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HOMOGENIZATION OF BOUNDARY OPTIMAL CONTROL PROBLEM

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Abstract. In this article, we study the asymptotic behavior of solutions to some optimal control problems, governed by an elliptic boundary value problem with Robin boundary conditions in a periodically perforated domain. The coefficients of the differential operator in the state equation and in the cost-functional are rapidly oscillating. We also study the boundary homogenization of some optimal control problems.

1. Introduction

In this article, we study the convergence of solutions of an optimal control problem governed by a second order elliptic boundary value problem in a periodically perforated domain. The sizes of the holes are same as the period, the holes can intersect the boundary of the domain. The coefficients of the state equation and the cost-functional are rapidly oscillating. The cost-functional involves a Dirichlet type integral of the state function. We prescribe a linear Robin condition on the boundary of holes and the homogeneous Dirichlet condition on the external boundary. The Robin conditions appear in several physical situations such as chemical reactive flows [12] or climatization [20]. We use periodic unfolding method for the homogenization.

Periodic unfolding method was introduced for the perforated domain in [8]. In general, one does not need any extension operator in this method, which makes things simpler while dealing with problems involving non-homogeneous boundary conditions.

A version of our problem was studied by Kesavan et al. [16] in a perforated domain. Then Muthukumar et al. [19] studied this problem in a periodically perforated domain using two-scale convergence. Recently Diaz et al. [13] considered an optimal control problem in a perforated domain for the case of critically small holes, which after homogenization, gives rise to strange terms in the limit equations. Cabarrubias [5] studied similar problems using unfolding method, where they prove the energy convergence with L^2-cost functional only. However in this paper, we establish the energy convergence associated with Dirichlet-cost functional.

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In homogenization of state and adjoint equation with of Robin's boundary condition in a periodically perforated domain, we face two main difficulties. The first one arises in obtaining the a-priori estimates for the state and adjoint variables. This is because of presence of some surface terms due to Robin boundary conditions, thanks to the boundary unfolding operators (cf. Section 2), we overcame this.

The second difficulty lies in the homogenization of the adjoint equation for which we used the special cell problems introduced by Allaire [1] and further used in [19]. The presence of the Robin conditions on the boundary of holes, and the cost-functional (1.3) involving gradients and the oscillating coefficients B_ε together led us to play repeatedly with these cell problems, which contributed the nontrivial terms in the homogenized equation. Our paper generalizes the existing results for the elliptic case in a periodically perforated domain with a more general condition (Robin conditions) on the boundary of holes. Our second main result comprises of studying the boundary homogenization of some optimal control problems.

The organization of this article is as follows: In Section 1, we introduce the problem and the method used. In Section 1.1, we give the preliminaries and briefly describe the setting of the problem and the optimality conditions. Section 2 is the brief review of the periodic unfolding method for the perforated domain. In Section 3, we introduce some cell problems, state and prove our first main result Theorem 3.2. We also establish Theorem 3.3, the ellipticity of the bilinear form defined by perturbed matrix $B^{\#}$ (see (3.6)). In Section 3.1, we observe the existence of unique optimal controls over certain convex sets. In Section 4, we study our second main result where we study the boundary homogenization of some optimal control problems, in a sense that control is acted upon on a part of the external boundary.

1.1. **Notation and problem setting.** Suppose Ω is an open bounded set of $\mathbb{R}^N$ $(N \geq 2)$ with a Lipschitz continuous boundary $\partial\Omega$ such that $|\partial\Omega| = 0$. Let Y, T and Y^* be as follows: $Y = (0,1)^N$ is a reference cell, or more generally a set having the paving property with respect to a basis $(b_1, \dots, b_N)$ defining the periods,

$$Y = \big\{ y \in \mathbb{R}^N : y = \sum_{i=1}^N y_i b_i,\ (y_1, y_2, \dots, y_N) \in (0,1)^N \big\}.$$

$T \subset \overline{Y}$ is an open set with Lipschitz boundary with finite number of connected components such that ∂T does not contain the summits of Y. Perforated reference cell $Y^* = Y \backslash \overline{T}$ is a connected open set. Let $\{\varepsilon\}$ be a positive sequence that converges to zero and we set

$$G = \{\xi \in \mathbb{R}^N : \xi = \sum_{i=1}^N k_i b_i\ (k_1, \dots, k_N) \in \mathbb{Z}^N\}, \quad \Xi_\varepsilon = \{\xi \in G : \varepsilon(\xi + Y) \subset \Omega\}.$$

The perforated domain Ω_ε^* is defined by

$$\Omega_\varepsilon^* = \Omega \backslash T_\varepsilon, \quad \text{where } T_\varepsilon = \cup_{\xi \in G} \varepsilon(\xi + T).$$

Following the notation introduced in [9] for the periodic unfolding method in perforated domains, we set

$$\widehat{\Omega}_\varepsilon = \text{interior}\Big(\cup_{\xi \in \Xi_\varepsilon} \varepsilon(\xi + \overline{Y}) \Big), \quad \Lambda_\varepsilon = \Omega \backslash \widehat{\Omega}_\varepsilon.$$

By construction, $\widehat{\Omega}_\varepsilon$ is the interior of the largest union of $\varepsilon(\xi + \overline{Y})$ cells such that $\varepsilon(\xi + Y)$ is included in Ω and Λ_ε is the subset of Ω containing the parts from the

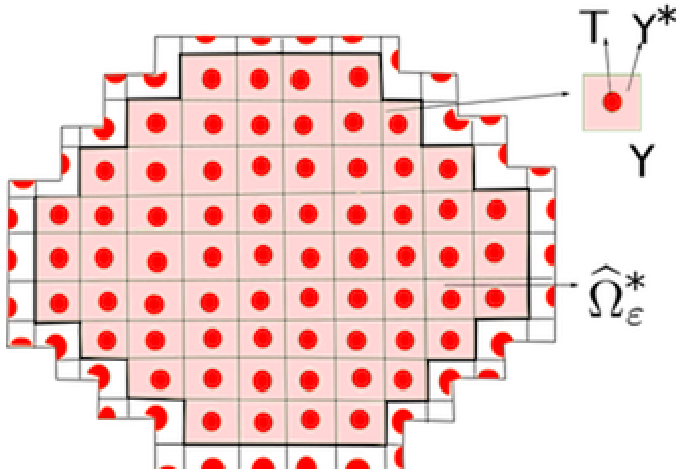

FIGURE 1. Perforated domain Ω_ε^* and the reference set Y.

$\varepsilon(\xi+\overline{Y})$ cells intersecting the boundary $\partial\Omega$. We define the corresponding perforated sets as

$$\widehat{\Omega}_\varepsilon^* = \widehat{\Omega}_\varepsilon \backslash T_\varepsilon, \quad \Lambda_\varepsilon^* = \Omega_\varepsilon^* \backslash \widehat{\Omega}_\varepsilon^*.$$

We decompose the boundary of the perforated domain Ω_ε^* as (see Figure 1)

$$\partial\Omega_\varepsilon^* = \Gamma_0^\varepsilon \cup \Gamma_1^\varepsilon, \quad \text{where } \Gamma_1^\varepsilon = \partial\widehat{\Omega}_\varepsilon^* \cap \partial T_\varepsilon \text{ and } \Gamma_0^\varepsilon = \partial\Omega_\varepsilon^* \backslash \Gamma_1^\varepsilon,$$

so that Γ_1^ε is the boundary of set of holes included in $\widehat{\Omega}_\varepsilon$.

Boundary of holes inside Ω_ε^* is Γ_1^ε, the remaining part including holes inside dark boundary is Λ_ε^* and the boundary of holes contained in this part together with the external boundary $\partial\Omega$ is Γ_0^ε. Hence Ω_ε^* is a periodically perforated domain where the size of the holes are of same order as the period.

We shall use the following notation throughout the paper.

- $|E|$ denotes the Lebesgue measure of the measurable set E.
- $Y^* = Y \setminus T$.
- $\Theta = |Y^*|/|Y|$ the proportion of the material.
- $\mathcal{M}_Y(v)$ is the mean value of v over the measurable set Y.
- χ_E is the characteristic function of the set E.
- $\widetilde{u}$ is the extension by zero on E of a function u defined on $E_\varepsilon(= E\cap\Omega_\varepsilon)$.
- $(n_\varepsilon) = (n_\varepsilon^i)_{i=1}^N$ the unit external normal vector with respect to Ω_ε.
- $A_\varepsilon(x) = A\Big(\frac{x}{\varepsilon}\Big)$ a.e. in Ω, for any $\varepsilon > 0$.
- $\|u\|_{0,E}$ and $\|u\|_{1,E}$, represents respectively L^2 and H^1-norms defined over the set E.
- $C^\infty_{\rm per}(Y)$ is a subset of $C^\infty(\mathbb{R}^N)$, and it consists of Y-periodic functions.
- $H^1_{\rm per}(Y^*)$ is the closure of $C^\infty_{\rm per}(Y)$ with respect to H^1-norm.
- $H^1_{\rm per}(Y^*)/\mathbb{R}$ is the space of equivalence classes defined by: '$u \simeq v' \Leftrightarrow u - v$ is constant, for all $u, v \in H^1_{\rm per}(Y^*)$.
- $L^2(\Omega; H^1_{\rm per}(Y^*))$ is the space of functions $\mathbf{f}$ taking $x \in \Omega$ to $f(x,\cdot) \in H^1_{\rm per}(Y^*)$ and $\|f(x,\cdot)\|_{H^1_{\rm per}(Y^*)} \in L^2(\Omega)$.

and the constants at different places are denoted by C, which are independent of ε. Note that

$$\chi_{\Omega_\varepsilon^*} \rightharpoonup \Theta = |Y^*|/|Y| \quad \text{weak}^* \text{ in } L^\infty(\Omega).$$

Now we discuss the setting of the problem. Let $U_{\rm ad}^\varepsilon \subset L^2(\Omega_\varepsilon^*)$ be a closed, convex subset. For given constants $0 < \alpha_m \leq \alpha_M$, we denote by $\mathcal{M}(\alpha_m, \alpha_M, \Omega)$ the set of all $N \times N$ matrices $A = A(x)$ such that

$$\begin{gathered} A \in L^\infty(\Omega)^{N\times N}, \\ (A(x)\xi, \xi) \geq \alpha_m|\xi|^2 \text{ and } |A(x)\xi| \leq \alpha_M|\xi|, \ \forall \xi \in \mathbb{R}^N \text{ and a.e. } x \in \Omega. \end{gathered} \tag{1.1}$$

Let $A_\varepsilon \in \mathcal{M}(\alpha_m, \alpha_M, \Omega)$, $B_\varepsilon \in \mathcal{M}(\beta_m, \beta_M, \Omega)$, where $0 < \beta_m \leq \beta_M$ are given constants.

Assume that B_ε is symmetric for every $\varepsilon > 0$. We consider the optimal control problem governed by the boundary value problem

$$\begin{gathered} -\operatorname{div}(A_\varepsilon \nabla u_\varepsilon) = f + \theta_\varepsilon \quad \text{in } \Omega_\varepsilon^*, \\ A_\varepsilon \nabla u_\varepsilon \cdot n_\varepsilon + h\varepsilon u_\varepsilon = \varepsilon g^\varepsilon \quad \text{on } \Gamma_1^\varepsilon, \\ u^\varepsilon = 0 \quad \text{on } \Gamma_0^\varepsilon, \end{gathered} \tag{1.2}$$

where $f \in L^2(\Omega)$, $\theta_\varepsilon \in L^2(\Omega_\varepsilon^*)$, n_ε is unit outward normal to Γ_1^ε, h is a real, positive number, $g^\varepsilon(x) = g(x/\varepsilon)$, where g is Y-periodic function in $L^2(\partial T)$. For $N_0 > 0$, the following cost functional is associated with (1.2),

$$J_\varepsilon(\theta_\varepsilon) = \frac{1}{2}\int_{\Omega_\varepsilon^*} B_\varepsilon \nabla u_\varepsilon \nabla u_\varepsilon \, dx + \frac{N_0}{2}\int_{\Omega_\varepsilon^*} \theta_\varepsilon^2 \, dx. \tag{1.3}$$

Then the optimal control problem given by (1.2) and (1.3) admits a unique solution $\theta_\varepsilon^* \in U_{\rm ad}^\varepsilon$ minimizing the cost-functional (1.3) over $U_{\rm ad}^\varepsilon$. We wish to study the limiting behavior of θ_ε^* as $\varepsilon \to 0$. Further if $\widetilde{\theta}_\varepsilon^* \rightharpoonup \theta_0^*$ weakly in $L^2(\Omega)$, we would like to characterize θ_0^* as the optimal control of a similar problem in a fixed domain Ω. Using periodic unfolding, we study the homogenization of state (1.2) and its adjoint equations in periodically perforated domain, when the holes are of same size as period.

