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ASYMPTOTIC FORMULA FOR DETECTING INCLUSIONS VIA BOUNDARY MEASUREMENTS

KHALIFA KHELIFI, MOHAMED ABDELWAHED, NEJMEDDINE CHORFI, MAATOUG HASSINE

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Abstract. In this article, we are concerned with a geometric inverse problem related to the Laplace operator in a three-dimensional domain. The aim is to derive an asymptotic formula for detecting an inclusion via boundary measurement. The topological sensitivity method is applied to calculate a high-order topological asymptotic expansion of the semi-norm Kohn-Vogelius functional, when a Dirichlet perturbation is introduced in the initial domain.

1. Introduction

The detection of an object from boundary measurements is used in several applications such as in fluid mechanics, electrical impedance tomography, electromagnetic casting, non-destructive testing [2, 11, 12, 15].

On the theoretical level, these applications correspond to geometric inverse problems. Among the methods to solve this type of problems, there exist a method based on the Kohn-Vogelius formulation and the topological sensitivity method [5, 6, 10, 12, 17, 18, 20, 21, 30]. The majority of works interested to this method are based on the first-order asymptotic expansion of the Kohn-Vogelius functional [3, 4, 8, 9, 10, 12, 13, 14, 19, 22, 23]. This method is sufficient in the case of small unknown object far from the boundary.

In general application case the size of the object to detect is finite. For this reason, we consider high-order terms in the asymptotic expansion of the Kohn-Vogelius functional formula.

In this article we apply the topological sensitivity method and the Kohn-Vogelius formulation, to derive a high-order asymptotic formula connecting the known boundary data and the unknown inclusion properties; its location , size and shape. More precisely in this paper we derive a high-order topological asymptotic expansion of the semi-norm Kohn-Vogelius functional associated to the Laplace operator in three-dimensional domain, when a Dirichlet perturbation is introduced in the initial domain.

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The proposed approach permit to calculate the topological gradient for any order for the semi-norm Kohn-Vogelius functional. We present a general approach applicable to various problems such as elasticity, Stokes equations, Navier-Stokes equation, Maxwell's equations, etc.

The remaining of this paper are organized as follows. We begin by presenting the inverse problem and the Kohn-Vogelius formulation in section 2. In section 3 we present the topological sensitivity method. In section 4, we establish a some preliminary results, where we derive an asymptotic formula describing the variation of the solutions of Neumann and Dirichlet problems when a Dirichlet perturbation is introduced in the initial domain. Section 5 presents the main result of the paper. Finally, section 6 contains the proofs of the different results. The paper ends by some concluding remarks.

2. Inverse problem and the Kohn-Vogelius formulation

The geometric inverse Laplace problem related to the Laplace operator in three-dimensional domain is considered in this paper. Let $\Omega \subset \mathbb{R}^3$ denote a bounded domain with smooth boundary $\partial\Omega$ and satisfies $\partial\Omega = \Gamma_1 \cup \Gamma_2$ with $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\Gamma_2 \neq \emptyset$.

We suppose that there exist a sub-domain $\mathcal{D}^*$ of Ω with a smooth boundary $\partial\mathcal{D}^*$. The studied inverse problem can be formulated:

For regular given data F, V and ψ_m, find the unknown domain $\mathcal{D}^*$ such that ψ is solution of the following over determined problem

$$\begin{gathered}
-\Delta\psi = F \quad \text{in } \Omega\backslash\mathcal{D}^*,\\
\nabla\psi\cdot\mathbf{n} = V \quad \text{on } \Gamma_1,\\
\psi = \psi_m \quad \text{on } \Gamma_1,\\
\psi = 0 \quad \text{on } \Gamma_2,\\
\psi = 0 \quad \text{on } \partial\mathcal{D}^*.
\end{gathered}$$

