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A FRACTIONAL GRONWALL INEQUALITY AND THE ASYMPTOTIC BEHAVIOUR OF GLOBAL SOLUTIONS OF CAPUTO FRACTIONAL PROBLEMS

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Abstract. We study the asymptotic behaviour of global solutions of some nonlinear integral equations related to some Caputo fractional initial value problems. We consider problems of fractional order between 0 and 1 and of order between 1 and 2, each in two cases: when the nonlinearity depends only on the function, and when the nonlinearity also depends on fractional derivatives of lower order. Our main tool is a new Gronwall inequality for integrals with singular kernels, which we prove here, and a related boundedness property of a fractional integral of an $L^1[0,\infty)$ function.

1. Introduction

We investigate the asymptotic behaviour of global solutions of problems of the form

$$u(t) = u_0 + I^\alpha g(t,u(t)) := u_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s,u(s))\,ds \tag{1.1}$$

where $0 < \alpha < 1$, $I^\alpha g$ is the Riemann-Liouville (R-L) fractional integral of g, and also in the case when g depends on lower order Caputo fractional derivatives, $g(t,u(t),D_*^\gamma u(t))$; precise definitions are given later in the paper. We also study

$$u(t) = u_0 + B_1 t^\beta + \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}(Gu)(t))\,ds, \tag{1.2}$$

where $(Gu)(t) = g(t,u(t),D_*^\gamma u(t))$ for $1 < \alpha+\beta < 2$ and $0 \le \gamma \le \beta$. The function g will be of the form $\phi(t)f(t,u,D_*^\gamma u(t))$ where f is continuous and $\phi \in L^1[0,\infty)$ or dominated by such a term. These problems are motivated by the study of initial value problems for Caputo fractional differential equations. Under certain conditions solution of (1.1) are also solutions of

$$D_*^\alpha u(t) = g(t,u(t)), \text{ for a.e. } t>0, \quad u(0)=u_0. \tag{1.3}$$

Similarly, solutions of

$$D_*^\alpha(D_*^\beta u)(t) = g(t,u(t),D_*^\gamma u(t)), \quad u(0)=u_0, \quad D_*^\beta u(0) = B_1\Gamma(\beta+1), \tag{1.4}$$

are closely related to solutions of (1.2).

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For problem (1.1) the kernel $(t-s)^{\alpha-1}$ has a singularity when $t=s$, whereas in problem (1.2) the term $(t-s)^{\alpha+\beta-1}$ is not singular. For the singular case, (1.1), some special inequality is needed, we will use a new Gronwall inequality suitable for this purpose. For the case (1.2) when f does not depend on $D_*^\gamma u(t)$, that is $\gamma=0$, we can use the classical Gronwall inequality, but when there is explicit dependence on $D_*^\gamma u(t)$ we again need the new Gronwall inequality to handle some fractional derivatives.

This paper was motivated by some previous papers, mainly those by Medveď and Pospíšil [14] and more recently by Kassim and Tatar [11].

2. Known Gronwall inequalities

We state one classical version of the Gronwall inequality that we shall use, which was proved by Bellman [1]. This, and many other versions, may be found in several books, for example [5, 15].

Theorem 2.1. *Suppose that $u \in C_+[0,T]$ satisfies $u(t) \le a(t) + \int_0^t \phi(s)u(s)\,ds$ for $t \in [0,T]$, where a is non-negative, continuous and non-decreasing, and $\phi \in L_+^1[0,T]$. Then*

$$u(t) \le a(t)\exp\Big(\int_0^t \phi(s)\,ds\Big) \quad \text{for } t \in [0,T]. \tag{2.1}$$

If also a is uniformly bounded for all $t>0$ and $\phi \in L^1[0,\infty)$ then u is uniformly bounded for all $t>0$.

Here, $L_+^1[0,T]$ denotes the Lebesgue integrable functions u defined on a finite interval $[0,T]$ with $u(t) \ge 0$ a.e., similarly $C_+[0,T]$ denotes the non-negative continuous functions.

For Gronwall type inequalities with singular kernels the pioneering work was done by Henry [8]. A version of one result is as follows.

Theorem 2.2. *Let a, g be non-decreasing and $g(t) \le C$ for all $t \in [0,T]$ and let $0<\beta<1$. If $x \in L_+^\infty[0,T]$ satisfies the inequality*

$$x(t) \le a(t) + g(t)\int_0^t (t-s)^{\beta-1}x(s)ds, \quad t \in [0,T],$$

then $x(t) \le a(t)E_\beta[g(t)\Gamma(\beta)(t^\beta)]$ where E_β is the Mittag-Leffler function.

The Mittag-Leffler function is an entire function of $z \in \mathbb{C}$ defined by a power series $E_\beta(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\beta k+1)}$. The special case $E_1(z)$ is the exponential function. Further information can be found in many places, for example in the books [3, 17].

The original result in [8] was proved by an iteration argument. Another proof was given by Dixon and McKee [4, Theorem 3.1] when g is a constant. It is simple to replace a constant by a non-decreasing function as in Corollary 5.4 below. Such a generalization of Henry's work was done by Ye, Gao and Ding [22] but using a modification of Henry's original proof. A different proof is given by Tisdell [19].

In a recent paper [20] we proved a new result for such an inequality by a different method and we obtained bounds in terms of the much better known exponential function. It is this result we will extend to enable us to handle our problems. We do not know of a result similar to this new one involving Mittag-Leffler functions.

There are other results with exponential bounds. For example the following integral inequality is a special case of Medveď [12, Theorem 2]. If $f \geq 0$ is continuous and $u \geq 0$ is continuous and satisfies

$$u(t) \leq a + \int_0^t (t-s)^{\beta-1} f(s)u(s)\,ds, \quad t \in [0,T],$$

then under certain hypotheses on f, $u(t)$ is bounded on $[0,T]$ by $a\exp(H(t)+t)$ for an explicit function H. (He also considers Bihari type inequalities.) The proofs involve applications of various inequalities such as Hölder's inequality. Further inequalities using the idea from [12] have been obtained by Zhu [23, 24]. These types of inequality are quite different from the one we will prove.

3. Definitions of fractional integrals and derivatives

All functions we consider will be real-valued and defined on an interval $[0,T]$ or on $[0,\infty)$. The space of continuous functions defined on $[0,T]$ is denoted $C[0,T]$ and is endowed with the norm $\|u\|_\infty := \sup_{t\in[0,T]} |u(t)|$. $AC[0,T]$ denotes the space of Absolutely Continuous functions on $[0,T]$. $L^1[0,T]$ denotes the space of Lebesgue integrable functions u, that is $\int_0^T |u(s)|\,ds < \infty$, and $L^1[0,\infty)$ are those functions such that $\int_0^\infty |u(s)|\,ds < \infty$. A subscript $+$ will denote the non-negative functions in the corresponding space. For $\eta > -1$ we define the space denoted $C_\eta = C_\eta[0,T]$ by

$$C_\eta[0,T] := \{u \in C(0,T] \text{ such that } \lim_{t\to 0+} t^{-\eta}u(t) \text{ exists}\},$$

then $u \in C_\eta$ if and only if $u(t) = t^\eta v(t)$ for some function $v \in C[0,T]$ and we define $\|u\|_\eta := \|v\|_\infty$. The spaces of functions with singularity at $t = 0$ are $C_{-\eta}$ where $\eta > 0$. The space C_0 coincides with the space $C^0 = C[0,T]$. Clearly, for $\eta > 0$ the space C_η is a subspace of $C[0,T]$.

The Gamma and Beta functions occur frequently so we recall them here. The Gamma function is, for $p > 0$, given by

$$\Gamma(p) := \int_0^\infty s^{p-1}\exp(-s)\,ds \tag{3.1}$$

which is an improper Riemann integral but is well defined as a Lebesgue integral, and is an extension of the factorial function: $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$. The Beta function is defined by

$$B(p,q) := \int_0^1 (1-s)^{p-1}s^{q-1}\,ds \tag{3.2}$$

which is a well defined Lebesgue integral for $p > 0, q > 0$ and it is well known, and proved in many calculus texts, that $B(p,q) = \dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$.

The Riemann-Liouville (R-L) fractional integral $I^\alpha u$ is 'defined' informally by:

$$I^\alpha u(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)\,ds, \quad \text{provided the integral exists.}$$

This does not specify to which space of functions u belongs, and leaves open whether the integral is to exist for all t, or for all nonzero t, or for almost every (a.e.) t. The most interesting case is when $0 < \alpha < 1$ then the integrand has a singularity. A precise definition for integrable functions is the following.

Definition 3.1. The Riemann-Liouville (R-L) fractional integral of order $\alpha > 0$ of a function $u \in L^1[0,T]$ is defined for a.e. t by

$$I^\alpha u(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds.$$

The integral $I^\alpha u$ is the convolution of the L^1 functions h, u, where $h(t) = t^{\alpha-1}/\Gamma(\alpha)$, so, by well known results on convolutions, $I^\alpha u$ is defined as an L^1 function, in particular $I^\alpha u(t)$ is finite for a.e. t. When $\alpha = 1$ this is the usual integration operator which we denote I. We set $I^0 u = u$. If $\alpha \in (0,1)$ and $f \in L^p[0,T]$ with $p > 1/\alpha$ then $I^\alpha f$ belongs to a Hölder space hence is defined at all points and is continuous, a result of Hardy and Littlewood [7, Theorem 12].

The Riemann-Liouville (R-L) fractional derivative of order $\alpha \in (0,1)$ is defined as follows. We write D for the usual derivative operator, that is, $Du = u'$.