We introduce the space

$$V_\varepsilon = \{v \in H^1(\Omega_\varepsilon^*) : v = 0 \text{ on } \Gamma_0^\varepsilon\}.$$

This is a Banach space equipped with

$$\|v\|_{V_\varepsilon} := \|\nabla v\|_{[L^2(\Omega_\varepsilon^*)]^N}, \quad \forall u \in V_\varepsilon. \tag{1.4}$$

The weak formulation of (1.2) is given as follows: Find $u_\varepsilon \in V_\varepsilon$ solution of

$$\int_{\Omega_\varepsilon^*} A_\varepsilon \nabla u_\varepsilon \nabla \varphi \, dx + h\varepsilon \int_{\Gamma_1^\varepsilon} y_\varepsilon \varphi \, d\sigma(x) = \int_{\Omega_\varepsilon^*} (f+\theta)\varphi \, dx + \varepsilon \int_{\Gamma_1^\varepsilon} g^\varepsilon \varphi \, d\sigma(x), \tag{1.5}$$

for all $\varphi \in V_\varepsilon$.

The assumption that ∂T is Lipschitz continuous, is necessary, in order to write surface integrals on the boundary of holes, appearing in the variational formulation of the problem. We obtain the following result as an application of Lax Milgram theorem.

Theorem 1.1. *There exists a unique solution of* (1.5) *when the assumptions on the data stated after* (1.2) *holds.*

1.2. **Optimality conditions.** The optimality conditions [18, Theorem 1.3] are given by $\int_{\Omega_\varepsilon^*} \frac{\partial J_\varepsilon}{\partial \theta}(\theta-\theta_\varepsilon^*) \geq 0$. Now, in view of [21, Chapter 2] and [15, page 140], these conditions can be rewritten as

$$\int_{\Omega_\varepsilon^*} (p_\varepsilon^* + M\theta_\varepsilon^*)(\theta-\theta_\varepsilon^*) \geq 0,$$

where $p_\varepsilon^* = p_\varepsilon(\theta_\varepsilon^*)$ is the solution of the adjoint equation of (1.2), for $\theta = \theta_\varepsilon^*$,

$$\begin{gathered} -\operatorname{div}({}^t A_\varepsilon \nabla p_\varepsilon) = -\operatorname{div}(B_\varepsilon \nabla u_\varepsilon) \quad \text{in } \Omega_\varepsilon^*, \\ ({}^t A_\varepsilon \nabla p_\varepsilon)\cdot n_\varepsilon - p_\varepsilon h\varepsilon = (B_\varepsilon \nabla u_\varepsilon)\cdot n_\varepsilon \quad \text{on } \Gamma_1^\varepsilon, \\ p_\varepsilon = 0 \quad \text{on } \Gamma_0^\varepsilon. \end{gathered} \tag{1.6}$$

We consider the following optimal control problem for the cost functional (1.3):

$$\big\{ \inf J_\varepsilon(u_\varepsilon,\theta) : (u_\varepsilon,\theta) \in V_\varepsilon \times L^2(\Omega_\varepsilon^*),\ (u_\varepsilon,\theta) \text{ satisfies } (1.2)\big\}. \tag{1.7}$$

We have the following results, which can be proved along the same lines as in [18] and [21, Chapter 2].

Theorem 1.2. *For each $\varepsilon > 0$, the optimal control problem* (1.7) *admits a unique solution.*

Let $u_\varepsilon^* = u_\varepsilon(\theta_\varepsilon^*)$ be the optimal state. Then the characterization of θ_ε^* is given below.

Theorem 1.3. *For $(u_\varepsilon^*, \theta_\varepsilon^*)$, the optimal solution of* (1.7)*, let $p_\varepsilon^* = p_\varepsilon(\theta_\varepsilon^*)$ be the optimal adjoint state. Then the optimal control is given by*

$$\theta_\varepsilon^* = -\frac{1}{M} p_\varepsilon^*. \tag{1.8}$$

2. The periodic unfolding method for perforated domain

In this section, we briefly recall the definitions and properties of the unfolding operator $\mathcal{T}_\varepsilon^*$ and the boundary unfolding operator $\mathcal{T}_\varepsilon^b$. For more details on this topic, we refer to [8].

Let $[x]_Y = \sum_{j=1}^N k_j b_j$ be the unique integer combination of periods such that for any $x \in \mathbb{R}^N$, $x - [x]_Y$ is in Y. Set $\{x\}_Y = x - [x]_Y$. In particular, for any $\varepsilon > 0$,

$$x = \varepsilon\Big(\big[\frac{x}{\varepsilon}\big]_Y + \big\{\frac{x}{\varepsilon}\big\}_Y\Big) \quad \text{for all } x \in \mathbb{R}^N.$$

Definition 2.1. For any Lebesgue-measurable function ϕ on Ω_ε^*, the unfolding operator $\mathcal{T}_\varepsilon^*$ is defined as

$$\mathcal{T}_\varepsilon^*(\phi)(x,y) = \begin{cases} \phi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right), & \text{a.e. for } (x,y) \in \widehat{\Omega}_\varepsilon \times Y^*, \\ 0, & \text{a.e. for } (x,y) \in \Lambda_\varepsilon \times Y^*. \end{cases}$$

In what follows, if ϕ is a function on a domain containing Ω_ε^*, we will use the notation $\mathcal{T}_\varepsilon^*(\phi)$ instead of $\mathcal{T}_\varepsilon^*(\phi|_{\Omega_\varepsilon^*})$.

Proposition 2.2. *Let $p \in (1,+\infty)$. Then*

1. *$\mathcal{T}_\varepsilon^*$ is linear and continuous from $L^p(\Omega_\varepsilon^*)$ to $L^p(\Omega \times Y^*)$.*
2. *$\mathcal{T}_\varepsilon^*(\phi\psi) = \mathcal{T}_\varepsilon^*(\phi)\mathcal{T}_\varepsilon^*(\psi)$ for every ϕ, $\psi \in L^p(\Omega_\varepsilon^*)$.*
3. *For $w \in L^p(\Omega)$, $\mathcal{T}_\varepsilon^*(w) \to w$ strongly in $L^p(\Omega \times Y^*)$.*

(4) *For all* $\phi \in L^1(\Omega_\varepsilon^*)$ *we have*
$$\int_{\widehat{\Omega}_\varepsilon^*} \phi(x)\,dx = \int_{\Omega_\varepsilon^*} \phi(x)\,dx - \int_{\Lambda_\varepsilon^*} \phi(x)\,dx = \frac{1}{|Y|}\int_{\Omega\times Y^*} \mathcal{T}_\varepsilon^*(\phi)(x,y)\,dx\,dy.$$
Moreover, if $\{\phi_\varepsilon\}$ *is a bounded sequence in* $L^r(\Omega_\varepsilon^*)$ *for some* $r>1$*, then*
$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon^*} \phi_\varepsilon(x)\,dx = \lim_{\varepsilon\to 0}\frac{1}{|Y|}\int_{\Omega\times Y^*} \mathcal{T}_\varepsilon^*(\phi_\varepsilon)(x,y)\,dx\,dy.$$
(5) *Let* $\phi_\varepsilon \in L^p(\Omega)$ *be such that* $\phi_\varepsilon \to \phi$ *strongly in* $L^p(\Omega)$*. Then* $\mathcal{T}_\varepsilon^*(\phi_\varepsilon) \to \phi$ *strongly in* $L^p(\Omega\times Y^*)$*.*
(6) *Let* $\varphi \in L^p(Y^*)$ *be a* Y*-periodic function and set* $\varphi_\varepsilon(x) = \varphi\left(\frac{x}{\varepsilon}\right)$*. Then* $\mathcal{T}_\varepsilon^*(\varphi_\varepsilon)(x,y) = \varphi(y)$ *a.e. in* $\widehat{\Omega}_\varepsilon \times Y^*$*.*

Proposition 2.3. *Suppose* Ω *is a bounded open subset of* $\mathbb{R}^N$ *with Lipschitz boundary* $\partial\Omega$*. Let* $w_\varepsilon \in W_0^{1,p}(\Omega_\varepsilon^*; \partial\Omega\cap\partial\Omega_\varepsilon^*)$ *satisfy*
$$\|\nabla w_\varepsilon\|_{L^p(\Omega_\varepsilon)} \le C,$$
where C *is a positive constant independent of* ε*. Then there exist* $w_0 \in W_0^{1,p}(\Omega)$ *and* $\widehat{w} \in L^p(\Omega; W_{per}^{1,p}(Y^*))$ *with* $\mathcal{M}_{Y^*}(\widehat{w}) = 0$*, such that up to a subsequence,*
$$\begin{gathered}\mathcal{T}_\varepsilon^*(w_\varepsilon) \to w_0 \quad \text{strongly in } L^p(\Omega; W^{1,p}(Y^*)),\\ \mathcal{T}_\varepsilon^*(\nabla w_\varepsilon) \rightharpoonup \nabla_x w_0 + \nabla_y \widehat{w} \quad \text{weakly in } \left(L^p(\Omega\times Y^*)\right)^N,\\ \widetilde{w_\varepsilon} \rightharpoonup \Theta w_0 \quad \text{weakly in } L^p(\Omega),\end{gathered} \tag{2.1}$$
as ε *tends to zero, where* $\Theta = |Y^*|/|Y|$*.*

We now recall the definition and properties of the boundary unfolding operator $\mathcal{T}_\varepsilon^b$.

Definition 2.4. For any function φ, Lebesgue-measurable on $\partial\widehat{\Omega}_\varepsilon^* \cap \partial T_\varepsilon$, the boundary unfolding operator is defined as
$$\mathcal{T}_\varepsilon^b(\varphi)(x,y) = \begin{cases}\varphi\left(\varepsilon[\frac{x}{\varepsilon}]_Y + \varepsilon y\right), & \text{a.e. for } (x,y)\in\widehat{\Omega}_\varepsilon\times\partial T,\\ 0, & \text{a.e. for } (x,y)\in\Lambda_\varepsilon\times\partial T.\end{cases}$$

Proposition 2.5 ([8])**.** *Let* $p\in(1,+\infty)$*. Then*
(1) $\mathcal{T}_\varepsilon^b$ *is a linear operator from* $L^p(\partial T_\varepsilon)$ *to* $L^p(\Omega\times\partial T)$*.*
(2) $\mathcal{T}_\varepsilon^b(\phi\psi) = \mathcal{T}_\varepsilon^b(\phi)\mathcal{T}_\varepsilon^b(\psi)$ *for every* ϕ*,* $\psi \in L^p(\partial T_\varepsilon)$*.*
(3) *Let* $\phi \in L^p(\partial T)$ *be a* Y*-periodic function. Set* $\phi_\varepsilon(x) = \phi(\frac{x}{\varepsilon})$*. Then*
$$\mathcal{T}_\varepsilon^b(\phi_\varepsilon)(x,y) = \phi(y) \quad \text{a.e. in } \Omega_\varepsilon^*\times\partial T.$$
(4) *For all* $\phi\in L^1(\partial T_\varepsilon)$*, the integration formula is given by*
$$\int_{\Gamma_1^\varepsilon} \phi(x)\,d\sigma(x) = \frac{1}{\varepsilon|Y|}\int_{\Omega\times\partial T} \mathcal{T}_\varepsilon^b(\phi)(x,y)\,dx\,d\sigma(y).$$
(5) *Let* $\phi\in L^p(\partial T_\varepsilon^*)$*. Then* $\mathcal{T}_\varepsilon^b(\phi)\to\phi$ *strongly in* $L^p(\Omega\times\partial T)$*.*

Proposition 2.6 (see [8, Proposition 5.7])**.** *We have the following convergence results:*
(1) *Let* $\varphi\in L^2(\Omega)$*. Then as* $\varepsilon\to 0$*, one has the convergence*
$$\int_{\mathbb{R}^N\times\partial T} \mathcal{T}_\varepsilon^b(\varphi)(x,y)\,dx\,d\sigma(y) \to \int_{\mathbb{R}^N\times\partial T} \widetilde{\varphi}\,dx\,d\sigma(y). \tag{2.2}$$

(2) *Let* $\varphi \in L^2(\Omega)$. *Then* $\mathcal{T}_\varepsilon^b(\varphi) \to \widetilde{\varphi}$ *strongly in* $L^2(\mathbb{R}^N \times \partial T)$.