To derive an asymptotic formula connecting the boundary measurements and the location of the unknown domain $\mathcal{D}^*$, we propose in this work a new technique based on the Kohn-Vogelius formulation and the topological sensitivity technique. The Kohn-Vogelius formulation is a self regularization method which transforms the geometric inverse problem to a shape optimization problem. It leads to define two problems for any given domain $\mathcal{D} \subset \Omega$. The first one, named the Neumann problem, is associated with the Neumann datum V:

$$\begin{gathered}
-\Delta\psi_n = F \quad \text{in } \Omega\backslash\overline{\mathcal{D}}\\
\nabla\psi_n\cdot\mathbf{n} = V \quad \text{on } \Gamma_1\\
\psi_n = 0 \quad \text{on } \Gamma_2\\
\psi_n = 0 \quad \text{on } \partial\mathcal{D}.
\end{gathered} \tag{2.1}$$

The second one is associated to the measured ψ_m, which will be named as the Dirichlet problem:

$$\begin{gathered}
-\Delta\psi_d = F \quad \text{in } \Omega\backslash\overline{\mathcal{D}}\\
\psi_d = \psi_m \quad \text{on } \Gamma_1\\
\psi_d = 0 \quad \text{on } \Gamma_2\\
\psi_d = 0 \quad \text{on } \partial\mathcal{D}.
\end{gathered} \tag{2.2}$$

We remark that if the domains $\mathcal{D}$ and $\mathcal{D}^*$ coincide then $\psi_n = \psi_d$. According to this observation, Kohn and Vogelius [25] proposed to change the inverse problem to the minimization of a function measuring the difference between the Dirichlet and Neumann solutions. We define the Kohn-Vogelius semi-norm function

$$\mathcal{J}(\Omega\backslash\overline{\mathcal{D}}) = \int_{\Omega\backslash\overline{\mathcal{D}}} |\nabla\psi_n - \nabla\psi_d|^2 \, d\mathbf{x},$$

where ψ_n (resp. ψ_d) is solution to the Neumann (resp. Dirichlet) perturbed problem.

3. Topological sensitivity method

To calculate a high-order topological asymptotic expansion of the semi-norm Kohn-Vogelius functional $\mathcal{J}$, we apply the topological sensitivity method. It consists in calculating the variation of $\mathcal{J}$ regarding to a small perturbation $B_{\mathbf{z},\epsilon}$ at the point $\mathbf{z}$ of the domain Ω. For $\mathbf{z} \in \Omega$ and $\epsilon > 0$, we define $B_{\mathbf{z},\epsilon} = \mathbf{z} + \epsilon B$, where $B \subset \mathbb{R}^3$ is a bounded fixed regular domain which contains the origin. We define the perturbed domain $\Omega_{\mathbf{z},\epsilon} = \Omega \setminus \overline{B_{\mathbf{z},\epsilon}}$ Let us consider the following overdetermined boundary value problem

$$\begin{gathered}
-\Delta\psi_\epsilon = F \quad \text{in } \Omega\backslash\overline{B_{\mathbf{z},\epsilon}}, \\
\nabla\psi_\epsilon \cdot \mathbf{n} = V \quad \text{on } \Gamma_1, \\
\psi_\epsilon = \psi_m \quad \text{on } \Gamma_1, \\
\psi_\epsilon = 0 \quad \text{on } \Gamma_2, \\
\psi_\epsilon = 0 \quad \text{on } \partial B_{\mathbf{z},\epsilon}.
\end{gathered} \tag{3.1}$$

We assume here that there exists $B_{\mathbf{z}^*,\epsilon} = \mathbf{z}^* + \epsilon B \subset \Omega$ such that there exists a solution to problem (3.1). Consequently, the following geometric inverse problem is considered:

> Find $B_{\mathbf{z},\epsilon} \subset \Omega$ such that the solution ψ_ϵ satisfies the overdetermined system (3.1).