Definition 3.2. *For $\alpha \in (0,1)$ and $u \in L^1[0,T]$ the R-L fractional derivative $D^\alpha u$ is defined when $I^{1-\alpha} u \in AC[0,T]$ by*

$$D^\alpha u(t) := D\, I^{1-\alpha} u(t), \quad \text{a.e. } t \in [0,T].$$

The condition $I^{1-\alpha} u \in AC$ is often not stated in published work, but it is necessary if we want to relate solutions of R-L fractional differential equations with solutions of a Volterra integral equation. It is not enough to assume that $I^{1-\alpha} u$ is differentiable for a.e. t. This has been noted in the monograph [17], see [17, Definition 2.4] and the related comments in the 'Notes to §2.6'.

The Caputo fractional derivative is frequently defined with the derivative and fractional integral taken in the reverse order to that of the R-L derivative.

Definition 3.3. For $\alpha \in (0,1)$ and $u \in AC[0,T]$ the Caputo fractional derivative $D_C^\alpha u$ is defined for a.e. t by

$$D_C^\alpha u(t) := I^{1-\alpha} Du(t).$$

For $u \in AC$, we have $Du \in L^1$ and so $D_C^\alpha u = I^{1-\alpha}(Du)$ is defined as an L^1 function. However, this definition has a severe disadvantage. It is often claimed that for $0 < \alpha < 1$ and f continuous

$$D_C^\alpha u(t) = f(t),\ u(0) = u_0 \text{ is equivalent to } u(t) = u_0 + I^\alpha f.$$

However, 'solution' means different things on each side of the equation. Usually solution of the integral equation is a function in $C[0,T]$, and it is never shown that such a function is in $AC[0,T]$ for the very good reason that it is **false** in general, so the derivative $D_C^\alpha u(t)$ is not shown to exist. Again this was proved by Hardy and Littlewood [7]. For $0 < \alpha < 1$, I^α maps $C_{-\eta}[0,T]$ into $C_{\alpha-\eta}[0,T]$, I^α maps $C[0,T]$ into $C[0,T]$ and maps $AC[0,T]$ into $AC[0,T]$ but does not map $C[0,T]$ into $AC[0,T]$ and does not map $C^1[0,T]$ into $C^1[0,T]$. Details can be found the paper [21] and its addendum.

To get an equivalence it is necessary to use the definition of Caputo differential operator as is used in Diethelm's book [3].

Definition 3.4. The Caputo differential operator of order $\alpha \in (0,1)$ is defined by $D_*^\alpha u := D^\alpha(u - u(0)) = D(I^{1-\alpha}(u - u(0)))$ whenever this R-L derivative exists, that is when $u(0)$ exists and $I^{1-\alpha} u \in AC$.

The two definitions of $D_C^\alpha u$ and $D_*^\alpha u$ coincide when $u \in AC$.

For $1 < \beta < 2$, the R-L derivative is $D^\beta u = D^2 I^{2-\beta} u$ if $D(I^{2-\beta}u) \in AC$, and the Caputo derivative is $D_*^\beta u(t) := D^2\big(I^{2-\beta}(u - u(0) - tu'(0))\big)$ provided that $u(0), u'(0)$ exist and $D(I^{2-\beta}u) \in AC$. Higher order derivatives are defined similarly, for details see for example the books [3, 17].

The main advantages of the Caputo derivative over the R-L derivative are that $D_*^\alpha(c) = 0$ (any $\alpha > 0$) when c is a constant function, whereas the R-L derivative of a constant has a singularity at zero, and initial value problems for the Caputo derivative are well posed when initial values are prescribed on the function and its ordinary derivatives, fractional integrals and derivatives should be prescribed in the R-L case.

An equivalence that can be proved is as follows, in fact a somewhat more general result holds for $f \in C_{-\eta}$ for $0 \le \eta < \alpha$, see [21, Theorem 5.1].

Theorem 3.5. *Let f be continuous on $[0,T] \times \mathbb{R}$, let $0 < \alpha < 1$. If $u \in C[0,T]$ satisfies $u(t) = u_0 + I^\alpha f$ then $I^{1-\alpha}(u - u_0) \in AC$, $D_*^\alpha u$ exists a.e. and satisfies $D_*^\alpha u(t) = f(t)$ a.e., $u(0) = u_0$. Conversely, if $u \in C[0,T]$, $I^{1-\alpha}(u - u_0) \in AC$ and u satisfies $D_*^\alpha u(t) = f(t)$ a.e., $u(0) = u_0$, then u satisfies $u(t) = u_0 + I^\alpha f$.*

This means that Caputo fractional equations can be studied in the space $C[0,T]$, but the corresponding R-L fractional equations for $\alpha \in (0,1)$ should be studied in a space that allows functions to be singular at 0 such as $C_{\alpha-1}$.

4. Some properties of fractional integrals

It is well known that fractional integral operators satisfy a semigroup property as follows.

Lemma 4.1. *Let $\alpha, \beta > 0$ and $u \in L^1[0,T]$. Then $I^\alpha I^\beta(u)(t) = I^{\alpha+\beta}(u)(t)$ for each t for which $I^{\alpha+\beta}|u|(t)$ exists (finite), that is a.e. $t \in [0,T]$. If u is continuous, or if $u \in C_{-\eta}$ and $\alpha + \beta \ge \eta$, this holds for all $t \in [0,T]$. If $u \in L^1$ and $\alpha + \beta \ge 1$ equality again holds for all $t \in [0,T]$.*

The proof uses Fubini's theorem, and all but the part concerning $C_{-\eta}$ is usually sketched in the texts, for example [3, Theorem 2.2], [17, Eq.(2.21)]. A detailed proof of Lemma 4.1 is given in [21].

We first prove some properties of fractional integrals that we will use, the first one is probably known.

Proposition 4.2. *Let $f \in L^1[0,T]$ and $\alpha \ge \beta > 0$. Then*

$$\frac{\Gamma(\alpha)(I^\alpha|f|)(t)}{t^\alpha} \le \frac{\Gamma(\beta)(I^\beta|f|)(t))}{t^\beta} \quad \text{for a.e. } t \in (0,T]. \tag{4.1}$$

Proof. Clearly it suffices to give the proof for $f \in L^1_+[0,T]$ and omit absolute value signs. By standard properties, each fractional integral exists as an L^1 function, hence exists for a.e. t. For a.e. t we have

$$\begin{aligned}\Gamma(\alpha) I^\alpha f(t) &= \int_0^t (t-s)^{\alpha-1} f(s)\,ds \\ &= \int_0^t t^{\alpha-1}(1 - s/t)^{\alpha-1} f(s)\,ds\end{aligned}$$

$$
\begin{aligned}
&\le \int_0^t t^{\alpha-1}(1-s/t)^{\beta-1} f(s)\,ds \\
&= \int_0^t t^{\alpha-\beta}(t-s)^{\beta-1} f(s)\,ds \\
&= t^{\alpha-\beta}\Gamma(\beta) I^\beta f(t).
\end{aligned}
$$

□

Corollary 4.3. *Let $f \in L^1[0,T]$, $\gamma > 0$, and $\beta > 0$. Then*

$$I^{\gamma+\beta}|f|(t)) \le t^\gamma \frac{\Gamma(\beta)}{\Gamma(\gamma+\beta)} I^\beta |f|(t), \quad \text{for a.e. } t \in [0,T]. \tag{4.2}$$

Proof. Replace α by $\gamma+\beta$ in Proposition 4.2. □

Corollary 4.4. *Let $f \in L^1[0,T]$ and $\beta > 0$. Then*

$$I^{1+\beta}|f|(t) \le \frac{t}{\beta} I^\beta |f|(t), \ \text{for a.e. } t \in [0,T]. \tag{4.3}$$

This corollary follows from Corollary 4.3 with $\gamma = 1$ noting that $\frac{\Gamma(\beta)}{\Gamma(1+\beta)} = \frac{1}{\beta}$. If the only information is that $f \in L^1[0,T]$ then Corollary 4.4 is optimal as the following example shows.

Example 4.5. Let $f(t) := t^p$ for $p > -1$, $t \in [0,T]$. It is known, and easy to show, that $I^\beta f(t) = \dfrac{\Gamma(1+p)}{\Gamma(1+\beta+p)} t^{\beta+p}$. Then, by the semigroup property of fractional integration Lemma 4.1, we obtain, for every $t \in [0,T]$,

$$I^{1+\beta} f(t) = I(I^\beta f)(t) = \frac{\Gamma(1+p)}{\Gamma(2+\beta+p)} t^{1+\beta+p}.$$

Hence, $I^{1+\beta} f(t) = \dfrac{1}{1+\beta+p} t\, I^\beta f(t)$ for $t > 0$ and the constant approaches $1/\beta$ as $p \to -1$.

We now consider some results given by Kassim-Tatar [11].