(3) *Let* $\varphi_\varepsilon \in L^2(\partial T_\varepsilon)$ *for every* ε, *such that* $\mathcal{T}_\varepsilon^b(\varphi_\varepsilon) \rightharpoonup \widehat{\varphi}$ *weakly in* $L^2(\mathbb{R}^N \times \partial T)$. *Then*

$$\varepsilon \int_{\partial T_\varepsilon} \varphi_\varepsilon \psi \, d\sigma(x) \to \frac{1}{|Y|} \int_{\mathbb{R}^N \times \partial T} \widehat{\varphi}(x,y)\psi(x)\, dx\, d\sigma(y),$$

for all $\psi \in H^1(\Omega)$.

(4) *Let* $\varphi_\varepsilon \in H^1(\Omega_\varepsilon)$ *for every* ε *and* $\widehat{\varphi} \in H^1(\Omega)$ *such that* $\mathcal{T}_\varepsilon^*(\varphi_\varepsilon) \rightharpoonup \widehat{\varphi}$ *weakly in* $L^2_{\rm loc}(\Omega; H^1(Y^*))$. *Then*

$$\mathcal{T}_\varepsilon^b(\varphi_\varepsilon) \rightharpoonup \widehat{\varphi} \quad \textit{weakly in } L^2_{\rm loc}(\Omega; H^{\frac{1}{2}}(\partial T)).$$

Let g be a Y-periodic function in $L^2(\partial T)$ and let $g^\varepsilon(x) = g(x/\varepsilon)$ for all x in the set $\mathbb{R}^N \setminus \cup_{\xi \in \mathbb{Z}^N} \varepsilon(\xi + T)$. Then we have the following two results (cf. [8, Cor 5.4, Prop. 5.6]), which will be needed for the homogenization of some problems.

Proposition 2.7. *For every* $\Phi \in V_\varepsilon$ *and* g^ε *as above, the following inequality holds*

$$\Big| \int_{\Gamma_1^\varepsilon} g\big(\frac{x}{\varepsilon}\big)\Phi(x)\, d\sigma(x)\Big| \le \frac{C}{\varepsilon}\left(|\mathcal{M}_{\partial T}(g)| + \varepsilon\right) \|\nabla \Phi\|_{[L^2(\Omega_\varepsilon^*)]^N}. \tag{2.3}$$

3. Main results

To state and prove our first main result, we need the following hypothesis:

(H1) Let $A_\varepsilon \in \mathcal{M}(\alpha_m, \alpha_M, \Omega)$, $B_\varepsilon \in \mathcal{M}(\beta_m, \beta_M, \Omega)$.

(H2) Let $A = A(x,y) \in \mathcal{M}(\alpha_m, \alpha_M, \Omega \times Y^*)$, $B \in \mathcal{M}(\beta_m, \beta_M, \Omega \times Y^*)$, such that

$$\begin{gathered} \mathcal{T}_\varepsilon^*(A_\varepsilon) \to A(x,y) \quad \text{a.e. in } \Omega \times Y^*, \\ \mathcal{T}_\varepsilon^*(B_\varepsilon) \to B(x,y) \quad \text{a.e. in } \Omega \times Y^*. \end{gathered}$$

(H3) A, B are Y-periodic with respect to the second variable y.

Now we define some cell problems, needed to state our first main results, and for the identification of the limits. Such cell problems were introduced by Allaire [1], and further used by Muthukumar et al. [19].

For $1 \le i \le N$, let $\mu_i \in H^1_{\rm per}(Y^*)/\mathbb{R}$ be the solution of the cell problem

$$\begin{gathered} -\operatorname{div}(A(x,y)[\nabla_y \mu_i(x,y) + e_i]) = 0 \quad \text{in } Y^*, \\ A(x,y)[\nabla_y \mu_i(x,y) + e_i]\cdot \nu = 0 \quad \text{on } \partial Y^* \setminus \partial Y, \\ y \mapsto \mu_i(x,y) \quad \text{is } Y\text{-periodic}. \end{gathered} \tag{3.1}$$

Let $\omega_i \in H^1_{\rm per}(Y^*)/\mathbb{R}$ be the solution of the adjoint cell problem

$$\begin{gathered} -\operatorname{div}({}^tA(x,y)[\nabla_y \omega_i(x,y) + e_i]) = 0 \quad \text{in } Y^*, \\ {}^tA(x,y)[\nabla_y \omega_i(x,y) + e_i]\cdot \nu = 0 \quad \text{on } \partial Y^* \setminus \partial Y, \\ y \mapsto \omega_i(x,y) \quad \text{is } Y\text{-periodic}. \end{gathered} \tag{3.2}$$

Let $\psi_i \in H^1_{\rm per}(Y^*)/\mathbb{R}$, be the solution of the cell problem

$$\begin{gathered} -\operatorname{div}\big({}^tA(x,y)\nabla_y \psi_i(x,y) - B(x,y)(\nabla_y \mu_i + e_i)\big) = 0 \quad \text{in } Y^*, \\ \big({}^tA(x,y)\nabla_y \psi_i(x,y) - B(x,y)(\nabla_y \mu_i + e_i)\big)\cdot \nu = 0, \quad \text{on } \partial Y^* \setminus \partial Y, \\ y \mapsto \mu_i(x,y) \quad \text{is } Y\text{-periodic}. \end{gathered} \tag{3.3}$$

Here $\{e_1, e_2, \dots, e_N\}$ is the standard basis of $\mathbb{R}^N$. The homogenized matrix A_0 is defined as

$$(A_0)_{ij} = \int_{Y^*} A(x,y)[\nabla_y \mu_i(x,y) + e_i] \cdot [\nabla_y \mu_j + e_j]\, dy. \tag{3.4}$$

The homogenized transposed matrix ${}^tA_0$ is

$$({}^tA_0)_{ij} = \int_{Y^*} {}^tA(x,y)[\nabla_y \omega_i(x,y) + e_i] \cdot [\nabla_y \omega_j(x,y) + e_j]\, dy, \tag{3.5}$$

and the perturbed matrix $B^{\#}$ is

$$(B^{\#})e_i = \int_{Y^*} \{B(x,y)[\nabla_y \mu_i(x,y) + e_i] - {}^tA(x,y)\nabla_y \psi_i(x,y)\}\, dy. \tag{3.6}$$

To see the forms of the above homogenized matrices, we refer to [7, Propositions 6.8, 6.9]. We now state a Poincaré inequality result for the perforated domain Ω_ε^* (when the holes can even meet the boundary).

Lemma 3.1 (see [2, Lemma A.4])**.** *There exists a positive constant C, independent of ε, such that*

$$\|u\|_{0,\Omega_\varepsilon^*} \le C\|\nabla u\|_{[L^2(\Omega_\varepsilon^*)]^N}. \tag{3.7}$$

The above lemma gives an equivalent norm on V_ε, as $\|u\|_{V_\varepsilon} = \|\nabla u\|_{0,\Omega_\varepsilon}$. Now, we state our first main result of the paper.

Theorem 3.2. *Assume* (H1)–(H3) *hold. Let $u_\varepsilon \in V_\varepsilon$ and $p_\varepsilon \in V_\varepsilon$ be the solutions of the following systems, respectively*

$$\begin{gathered} -\operatorname{div}(A_\varepsilon \nabla u_\varepsilon) = f + \theta_\varepsilon \quad \text{in } \Omega_\varepsilon^*, \\ A_\varepsilon \nabla u_\varepsilon \cdot n_\varepsilon + h\varepsilon u_\varepsilon = \varepsilon g^\varepsilon \quad \text{on } \Gamma_1^\varepsilon, \\ u^\varepsilon = 0 \quad \text{on } \Gamma_0^\varepsilon, \end{gathered} \tag{3.8}$$

and

$$\begin{gathered} -\operatorname{div}\big({}^tA_\varepsilon \nabla p_\varepsilon - B_\varepsilon \nabla u_\varepsilon\big) = 0 \quad \text{in } \Omega_\varepsilon^*, \\ ({}^tA_\varepsilon \nabla p_\varepsilon - B_\varepsilon \nabla u_\varepsilon) \cdot n_\varepsilon = p_\varepsilon h\varepsilon \quad \text{on } \Gamma_1^\varepsilon, \\ p_\varepsilon = 0 \quad \text{on } \Gamma_0^\varepsilon, \end{gathered} \tag{3.9}$$

where $f \in L^2(\Omega)$ is fixed, $\theta_\varepsilon \in L^2(\Omega_\varepsilon)$, $h > 0$ is a real number, $g^\varepsilon(x) = g(x/\varepsilon)$, where g is Y-periodic function in $L^2(\partial T)$. Then there exist functions u_0, $p_0 \in H_0^1(\Omega)$ and $\theta_0 \in L^2(\Omega)$ such that for a subsequence, we have

$$\begin{gathered} \widetilde{u}_\varepsilon \rightharpoonup \Theta u_0 \quad \text{weakly in } L^2(\Omega), \\ \widetilde{p}_\varepsilon \rightharpoonup \Theta p_0 \quad \text{weakly in } L^2(\Omega), \\ \widetilde{\theta}_\varepsilon \rightharpoonup \Theta \theta_0 \quad \text{weakly in } L^2(\Omega). \end{gathered} \tag{3.10}$$

The functions u_0 and p_0 satisfy the following systems respectively,

$$\begin{gathered} -\operatorname{div}(A_0 \nabla u_0) + h\frac{|\partial T|}{|Y|} u_0 = \Theta(f + \theta_0) + \frac{|\partial T|}{|Y|} \mathcal{M}_{\partial T}(g) \quad \text{in } \Omega, \\ u_0 = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.11}$$

and

$$\begin{gathered} -\operatorname{div}({}^tA_0 \nabla p_0 - B^{\#} \nabla u_0) = h\frac{|\partial T|}{|Y|} p_0 \quad \text{in } \Omega, \\ p_0 = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.12}$$

where $\Theta = |Y^*|/|Y|$, the matrices A_0, ${}^tA_0$ and $B^\#$ are defined by (3.4), (3.5), *and* (3.6) *respectively.*

Proof. We prove this theorem in three main steps. In the first step, we establish the convergence of solutions (3.10). In the second step, we homogenize the state equation, and in the third step, we homogenize the adjoint equation using the cell problems introduced in the beginning of this section.

Step 1: Convergence of solutions of (3.8) and (3.9). Taking $\varphi \in \mathcal{D}(\Omega)$ as a test function in (3.8), we have

$$\int_{\Omega_\varepsilon^*} A_\varepsilon \nabla u_\varepsilon \nabla\varphi\,dx + h\varepsilon\int_{\partial T_\varepsilon} u_\varepsilon\varphi\,d\sigma(x) = \int_{\Omega_\varepsilon^*}(f+\theta_\varepsilon)\varphi\,dx + \varepsilon\int_{\partial T_\varepsilon} g^\varepsilon\varphi\,d\sigma(x). \tag{3.13}$$

We first establish "a priori estimates" for u_ε. Considering u_ε as a test-function in (3.13), we have

$$\begin{aligned}&\alpha_m\|\nabla u_\varepsilon\|^2_{[L^2(\Omega_\varepsilon^*)]^N} + h\varepsilon\|u_\varepsilon\|^2_{L^2(\partial T_\varepsilon)}\\ &\leq (\|f\|_{L^2(\Omega)} + \|\theta_\varepsilon\|_{L^2(\Omega_\varepsilon^*)})\|u_\varepsilon\|_{0,\Omega_\varepsilon^*} + \varepsilon\Big|\int_{\partial T_\varepsilon} g^\varepsilon u_\varepsilon\,d\sigma\Big|.\end{aligned} \tag{3.14}$$

Using the Poincaré inequality (3.7) and Proposition 2.7 in (3.14), we obtain

$$\alpha_m\|\nabla u_\varepsilon\|^2_{[L^2(\Omega_\varepsilon^*)]^N} \leq C(1+\varepsilon+|\mathcal{M}_{\partial T}(g)|)\|\nabla u_\varepsilon\|_{[L^2(\Omega_\varepsilon^*)]^N}.$$

Hence we have

$$\|u_\varepsilon\|_{H^1(\Omega_\varepsilon^*)} \leq \widetilde{C}. \tag{3.15}$$

Thus we deduce that there exists $U_0 \in H^1(\Omega)$ such that

$$\widetilde{u_\varepsilon} \rightharpoonup U_0 \quad \text{weakly in } L^2(\Omega).$$

By [8, Theorem 3.2 (2, 3, 4)], there exist $u_0 \in H^1_0(\Omega)$ and $\widehat{u} \in L^2(\Omega, H^1_{\rm per}(Y^*))$ such that

- (i) $\mathcal{T}_\varepsilon^*(u_\varepsilon) \rightharpoonup u_0$ weakly in $L^2_{\rm loc}(\Omega; H^1(Y^*))$,
- (ii) $\mathcal{T}_\varepsilon^*(\nabla_x(u_\varepsilon)) \rightharpoonup \nabla_x u_0 + \nabla_y\widehat{u}$ weakly in $L^2_{\rm loc}(Omega\times Y^*)$.

