{\omega(\xi)} \\ &\geq C_1 - 2\mu - f'(0)e^{\lambda cr}(e^{2\mu r}-1)e^{2\lambda(\xi-\xi_*)} \\ &\geq C_1 - 2\mu - f'(0)e^{\lambda cr}(e^{2\mu r}-1) \\ &=: C_{22} > 0 \quad \text{for } 0 < \mu < \mu_2 \end{aligned} \tag{3.28}$$

where μ_2 is the unique root of the equation

$$C_1 - 2\mu - f'(0)e^{\lambda cr}(e^{2\mu r}-1) = 0.$$

Case 3: For $\xi \geq \xi_*, \omega(\xi) = \omega(\xi+cr) = 1$,

$$\begin{aligned} B_{\eta,\mu,\omega}(\xi) &= A_{\eta,\omega}(\xi) - 2\mu - \frac{1}{\eta}(e^{2\mu r}-1)|f'(\phi(\xi))|\frac{\omega(\xi+cr)}{\omega(\xi)} \\ &\geq C_1 - 2\mu - f'(0)e^{\lambda cr}(e^{2\mu r}-1) \\ &=: C_{22} > 0 \quad \text{for } 0 < \mu < \mu_2. \end{aligned} \tag{3.29}$$

Combining (3.27), (3.28) and (3.29), we obtain $B_{\eta,\mu,\omega}(\xi) \geq C_2$, for $0 < \mu < \mu_*$, where $C_2 := \min\{C_{21}, C_{22}\}$ and $\mu_* = \min\{\mu_1, \mu_2\}$. It follows from (3.10) that

$$\begin{aligned} &\|u(t)\|^2_{L^2_\omega} + \int_0^t e^{-2\mu(t-s)} \int_{\mathbb{R}} [C_2 - CM(t)]\omega(\xi)u^2(s,\xi)\,d\xi\,ds \\ &\leq Ce^{-2\mu t}\Big(\|u_0(0)\|^2_{L^2_\omega} + \int_{-r}^0 \|u_0(s)\|^2_{L^2_\omega}\,d\xi\,ds\Big), \end{aligned}$$

which implies (3.26) by letting $M(t) \ll 1$. This completes the proof. □

Next, we shall establish the energy estimate for u_ξ, which is similar to (3.26).

Lemma 3.7. *Let $u(t,\xi) \in X(-r,T)$. Then it holds*

$$\begin{aligned} &\|u_\xi(t)\|^2_{L^2_\omega} + \int_0^t e^{-2\mu(t-s)}\|u_\xi(t)\|^2_{L^2_\omega}\,ds \\ &\le Ce^{-2\mu t}\Big(\|u_0(0)\|^2_{H^1_\omega} + \int_{-r}^0 \|u_0(s)\|^2_{H^1_\omega}\,ds\Big). \end{aligned} \tag{3.30}$$

provided $M(t) \ll 1$.

Proof. Differentiating (3.2) with respect to ξ and multiplying the obtained equation by $e^{2\mu t}\omega(\xi)u_\xi(t,\xi)$, we have

$$\begin{aligned} &\Big\{\frac{1}{2}e^{2\mu t}\omega u_\xi^2\Big\}_t + e^{2\mu t}\Big\{\frac{1}{2}c\omega u_\xi^2 - D\omega u_\xi u_{\xi\xi}\Big\}_\xi + De^{2\mu t}\omega u_{\xi\xi}^2 + De^{2\mu t}\omega' u_\xi u_{\xi\xi} \\ &+ \Big\{-\frac{c}{2}\frac{\omega'}{\omega} + d'(\phi(\xi)) - \mu\Big\}e^{2\mu t}\omega u_\xi^2 - e^{2\mu t}\omega(\xi)u_\xi(t,\xi)f'(\phi(\xi-cr)) \\ &\quad\times u_\xi(t-r,\xi-cr) \\ &= e^{2\mu t}\omega(\xi)u_\xi(t,\xi)[G_2(u)+G_1(u)] - e^{2\mu t}\omega(\xi)u_\xi(t,\xi)[E_2(u)+E_1(u)], \end{aligned} \tag{3.31}$$