Proposition 4.6. *Let $\rho > 1$ and let F be a function such that F' exists a.e. and $F' \in L^1([0,\infty)$, that is, $\int_0^\infty |F'(t)|\,dt$ exists and is finite. Then we have*

(a)
$$\lim_{t\to\infty} \frac{I^\rho F'(t)}{t^{\rho-1}} = \frac{1}{\Gamma(\rho)} \int_0^\infty F'(t)dt. \tag{4.4}$$

(b) *If, in addition, $F \in AC[0,T]$ for every $T > 0$ then*
$$\lim_{t\to\infty} \frac{I^\rho F'(t)}{t^{\rho-1}} = \frac{1}{\Gamma(\rho)} \lim_{t\to\infty} (F(t) - F(0)). \tag{4.5}$$

Moreover in this case
$$\lim_{t\to\infty} \frac{I^{\rho-1} F(t)}{t^{\rho-1}} = \frac{1}{\Gamma(\rho)} \lim_{t\to\infty} F(t). \tag{4.6}$$

Proof. (a) Let χ_J denote the characteristic function of an interval J. Then we have

$$\begin{aligned}\frac{I^\rho F'(t)}{t^{\rho-1}} &= \frac{1}{\Gamma(\rho)}\int_0^t (1-s/t)^{\rho-1}F'(s)\,ds\\ &= \frac{1}{\Gamma(\rho)}\int_0^\infty \chi_{[0,t]}(s)(1-s/t)^{\rho-1}F'(s)\,ds\end{aligned}$$

For each t, since $\rho>1$, $s\mapsto |\chi_{[0,t]}(s)(1-s/t)^{\rho-1}F'(s)|$ is dominated by the integrable function $|F'(s)|$, so the limit as $t\to\infty$ exists by the dominated convergence theorem, and taking the limit proves (4.4).

(b) As $F\in AC[0,T]$ for every $T>0$, we have $F(t)-F(0)=\int_0^t F'(s)ds$ for every $t>0$. Since the right hand side in this equation has a limit as $t\to\infty$, the limit of the left hand side exists and equals that limit, so using (a), we see that (4.5) holds.

For the last part, we note that $I^\rho F'(t)=I^{\rho-1}IF'(t)$, for all t, by the semigroup property of fractional integral, as given in Lemma 4.1. Since $F\in AC[0,T]$ we have $IF'(t)=F(t)-F(0)$ and, by a simple calculation, $I^\alpha(F(0))(t)=\frac{t^\alpha}{\Gamma(1+\alpha)}F(0)$ for any $\alpha>0$. Therefore, applying (4.5) we obtain

$$\frac{1}{\Gamma(\rho)}\lim_{t\to\infty}(F(t)-F(0))=\lim_{t\to\infty}\frac{I^\rho F'(t)}{t^{\rho-1}}=\lim_{t\to\infty}\Big(\frac{I^{\rho-1}F(t)}{t^{\rho-1}}-\frac{1}{\Gamma(\rho)}F(0)\Big),$$

that is

$$\frac{1}{\Gamma(\rho)}\lim_{t\to\infty}F(t)=\lim_{t\to\infty}\frac{I^{\rho-1}F(t)}{t^{\rho-1}}.$$

□

Remark 4.7. The above results hold if we have $f\in L^1[0,\infty)$ and take $F(t):=\int_0^t f(s)ds$ for then $F\in AC[0,T]$ and $F'=f$ a.e.. However, $F'\in L^1$ does not imply $F\in AC$ as shown by the well-known Lebesgue's singular function F (also known as the Cantor-Vitali function, or Devil's staircase).

Remark 4.8. Result (4.4) is stated as [11, Lemma 2.10] with $\rho=1+\alpha$ for $\alpha>0$, for the case $f\in L^1$ as in above Remark 4.7, and was previously proved in [10, Lemma 7] by a more detailed version of the proof we gave. The result (4.6) is essentially the same result as claimed in [11, Lemma 2.11] and proved in [10, Lemma 18] which is stated when $f(t)=D_C^\alpha x(t)$ for a function $x\in AC$, but those lemmas statements and proof have omitted the necessary $L^1[0,\infty)$ hypotheses. Moreover, our different proof is shorter.

In part (a) it is necessary that $F'\in L^1[0,\infty)$ otherwise both sides can be infinite, but the result does not have a weaker meaning in that case, it can happen that

$$\frac{I^{1+\alpha}F'(t)/t^\alpha}{\int_0^t F'(s)\,ds/\Gamma(1+\alpha)}\not\to 1,\quad \text{as } t\to\infty.$$

Example 4.9. *Let $0<\alpha<1$ and let $F(t)=t^\gamma/\gamma$ with $\gamma>0$ so that $F\in AC$ and $F'(t)=t^{\gamma-1}$ for $t>0$ and $F'\in L^1[0,T]$ for every $T>0$ but $F'\notin L^1[0,\infty)$. Then the numerator is*

$$\begin{aligned}\frac{I^{1+\alpha}F'(t)}{t^\alpha} &= \frac{1}{t^\alpha\Gamma(1+\alpha)}\int_0^t (t-s)^\alpha s^{\gamma-1}\,ds\\ &= \frac{1}{t^\alpha\Gamma(1+\alpha)}t^{\alpha+\gamma}B(1+\alpha,\gamma)\end{aligned}$$

$$= t^{\gamma}\frac{\Gamma(\gamma)}{\Gamma(1+\alpha+\gamma)}.$$

The denominator is

$$\frac{1}{\Gamma(1+\alpha)}\int_0^t s^{\gamma-1}\,ds = \frac{1}{\Gamma(1+\alpha)}\frac{t^{\gamma}}{\gamma}.$$

The ratio

$$\frac{\textit{numerator}}{\textit{denominator}} = \frac{\Gamma(\gamma)}{\Gamma(1+\alpha+\gamma)}\gamma\Gamma(1+\alpha) = \frac{\Gamma(1+\gamma)\Gamma(1+\alpha)}{\Gamma(1+\alpha+\gamma)},$$

and this is not equal to 1, except in the excluded cases $\gamma = 0$ *or* $\alpha = 0$.

5. New Gronwall type of inequality

To discuss the asymptotic behaviour of solutions of some fractional integral equations of order less than 1, or higher order with nonlinearities depending on fractional derivatives of order less than 1, it is useful to have an inequality of Gronwall type appropriate for discussing fractional integrals with singular kernels.

We will prove the following inequality which has similar conclusions to those of the classical Gronwall inequality, with extra restrictions necessary. We believe this result to be new.

Theorem 5.1. *Let* $a > 0$, $b > 0$, $0 < \beta < 1$ *and let* ϕ *be non-increasing,* $\phi \in L^1_+[0,T]$ *for all* $T > 0$. *Suppose that* $u \in C_+[0,\infty)$ *satisfies the inequality*

$$u(t) \le a + b\int_0^t (t-s)^{-\beta}\phi(s)u(s)\,ds, \quad \text{for } t > 0, \tag{5.1}$$

If there exists $r \in (0,1)$ *and* $t_r > 0$ *such that* $b\Gamma(1-\beta)(I^{1-\beta}\phi)(t) \le r$ *for* $0 \le t \le t_r$, *then*

$$u(t) \le \frac{a}{1-r}\exp\Big(\frac{b}{t_r^{\beta}(1-r)}\int_0^t \phi(s)\,ds\Big), \quad \text{for every } t > 0. \tag{5.2}$$

Moreover, if, in addition, $\phi \in L^1_+[0,\infty)$ *then*

$$u(t) \le \frac{a}{1-r}\exp\Big(\frac{b}{t_r^{\beta}(1-r)}\int_0^{\infty} \phi(s)\,ds\Big), \tag{5.3}$$

so that $u(t)$ *is uniformly bounded.*

Proof. We will use the notation $u^*(t) := \max_{s\in[0,t]} u(s)$. Let $t > 0$ and let $\tau \in (0,t]$ be arbitrary. For $\tau \le t_r$ we have

$$\begin{aligned} u(\tau) &\le a + b\int_0^{\tau}(\tau-s)^{-\beta}\phi(s)u(s)\,ds \le a + bu^*(t)\int_0^{\tau}(\tau-s)^{-\beta}\phi(s)\,ds, \\ &= a + b\Gamma(1-\beta)\big(I^{1-\beta}\phi\big)(\tau)u^*(t) \le a + ru^*(t). \end{aligned} \tag{5.4}$$

Now we consider the case when $\tau > t_r$. We have

$$u(\tau) \le a + b\int_0^{\tau-t_r}(\tau-s)^{-\beta}\phi(s)u(s)\,ds + b\int_{\tau-t_r}^{\tau}(\tau-s)^{-\beta}\phi(s)u(s)\,ds.$$

In the first integral we use the fact that $\tau - s \ge t_r$ so that $(\tau-s)^{-\beta} \le t_r^{-\beta}$, while in the second integral we use $s \ge s-(\tau-t_r) \ge 0$ and the fact that ϕ is non-increasing.