To identify U_0, let $\varphi\in\mathcal{D}(\Omega)$ and consider

$$\int_\Omega \widetilde{u}_\varepsilon\varphi\,dx = \int_{\Omega_\varepsilon^*} u_\varepsilon\varphi\,dx = \frac{1}{|Y|}\int_{\Omega\times Y^*}\mathcal{T}_\varepsilon^*(u_\varepsilon)\mathcal{T}_\varepsilon^*(\varphi)\,dx\,dy.$$

The former convergences yield

$$\int_{\Omega_\varepsilon^*} u_\varepsilon\varphi\,dx \to \frac{1}{|Y|}\int_{\Omega\times Y^*} u_0(x)\varphi(x)\,dx\,dy = \frac{|Y^*|}{|Y|}\int_\Omega u_0\varphi\,dx = \Theta\int_\Omega u_0\varphi\,dx.$$

But we know that $\int_\Omega \widetilde{u}_\varepsilon\varphi\,dx \rightharpoonup \int_\Omega U_0\varphi\,dx$, as $\varepsilon\to 0$. Consequently, we have $U_0 = \Theta u_0$ and we obtain the first convergence of (3.10). We note that u_0 is a function of x only. Letting p_ε as a test function in the adjoint eq. (3.9), we have

$$\int_{\Omega_\varepsilon^*} {}^tA_\varepsilon\nabla p_\varepsilon\nabla p_\varepsilon\,dx - \int_{\Omega_\varepsilon^*} B_\varepsilon\nabla u_\varepsilon\nabla p_\varepsilon\,dx = h\varepsilon\int_{\partial T_\varepsilon} p_\varepsilon\,d\sigma(x).$$

By ellipticity of A_ε, boundedness of B_ε, and Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\beta_m\|\nabla p_\varepsilon\|^2_{[L^2(\Omega_\varepsilon^*)]^N}\\ &\leq \beta_M\|\nabla u_\varepsilon\|_{[L^2(\Omega_\varepsilon^*)]^N}\|\nabla p_\varepsilon\|_{[L^2(\Omega_\varepsilon^*)]^N} + h\varepsilon\Big|\int_{\partial T_\varepsilon} 1.p_\varepsilon\,d\sigma(x)\Big|.\end{aligned} \tag{3.16}$$

It follows by (3.15), Proposition 2.7 (with $g^\varepsilon{=}1$) and Poincaré inequality (3.7) in the last term, that

$$\beta_m \|\nabla p_\varepsilon\|^2_{[L^2(\Omega^*_\varepsilon)]^N} \le (\beta_M C + h\varepsilon)\|\nabla p_\varepsilon\|_{[L^2(\Omega^*_\varepsilon)]^N}.$$

This implies that

$$\|p_\varepsilon\|_{H^1(\Omega^*_\varepsilon)} \le C. \tag{3.17}$$

A similar argument, as to get (3.10)(i), would give us the existence of $p_0 \in H^1_0(\Omega)$ such that (3.10)(ii) holds. Now from (1.8) and (3.17), we have

$$\|\widetilde{\theta}_\varepsilon\|_{L^2(\Omega^*_\varepsilon)} \le C,$$

where is a positive constant. Thus there exists $\theta_0 \in L^2(\Omega)$ such that the convergence (3.10)(iii) holds.

Step2: Homogenization of the state equation (3.8). We pass to the limit in (3.13), as $\varepsilon \to 0$. Using the Proposition 2.2, Proposition 2.5(4) and by unfolding in (3.13), we obtain

$$\begin{aligned} &\int_{\widetilde{\Omega}\times Y^*} \mathcal{T}^*_\varepsilon(A_\varepsilon)\mathcal{T}^*_\varepsilon(\nabla u_\varepsilon)\mathcal{T}^*_\varepsilon(\nabla\varphi)\,dx\,dy + h\int_{\widetilde{\Omega}\times\partial T} \mathcal{T}^b_\varepsilon(u_\varepsilon)\mathcal{T}^b_\varepsilon(\varphi)\,dx\,d\sigma(y)\\ &= \int_{\widetilde{\Omega}\times Y^*} \mathcal{T}^*_\varepsilon(f)\mathcal{T}^*_\varepsilon(\varphi)\,dx\,dy + \int_{\widetilde{\Omega}\times Y^*} \mathcal{T}^*_\varepsilon(\theta_\varepsilon)\mathcal{T}^*_\varepsilon(\varphi)\,dx\,dy\\ &\quad + \int_{\widetilde{\Omega}\times\partial T} \mathcal{T}^b_\varepsilon(g^\varepsilon)\mathcal{T}^b_\varepsilon(\varphi)\,dx\,d\sigma(y). \end{aligned} \tag{3.18}$$

From Proposition 2.2(3) and Proposition 2.6(2, 4), we obtain by passing to the limit that

$$\begin{aligned} &\int_{\Omega\times Y^*} A(x,y)(\nabla_x u_0 + \nabla_y \widehat{u})\nabla\varphi(x)\,dx\,dy + h\int_{\Omega\times\partial T} u_0\varphi\,dx\,d\sigma(y)\\ &= \int_{\Omega\times Y^*} (f+\theta_0)\varphi\,dx\,dy + \int_{\Omega\times\partial T} g\varphi\,dx\,d\sigma(y). \end{aligned}$$

Since u_0, f, θ_0, and φ are functions of x only, we have

$$\begin{aligned} &\int_{\Omega\times Y^*} A(x,y)(\nabla_x u_0 + \nabla_y \widehat{u})\nabla\varphi(x)\,dx\,dy + h|\partial T|\int_{\Omega\times\partial T} u_0\varphi\,dx\,d\sigma(y)\\ &= |Y^*|\int_\Omega (f+\theta_0)\varphi\,dx + \int_\Omega \varphi\,dx\int_{\partial T} g\,d\sigma(y). \end{aligned} \tag{3.19}$$

By denseness, this result is valid for all $\varphi \in H^1_0(\Omega)$. We define a new test-function

$$v_\varepsilon = \varepsilon\varphi(x)\xi\big(\frac{x}{\varepsilon}\big), \quad \text{where } \varphi \in \mathcal{D}(\Omega),\ \xi \in H^1_{\rm per}(Y^*).$$

We observe that $\mathcal{T}^*_\varepsilon(v_\varepsilon) = \varepsilon\mathcal{T}^*_\varepsilon(\varphi)\xi$ and $\nabla v_\varepsilon = \varepsilon\nabla_x\varphi\xi\big(\frac{\cdot}{\varepsilon}\big) + \varphi\nabla_y\xi\big(\frac{\cdot}{\varepsilon}\big)$. Hence

$$\begin{gathered} \mathcal{T}^*_\varepsilon(v_\varepsilon) \rightharpoonup 0 \quad \text{weakly in } L^2(\Omega; H^1(Y^*)),\\ \mathcal{T}^*_\varepsilon(\nabla v_\varepsilon) \rightharpoonup \varphi\nabla_y\xi \quad \text{weakly in } L^2(\Omega\times Y^*). \end{gathered}$$

Taking v_ε as a test-function in (3.8) and by unfolding operator in the resulting variational-formulation, we obtain

$$\begin{aligned} &\int_{\widetilde{\Omega}\times Y^*} \mathcal{T}^*_\varepsilon(A_\varepsilon)\mathcal{T}^*_\varepsilon(\nabla u_\varepsilon)\mathcal{T}^*_\varepsilon(\nabla v_\varepsilon)\,dx\,dy + h\int_{\widetilde{\Omega}\times\partial T} \mathcal{T}^b_\varepsilon(u_\varepsilon)\mathcal{T}^b_\varepsilon(v_\varepsilon)\,dx\,d\sigma(y)\\ &= \int_{\widetilde{\Omega}\times Y^*} \mathcal{T}^*_\varepsilon(f)\mathcal{T}^*_\varepsilon(v_\varepsilon)\,dx\,dy + \int_{\widetilde{\Omega}\times Y^*} \mathcal{T}^*_\varepsilon(\theta_\varepsilon)\mathcal{T}^*_\varepsilon(v_\varepsilon)\,dx\,dy \end{aligned}$$

$$+\int_{\widetilde{\Omega}\times\partial T}\mathcal{T}_\varepsilon^b(g^\varepsilon)\mathcal{T}_\varepsilon^b(v_\varepsilon)\,dx\,d\sigma(y).$$

When passing to the limit, we obtain

$$\int_{\Omega\times Y^*}A(x,y)(\nabla_x u_0+\nabla_y\widehat{u})\varphi(x)\nabla_y\xi(y)\,dx\,dy=0.$$

It is known from [4], that the space $C_c^\infty(\Omega)\bigotimes C_{\rm per}^\infty(Y)$ consisting of all finite sums of functions of the form $g(x)h(y)$ ($g\in C_c^\infty(\Omega), h\in C_{\rm per}^\infty(Y)$) is dense in $L^p(\Omega\times Y)$, for $1\le p<\infty$. It follows by denseness that for all $\zeta\in L^2(\Omega;H^1_{\rm per}(Y^*))$, we have

$$\int_{\Omega\times Y^*}A(x,y)(\nabla_x u_0+\nabla_y\widehat{u})\nabla_y\zeta(x,y)\,dx\,dy=0. \tag{3.20}$$

Finally adding (3.19) (for $\varphi\in H_0^1(\Omega)$) and (3.20), we obtain the following variational formulation of (3.8): For all $\varphi\in H_0^1(\Omega)$ and all $\zeta\in L^2(\Omega;H^1_{\rm per}(Y^*))$,

$$\begin{aligned}&\int_{\Omega\times Y^*}{}^tA(x,y)(\nabla_x u_0+\nabla_y\widehat{u})(\nabla_x\varphi+\nabla_y\zeta(x,y))\,dx\,dy+h|\partial T|\int_\Omega u_0\varphi\,dx\\&=|Y^*|\int_\Omega(f+\theta_0)\varphi\,dx+\int_\Omega\varphi\,dx\int_{\partial T}g\,d\sigma(y),\end{aligned}\tag{3.21}$$

where $(u_0,\widehat{u})$ is the unique solution of (3.21). The proof that u_0 is the solution of (3.11) follows along the same lines as in the proof given in [7, Chapter 9].

Taking successively $\varphi=0$ and $\zeta=0$ in (3.21) yields

$$\widehat{u}(x,y)=\sum_{i=1}^N\mu_i(x,y)\frac{\partial u_0}{\partial x_i}(x), \tag{3.22}$$

where μ_i satisfies the cell problem (3.1). Substituting the value of $\widehat{u}$ obtained in (3.22) in the variational formulation (3.21) and integrating by parts with respect to x, after putting $\zeta=0$, we obtain the expression for A_0. By a standard argument [7], it is easily seen that the matrix A_0 is elliptic. Then the uniqueness of u_0 as a solution of state equation (3.11), follows from the Lax-Milgram theorem.