where

$$\begin{gathered} G_1(u)(t,\xi) = [f'(u(t-r,\xi-cr)+\phi(\xi-cr)) - f'(\phi(\xi-cr))]\phi'(\xi-cr), \\ G_2(u)(t,\xi) = [f'(u(t-r,\xi-cr)+\phi(\xi-cr)) - f'(\phi(\xi-cr))]u_\xi(t-r,\xi-cr), \\ E_1(u)(t,\xi) = [d'(u(t,\xi)+\phi(\xi)) - d'(\phi(\xi))]\phi'(\xi), \\ E_2(u)(t,\xi) = [d'(u(t,\xi)+\phi(\xi)) - d'(\phi(\xi))]u_\xi(t,\xi). \end{gathered}$$

Using the Cauchy-Schwarz inequality

$$|De^{2\mu t}\omega' u_\xi u_{\xi\xi}| \le De^{2\mu t}\omega u_{\xi\xi}^2 + \frac{D}{4}e^{2\mu t}\Big(\frac{\omega'}{\omega}\Big)^2\omega u_\xi^2,$$

it follows from (3.31) that

$$\begin{aligned} &\Big\{\frac{1}{2}e^{2\mu t}\omega u_\xi^2\Big\}_t + e^{2\mu t}\Big\{\frac{1}{2}c\omega u_\xi^2 - D\omega u_\xi u_{\xi\xi}\Big\}_\xi \\ &+ \Big\{-\frac{c}{2}\frac{\omega'}{\omega} + d'(\phi(\xi)) - \mu - \frac{D}{4}\Big(\frac{\omega'}{\omega}\Big)^2\Big\}e^{2\mu t}\omega u_\xi^2 \\ &- e^{2\mu t}\omega(\xi)u_\xi(t,\xi)f'(\phi(\xi-cr))u_\xi(t-r,\xi-cr) \\ &\le e^{2\mu t}\omega(\xi)u_\xi(t,\xi)[G_2(u)+G_1(u)] - e^{2\mu t}\omega(\xi)u_\xi(t,\xi)[E_2(u)+E_1(u)]. \end{aligned} \tag{3.32}$$

Integrating the above inequality over $\mathbb{R}\times[0,t]$ with respect to ξ and t, we have

$$\begin{aligned} &e^{2\mu t}\|u_\xi(t)\|^2_{L^2_\omega} + \int_0^t\int_{\mathbb{R}} e^{2\mu s}\Big\{-c\frac{\omega'(\xi)}{\omega(\xi)} + 2d'(\phi(\xi)) - 2\mu - \frac{D}{2}\Big(\frac{\omega'(\xi)}{\omega(\xi)}\Big)^2\Big\} \\ &\quad\times\omega(\xi)u_\xi^2(s,\xi)\,d\xi\,ds \\ &\quad- 2\int_0^t\int_{\mathbb{R}} e^{2\mu s}\omega(\xi)f'(\phi(\xi-cr))u_\xi(s,\xi)u_\xi(s-r,\xi-cr)\,d\xi\,ds \end{aligned} \tag{3.33}$$

$$
\begin{aligned}
&\le \|u_0(0)\|_{H^1_\omega}^2 + 2\int_0^t \int_{\mathbb{R}} e^{2\mu s}\omega(\xi)u_\xi(s,\xi)[G_2(u)(s,\xi)+G_1(u)(s,\xi)]\,d\xi\,ds \\
&\quad - 2\int_0^t \int_{\mathbb{R}} e^{2\mu s}\omega(\xi)u_\xi(s,\xi)[E_2(u)(s,\xi)+E_1(u)(s,\xi)]\,d\xi\,ds.
\end{aligned}
$$