This gives

$$u(\tau) \le a + b\int_0^{\tau - t_r} t_r^{-\beta}\phi(s)u^*(s)\,ds + bu^*(t)\int_{\tau-t_r}^{\tau}(\tau-s)^{-\beta}\phi(s-(\tau-t_r))\,ds \quad (5.5)$$

In the second integral we now let $\sigma = s - (\tau - t_r)$ and it becomes

$$\int_0^{t_r}(t_r-\sigma)^{-\beta}\phi(\sigma)d\sigma = \Gamma(1-\beta)(I^{1-\beta}\phi)(t_r) \le r/b.$$

Thus (5.5) gives

$$\begin{aligned} u(\tau) &\le a + b\int_0^{\tau-t_r} t_r^{-\beta}\phi(s)u^*(s)\,ds + ru^*(t) \\ &\le a + b\int_0^{\tau} t_r^{-\beta}\phi(s)u^*(s)\,ds + ru^*(t). \end{aligned} \quad (5.6)$$

As (5.4) holds, (5.6) holds for all $\tau \in (0,t]$, and taking the $\sup_{\tau\in(0,t]}$, we obtain

$$u^*(t) \le a + b\int_0^t t_r^{-\beta}\phi(s)u^*(s)\,ds + ru^*(t).$$

Hence we have

$$u^*(t) \le \frac{a}{1-r} + \frac{b}{1-r}\int_0^t t_r^{-\beta}\phi(s)u^*(s)\,ds,$$

and this holds for all $t > 0$. This is now a classical Gronwall inequality, Theorem 2.1, and we can immediately deduce that

$$u(t) \le u^*(t) \le \frac{a}{1-r}\exp\Big(\frac{b}{1-r}\,t_r^{-\beta}\int_0^t \phi(s)ds\Big), \quad \text{for all } t > 0.$$

The last assertion is now obvious. □

Remark 5.2. The proof of Theorem 5.1 is a modification of the proof of Theorem 3.2 in [20] which considered the Gronwall inequality on a finite interval $[0,T]$ for the special case where $\phi(t) = t^{-\eta}$ for $\eta < 1-\beta$. We allow ϕ to be possibly singular at 0 with essentially the same type of singularity. The proof in [20] was based on the proof in Haraux [6, Lemma 6, p.33], which had $\beta = 1/2$, he attributed the method to Pazy [16]. In the paper [20] it was possible to show that $r = \beta/(1-\eta)$ (when $\phi(t) = t^{-\eta}$) had an optimal property; nothing similar seems possible in the general case here. Clearly t_r should be chosen as large as possible to give a better estimate in (5.2).

A simpler case is when $r_1 = 1/2$ is an allowed value, the conclusion is then

$$u(t) \le 2a\exp\Big(2\,b\,t_{r_1}^{-\beta}\int_0^t \phi(s)ds\Big). \quad (5.7)$$

Remark 5.3. The hypotheses that there exist $r \in (0,1)$ and $t_r > 0$ such that

$$b\Gamma(1-\beta)(I^{1-\beta}\phi)(t) \le r \text{ for } 0 \le t \le t_r,$$

can be replaced by the hypothesis $(I^{1-\beta}\phi)(t) \to 0$ as $t \to 0+$ and then we may choose any $r \in (0,1)$. The property $(I^{1-\beta}\phi)(t) \to 0$ as $t \to 0+$ holds in the following two cases:

(a) $\phi \in L^p[0,\tau]$ for some $p > 1/(1-\beta)$ and some $\tau > 0$.
(b) $\phi \in C_{-\eta}[0,\tau]$ for some $\eta < 1-\beta$ and some $\tau > 0$.

In case (a), a result of Hardy and Littlewood [7] shows that if $p > 1/(1-\beta)$, then $I^{1-\beta}$ maps $L^p[0,\tau]$ into a Hölder space $C^{0,1-\beta-1/p}$ (hence $I^{1-\beta}\phi$ is continuous) and also that $I^{1-\beta}\phi(t) \to 0$ as $t \to 0+$. Further information may be found in [21, Proposition 3.2 (3)].

Case (b) is a special case of case (a) but more can be easily proved in this case. It is shown in [21, Proposition 3.2 (5)] that $I^{1-\beta}$ maps $C_{-\eta}[0,\tau]$ into $C_{1-\beta-\eta}[0,\tau] \subset C[0,\tau]$ and that $I^{1-\beta}\phi(t) \to 0$ as $t \to 0+$. In fact, if $\phi(t) = t^{-\eta}v(t)$ where v is continuous, then

$$\begin{aligned} I^{1-\beta}\phi(t) &= \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} s^{-\eta} v(s)\, ds \\ &= \frac{1}{\Gamma(1-\beta)} t^{1-\beta-\eta} \int_0^1 (1-\sigma)^{-\beta}\sigma^{-\eta} v(t\sigma)\, d\sigma, \end{aligned}$$

which proves the result since $\int_0^1 (1-\sigma)^{-\beta}\sigma^{-\eta}\, d\sigma = B(1-\beta, 1-\eta)$.

Of course in this remark τ can be as small as we wish. In specific cases, t_r in Theorem 5.1 can be determined explicitly.

The constants may be replaced by non-decreasing functions as we now show by a simple method.

Corollary 5.4. *Suppose a, b are positive non-decreasing functions, $0 < \beta < 1$ and $u \in C_+[0,\infty)$ satisfies the inequality*

$$u(t) \le a(t) + b(t) \int_0^t (t-s)^{-\beta}\phi(s)u(s)\, ds, \quad \textit{for } t > 0,$$

where r, t_r, ϕ are as in Theorem 5.1. Then we have

$$u(t) \le \frac{a(t)}{1-r} \exp\Big(\frac{b(t)}{1-r} t_r^{-\beta} \int_0^t \phi(s) ds\Big), \quad \textit{for every } t > 0$$

In particular, if $r_1 = 1/2$ is an allowed value then using t_{r_1}, we obtain

$$u(t) \le 2a(t) \exp\Big(\frac{2b(t)}{t_{r_1}^{\beta}} \int_0^t \phi(s)\, ds\Big), \quad \textit{for every } t > 0.$$

Proof. In the proof of Theorem 5.1, for a fixed $\bar{t} > 0$, first replace the inequality by

$$u(t) \le a(\bar{t}) + b(\bar{t}) \int_0^t (t-s)^{-\beta}\phi(s)u(s)\, ds, \quad \text{for } t \le \bar{t}.$$

Applying the result proved, and noting that the final inequality holds for $t = \bar{t}$, gives the conclusion since $\bar{t}$ is arbitrary. $\square$

The following fact will be important in the following discussions. It seems to be a new result concerning properties of fractional integrals for it was shown by Hardy and Littlewood [7, §3.5,(iv)] that, for $p > 1$, $I^{1/p}$ does not map L^p into L^∞. The non-increasing property of the function is important for our proof and prevents ϕ having any spikes.

Theorem 5.5. *Let $0 < \alpha < 1$ and suppose that ϕ is non-increasing, $\phi \in L^1_+(0,\infty)$ and there exist $t_1 > 0$ and a constant $M > 0$ such that $(I^\alpha\phi)(t) \le M$ for $0 \le t \le t_1$. Then $I^\alpha\phi(t)$ is uniformly bounded for all $t > 0$.*

Proof. Take $t_1 \in (0,1)$ such that $I^\alpha \phi(t) \le M$ for every $t \le t_1$. For $t > t_1$ we argue as in Theorem 5.1 and write

$$\begin{aligned}
I^\alpha \phi(t) &= \frac{1}{\Gamma(\alpha)} \int_0^{t-t_1} (t-s)^{\alpha-1} \phi(s)\, ds + \frac{1}{\Gamma(\alpha)} \int_{t-t_1}^{t} (t-s)^{\alpha-1} \phi(s)\, ds \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^{t-t_1} t_1^{\alpha-1} \phi(s)\, ds + \frac{1}{\Gamma(\alpha)} \int_{t-t_1}^{t} (t-s)^{\alpha-1} \phi(s-(t-t_1))\, ds \\
&\le \frac{t_1^{\alpha-1}}{\Gamma(\alpha)} \int_0^{\infty} \phi(s)\, ds + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-\sigma)^{\alpha-1} \phi(\sigma)\, d\sigma \\
&\le \frac{t_1^{\alpha-1}}{\Gamma(\alpha)} \int_0^{\infty} \phi(s)\, ds + M =: C_0.
\end{aligned}$$

Thus $I^\alpha \phi(t) \le C_0$ for every $t > 0$. $\square$

Remark 5.6. We could assume $\phi \in C_{-\eta}[0, t_2]$ for some $\eta < \alpha$ and some $t_2 > 0$, or that $\phi \in L^p[0, t_2]$ for some $p > 1/(1-\alpha)$, which imply that $\lim_{t\to 0+} I^\alpha \phi(t) \to 0$ as in Remark 5.3. The hypotheses do not imply that $I^\alpha \phi \in L^1[0,\infty)$ as the following example shows.

Example 5.7. Let $f(t) = \frac{t^{-\gamma}}{1+t}$ for $0 < \gamma < 1$. Then $f \in L^1[0,\infty)$ and for $0 < \gamma < \alpha < 1$ we have $I^\alpha f(t) \to 0$ as $t \to 0+$, $I^\alpha f$ is uniformly bounded but $I^\alpha f \notin L^1[0,\infty)$.

Clearly $f \in L^1[0,\infty)$, in fact, $\int_0^\infty f(t)\, dt = \frac{\pi}{\sin(\gamma\pi)}$ (Beta function). Firstly, we have

$$\begin{aligned}
I^\alpha f(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{s^{-\gamma}}{1+s}\, ds \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{-\gamma}\, ds \\
&= \frac{t^{\alpha-\gamma}}{\Gamma(\alpha)} B(\alpha, 1-\gamma).
\end{aligned} \tag{5.8}$$

This shows that $I^\alpha f(t) \to 0$ as $t \to 0+$. Secondly we will show that $I^\alpha f$ is uniformly bounded. For $t \le 1$ the above shows that $I^\alpha f(t) \le C_1$. For $t > 1$ we write

$$\begin{aligned}
I^\alpha f(t) &\le C_1 + \frac{1}{\Gamma(\alpha)} \int_1^t (t-s)^{\alpha-1} \frac{s^{-\gamma}}{1+s}\, ds \\
&\le C_1 + \frac{1}{\Gamma(\alpha)} \int_1^t (t-s)^{\alpha-1} \frac{1}{1+s}\, ds \\
&\le C_1 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{1}{1+s}\, ds \\
&\le C_2.
\end{aligned}$$

The integral here, according to Maple, involves hypergeometric functions in general. For the special case when $\alpha = 1/2$ we have

$$\int_0^t (t-s)^{-1/2} \frac{1}{1+s}\, ds = 2\tanh^{-1}\big(\frac{\sqrt{t}}{\sqrt{t+1}}\big) \frac{1}{\sqrt{t+1}},$$

with the inverse of the tanh function; this is bounded by 4/3. Thirdly, we have

$$\begin{aligned} I^\alpha f(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{s^{-\gamma}}{1+s}\, ds \\ &\geq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{t^{-\gamma}}{1+t}\, ds \\ &= \frac{t^{\alpha-\gamma}}{\Gamma(\alpha+1)(1+t)}, \end{aligned}$$

which shows that $I^\alpha f \notin L^1[0,\infty)$ since $\alpha - \gamma > 0$.