Step3: Homogenization of the adjoint equation. Taking $\varphi\in\mathcal{D}(\Omega)$ as a test-function in (3.9), we have

$$\int_{\Omega_\varepsilon^*}{}^tA_\varepsilon\nabla p_\varepsilon\nabla\varphi\,dx-\int_{\Omega_\varepsilon^*}B_\varepsilon\nabla u_\varepsilon\nabla\varphi\,dx-\int_{\partial\Omega_\varepsilon^*}p_\varepsilon h\varepsilon\varphi\,dx=0.$$

By unfolding and using Propositions 2.5 and 5.2, we obtain

$$\begin{aligned}&\int_{\widetilde{\Omega}_\varepsilon^*\times Y^*}\mathcal{T}_\varepsilon^*({}^tA_\varepsilon)\mathcal{T}_\varepsilon^*(\nabla p_\varepsilon)\mathcal{T}_\varepsilon^*(\nabla\varphi)\,dx\,dy-\int_{\widetilde{\Omega}_\varepsilon^*\times Y^*}\mathcal{T}_\varepsilon^*(B_\varepsilon)\mathcal{T}_\varepsilon^*(\nabla u_\varepsilon)\mathcal{T}_\varepsilon^*(\nabla\varphi)\,dx\,dy\\&-\int_{\widetilde{\Omega}_\varepsilon^*\times\partial T}\mathcal{T}_\varepsilon^b(p_\varepsilon)h\mathcal{T}_\varepsilon^b(\varphi)\,dx\,d\sigma(y)=0.\end{aligned}$$

We pass the limit as $\varepsilon\to0$ and obtain

$$\begin{aligned}&\int_{\Omega\times Y^*}{}^tA(x,y)(\nabla_x p_0+\nabla_y\widehat{p})\nabla\varphi\,dx\,dy\\&-\int_{\Omega\times Y^*}B(x,y)(\nabla_x u_0+\nabla_y\widehat{u})\nabla\varphi\,dx\,dy\\&=h|\partial T|\int_\Omega p_0\varphi\,dx.\end{aligned}\tag{3.23}$$

Next we take $v_\varepsilon = \varepsilon\varphi\xi\Big(\frac{x}{\varepsilon}\Big)$ as a test-function in (3.9), where $\varphi \in \mathcal{D}(\Omega)$ and $\xi \in H^1_{\rm per}(Y^*)$,

$$\begin{aligned}
&\int_{\widetilde{\Omega}^*_\varepsilon\times Y^*} \mathcal{T}^*_\varepsilon({}^tA_\varepsilon)\mathcal{T}^*_\varepsilon(\nabla p_\varepsilon)\mathcal{T}^*_\varepsilon(\nabla v_\varepsilon)\,dx\,dy - \int_{\widetilde{\Omega}^*_\varepsilon\times Y^*} \mathcal{T}^*_\varepsilon(B_\varepsilon)\mathcal{T}^*_\varepsilon(\nabla u_\varepsilon)\mathcal{T}^*_\varepsilon(\nabla v_\varepsilon)\,dx\,dy\\
&- h\int_{\widetilde{\Omega}^*_\varepsilon\times \partial T} \mathcal{T}^b_\varepsilon(p_\varepsilon)\mathcal{T}^b_\varepsilon(v_\varepsilon)\,dx\,d\sigma(y) = 0,
\end{aligned}$$

which by passing to the limit gives

$$\begin{aligned}
&\int_{\Omega\times Y^*} {}^tA(x,y)(\nabla_x p_0 + \nabla_y \widehat{p})\varphi(x)\nabla_y\xi(x,y)\,dx\,dy\\
&= \int_{\Omega\times Y^*} B(x,y)(\nabla_x u_0 + \nabla_y \widehat{u})\varphi(x)\nabla_y\xi(x,y)\,dx\,dy.
\end{aligned}$$

Using denseness arguments as before, for all $\zeta \in L^2(\Omega; H^1_{\rm per}(Y^*)$, we have

$$\begin{aligned}
&\int_{\Omega\times Y^*} {}^tA(x,y)(\nabla_x p_0 + \nabla_y \widehat{p})\nabla_y\zeta(x,y)\,dx\,dy\\
&= \int_{\Omega\times Y^*} B(x,y)(\nabla_x u_0 + \nabla_y \widehat{u})\nabla_y\zeta(x,y)\,dx\,dy.
\end{aligned} \tag{3.24}$$

Adding (3.23) and (3.24), we obtain the variational formulation of the adjoint problem: For all $\varphi \in H^1_0(\Omega)$ and all $\zeta \in L^2(\Omega; H^1_{\rm per}(Y^*))$,

$$\begin{aligned}
&\int_{\Omega\times Y^*} {}^tA(x,y)(\nabla_x p_0 + \nabla_y \widehat{p})(\nabla_x\varphi + \nabla_y\zeta(x,y))\,dx\,dy - h|\partial T|\int_\Omega p_0\varphi\,dx\\
&= \int_{\Omega\times Y^*} B(x,y)(\nabla_x u_0 + \nabla_y \widehat{u})(\nabla_x\varphi + \nabla_y\zeta(x,y)\,dx\,dy,
\end{aligned} \tag{3.25}$$

where $(p_0, \widehat{p})$ is the unique solution of (3.25).

Now, taking u_0 as a test-function in (3.3), we have

$$\int_{\Omega\times Y^*} B(x,y)(\nabla_y\mu_i + e_i)\nabla u_0 = \int_{\Omega\times Y^*} {}^tA(x,y)\nabla_y\psi_i(x,y)\nabla u_0. \tag{3.26}$$

Letting $\varphi = 0$ in the variational-formulation (3.25), and putting $\widehat{u} = \sum_{i=1}^N \mu_i \frac{\partial u_0}{\partial x_i}$, we have

$$\begin{aligned}
&\int_{\Omega\times Y^*} {}^tA(x,y)(\nabla_x p_0 + \nabla_y \widehat{p})\nabla_y\zeta(x,y)\,dx\,dy\\
&= \int_{\Omega\times Y^*} B(x,y)(e_i + \nabla_y\mu_i)\nabla_x u_0\nabla_y\zeta(x,y)\,dx\,dy.\\
&= \int_{\Omega\times Y^*} {}^tA(x,y)\nabla_y\psi_i(x,y)\nabla_x u_0\nabla_y\zeta(x,y)\,dx\,dy\,,
\end{aligned}$$

which follows by (3.26). This implies that

$$\begin{aligned}
&\int_{\Omega\times Y^*} {}^tA(x,y)\nabla_y\widehat{p}\;\nabla_y\zeta(x,y)\,dx\,dy\\
&= \int_{\Omega\times Y^*} {}^tA(x,y)\nabla_y\psi_i(x,y)\nabla_x u_0\nabla_y\zeta(x,y)\,dx\,dy\\
&\quad - \int_{\Omega\times Y^*} {}^tA(x,y)\nabla_x p_0\nabla_y\zeta(x,y)\,dx\,dy.
\end{aligned} \tag{3.27}$$

Taking p_0 as a test-function in (3.2), we obtain

$$\int_{\Omega\times Y^*} {}^tA(x,y)(\nabla_y\omega_i+e_i)\nabla p_0 = 0. \tag{3.28}$$

Adding and subtracting $\int_{\Omega\times Y^*} {}^tA(x,y)(\nabla_y\omega_i+e_i)\nabla_x p_0\nabla_y\zeta(x,y)\,dx\,dy$ from (3.27), we have

$$\begin{aligned}
&\int_{\Omega\times Y^*} {}^tA(x,y)\nabla_y\widehat{p}\,\nabla_y\zeta(x,y)\,dx\,dy\\
&= \int_{\Omega\times Y^*} {}^tA(x,y)\nabla_y\psi_i(x,y)\nabla_x u_0\nabla_y\zeta(x,y)\,dx\,dy\\
&\quad - \int_{\Omega\times Y^*} {}^tA(x,y)\nabla_y(\omega_i+e_i)\nabla_x p_0\nabla_y\zeta(x,y)\,dxdy\\
&\quad + \int_{\Omega\times Y^*} {}^tA(x,y)\nabla_y\omega_i\nabla_x p_0\nabla_y\zeta(x,y)\,dx\,dy\,.
\end{aligned}$$

Integrating by parts with respect to y and using (3.28), we have

$$\begin{aligned}
-\operatorname{div}_y({}^tA(x,y)\nabla_y\widehat{p}) &= -\operatorname{div}_y({}^tA(x,y)\nabla_y\psi_i(x,y))\nabla_x u_0\\
&\quad - \operatorname{div}_y({}^tA(x,y)\nabla_y\omega_i(x,y))\nabla_x p_0.
\end{aligned} \tag{3.29}$$

This equality shows that

$$\widehat{p} = \sum_{i=1}^N \omega_i\frac{\partial p_0}{\partial x_i} + \sum_{i=1}^N \psi_i\frac{\partial u_0}{\partial x_i}.$$

Further taking $\zeta = 0$ in variational-formulation (3.25) and integrating by parts with respect to x, we obtain the desired homogenized adjoint equation:

$$\begin{gathered}
-\operatorname{div}({}^tA_0\nabla p_0 - B^{\#}\nabla u_0) = h\frac{|\partial T|}{|Y|}p_0 \quad \text{in } \Omega,\\
p_0 = 0 \quad \text{on } \partial\Omega.
\end{gathered}$$

□

Below we prove that the bilinear form defined by $B^{\#}$ (the perturbed matrix of B_0),is $H_0^1(\Omega)$-elliptic. A key step involves in establishing that

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon^*} B_\varepsilon\nabla w_\varepsilon\nabla w_\varepsilon\,dx = \int_\Omega B^{\#}\nabla w\nabla w\,dx, \tag{3.30}$$

where w_ε satisfies (3.33) below. The Robin condition on the boundary of holes in (3.8) and hence the limit equation with nontrivial terms, leads us to play with the various integration by parts, for passing to the limit in (3.30). An analogous result with Neumann condition on the boundary of holes can be seen [16, Theorem 3.3].

To prove the ellipticity of the bilinear form, we shall need [8, Proposition 6.6] involving extension operator and the unfolding method. We have the existence of sequence $\{P^\varepsilon\}$ of linear extension operators [10], such that for any $\varepsilon > 0$,

$$\begin{gathered}
P^\varepsilon \in \mathcal{L}(V_\varepsilon, H_0^1(\Omega)),\\
P^\varepsilon v = v \quad \text{in } \Omega_\varepsilon^*, \quad \forall v\in V_\varepsilon\\
\|P^\varepsilon v\|_{L^2(\Omega)} \le C\|v\|_{L^2(\Omega_\varepsilon^*)}, \quad \forall v\in V_\varepsilon\\
\|\nabla P^\varepsilon v\|_{[L^2(\Omega)]^N} \le C\|\nabla v\|_{[L^2(\Omega_\varepsilon^*)]^N} \quad \forall v\in V_\varepsilon.
\end{gathered} \tag{3.31}$$

Although in [10], the set of holes do not intersect the external boundary Γ_0^ε, it is easy to check (see [6, page 6]) that if we extend a function in V_ε as in [10] inside the holes contained in $\widehat{\Omega}_\varepsilon^*$ and extend it by zero elsewhere, we obtain an extension operator that still verify (3.31).

Theorem 3.3. *For every $w \in H_0^1(\Omega)$, we have*

$$\int_\Omega B^\# \nabla w \nabla w \, dx \geq \beta_m C^{-2} \|w\|^2_{H_0^1(\Omega)},$$

where $B_\varepsilon \in \mathcal{M}(\beta_m, \beta_M, \Omega)$ and $C > 0$ is a constant appearing in (3.31).

Proof. Let

$$F = -\operatorname{div}(A_0 \nabla w) - \frac{|\partial T|}{|Y|} \mathcal{M}_{\partial T}(g) + h \frac{|\partial T|}{|Y|} w. \tag{3.32}$$

Then $F \in H^{-1}(\Omega)$. Let $w_\varepsilon \in V_\varepsilon$, be the solution of the problem:

$$\begin{gathered} -\operatorname{div}(A_\varepsilon \nabla w_\varepsilon) = (P^\varepsilon)^* F \quad \text{in } \Omega_\varepsilon^*, \\ A_\varepsilon \nabla w_\varepsilon \cdot n_\varepsilon + h \varepsilon u_\varepsilon = \varepsilon g^\varepsilon \quad \text{on } \Gamma_1^\varepsilon, \\ u_\varepsilon = 0 \quad \text{on } \Gamma_0^\varepsilon. \end{gathered} \tag{3.33}$$

Using ellipticity of A_ε and bounds of (3.32), we obtain $\|\nabla w_\varepsilon\|_{[L^2(\Omega_\varepsilon^*)]^N} \leq C$. It follows by (3.31)(iv) and [8, Proposition 6.6] (also see [10]) that

$$P^\varepsilon w_\varepsilon \rightharpoonup w \quad \text{weakly in } H_0^1(\Omega).$$

Let us define $q_\varepsilon \in V_\varepsilon$ as the solution of the problem

$$\begin{gathered} -\operatorname{div}\big({}^t A_\varepsilon \nabla q_\varepsilon - B_\varepsilon \nabla w_\varepsilon\big) = 0 \quad \text{in } \Omega_\varepsilon^*, \\ ({}^t A_\varepsilon \nabla q_\varepsilon - B_\varepsilon \nabla w_\varepsilon) \cdot n_\varepsilon = q_\varepsilon h \varepsilon \quad \text{on } \Gamma_1^\varepsilon, \\ q_\varepsilon = 0 \quad \text{on } \Gamma_0^\varepsilon. \end{gathered} \tag{3.34}$$

Its homogenized equation is

$$\begin{gathered} -\operatorname{div}({}^t A_0 \nabla q_0 - B^\# \nabla w) = h \frac{|\partial T|}{|Y|} q_0 \quad \text{in } \Omega, \\ q_0 = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3.35}$$

Multiplying (3.34) by w_ε and integrating by parts, we obtain

$$\int_{\Omega_\varepsilon^*} {}^t A_\varepsilon \nabla q_\varepsilon \nabla w_\varepsilon = \int_{\Omega_\varepsilon^*} B_\varepsilon \nabla w_\varepsilon \nabla w_\varepsilon \, dx + \int_{\Gamma_1^\varepsilon} \varepsilon w_\varepsilon h q_\varepsilon. \tag{3.36}$$