By the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
&2\Big|\int_0^t \int_{\mathbb{R}} e^{2\mu s}\omega(\xi)f'(\phi(\xi-cr))u_\xi(s,\xi)u_\xi(s-r,\xi-cr)\,d\xi\,ds\Big| \\
&\le \eta\int_0^t \int_{\mathbb{R}} e^{2\mu s}\omega(\xi)|f'(\phi(\xi-cr))|u_\xi^2(s,\xi)\,d\xi\,ds \\
&\quad + \frac{1}{\eta}e^{2\mu r}\int_0^t \int_{\mathbb{R}} e^{2\mu s}\omega(\xi+cr)|f'(\phi(\xi))|u_\xi^2(s,\xi)\,d\xi\,ds \\
&\quad + \frac{1}{\eta}e^{2\mu r}\int_{-r}^0 \int_{\mathbb{R}} e^{2\mu s}\omega(\xi+cr)|f'(\phi(\xi))|u_{0\xi}^2(s,\xi)\,d\xi\,ds.
\end{aligned}
\tag{3.34}
$$

It follows from (3.33) and (3.34) that

$$
\begin{aligned}
&e^{2\mu t}\|u_\xi(t)\|_{L^2_\omega}^2 + \int_0^t \int_{\mathbb{R}} e^{2\mu s}B_{\eta,\mu,\omega}(\xi)\omega(\xi)u_\xi^2(s,\xi)\,d\xi\,ds \\
&\le \|u_0(0)\|_{H^1_\omega}^2 + \frac{e^{2\mu r}}{\eta}\int_{-r}^0 \int_{\mathbb{R}} e^{2\mu s}\omega(\xi+cr)|f'(\phi(\xi))|u_{0\xi}^2(s,\xi)\,d\xi\,ds \\
&\quad + 2\int_0^t \int_{\mathbb{R}} e^{2\mu s}\omega(\xi)u_\xi(s,\xi)[G_2(u)(s,\xi)+G_1(u)(s,\xi)]\,d\xi\,ds \\
&\quad - 2\int_0^t \int_{\mathbb{R}} e^{2\mu s}\omega(\xi)u_\xi(s,\xi)[E_2(u)(s,\xi)+E_1(u)(s,\xi)]\,d\xi\,ds.
\end{aligned}
\tag{3.35}
$$

Again, by the Taylor expansion,

$$
\begin{aligned}
|G_2(u)(t,\xi)| &= \Big|[f'(u(t-r,\xi-cr)+\phi(\xi-cr))-f'(\phi(\xi-cr))]\Big|u_\xi(t-r,\xi-cr) \\
&\le C|u(t-r,\xi-cr)|u_\xi(t-r,\xi-cr),
\end{aligned}
$$

we can estimate the nonlinear term as

$$
\begin{aligned}
&2\Big|\int_0^t \int_{\mathbb{R}} e^{2\mu s}\omega(\xi)u_\xi(s,\xi)G_2(u)(s,\xi)\,d\xi\,ds\Big| \\
&\le C\int_0^t \int_{\mathbb{R}} e^{2\mu s}\omega(\xi)u_\xi(s,\xi)|u(s-r,\xi-cr)|u_\xi(s-r,\xi-cr)\,d\xi\,ds \\
&\le CM(t)\int_0^t \int_{\mathbb{R}} e^{2\mu s}\omega(\xi)u_\xi^2(s,\xi)\,d\xi\,ds \\
&\quad + CM(t)\Big(\int_{-r}^0 \int_{\mathbb{R}} + \int_0^t \int_{\mathbb{R}}\Big)e^{2\mu(s+r)}\omega(\xi+cr)u_\xi^2(s,\xi)\,d\xi\,ds \\
&\le CM(t)\int_0^t \int_{\mathbb{R}} e^{2\mu s}\omega(\xi)u_\xi^2(s,\xi)\,d\xi\,ds \\
&\quad + CM(t)\int_{-r}^0 \int_{\mathbb{R}} e^{2\mu(s+r)}\omega(\xi+cr)u_{0\xi}^2(s,\xi)\,d\xi\,ds,
\end{aligned}
\tag{3.36}
$$