Remark 5.8. *It is a basic fact that I^α maps $L^1[0,T]$ into $L^1[0,T]$ for every finite $T > 0$ but that does not hold for $T = \infty$; the above example is an explicit proof of this fact.*

6. Asymptotic behaviour for equations of order $\alpha < 1$

In this section we investigate the asymptotic behaviour of solutions of the integral equation corresponding to solutions of the initial value problem

$$D_*^\alpha u(t) = \phi(t) f(t, u(t), D_*^\gamma u(t)), \quad \text{for a.e. } t > 0, \quad u(0) = u_0, \tag{6.1}$$

when $0 < \gamma < \alpha < 1$.

6.1. **When f does not depend on fractional derivatives.** We first consider the simpler case

$$D_*^\alpha u(t) = \phi(t) f(t, u(t)), \quad \text{for a.e. } t > 0, \quad u(0) = u_0, \tag{6.2}$$

and will study the asymptotic behaviour of solutions of the corresponding integral equation

$$u(t) = u_0 + I^\alpha(\phi(t) f(t, u(t))) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi(s) f(s, u(s))\, ds. \tag{6.3}$$

with f continuous. These problems are essentially equivalent on a finite interval as in Theorem 3.5. Thus we will study solutions of (6.3) in the space $C[0,T]$ that are assumed to exist for an arbitrary $T > 0$ and consider the behaviour of any solution as $T \to \infty$.

Remark 6.1. When $|f(t,u)| \leq C(1 + |u|)$ problems such as (6.3) have global solutions, that is solutions that exist on intervals $[0,T]$ for all $T > 0$, such solutions will often grow exponentially so to have a linear or polynomial decay $\phi(t)$ will need some suitable smallness property for large t.

Global existence for this problem can be proved exactly as in [20, Theorem 4.8] under the assumptions we use below by using the estimates from the fractional Gronwall inequality of Theorem 5.1.

The following remark can be used to reduce the number of hypotheses that are stated in the following results, but we have not necessarily done this in every case.

Remark 6.2.

(1) If $\phi(s)$ and $s^p\phi(s)$ belong to $L^1[0,\infty)$ for some $p < 1$ then $s^r\phi(s) \in L^1[0,\infty)$ for each $r \in [p,1]$. This follows from the fact that, for each $s > 0$, $s^r = \exp(r\ln(s))$ is a convex function of r, and every $r \in [p,1]$ can be written $r = (1-\lambda)p + \lambda$ for some $\lambda \in [0,1]$, hence, by the convexity, $s^r \leq (1-\lambda)s^p + \lambda s$.

(2) If $s^q\phi(s)$ is non-increasing for some $q > 0$ then $s^r\phi(s)$ is also non-increasing for each $r \in [0, q]$.
(3) If $\delta > 0$ and $I^\delta\phi(t)$ is bounded on a neighbourhood of 0 then also $I^\delta(t^p\phi(t))$ is bounded on a neighbourhood of 0 for any $p > 0$.
(4) Recall Proposition 4.2; if $\alpha > \beta$ then $I^\alpha\phi(t) \le Ct^{\alpha-\beta}I^\beta\phi(t)$, thus $I^\beta\phi$ bounded near 0 implies $I^\alpha\phi(t) \to 0$ as $t \to 0$.

Our result for this problem is the following.

Theorem 6.3. *Let $0 < \alpha < 1$, $\phi \in L^1_+[0,\infty)$, and let ϕ be non-increasing and suppose that $I^\alpha\phi(t) \to 0$ as $t \to 0$. Let f be continuous and satisfy $|f(t,u)| \le C(1 + |u|)$ for a constant $C > 0$, all $t \in [0,\infty)$ and all $u \in \mathbb{R}$. If u is a global solution of* (6.3)*, then $|u|$ is uniformly bounded on $[0,\infty)$.*

Proof. Since $C\phi$ satisfies the same assumptions as ϕ, without loss of generality we can and do take $C = 1$. We note that $I^\alpha\phi(t) \le C_0$ for every $t > 0$ by Theorem 5.5 and then we have

$$
\begin{aligned}
|u(t)| &\le |u_0| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi(s)|f(s,u(s))|\,ds \\
&\le |u_0| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi(s)(1+|u(s)|)\,ds \\
&\le |u_0| + C_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi(s)|u(s)|)\,ds.
\end{aligned}
\tag{6.4}
$$

By the fractional Gronwall inequality (5.7), for t_1 chosen so that $\Gamma(\alpha)(I^\alpha\phi)(t) < 1/2$ for $t \le t_1$, we obtain

$$
u(t) \le 2(|u_0| + C_0)\exp\Big(2t_1^{\alpha-1}\int_0^t \phi(s)\,ds\Big), \quad \text{for every } t > 0, \tag{6.5}
$$

hence $|u(t)| \le 2(|u_0| + C_0)\exp\big(2t_1^{\alpha-1}\int_0^\infty \phi(s)\,ds\big)$, that is $|u|$ is uniformly bounded since $\phi \in L^1[0,\infty)$. □

Remark 6.4. The result applies to the slightly more general case of a function f_1 replacing ϕf where we suppose that

$$
|f_1(t,u(t))| \le \phi(t)|f(t,u(t))|,
$$

with the given hypotheses, and no changes in the proof.

6.2. **When f depends on fractional derivatives.** We now turn to the much trickier case

$$
D_*^\alpha u(t) = \phi(t)f(t,u(t),D_*^\gamma u(t)), \quad \text{for a.e. } t > 0, \quad u(0) = u_0, \tag{6.6}
$$

when $0 < \gamma < \alpha < 1$ and f is continuous. If u is a sufficiently regular solution of (6.6) then u is a solution of the integral equation

$$
\begin{aligned}
u(t) &= u_0 + I^\alpha\phi(t)f(t,u(t),D_*^\gamma u(t)) \\
&= u_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi(s)f(s,u(s),D_*^\gamma u(s))\,ds.
\end{aligned}
\tag{6.7}
$$

However, the two problems are not necessarily equivalent. Equivalence depends on how 'solution' of each problem is defined. It is not entirely clear in what space we should seek solutions of (6.6) and of (6.7). A solution in $C[0,T]$ of (6.7) is not enough since it is necessary that $D_*^\gamma u$ exists at least a.e. for the problem to have a meaning. Solutions in $C^1[0,T]$ would certainly be sufficient and (6.6) would make sense, but this requires knowledge of u' which formally would be given by $I^{\alpha-1}\phi(t)f(t,u(t),D_*^\gamma u(t))$, unfortunately that expression might not have a finite value for every t. An appropriate subspace of continuous functions is

$$X := \{u \in C[0,T], D_*^\gamma u \in C[0,T]\}$$

endowed with the norm $\|u\|_X := \|u\|_\infty + \|D_*^\gamma u\|_\infty$. Thus $u \in X$ is equivalent to $u \in C[0,T]$ and $I^{1-\gamma}(u-u_0) \in C^1[0,T]$.

Definition 6.5. *For a continuous function f we say that $u \in X$ is a solution of* (6.6)*, on an interval* $[0,T]$*, if $D_*^\alpha u$ exists and equals ϕf a.e. We say that $u \in X$ is a solution of* (6.7) *if u satisfies* (6.7) *for all $t \in [0,T]$. A global solution is one that exists on* $[0,T]$ *for every* $T>0$.

We now show the equivalence of the two problems.

Proposition 6.6. *Let f be continuous. Then $u \in X$ is a solution of* (6.6) *if and only if $u \in X$ is a solution of* (6.7).

Proof. Firstly suppose that $u \in X$ is a solution of (6.7). Then

$$F(t) := \phi(t)f(t,u(t),D_*^\gamma u(t)) \in L^1[0,T]$$

so that

$$I^{1-\alpha}(u-u_0)(t) = I^{1-\alpha}I^\alpha F(t) = IF(t)$$

where $IF \in AC$ hence $D_*^\alpha u(t) = F(t)$ exists a.e., that is u is a solution of (6.6). Conversely, if $u \in X$ is a solution of (6.6) then $D_*^\alpha u(t) = F(t)$ a.e., that is $D(I^{1-\alpha}(u-u_0)(t) = F(t)$ a.e. and $I^{1-\alpha}(u-u_0) \in AC[0,T]$. Thus we have $I^{1-\alpha}(u-u_0)(t) = IF(t)$ (note that $I^{1-\alpha}(u-u_0)(0) = 0$ by Remark 5.3). Applying I^α, and using the semigroup property of fractional integrals, we obtain

$$I(u-u_0)(t) = I^\alpha I^{1-\alpha}(u-u_0)(t) = I^\alpha IF = II^\alpha F(t), \quad \text{for every } t.$$

The left hand side is a C^1 function, the right hand side is in AC. Therefore, the derivatives exist a.e. and $u(t)-u_0 = I^\alpha F(t)$ for all a.e. t, so u is a solution of (6.6). □

It is useful to know that X is Banach space, that is, is complete. This is a known result proved for the Riemann-Liouville derivative in Su [18], but we include a proof for completeness.