Taking $q_\varepsilon \in V_\varepsilon$ as a test-function in (3.33), we have

$$\int_{\Omega_\varepsilon^*} A_\varepsilon \nabla w_\varepsilon \nabla q_\varepsilon - \int_{\Gamma_1^\varepsilon} A_\varepsilon \nabla w_\varepsilon \cdot n_\varepsilon \nabla q_\varepsilon = \langle F, P^\varepsilon q_\varepsilon \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}. \tag{3.37}$$

Substituting F from (3.32) and using (3.36) in the above equation, we obtain

$$\begin{aligned} \int_{\Omega_\varepsilon^*} B_\varepsilon \nabla w_\varepsilon \nabla w_\varepsilon \, dx &= \langle -\operatorname{div}(A_0 \nabla w), P^\varepsilon q_\varepsilon \rangle - \varepsilon \int_{\Gamma_1^\varepsilon} (2h w_\varepsilon - g^\varepsilon) q_\varepsilon \\ &\quad + \frac{|\partial T|}{|Y|} \left\langle h w - \mathcal{M}_{\partial T}(g), P^\varepsilon q_\varepsilon \right\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}. \end{aligned} \tag{3.38}$$

By unfolding, the second term of the right-hand side of (3.38), it can be written as

$$\lim_{\varepsilon \to 0} \int_{\widetilde{\Omega} \times \partial T} (2h \mathcal{T}_\varepsilon^b(w_\varepsilon) - \mathcal{T}_\varepsilon^b(g^\varepsilon)) \mathcal{T}_\varepsilon^b(q_\varepsilon) \, dx d\sigma(y) = \frac{|\partial T|}{|Y|} \big(2hw - \mathcal{M}_{\partial T}(g)\big). \tag{3.39}$$

Now passing to the limit as $\varepsilon \to 0$ in (3.38), and using (3.39) we obtain

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon^*} B_\varepsilon \nabla w_\varepsilon \nabla w_\varepsilon \,dx = \int_\Omega A_0 \nabla w \nabla q_0 - h\frac{|\partial T|}{|Y|} q_0. \tag{3.40}$$

Multiplying (3.35) by w and integrating by parts, we obtain

$$\int_\Omega {}^t A_0 \nabla q_0 \nabla w = \int_\Omega B^\# \nabla w \nabla w + h\frac{|\partial T|}{|Y|} q_0 w. \tag{3.41}$$

Substituting this value of $\int_\Omega {}^t A_0 \nabla q_0 \nabla w$ in (3.40), we obtain (3.30).

We use the ellipticity of B_ε and (3.31) to obtain

$$\int_{\Omega_\varepsilon^*} B_\varepsilon \nabla w_\varepsilon \nabla w_\varepsilon \,dx \geq \beta_m \|w_\varepsilon\|^2_{V_\varepsilon} \geq \beta_m C^{-2} \|P^\varepsilon w_\varepsilon\|^2_{H_0^1(\Omega)}. \tag{3.42}$$

Passing to the limit in (3.42) as $\varepsilon \to 0$, in view of (3.30) and [8, Proposition 6.6], we obtain

$$\int_\Omega B^\# \nabla w \nabla w \,dx \geq \liminf_{\varepsilon\to 0} \beta_m C^{-2} \|P^\varepsilon w_\varepsilon\|^2_{H_0^1(\Omega)} \geq \beta_m C^{-2} \|w\|^2_{H_0^1(\Omega)}, \tag{3.43}$$

which shows the ellipticity of $B^\#$. $\square$

Remark 3.4. A byproduct of the above proof is

$$\begin{gathered} -\operatorname{div}(A_\varepsilon \nabla w_\varepsilon) = (P^\varepsilon)^* f_\varepsilon \quad \text{in } \Omega_\varepsilon^*, \\ A_\varepsilon \nabla w_\varepsilon \cdot n_\varepsilon + h\varepsilon u_\varepsilon = \varepsilon g^\varepsilon \quad \text{on } \Gamma_1^\varepsilon, \\ u_\varepsilon = 0 \quad \text{on } \Gamma_0^\varepsilon, \end{gathered}$$

where $f_\varepsilon \to f$ strongly in $H^{-1}(\Omega)$. Then the proof of Theorem 3.3 can be adapted to prove that if $P^\varepsilon w_\varepsilon \rightharpoonup w_0$ weakly in $H_0^1(\Omega)$, then we have

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon^*} B_\varepsilon \nabla w_\varepsilon \nabla w_\varepsilon \,dx = \int_\Omega B^\# \nabla w \nabla w \,dx. \tag{3.44}$$

3.1. **Convergence of optimal controls.** Now we consider some optimal control problems, where we take the convex set $U_{\rm ad}^\varepsilon \subset L^2(\Omega_\varepsilon^*)$, to be of obstacle type. Here we will use the notation χ_ε for $\chi_{\Omega_\varepsilon^*}$.

We consider $U_{\rm ad}^\varepsilon$ as one of the following sets:

$$U_{\rm ad}^\varepsilon = L^2(\Omega_\varepsilon^*), \tag{3.45}$$

$$\begin{aligned} U_{\rm ad}^\varepsilon &= \{\theta \in L^2(\Omega_\varepsilon^*) | \theta \geq \psi \text{ a.e. in } \Omega_\varepsilon^*\} \\ &= \{\theta \in L^2(\Omega_\varepsilon^*) | \tilde\theta \geq \chi_\varepsilon \psi \text{ a.e. in } \Omega\}, \end{aligned} \tag{3.46}$$

$$U_{\rm ad}^\varepsilon = \{\theta \in L^2(\Omega_\varepsilon^*) | \chi_\varepsilon \psi_1 \leq \tilde\theta \leq \chi_\varepsilon \psi_2 \text{ a.e. in } \Omega\}, \tag{3.47}$$

where ψ, ψ_1 and ψ_2 are given functions in $L^2(\Omega)$. The first case considered above is unconstrained, the second one is unilateral and the third one corresponds to bilateral constraint. We define the limiting set $U_{\rm ad} \subset L^2(\Omega)$ corresponding to (3.45), (3.46) and (3.47) as follows:

$$U_{\rm ad} = L^2(\Omega), \tag{3.48}$$

$$U_{\rm ad} = \{\theta \in L^2(\Omega) |\ \theta \geq \Theta\psi \text{ a.e. in } \Omega\}, \tag{3.49}$$

$$U_{\rm ad} = \{\theta \in L^2(\Omega_\varepsilon^*) |\ \Theta\psi_1 \leq \tilde\theta \leq \Theta\psi_2 \text{ a.e. in } \Omega\}. \tag{3.50}$$

Now, in view of classical linear control theory of Lions [18], there exists a unique minimal-norm control $\theta_\varepsilon^* \in U_{\rm ad}^\varepsilon$, minimizing the cost-functional (1.3) over any of the closed convex sets (3.45)-(3.47). So that we have

$$\frac{N_0}{2}\int_{\Omega_\varepsilon^*} {\theta_\varepsilon^*}^2\,dx \le J_\varepsilon(\theta_\varepsilon^*) \le J_\varepsilon(\Phi_\varepsilon), \tag{3.51}$$

where

$$\Phi_\varepsilon = \begin{cases} \chi_\varepsilon & \text{in case of (3.45)},\\ \chi_\varepsilon\psi & \text{in case of (3.46)},\\ \chi_\varepsilon\psi_2 & \text{in case of (3.47)}. \end{cases}$$

In all the cases defined in (3.45)-(3.47), $J_\varepsilon(\Phi_\varepsilon)$ is uniformly bounded, since $\|u\|_{V_\varepsilon}$ of the solution u_ε of (3.8) is bounded independent of ε and $B_\varepsilon \in \mathcal{M}(\beta_m, \beta_M, \Omega)$. Thus $\{\tilde{\theta}_\varepsilon^*\} \subset L^2(\Omega)$ will be bounded sequence, hence there exists a subsequence converging weakly to θ_0^* in $L^2(\Omega)$. By (3.7), $\{\widetilde{u}_\varepsilon^*\}$ and $\{\widetilde{p}_\varepsilon^*\}$ ($u_\varepsilon^*,\ p_\varepsilon^*$ are solutions of (3.8) and of (3.9) with $\theta_\varepsilon = \theta_\varepsilon^*$) will be bounded independent of ε. Therefore by extracting a further subsequence (if necessary) we have $\widetilde{u}_\varepsilon^* \rightharpoonup \Theta u_0^*$ weakly in $L^2(\Omega)$ and $\widetilde{p}_\varepsilon^* \rightharpoonup \Theta p_0^*$ weakly in $L^2(\Omega)$.

We define below the limiting optimal control problem. For that let $\theta_0 \in U_{\rm ad}$, and let $u_0 \in H_0^1(\Omega)$ be the solution of

$$\begin{gathered} -\operatorname{div}(A_0\nabla u_0) + h\frac{|\partial T|}{|Y|}u_0 = \Theta(f+\theta_0) + \frac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g) \quad \text{in } \Omega,\\ u_0 = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

and the cost-functional by

$$J_0(\theta) = \frac{1}{2}\int_\Omega B^{\#}\nabla u_0\nabla u_0\,dx + \frac{N_0}{2}\int_\Omega \frac{\theta_0^2}{\Theta}\,dx,$$

where A_0 and $B^{\#}$ are same as defined in (3.4) and (3.6). With this framework, we state the results on convergence of optimal controls. A proof of which follows by Remark 3.4, (also see Kesavan et al. [16, Theorem 4.1]).

Theorem 3.5. *The limit θ_0^* satisfies the optimality conditions*

$$\begin{gathered} J_0(\theta_0^*) = \min_{\theta\in U_{\rm ad}} J_0(\theta),\\ \lim_{\varepsilon\to 0} J_\varepsilon(\theta_\varepsilon^*) = J_0(\theta_0^*). \end{gathered}$$

4. Boundary homogenization of optimal control problems

In this section, we study the boundary homogenization of some boundary control problems. Let $\Omega_\varepsilon = \Omega\backslash T_\varepsilon$, where $\{T_\varepsilon\}$ is a sequence of admissible holes in the sense of H_0-convergence and T_ε do not intersect the external boundary $\partial\Omega$. We assume that $\partial\Omega$ is partitioned into two parts $\Gamma^{0,\varepsilon}$ and $\Gamma^{1,\varepsilon}$ (corresponding to an ε-periodic structure for example) such that each connected component of $\Gamma^{0,\varepsilon}$ and/ or $\Gamma^{1,\varepsilon}$ has diameter going to zero with ε or a directional thickness going to zero with ε for some given linear subspace direction. These kinds of problems (considered below) arise in oil exploitation.

Let A_ε satisfy (1.1), $g_\varepsilon \in L^2(\partial\Omega)$, and $\mathcal{U}_{\rm ad}^\varepsilon \subset L^2(\Gamma^{1,\varepsilon})$ be a closed, convex subset representing the admissible controls. For $\theta \in \mathcal{U}_{\rm ad}^\varepsilon$ the state equation is

$$\begin{gathered} -\operatorname{div}(A_\varepsilon \nabla u_\varepsilon) + u_\varepsilon = 0 \quad \text{in } \Omega_\varepsilon, \\ A_\varepsilon \nabla u_\varepsilon \cdot n_\varepsilon = g_\varepsilon + \theta \quad \text{on } \Gamma^{1,\varepsilon}, \\ u_\varepsilon = c_\varepsilon \quad \text{(unknown constant)} \quad \text{on } \Gamma^{0,\varepsilon} \\ \int_{\Gamma^{0,\varepsilon}} A_\varepsilon \nabla u_\varepsilon \cdot n_\varepsilon \, ds = \int_{\Gamma^{0,\varepsilon}} g_\varepsilon \, ds \\ A_\varepsilon \nabla u_\varepsilon \cdot n_\varepsilon = 0 \quad \text{on } \partial T_\varepsilon. \end{gathered} \tag{4.1}$$

We refer [14] for the physical significance of the boundary conditions on $\Gamma^{0,\varepsilon}$. For $N_0 > 0$, we associate the cost functional

$$J_\varepsilon(\theta) = \frac{1}{2} \int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla u_\varepsilon \, dx + \frac{N_0}{2} \int_{\Gamma^{1,\varepsilon}} \theta^2 \, ds \tag{4.2}$$