and

$$
\begin{aligned}
&2\Big|\int_0^t\int_{\mathbb{R}} e^{2\mu s}\omega(\xi)u_\xi(s,\xi)G_1(u)(s,\xi)\,d\xi\,ds\Big| \\
&\le C\Big|\int_0^t\int_{\mathbb{R}} e^{2\mu s}\omega(\xi)u_\xi(s,\xi)u(s-r,\xi-cr)\phi'(\xi-cr)\,d\xi\,ds\Big| \\
&\le C\int_0^t\int_{\mathbb{R}} e^{2\mu s}\omega(\xi)u_\xi(s,\xi)|u(s-r,\xi-cr)|\,d\xi\,ds \\
&\le C\int_0^t\int_{\mathbb{R}} e^{2\mu s}\omega(\xi)u_\xi^2(s,\xi)\,d\xi\,ds \\
&\quad + C\Big(\int_{-r}^0\int_{\mathbb{R}} + \int_0^t\int_{\mathbb{R}}\Big)e^{2\mu(s+r)}\omega(\xi+cr)u^2(s,\xi)\,d\xi\,ds.
\end{aligned}
\tag{3.37}
$$

Similarly, we obtain

$$
2\Big|\int_0^t\int_{\mathbb{R}} e^{2\mu s}\omega(\xi)u_\xi(s,\xi)E_2(u)(s,\xi))\,d\xi\,ds\Big| \le CM(t)\int_0^t\int_{\mathbb{R}} e^{2\mu s}\omega(\xi)u_\xi^2(s,\xi)\,d\xi\,ds
$$

and

$$
\begin{aligned}
&2\Big|\int_0^t\int_{\mathbb{R}} e^{2\mu s}\omega(\xi)u_\xi(s,\xi)E_1(u)(s,\xi))\,d\xi\,ds\Big| \\
&\le C\Big|\int_0^t\int_{\mathbb{R}} e^{2\mu s}\omega(\xi)u_\xi(s,\xi)u(s,\xi))\phi'(\xi-cr)\,d\xi\,ds\Big| \\
&\le C\int_0^t\int_{\mathbb{R}} e^{2\mu s}\omega(\xi)u_\xi^2(s,\xi)\,d\xi\,ds \\
&\quad + C\int_0^t\int_{\mathbb{R}} e^{2\mu s}\omega(\xi)u^2(s,\xi)\,d\xi\,ds.
\end{aligned}
\tag{3.38}
$$

It then follows from (3.36)-(3.38) and Lemma 3.6 that

$$
\|u_\xi(t)\|_{L^2_\omega}^2 + \int_0^t e^{-2\mu(t-s)}\|u_\xi(t)\|_{L^2_\omega}^2\,ds \le Ce^{-2\mu t}\Big(\|u_0(0)\|_{H^1_\omega}^2 + \int_{-r}^0\|u_0(s)\|_{H^1_\omega}^2\,ds\Big).
$$

Combining (3.26) and (3.30), for some constant C, which is independent of T and $u(t,\xi)$, we have

$$
\|u(t)\|_{H^1_\omega}^2 \le Ce^{-2\mu t}\Big(\|u_0(0)\|_{H^1_\omega}^2 + \int_{-r}^0\|u_0(s)\|_{H^1_\omega}^2\,ds\Big), \quad \text{for all } 0\le t\le T.
$$

This completes the proof. □

Proof of Theorem 2.3. It is based on the existence of a local solution and the estimate obtained above. The process is similar to the one in [20, 21], using the continuity extension method, so we omit it. □

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