Proposition 6.7. $(X, \|\cdot\|)$ *is a Banach space.*

Proof. Let $\{u_n\}$ be a Cauchy sequence in X, then $\{u_n\}$ and $\{D_*^\gamma u_n\}$ are Cauchy sequences in $C[0,T]$ hence are uniformly convergent, say $u_n \to u$ and $D_*^\gamma u_n \to v$ uniformly on $[0,T]$ with u and v continuous. We have to prove that $D_*^\gamma u$ exists and $v = D_*^\gamma u$. Let $u_{n,0}$ denote the constant function with value $u_n(0)$. As $I^{1-\gamma}$ is a bounded linear operator from $C[0,T]$ to itself, we have $I^{1-\gamma}(u_n-u_{n,0}) \to I^{1-\gamma}(u-u_0)$ uniformly on $[0,T]$. By definition, $D_*^\gamma u_n = D(I^{1-\gamma}(u_n-u_{n,0}))$ so that $I^{1-\gamma}(u_n-u_{n,0}) \in C^1$ for $u_n \in X$. Therefore we have, for every n,

$$I^{1-\gamma}(u_n-u_{n,0})(t) = I^{1-\gamma}(u_n-u_{n,0})(0) + \int_0^t D(I^{1-\gamma}(u_n-u_{n,0}))(s)\,ds.$$

Passing to the limit on both sides gives

$$I^{1-\gamma}(u-u_0)(t) = I^{1-\gamma}(u-u_0)(0) + \int_0^t v(s)\,ds.$$

This proves that $v(t) = D(I^{1-\gamma}(u-u_0))(t) = D_*^\gamma u(t)$, as required. $\square$

We now assume that a global solution of (6.7) exists and we discuss its asymptotic behaviour. In doing this we establish some *a priori* bounds which could be used to prove an existence result.

Theorem 6.8. *Let $0<\gamma<\alpha<1$ and let ϕ be non-increasing and $(I^{\alpha-\gamma}\phi)(t)\to 0$ as $t\to 0+$. Suppose also that $\phi(s)$ and $s^\gamma\phi(s)$ are in $L^1[0,\infty)$. Let f be continuous and satisfy $|f(t,u,p)|\le C(1+|u|+|p|)$ for all $t\in[0,\infty)$ and all $u,p\in\mathbb{R}$. If u is a global solution of (6.7), then $|u|$ and $|D_*^\gamma u|$ are uniformly bounded on $[0,\infty)$.*

Proof. Without loss of generality we take $C=1$. We first note that if u is a global solution of (6.7), that is, $u\in X$ and $u(t) = u_0 + I^\alpha\big(\phi(t)f(t,u(t),D_*^\gamma u(t))\big)$, then from Proposition 4.2 we obtain

$$|u-u_0| = |I^\alpha(\phi f)| \le I^\alpha|\phi f| \le t^\gamma\Gamma(\alpha-\gamma)I^{\alpha-\gamma}|\phi f|,$$

hence

$$|u| \le |u_0| + t^\gamma\Gamma(\alpha-\gamma)I^{\alpha-\gamma}|\phi f|.$$

Also we have, by definition, and the semigroup property that

$$D_*^\gamma u = D(I^{1-\gamma})(u-u_0) = D(I^{1-\gamma}I^\alpha)\phi f = DII^{\alpha-\gamma}\phi f = I^{\alpha-\gamma}\phi f,$$

and hence $|D_*^\gamma u(t)| \le I^{\alpha-\gamma}|\phi f|$ a.e. We now have

$$\begin{aligned}
&I^{\alpha-\gamma}|\phi f|\\
&= \frac{1}{\Gamma(\alpha-\gamma)}\int_0^t (t-s)^{\alpha-\gamma-1}|\phi(s)f(s,u(s),D_*^\gamma u(s))|\,ds\\
&\le \frac{1}{\Gamma(\alpha-\gamma)}\int_0^t (t-s)^{\alpha-\gamma-1}\phi(s)(1+|u(s)|+|D_*^\gamma u(s))|\,ds\\
&\le \frac{1}{\Gamma(\alpha-\gamma)}\int_0^t (t-s)^{\alpha-\gamma-1}\phi(s)\big(1+|u_0|+s^\gamma\Gamma(\alpha-\gamma)I^{\alpha-\gamma}|\phi f| + I^{\alpha-\gamma}|\phi f|\big)\,ds.
\end{aligned}$$

Write $v(t) := I^{\alpha-\gamma}|\phi f|(t)$, then v satisfies the inequality

$$\begin{aligned}
&v(t)\\
&\le \frac{1}{\Gamma(\alpha-\gamma)}\int_0^t (t-s)^{\alpha-\gamma-1}\phi(s)\big(1+|u_0|+s^\gamma\Gamma(\alpha-\gamma)v(s)+v(s)\big)\,ds,\\
&= (1+|u_0|)(I^{\alpha-\gamma}\phi)(t) + \frac{1}{\Gamma(\alpha-\gamma)}\int_0^t (t-s)^{\alpha-\gamma-1}\phi(s)\big(\Gamma(\alpha-\gamma)s^\gamma+1)\big)v(s)\,ds
\end{aligned}$$

Now we note that $I^{\alpha-\gamma}\phi$ is bounded on $[0,\infty)$ by Theorem 5.5, say $I^{\alpha-\gamma}\phi(t)\le M$ for all $t\ge 0$. We then have

$$v(t) \le M(1+|u_0|) + \int_0^t (t-s)^{\alpha-\gamma-1}\phi(s)\big(s^\gamma + \frac{1}{\Gamma(\alpha-\gamma)}\big)v(s)\,ds.$$

By our assumption, $\phi(s)(s^\gamma + \frac{1}{\Gamma(\alpha-\gamma)}) \in L^1[0,\infty)$, and $(I^{\alpha-\gamma}\phi)(t) \to 0$ as $t \to 0+$ gives $I^{\alpha-\gamma}(t^\gamma\phi(t)) \to 0$ also, so we may apply the fractional Gronwall inequality of Theorem 5.1 to deduce that, for every $t > 0$,

$$\begin{aligned} v(t) &\le 2M(1+|u_0|)\exp\Big(2\,t_1^{1+\gamma-\alpha}\int_0^t \phi(s)\big(s^\gamma + \frac{1}{\Gamma(\alpha-\gamma)}\big)\,ds\Big), \\ &\le 2M(1+|u_0|)\exp\Big(2\,t_1^{1+\gamma-\alpha}\int_0^\infty \phi(s)\big(s^\gamma + \frac{1}{\Gamma(\alpha-\gamma)}\big)\,ds\Big) := M_1. \end{aligned} \tag{6.8}$$

Since $|D_*^\gamma u(t)| \le I^{\alpha-\gamma}|\phi f| = v(t)$ this has proved that $|D_*^\gamma u(t)| \le M_1$.

Next we have $u(t) = u_0 + I^\alpha \phi f$ thus $|u(t)| \le |u_0| + I^\alpha|\phi f|$. This gives

$$\begin{aligned} |u(t) &\le |u_0| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|\phi(s)f(s,u(s),D_*^\gamma u(s))|\,ds \\ &\le |u_0| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi(s)(1+|u(s)|+M_1)\,ds \\ &= |u_0| + (1+M_1)I^\alpha\phi(t) + \int_0^t (t-s)^{\alpha-1}\phi(s)|u(s)|\,ds. \end{aligned} \tag{6.9}$$

By Theorem 5.5, $I^\alpha\phi(t)$ is uniformly bounded, so that

$$|u(t)| \le M_2 + \int_0^t (t-s)^{\alpha-1}\phi(s)|u(s)|\,ds.$$

By the fractional Gronwall inequality of Theorem 5.1, we deduce that $|u(t)| \le M_3$ for all $t > 0$. This completes the proof. □

Remark 6.9. Of course the result applies to the case of a function f_1 replacing ϕf where we suppose that

$$|f_1(t,u(t),D_*^\gamma u(t))| \le \phi(t)|f(t,u(t),D_*^\gamma u(t))|,$$

with the given hypotheses.

The apparently more general case when $|f(t,u,p)| \le a + b|u| + c|p|$ is really the same since $a + b|u| + c|p| \le \max\{a,b,c\}(1+|u|+|p|)$.

Remark 6.10. Medveď and Pospíšil [14, Theorem 1] studied this problem under different growth assumptions on f, not necessarily linear but with some integrability conditions. They use Hölder's inequality and the Bihari inequality to prove that $|u(t)|/t^\gamma$ and $|D_*^\gamma u(t)|$ are bounded and the dominated convergence theorem to show that $|u(t)|/t^\gamma$ has a limit L as $t \to \infty$. In our case we have $L = 0$. It is not easy to make a comparison of these results.

This problem was also recently studied by Kassim-Tatar [11, Theorem 6.2] and they proved boundedness of $|u|$ and $|D_*^\gamma u|$ assuming f satisfies a multiplicative type inequality which appears more restrictive than our sum inequality. Their example 6.3 also follows from our result, where they essentially have $\phi(t) = t^{-1/3}\exp(-t)$, which is smaller than necessary, but it is an example and optimal conditions are unknown.