For the existence and uniqueness of solutions of (4.1), (4.2), we refer to [21, Section 2.7, 2.2], [17, Section 5, p. 265], and [18]. To characterize the optimal control θ_ε^*, minimizing (4.2), we introduce the adjoint state which is the solution of

$$\begin{gathered} -\operatorname{div}({}^t A_\varepsilon \nabla p_\varepsilon) + p_\varepsilon = -\Delta u_\varepsilon \quad \text{in } \Omega_\varepsilon, \\ {}^t A_\varepsilon \nabla p_\varepsilon \cdot n_\varepsilon = \nabla u_\varepsilon \cdot n_\varepsilon \quad \text{on } \Gamma^{1,\varepsilon}, \\ p_\varepsilon = d_\varepsilon \quad \text{(unknown constant)} \quad \text{on } \Gamma^{0,\varepsilon}, \\ \int_{\Gamma^{0,\varepsilon}} {}^t A_\varepsilon \nabla p_\varepsilon \cdot n_\varepsilon \, ds = \int_{\Gamma^{0,\varepsilon}} \nabla u_\varepsilon \cdot n_\varepsilon \, ds \quad \text{on } \Gamma^{1,\varepsilon} \\ {}^t A_\varepsilon \nabla p_\varepsilon \cdot n_\varepsilon = \nabla u_\varepsilon \cdot n_\varepsilon \quad \text{on } \partial T_\varepsilon. \end{gathered} \tag{4.3}$$

Let $u_\varepsilon^* \in H^1(\Omega_\varepsilon)$, $p_\varepsilon^* \in H^1(\Omega_\varepsilon)$ be the state and the adjoint state for $\theta = \theta_\varepsilon^*$. Then θ_ε^* satisfies the extremality condition

$$\int_{\Gamma^{1,\varepsilon}} (p_\varepsilon^* + N_0 \theta_\varepsilon^*)(\theta - \theta_\varepsilon^*) \, ds \geq 0 \quad \forall\, \theta \in \mathcal{U}_{\rm ad}^\varepsilon \tag{4.4}$$

We need the following hypotheses:

(H4) The characteristic function $\chi_{\Gamma^{0,\varepsilon}}$ of $\Gamma^{0,\varepsilon}$ satisfies

$$\begin{gathered} \exists \chi \in L^\infty(\partial\Omega),\ 0 < \chi(x) \text{ for a.e. } x \in \partial\Omega \text{ and } (1-\chi)^{-1} \in L^\infty(\partial\Omega). \\ \text{and } \chi_{\Gamma^{0,\varepsilon}} \rightharpoonup \chi \text{ weak}^* \ L^\infty(\partial\Omega). \end{gathered} \tag{4.5}$$

(H5) g_ε is bounded in $L^2(\partial\Omega)$ and there exists $g_0 \in \mathbb{R}$,

$$\int_{\partial\Omega} g_\varepsilon \, ds \to g_0.$$

(H6) A_ε H_0-converges to A_0 in the sense of Briane et al. [3].

(H7) Every weak* limit point in $L^\infty(\Omega)$ of $\chi_{\Omega_\varepsilon}$ is positive a.e. in Ω, that is $\chi_{\Omega_\varepsilon} \rightharpoonup \chi_0$ weak* $L^\infty(\Omega)$, then $\chi_0^{-1} \in L^\infty(\Omega)$.

(H58 For each $\varepsilon > 0$, there exists an extension operator $P^\varepsilon \in \mathcal{L}(V_\varepsilon, H^1(\Omega))$ such that for all $u \in V_\varepsilon$,

$$P^\varepsilon u|_{\Omega_\varepsilon} = u, \quad \|P^\varepsilon u\|_{1,\Omega} \leq C\|u\|_{1,\Omega_\varepsilon},$$

where

$$V_\varepsilon = \{u \in H^1(\Omega_\varepsilon);\ u = M \text{ (constant)} \quad \text{on } \Gamma^{0,\varepsilon}\}.$$

Here and in the sequel, C denotes the several positive constants independent of ε. Then in view of assumptions (H4)–(H8), the state and adjoint equations (4.1) and (4.3) can be homogenized to

$$\begin{gathered}
-\operatorname{div}(A_0\nabla u_0)+\chi_0 u_0=0 \quad \text{in } \Omega,\\
-\operatorname{div}({}^tA_0\nabla p_0-B_I^{\#}\nabla u_0)+\chi_0 p_0=0 \quad \text{in } \Omega,\\
u_0= \text{ constant on } \partial\Omega,\\
\int_{\partial\Omega} A_0\nabla u_0\cdot n\,ds,=g_0+\int_{\partial\Omega}(1-\chi)\theta\,ds,\\
\int_{\partial\Omega}({}^tA_0\nabla p_0-B_I^{\#}\nabla u_0)\cdot n\,ds=0.
\end{gathered} \tag{4.6}$$

The homogenization is known to us by Prof C. Conca [11], (cf. see Section 5).

4.1. **Optimal controls.** For each $\varepsilon>0$, let the optimal control be denoted by θ_ε^* which is characterized by (4.4). Let $\sigma\in L^2(\partial\Omega)$ be a given function, let $\mathcal{U}_{\rm ad}^\varepsilon$ and $\mathcal{U}_{\rm ad}$ be the following sets of admissible controls

$$\begin{gathered}
\mathcal{U}_{\rm ad}^\varepsilon=\big\{\theta_\varepsilon\in L^2(\Gamma^{1,\varepsilon}):\theta_\varepsilon(x)\geq\sigma(x)|_{\Gamma^{1,\varepsilon}} \text{ a.e. } x\in\partial\Omega\big\},\\
\mathcal{U}_{\rm ad}=\big\{\theta_\varepsilon\in L^2(\partial\Omega):\theta(x)\geq(1-\chi)\sigma(x) \text{ for a.e. } x\in\Gamma^{1,\varepsilon}\big\}.
\end{gathered}$$

Let $\theta\in\mathcal{U}_{\rm ad}$ and let us define

$$\theta_\varepsilon=\chi_{\Gamma^{1,\varepsilon}}\frac{\theta}{1-\chi}\in\mathcal{U}_{\rm ad}^\varepsilon. \tag{4.7}$$

Theorem 4.1. *Let J_ε and θ_ε be given by* (4.2), *and* (4.7). *Then we can pass to the limit in the cost functional*

$$\lim_{\varepsilon\to0}J_\varepsilon(\theta_\varepsilon)=J_0(\theta),$$

where

$$J_0(\theta)=\frac12\int_\Omega B_I^{\#}\nabla u_0\nabla u_0\,dx+\frac{N_0}{2}\int_{\partial\Omega}\frac{\theta^2}{1-\chi}\,ds\,. \tag{4.8}$$

Then $\theta_0^*\in\mathcal{U}_{\rm ad}$ is the unique minimizer of the above cost-functional over $\mathcal{U}_{\rm ad}$, that is

$$J_0(\theta_0^*)\leq J_0(\theta)\quad\forall\theta\in\mathcal{U}_{\rm ad}. \tag{4.9}$$

To prove the above theorem, we shall need the following key lemma.

Lemma 4.2. *Let*

$$\theta_\varepsilon=\chi_{\Gamma^{1,\varepsilon}}\frac{\theta}{1-\chi}\rightharpoonup\theta\quad\text{weakly in } L^2(\partial\Omega).$$

Then

$$\liminf_{\varepsilon\to0}\int_{\partial\Omega}\big|\chi_{\Gamma^{1,\varepsilon}}\frac{\theta}{1-\chi}\big|^2ds\geq\int_{\partial\Omega}\frac{\theta^2}{1-\chi}\,ds.$$

Proof. By (H4), $\frac{\chi_{\Gamma^{1,\varepsilon}}}{(1-\chi)}\theta_0\in L^2(\Omega)$. We define the convex functional $\varphi:L^2(\Omega)\to\mathbb{R}$ as

$$\varphi(\theta)=\int_{\partial\Omega}|\theta|^2\,ds.$$

Using convexity arguments and some simple calculations, we obtain

$$\varphi(\widetilde{\theta}_\varepsilon)-\varphi\Big(\frac{\chi_{\Gamma^{1,\varepsilon}}}{1-\chi}\theta_0\Big)\geq2\int_{\partial\Omega}\frac{\chi_{\Gamma^{1,\varepsilon}}}{1-\chi}\theta_0\Big(\widetilde{\theta}_\varepsilon-\frac{\chi_{\Gamma^{1,\varepsilon}}}{1-\chi}\theta_0\Big)\,ds. \tag{4.10}$$

Using that $\chi_{\Gamma^{1,\varepsilon}}\widetilde{\theta}_\varepsilon = \widetilde{\theta}_\varepsilon$ and $\chi^r_{\Gamma^{1,\varepsilon}} = \chi_{\Gamma^{1,\varepsilon}}$ for all $r>0$, the right-hand side of (4.10) becomes

$$2\int_{\partial\Omega} \frac{\theta_0}{1-\chi}\Big(\widetilde{\theta}_\varepsilon - \frac{\chi_{\Gamma^{1,\varepsilon}}}{1-\chi}\theta_0\Big)\,ds,$$

which tends to 0 as $\varepsilon\to 0$, by the weak convergence of $\widetilde{\theta}_\varepsilon \rightharpoonup \theta_0$ $L^2(\partial\Omega)$-weakly and the convergence $\chi_{\Gamma^{1,\varepsilon}} \xrightarrow[\text{weak}^*]{L^\infty} (1-\chi)$. Further we have

$$\begin{aligned}
\varphi\Big(\frac{\chi_{\Gamma^{1,\varepsilon}}}{1-\chi}\theta_0\Big) &= \int_{\partial\Omega} \frac{\chi^2_{\Gamma^{1,\varepsilon}}}{(1-\chi)^2}|\theta_0|^2 ds\\
&= \int_{\partial\Omega} \frac{1+\chi^2_{\Gamma^{0,\varepsilon}} - 2\chi_{\Gamma^{0,\varepsilon}}}{(1-\chi)^2}|\theta_0|^2 ds\\
&= \int_{\partial\Omega} \frac{1-\chi_{\Gamma^{0,\varepsilon}}}{(1-\chi)^2}|\theta_0|^2 ds\\
&\to \int_{\partial\Omega} \frac{|\theta_0|^2}{(1-\chi)} ds.
\end{aligned}$$

Then the result follows from the above observations. □

Proof of Theorem 4.1. Using the state equation associated with θ_ε, and [16, Remark 3.3], we have

$$J_\varepsilon\Big(\chi_{\Gamma^{1,\varepsilon}}\frac{\theta}{1-\chi}\Big) = \frac{1}{2}\int_{\Omega_\varepsilon} \nabla u_\varepsilon\cdot\nabla u_\varepsilon + \frac{N_0}{2}\int_{\Gamma^{1,\varepsilon}} \chi_{\Gamma^{1,\varepsilon}}\frac{\theta^2}{(1-\chi)^2}ds \longrightarrow J_0(\theta) \quad (4.11)$$

where $J_0(\theta)$ is defined by (4.8).

On the other hand, using [16, Remark 3.3] (when $B_\varepsilon = I$ for all $\varepsilon>0$), we have

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} |\nabla u^*_\varepsilon|^2\,dx = \int_\Omega B^{\#}_I \nabla u^*_0\nabla u^*_0\,dx.$$

Now we pass to the limit in the inequality

$$J_\varepsilon(\theta^*_\varepsilon) \le J_\varepsilon\Big(\chi_{\Gamma^{1,\varepsilon}}\frac{\theta}{1-\chi}\Big),$$

in view of (4.11), we have

$$J_0(\theta) \ge \frac{1}{2}\int_\Omega B^{\#}_I\nabla u^*_0\cdot\nabla u^*_0 + \limsup_{\varepsilon\to 0}\frac{N_0}{2}\int_{\partial\Omega}\chi_{\Gamma^{1,\varepsilon}}(\theta^*_\varepsilon)^2\,ds. \quad (4.12)$$

Thus taking $\theta = \theta^*_0$, we have

$$\theta^*_\varepsilon = \chi_{\Gamma^{1,\varepsilon}}\frac{\theta^*_0}{1-\chi}.$$

Now putting θ^*_ε in (4.12) we have

$$\limsup_{\varepsilon\to 0}\int_{\partial\Omega}\chi_{\Gamma^{1,\varepsilon}}\Big(\chi_{\Gamma^{1,\varepsilon}}\frac{\theta^*_0}{(1-\chi)}\Big)^2 ds \le \int_{\partial\Omega}\frac{(\theta^*_0)^2}{1-\chi}\,ds \quad (4.13)$$

which implies

$$\limsup_{\varepsilon\to 0}\int_{\partial\Omega}\chi_{\Gamma^{1,\varepsilon}}\Big(\chi_{\Gamma^{1,\varepsilon}}\frac{\theta^*_0}{(1-\chi)}\Big)^2 ds \le \int_{\partial\Omega}\frac{(\theta^*_0)^2}{1-\chi}\,ds. \quad (4.14)$$

Using $\chi_{\Gamma^{1,\varepsilon}} = 1-\chi_{\Gamma^{0,\varepsilon}}$, we have

$$(1-\chi_{\Gamma^{0,\varepsilon}})^2 = 1 + (\chi_{\Gamma^{0,\varepsilon}})^2 - 2\chi_{\Gamma^{0,\varepsilon}} = 1-\chi_{\Gamma^{0,\varepsilon}}$$

Similarly,

$$\begin{aligned}(1-\chi_{\Gamma^{0,\varepsilon}})^3 &= (1-\chi_{\Gamma^{0,\varepsilon}})(1-\chi_{\Gamma^{0,\varepsilon}})^2 = (1-\chi_{\Gamma^{0,\varepsilon}})(1-\chi_{\Gamma^{0,\varepsilon}})\\ &= (1-\chi_{\Gamma^{0,\varepsilon}}) = \chi_{\Gamma^{1,\varepsilon}}.\end{aligned}$$

With the above observations, (4.14) implies

$$\limsup_{\varepsilon\to 0}\int_{\partial\Omega}\chi_{\Gamma^{1,\varepsilon}}\Big(\frac{\theta_0^*}{1-\chi}\Big)^2\,ds \le \int_{\partial\Omega}\frac{(\theta_0^*)^2}{1-\chi}\,ds. \tag{4.15}$$

On the other hand, by a convexity argument, see Lemma 4.2, also see [16, Proposition 2.2], we have

$$\liminf_{\varepsilon\to 0}\int_{\partial\Omega}\chi_{\Gamma^{1,\varepsilon}}\Big(\frac{\theta_0^*}{1-\chi}\Big)^2\,ds \ge \int_{\partial\Omega}\frac{(\theta_0^*)^2}{1-\chi}\,ds. \tag{4.16}$$

From (4.12) and the last two inequalities (4.15) and (4.16), we have

$$J_0(\theta)\ge J_0(\theta_0^*)\quad \forall\theta\in\mathcal{U}_{\rm ad}. \tag{4.17}$$

This completes the proof . ☐

5. Appendix by C. Conca

The variational formulation of (4.1) is as follows: Find $u_\varepsilon\in V_\varepsilon$ such that for all $\varphi\in V_\varepsilon$,

$$\int_{\Omega_\varepsilon}A_\varepsilon\nabla u_\varepsilon\nabla\varphi\,dx+\int_{\Omega_\varepsilon}u_\varepsilon\varphi\,dx=\int_{\Gamma^{1,\varepsilon}}(g_\varepsilon+\theta)\varphi\,ds+\varphi|_{\Gamma^{0,\varepsilon}}\int_{\Gamma^{0,\varepsilon}}g_\varepsilon\,ds \tag{5.1}$$

Taking $\varphi=u_\varepsilon$ as a test function in (5.1),

$$\int_{\Omega_\varepsilon}A_\varepsilon\nabla u_\varepsilon\nabla u_\varepsilon\,dx+\int_{\Omega_\varepsilon}u_\varepsilon\cdot u_\varepsilon=\int_{\partial\Omega}g_\varepsilon u_\varepsilon+\theta|_{\Gamma^{1,\varepsilon}}\int_{\Gamma^{1,\varepsilon}}u_\varepsilon\,ds. \tag{5.2}$$

Since u_ε is constant on $\Gamma^{0,\varepsilon}$, and using (1.1), (H8)–(H10), we have

$$\begin{aligned}\alpha_m\|u_\varepsilon\|_{1,\Omega_\varepsilon}^2 &\le \|u_\varepsilon\|_{0,\partial\Omega}\big(\|g_\varepsilon\|_{0,\Omega_\varepsilon}+\|\theta\|_{0,\Gamma^{1,\varepsilon}}\big)\\ &\le C\|u_\varepsilon\|_{1,\Omega_\varepsilon}\big(\|g_\varepsilon\|_{0,\Omega_\varepsilon}+\|\theta\|_{0,\Gamma^{1,\varepsilon}}\big)\\ \|u_\varepsilon\|_{1,\Omega_\varepsilon} &\le C\big(g_0+\|\theta\|_{0,\Gamma^{1,\varepsilon}}\big)\,;\end{aligned}$$

thus we can extract a subsequence (still denoted by ε) $\{u_\varepsilon\}$ such that

$$\begin{gathered}P^\varepsilon u_\varepsilon\rightharpoonup u_0\quad\text{weakly in } H^1(\Omega),\\ (\widetilde{A_\varepsilon\nabla u_\varepsilon})\rightharpoonup(A_0\nabla u_0)\quad\text{weakly in } [L^2(\Omega)]^N\end{gathered} \tag{5.3}$$

where u_0 satisfies

$$-\operatorname{div}(A_0\nabla u_0)+\chi_0u_0=0\quad\text{in }\Omega. \tag{5.4}$$

We set

$$V=\{u\in H^1(\Omega): u\in H^1(\Omega),\ u=M\ (\text{constant})\text{ on }\partial\Omega\}.$$

Clearly, $V\subset V_\varepsilon$ for all $\varepsilon>0$. Then for all $\varphi\in V$, we have

$$\begin{aligned}&\int_{\Omega_\varepsilon}(\widetilde{A_\varepsilon\nabla u_\varepsilon})\cdot\nabla\varphi\,dx+\int_\Omega\chi_{\Omega_\varepsilon}P^\varepsilon u_\varepsilon\varphi\,dx\\ &=\varphi|_{\partial\Omega}\int_{\partial\Omega}g_\varepsilon\,ds+\varphi|_{\partial\Omega}\int_{\partial\Omega}(1-\chi_{\Gamma^{0,\varepsilon}})\theta\,ds.\end{aligned}$$

Passing to the limit in the above eq as $\varepsilon \to 0$, and using (5.3), (H4), (H5) and (H9), we obtain

$$-\int_\Omega \operatorname{div}(A_0\nabla u_0)\varphi\,dx + \int_\Omega \chi_0 u_0\varphi\,dx = \varphi|_{\partial\Omega}\; g_0 + \int_{\partial\Omega}(1-\chi)\theta\varphi\,ds.$$

Integrating by parts

$$-\varphi\int_{\partial\Omega}(A_0\nabla u_0)\cdot n\,dx + \int_\Omega A_0\nabla u_0\nabla\varphi + \int_\Omega \chi_0 u_0\varphi\,dx = \varphi|_{\partial\Omega}\; g_0 + \int_{\partial\Omega}(1-\chi)\theta\varphi\,ds.$$

Comparing the coefficients of φ, then again integrate by parts and combine with (5.4) we obtain

$$\int_{\partial\Omega} A_0\nabla u_0\cdot n\,ds = g_0 + \int_{\partial\Omega}(1-\chi)\theta\,ds.$$

Now we show that u_0 is constant on $\partial\Omega$. Since $P^\varepsilon u_\varepsilon \rightharpoonup u_0$ weakly in $H^1(\Omega)$. As in the proof of [14, Theorem 1, p. 354], we have

$$\chi_{\Gamma^{0,\varepsilon}} P^\varepsilon u_\varepsilon\big|_{\partial\Omega} \rightharpoonup \chi u_0|_{\partial\Omega} \quad \text{weakly } L^2(\partial\Omega).$$

On the other hand, since $u_\varepsilon = c_\varepsilon$ on $\Gamma^{0,\varepsilon}$, and

$$\chi_{\Gamma^{0,\varepsilon}} u_\varepsilon|_{\Gamma^{0,\varepsilon}} = \chi_{\Gamma^{0,\varepsilon}} c_\varepsilon \rightharpoonup c\cdot\chi$$

where c is just the limit of sequence of real numbers c_ε. Comparing both the limits of $\chi_{\Gamma^{0,\varepsilon}} u_\varepsilon|_{\Gamma^{0,\varepsilon}}$, we obtain

$$\chi u_0|_{\partial\Omega} = \chi\cdot c \Longrightarrow u_0 = c \text{ on } \partial\Omega, \quad \text{as } \chi\neq 0 \text{ on } \partial\Omega.$$

Similarly $P^\varepsilon p_\varepsilon \rightharpoonup p_0$ weakly in $H^1(\Omega)$.

Set $z_\varepsilon = {}^tA_\varepsilon\nabla p_\varepsilon - \nabla u_\varepsilon$. Then up to a subsequence

$$z_\varepsilon \rightharpoonup z_0 \quad \text{weakly in } [L^2(\Omega)]^N,$$

where z_0 satisfies

$$-\operatorname{div}(z_0) + \chi_0 p_0 = 0 \quad \text{in } \Omega. \tag{5.5}$$

Arguing as we did for the state equation, we prove that

$$p_0|_{\partial\Omega} = d_0 \text{ (constant), and } \int_{\partial\Omega} z_0\cdot n\,ds = 0.$$

To identify the limits z_0 and p_0, we introduce the auxiliary functions μ_k^ε and ψ_k^ε, $k=1,\dots,N$ as done in [16, p. 572].

$$\begin{gathered}
\mu_k^\varepsilon - x_k \rightharpoonup 0 \quad \text{weakly in } H^1(\Omega),\\
\operatorname{div}(A_\varepsilon\nabla\mu_k^\varepsilon) = (P^\varepsilon)^*\operatorname{div}(A_0e_k) \quad \text{in } \Omega_\varepsilon,\\
\big(A_\varepsilon\nabla(-\mu_k^\varepsilon + x_k)\big)\cdot n_\varepsilon = 0 \quad \text{on } \partial T_\varepsilon,\\
(\widetilde{A_\varepsilon\nabla\mu_k^\varepsilon}) \rightharpoonup A_0e_k \quad \text{weakly in } [L^2(\Omega)]^N,
\end{gathered} \tag{5.6}$$

where $(P_\varepsilon)^*$ is the adjoint of P^ε, and x_k is kth co-ordinate function and e_k is the kth standard basis vector. Similarly introducing $\psi_k^\varepsilon \in V_\varepsilon$, $k=1,\dots,N$ satisfy

$$\begin{gathered}
\operatorname{div}({}^tA_\varepsilon\nabla\psi_k^\varepsilon + \nabla\mu_k^\varepsilon) = 0 \quad \text{in } \Omega_\varepsilon,\\
({}^tA_\varepsilon\nabla\psi_k^\varepsilon + \nabla\mu_k^\varepsilon) = 0 \quad \text{on } \partial T_\varepsilon.
\end{gathered} \tag{5.7}$$

It follows that $P^\varepsilon\psi_k^\varepsilon \rightharpoonup \psi_k^0$ weakly in $H^1(\Omega)$. Let $\varphi \in \mathcal{D}(\Omega)$, then multiplying the first equation of (4.1) by $\varphi\psi_k^\varepsilon$ and adjoint equation (4.3) by $\varphi\mu_k^\varepsilon$, integrating by parts, and subtracting the second identity from the first, we obtain

$$\begin{aligned} 0 = &-\int_\Omega \widetilde{z}_\varepsilon \mu_k^\varepsilon \nabla\varphi\,dx - \int_\Omega \nabla(P^\varepsilon p_\varepsilon)(\widetilde{A_\varepsilon\nabla\mu_k^\varepsilon})\varphi + \int_\Omega \widetilde{\xi}_\varepsilon \nabla\varphi(P^\varepsilon\psi_k^\varepsilon)\,dx \\ &- \int_\Omega (P^\varepsilon u_\varepsilon) b_k^\varepsilon \nabla\varphi\,dx + \int_\Omega \chi_{\Omega_\varepsilon}\varphi P^\varepsilon\psi_k^\varepsilon P^\varepsilon u_\varepsilon\,dx. \end{aligned}$$

Let us denote by b_k^0 the weak limit in $[L^2(\Omega)]^N$ of $\big({}^tA_\varepsilon\nabla\psi_k^\varepsilon + \nabla\mu_k^\varepsilon\big)$. It is clear that

$$\operatorname{div}(b_k^0) = 0 \quad \text{in } \Omega. \tag{5.8}$$

Passing to the limit in (5.8) using div-curl lemma [3, Lemma 1.1], we obtain

$$\begin{aligned} 0 = &-\int_\Omega z_0 x_k \nabla\varphi\,dx - \int_\Omega \nabla p_0(A_0 e_k)\varphi\,dx + \int_\Omega (A_0\nabla u_0)\nabla\varphi\psi_0^k\,dx \\ &- \int_\Omega u_0 b_k^0 \nabla\varphi\,dx + \int_\Omega \chi_0\varphi\psi_k^0 u_0\,dx - \int_\Omega \chi_0 p_0\varphi x_k\,dx \end{aligned}$$

From (5.4), (5.5), and (5.8), integrating by parts, we obtain

$$z_0\cdot e_k = {}^tA_0\nabla p_0\cdot e_k + {}^tA_0\nabla\psi_k^0\nabla u_0 - b_k^0\cdot\nabla u_0. \tag{5.9}$$

Hence $z_0 = {}^tA_0\nabla p_0 - B_I^{\#}\nabla u_0$, and $B^{\#}e_k = b_k^0 - {}^tA_0\nabla\psi_k^0$. Combining with (5.5), we obtain second equation of (4.6). Thus we obtain the homogenized equation (4.6).

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