7. Higher order equation with non derivative dependent nonlinearity

We will investigate the asymptotic behaviour of global solutions of the integral equation

$$u(t) = u_0 + a(t)u_1 + I^{\alpha+\beta}f(t,u(t)),\ t > 0, \tag{7.1}$$

where a is continuous, u_0, u_1 are constants, $0 < \alpha, \beta \le 1$ with $1 < \alpha + \beta < 2$; here the ϕ is essentially included in f term. By a solution we mean that $u \in C[0, T]$ for all $T > 0$ and satisfies equation (7.1) on $[0, T]$; a global solution is one which exists on $[0, T]$ for all $T > 0$.

The motivation for studying this form is that for $a(t) = t$ this arises from seeking solutions of the IVP

$$D_*^{\alpha+\beta}(t) = f(t, u(t)), \quad u(0) = u_0, \quad u'(0) = u_1, \tag{7.2}$$

and for $a(t) = t^\beta$ with $0 < \beta < 1$ it arises from the similar problem

$$D_*^\alpha(D_*^\beta u)(t) = f(t, u(t)), \quad u(0) = u_0, D_*^\beta(0) = u_1\Gamma(\beta + 1). \tag{7.3}$$

Note: $D_*^{\alpha+\beta}u$ is not the same as $D_*^\alpha(D_*^\beta u)$. For example, if $\alpha + \beta < 1$, the problem $D_*^{\alpha+\beta}(t) = f(t)$ requires only one initial condition $u(0) = u_0$ to be well posed, whereas $D_*^\alpha(D_*^\beta u)(t) = f(t)$ requires two initial conditions; see also discussion and examples in Diethelm [3].

The special case of the problem (7.3) when $\alpha = 1$ was recently studied by Kassim-Tatar [11, Theorem 4.1] where a special inequality was used. We will show that the more general problem (7.1) can be tackled by a single method which involves using the classical Gronwall inequality Theorem 2.1.

Clearly the asymptotic behaviour of a solution u of (7.1) depends in an essential way on the behaviour of $I^{\alpha+\beta}f(t, u(t))$ as $t \to \infty$.

Theorem 7.1. *Let $a \in C[0, \infty)$. Suppose that u is continuous and is a global solution of the equation*

$$u(t) = u_0 + a(t)u_1 + I^{\alpha+\beta}f(t, u(t)), \quad t > 0, \tag{7.4}$$

where u_0, u_1 are constants and $0 < \alpha, \beta \le 1$ with $1 < \alpha + \beta < 2$. Write $v(t) := I^{\alpha+\beta}f(t, u(t))$. If f satisfies the growth assumption

$$|f(t, u)| \le \phi(t)(1 + |u|), \tag{7.5}$$

where $\phi \in L^1[0, \infty), a,\ \phi \in L^1[0, \infty)$, and $t^{\alpha+\beta-1}\phi(t) \in L^1[0, \infty)$, then there is a constant $C > 0$ such that $|v(t)|/t^{\alpha+\beta-1} \le C$. Moreover, $v(t)/t^{\alpha+\beta-1} \to L$ as $t \to \infty$ and therefore $u(t) - \big(u_0 + a(t)u_1 + Lt^{\alpha+\beta-1}\big) \to 0$ as $t \to \infty$, where $L = \frac{1}{\Gamma(\alpha+\beta)} \int_0^\infty f(s, u(s))\, ds$.

Remark 7.2. We do not need any non-increasing property of ϕ in this result.

Proof of Theorem 7.1. We have, since $\alpha + \beta > 1$,

$$\begin{aligned}
\frac{|v(t)|}{t^{\alpha+\beta-1}} &\le \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (1 - s/t)^{\alpha+\beta-1} |f(s, u(s))|\, ds \\
&\le \frac{1}{\Gamma(\alpha+\beta)} \int_0^t |f(s, u(s))|\, ds \\
&\le \frac{1}{\Gamma(\alpha+\beta)} \int_0^t \phi(s)(1 + |u_0| + |a(s)u_1| + |v(s)|)\, ds \\
&\le \frac{1}{\Gamma(\alpha+\beta)} \int_0^\infty (1 + |u_0|)\phi(s) + |u_1 a(s)|\phi(s)\, ds \\
&\quad + \frac{1}{\Gamma(\alpha+\beta)} \int_0^t s^{\alpha+\beta-1}\phi(s) \frac{|v(s)|}{s^{\alpha+\beta-1}}\, ds
\end{aligned}$$

$$\leq C_1 + \frac{1}{\Gamma(\alpha+\beta)} \int_0^t s^{\alpha+\beta-1}\phi(s)\frac{|v(s)|}{s^{\alpha+\beta-1}}\, ds,$$

using (7.5). By the classical Gronwall inequality of Theorem 2.1, this gives

$$\begin{aligned} \frac{|v(t)|}{t^{\alpha+\beta-1}} &\leq C_1 \exp\Big(\frac{1}{\Gamma(\alpha+\beta)} \int_0^t s^{\alpha+\beta-1}\phi(s) ds\Big) \\ &\leq C_1 \exp\Big(\frac{1}{\Gamma(\alpha+\beta)} \int_0^\infty s^{\alpha+\beta-1}\phi(s) ds\Big) \leq C. \end{aligned}$$

From above we have

$$\begin{aligned} |f(s,u(s))| &\leq \phi(s)(1+|u_0|+|a(s)u_1+|v(s)|) \\ &\leq \phi(s)(1+|u_0|+|a(s)u_1|+Cs^{\alpha+\beta-1}), \end{aligned}$$

so $f(s,u(s))$ is an $L^1[0,\infty)$ function, and hence, by the dominated convergence theorem,

$$\lim_{t\to\infty} \frac{v(t)}{t^{\alpha+\beta-1}} = \frac{1}{\Gamma(\alpha+\beta)} \int_0^\infty f(s,u(s))\, ds =: L.$$

As $u(t) = u_0 + a(t)u_1 + v(t)$ this gives $u(t) \sim u_0 + a(t)u_1 + Lt^{\alpha+\beta-1}$ asymptotically. $\square$

Remark 7.3. *If f does not depend on u then we can apply Proposition 4.6(a) to immediately deduce that if $f \in L^1[0,\infty)$ then*

$$u \sim u_0 + a(t)u_1 + \frac{\int_0^\infty f(s)ds}{\Gamma(\alpha+\beta)} t^{\alpha+\beta-1},$$

which confirms that our result is giving a correct answer. The main requirement of our proof is to prove $f(s,u(s))$ is an $L^1[0,\infty)$ function.

Corollary 7.4. *(1) For $a(t) = t$, we have $u(t) \sim u_0 + t\, u_1$ when $\phi \in L^1[0,\infty), t\phi(t) \in L^1[0,\infty)$ since $t^{\alpha+\beta-1} \leq t$ for $t \geq 1$.*
(2) For $a(t) = t^\beta$ we have $u(t) \sim u_0 + C_2 t^\beta$ since $\alpha+\beta-1 \leq \beta$, provided that $\phi \in L^1[0,\infty), t^\beta\phi(t) \in L^1[0,\infty)$.
(3) For $a(t) = t^\gamma$ and $\gamma \in (0,1)$, we have, under the corresponding conditions on ϕ, $u(t) \sim u_0 + u_1 t^\gamma$ when $\alpha+\beta-1 < \gamma$, and we have $u(t) \sim u_0 + Lt^{\alpha+\beta-1}$ when $\alpha+\beta-1 > \gamma$.

Case (1) is essentially studied in [13] when $\beta = 1$ with the same conclusion under different multiplicative type assumptions using some ideas from his own paper [12]. Case (2) with $\alpha = 1$ is the case studied in [11]. Their Theorem 4.1 proves $D_C^\beta u(t)$ and $\frac{u(t)}{t^\beta}$ have limits at $t \to \infty$ under some different multiplicative type hypotheses. Using the inequality $|xy| \leq |x|^p/p + |y|^{p'}/p'$, $p > 1$, $1/p + 1/p' = 1$, their hypotheses seem to imply ours; in [11, Example 4.2] they have $f(t,u) = \exp(-t)u^r$ where $0 < r \leq 1$ so that $|f(t,u)| \leq \phi(t)(1+|u|)$ for $\phi(t) = \exp(-t)$.

8. Higher order equation with derivative dependence

We now consider the problem

$$u(t) = u_0 + I^\beta b_1 + I^{\alpha+\beta} f(t, u(t), D_*^\gamma u(t)), \tag{8.1}$$

for $0 < \gamma \leq \beta \leq 1$, $0 < \alpha \leq \alpha+\beta-\gamma < 1$ and $\alpha+\beta > 1$. We do not impose an ordering between α and β. Here $I^\beta b_1 = b_1 t^\beta/\Gamma(\beta+1)$ but it is convenient to often write $I^\beta b_1$. This is the integral equation version of the initial value problem of the sequential fractional differential problem

$$\begin{gathered} D_*^\alpha\big(D_*^\beta u(t)\big) = f(t,u(t),D_*^\gamma u(t)), a.e. \\ u(0)=u_0,\ D_*^\beta u(0)=b_1. \end{gathered} \tag{8.2}$$

The special case $\alpha=1$ is studied in [11], but the general sequential problem here is not studied in either [11] or [14]. Writing $F(t):=f(t,u(t),D_*^\gamma u(t))$, informally we can 'solve' (8.2) as follows:

$$D_*^\alpha\big(D_*^\beta u(t)\big) = F(t), \tag{8.3}$$

$$\implies D_*^\beta u(t) = (D_*^\beta u)(0) + I^\alpha F = b_1 + I^\alpha F \tag{8.4}$$

$$\implies u(t) = u_0 + I^\beta b_1 + I^{\alpha+\beta}F = u_0 + \frac{b_1}{\Gamma(\beta+1)}t^\beta + I^{\alpha+\beta}F. \tag{8.5}$$

Equation (8.1) should be studied in a space where at least u and $D_*^\beta u$ are continuous but we will not study existence of solutions here. We suppose that a global solution u exists, that is u and $D_*^\beta u$ exist and are continuous on $[0,T]$ for all $T>0$ and (8.4) and (8.5) are satisfied. We investigate the asymptotic behaviour of u and fractional derivatives, under suitable conditions on f similar to the ones used previously.

We require a fact about the relationship between $D_*^\gamma u(t)$ and other terms in (8.1) as follows.

$$\begin{aligned} D_*^\gamma u(t) &= D^\gamma(u-u_0)(t) \\ &= D\big(I^{1-\gamma}(I^\beta b_1 + I^{\alpha+\beta}F)\big)(t) \\ &= I^{\beta-\gamma}b_1 + I^{\alpha+\beta-\gamma}F(t). \end{aligned} \tag{8.6}$$

Theorem 8.1. *Suppose that there exists ϕ such that*

$$|f(t,u,p)| \le \phi(t)(1+|u|+|p|),\ \text{for all}\ t\ge 0, u,p\in\mathbb{R}, \tag{8.7}$$

where $\phi(s)$ and $s^\beta\phi(s)$ are non-increasing, $I^{\alpha+\beta-\gamma}\phi(t)\le M$ for t near 0, and $\phi(s)$, $s^\gamma\phi(s)$, $s^{\alpha+\beta-\gamma}\phi(s)$ are all $L^1[0,\infty)$ functions. Then, for a global solution $u(t)$ of (8.4) and (8.5), there is a constant L such that

$$u(t) - \big(u_0 + \frac{b_1}{\Gamma(\beta+1)}t^\beta + Lt^{\alpha+\beta-1}\big) \to 0 \ \text{as}\ t\to\infty.$$

In fact, writing $F(t):=f(t,u(t),D_*^\gamma u(t))$, L is given by $L=\lim_{t\to\infty}\frac{I^{\alpha+\beta}F(t)}{t^{\alpha+\beta-1}}$.

Proof. Let u satisfy (8.4) and (8.5) and let $F(t):=f(t,u(t),D_*^\gamma u(t))$ and $\rho:=\alpha+\beta-\gamma$, so $\rho<1$ by assumption. We note that

$$\begin{aligned} &I^\rho|F|(t) \le \frac{1}{\Gamma(\rho)}\int_0^t (t-s)^{\rho-1}|f(s,u(s),D_*^\gamma u(s))|\,ds \\ &\le \frac{1}{\Gamma(\rho)}\int_0^t (t-s)^{\rho-1}\phi(s)\big(1+|u(s)|+|D_*^\gamma u(s)|\big)\,ds \\ &\le \frac{1}{\Gamma(\rho)}\int_0^t (t-s)^{\rho-1}\phi(s)\big(1+|u_0|+|I^\beta b_1|+I^{\alpha+\beta}|F|+|I^{\beta-\gamma}b_1|+I^\rho|F|\big)\,ds, \end{aligned} \tag{8.8}$$

where we have used (8.4) and (8.6). Since $\phi(s)$, $s^\beta\phi(s)$ and $s^{\beta-\gamma}\phi(s)$ are non-increasing $L^1[0,\infty)$ functions, and $I^\rho\phi(t)$ is assumed to be bounded for t near 0, the terms $\int_0^t (t-s)^{\rho-1}\phi(s)\big(1+|u_0|\big)\,ds$ and $\int_0^t (t-s)^{\rho-1}\phi(s)\big(|I^\beta b_1| + |I^{\beta-\gamma} b_1|\big)\,ds$ are uniformly bounded by Theorem 5.5. By Proposition 4.2 we have

$$I^{\alpha+\beta}|F|(s) \le \frac{\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha+\beta)} s^\gamma I^{\alpha+\beta-\gamma}|F|(s) = \frac{\Gamma(\rho)}{\Gamma(\alpha+\beta)} s^\gamma I^\rho |F|(s) \tag{8.9}$$

so from (8.8) we obtain

$$I^\rho|F|(t) \le C_1 + \frac{1}{\Gamma(\rho)}\int_0^t (t-s)^{\rho-1}\Phi(s) I^\rho|F|(s)\,ds, \tag{8.10}$$

where $\Phi(s) = \phi(s)\big(1 + s^\gamma \frac{\Gamma(\rho)}{\Gamma(\alpha+\beta)}\big)$ which is an $L^1[0,\infty)$ non-increasing function by our hypotheses. Since $0<\rho<1$ we may apply the fractional Gronwall inequality Theorem 5.1 to give $I^\rho|F|$ is uniformly bounded, by M_1 say, which proves that $|D_*^\gamma u(t)| \le |I^{\beta-\gamma}b_1| + M_1$.
Now we consider $I^{\alpha+\beta}|F|$ which is related to u by (8.5). Since $\alpha+\beta>1$ this is not the case of singular kernel as in Theorem 5.1. We argue similarly to Theorem 7.1. We consider $I^{\alpha+\beta}|F(t)|/t^{\alpha+\beta-1}$ and we have

$$\begin{aligned}
\frac{I^{\alpha+\beta}|F|(t)}{t^{\alpha+\beta-1}} &= \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (1-s/t)^{\alpha+\beta-1}|f(s,u(s),D_*^\gamma u(s))|\,ds,\\
&\le \frac{1}{\Gamma(\alpha+\beta)}\int_0^t \phi(s)\big(1+|u_0| + |I^\beta b_1| + I^{\alpha+\beta}|F|(s) + |I^{\beta-\gamma}b_1| + M_1\big)\,ds\\
&\le C_2 + \frac{1}{\Gamma(\alpha+\beta)}\int_0^t \phi(s) I^{\alpha+\beta}|F|(s)\,ds,\\
&\le C_2 + \frac{1}{\Gamma(\alpha+\beta)}\int_0^t \phi(s)s^{\alpha+\beta-1}\frac{I^{\alpha+\beta}|F|(s)}{s^{\alpha+\beta-1}}\,ds,
\end{aligned} \tag{8.11}$$

where C_2 comes from using Theorem 5.5 similarly to above, since $\phi(s), s^\beta\phi(s), s^{\beta-\gamma} \in L^1[0,\infty)$ and are non-increasing. This is a classical Gronwall inequality and by the hypothesis $s^{\alpha+\beta-1}\phi(s) \in L^1[0,\infty)$ we deduce that $\frac{I^{\alpha+\beta}|F|(t)}{t^{\alpha+\beta-1}}$ is uniformly bounded for all $t>0$, by M_2 say.

Now we note that

$$\frac{I^{\alpha+\beta}F(t)}{t^{\alpha+\beta-1}} = \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (1-s/t)^{\alpha+\beta-1}F(s)\,ds$$

and, when $F\ge 0$, the right side is a non-decreasing function of t since $\alpha+\beta\ge 1$. Write $F = F^+ - F^-$, the positive and negative parts, so that $F^+ + F^- = |F|$. Each of $\frac{I^{\alpha+\beta}F^\pm(t)}{t^{\alpha+\beta-1}}$ is a non-decreasing function of t, bounded above by $\frac{I^{\alpha+\beta}|F|(t)}{t^{\alpha+\beta-1}} \le M_2$, hence each has a limit as $t\to\infty$, say $L^\pm$. Thus $\frac{I^{\alpha+\beta}F(t)}{t^{\alpha+\beta-1}} \to L := L^+ - L^-$ as $t\to\infty$. (This is simpler than applying the Dominated Convergence Theorem which also works.)
In total we have, for $L = \frac{1}{\Gamma(\alpha+\beta)}\int_0^\infty F(s)\,ds$,

$$u(t) - \big(u_0 + \frac{b_1}{\Gamma(\beta+1)}t^\beta + Lt^{\alpha+\beta-1}\big) \to 0 \text{ as } t\to\infty. \tag{8.12}$$

□

Remark 8.2. *The asymptotic behaviour does not depend explicitly on γ but the hypotheses on ϕ have some γ dependence. The term with t^β dominates in* (8.12) *as*

$t \to \infty$ since $\alpha \le 1$ so we have $u(t) \sim \frac{b_1}{\Gamma(\beta+1)} t^{\beta}$ when $\alpha < 1$, but (8.12) *is a more precise conclusion.*

Remark 8.3. *A similar problem with $f = f(t, u(t), u'(t), D_*^{\beta} u(t))$ is studied in* [14, Theorem 2] *under some hypotheses and it is shown, for that case, that, $u(t)/t$ and $x'(t)$ have a limit as $t \to \infty$. The result uses Hölder's inequality and the Bihari inequality. Also higher order equations are studied in this paper.*
The special case of $\alpha = 1$ and $0 < \gamma < \beta < 1$ is studied in [11, Theorem 5.2] *under different hypotheses and they conclude that $u(t)/t^{\beta}$ has a limit as $t \to \infty$, which is the same as our conclusion.*

9. Conclusion

We have shown that, by using a new fractional Gronwall inequality (in terms of exponential functions) and the classical Gronwall inequality, we can tackle many problems concerning the asymptotic behaviour of solutions of equations involving fractional integrals with nonlinearities possibly depending on fractional derivatives, under realistic and reasonable hypotheses. We do not need a different type of inequality for each problem.

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