The Kohn-Vogelius functional for the perturbed domain is defined by

$$\mathcal{J}(\Omega_{\mathbf{z},\epsilon}) = \int_{\Omega_{\mathbf{z},\epsilon}} |\nabla\psi_{n,\epsilon} - \nabla\psi_{d,\epsilon}|^2 d\mathbf{x},$$

where $\psi_{n,\epsilon}$ is the solution to the perturbed Neumann problem

$$\begin{gathered}
-\Delta\psi_{n,\epsilon} = F \quad \text{in } \Omega\backslash\overline{B_{\mathbf{z},\epsilon}}, \\
\psi_{n,\epsilon} = 0 \quad \text{on } \partial B_{\mathbf{z},\epsilon}, \\
\nabla\psi_{n,\epsilon}\mathbf{n} = V \quad \text{on } \Gamma_1, \\
\psi_{n,\epsilon} = 0 \quad \text{on } \Gamma_2.
\end{gathered} \tag{3.2}$$

and $\psi_{d,\epsilon}$ is the solution to the perturbed Dirichlet problem

$$\begin{gathered}
-\Delta\psi_{d,\epsilon} = F \quad \text{in } \Omega\backslash\overline{B_{\mathbf{z},\epsilon}}, \\
\psi_{d,\epsilon} = 0 \quad \text{on } \partial B_{\mathbf{z},\epsilon}, \\
\psi_{d,\epsilon} = \psi_m \quad \text{on } \Gamma_1, \\
\psi_{d,\epsilon} = 0 \quad \text{on } \Gamma_2.
\end{gathered} \tag{3.3}$$

We remark that if $\epsilon = 0$, $\Omega_{\mathbf{z},0} = \Omega$ and ψ_0 satisfies

$$-\Delta\psi_0 = F \quad \text{in } \Omega,$$

$$
\begin{aligned}
\nabla\psi_0\cdot\mathbf{n} &= V \quad \text{on } \Gamma_1,\\
\psi_0 &= \psi_m \quad \text{on } \Gamma_1,\\
\psi_0 &= 0 \quad \text{on } \Gamma_2,
\end{aligned}
$$

$\psi_{n,0}$ is solution to

$$
\begin{aligned}
-\Delta\psi_{n,0} &= F \quad \text{in } \Omega\\
\nabla\psi_{n,0}\mathbf{n} &= V \quad \text{on } \Gamma_1\\
\psi_{n,0} &= 0 \quad \text{on } \Gamma_2,
\end{aligned} \tag{3.4}
$$

and $\psi_{d,0}$ is solution to

$$
\begin{aligned}
-\Delta\psi_{d,0} &= F \quad \text{in } \Omega\\
\psi_{d,0} &= \psi_m \quad \text{on } \Gamma_1\\
\psi_{d,0} &= 0 \quad \text{on } \Gamma_2.
\end{aligned} \tag{3.5}
$$

As mentioned in the introduction the majority of works interested to this method are based on the first-order asymptotic expansions of the functional $\mathcal{J}$ presented by

$$\mathcal{J}(\Omega_{\mathbf{z},\epsilon}) = \mathcal{J}(\Omega) + f(\epsilon)\delta\mathcal{J}(\mathbf{z}) + o(f(\epsilon)),$$

where $\delta\mathcal{J}$ is the topological gradient and f is a positive scalar function with $\lim_{\epsilon\to 0} f(\epsilon) = 0$. Then, for small ϵ the solution of the minimization problem

$$\min_{B_{\mathbf{z},\epsilon}\subset\Omega} \mathcal{J}(\Omega\setminus\overline{\omega_{\mathbf{z},\epsilon}}),$$

is given by $B_{\mathbf{z}^*,\epsilon}$, with $\mathbf{z}^*\in\Omega$ where $\delta\mathcal{J}$ is the most negative. This is due to the fact that if $\delta\mathcal{J}(\mathbf{z}^*) < \delta\mathcal{J}(\mathbf{z})$, we obtain $\mathcal{J}(\Omega_{\mathbf{z}^*,\epsilon}) < \mathcal{J}(\Omega_{\mathbf{z},\epsilon})$. The purpose of this work is to obtain an asymptotic expansion of higher order for the Kohn-Vogelius functional $\mathcal{J}$ to detect an object of finite size and valid when the topological gradient $\delta\mathcal{J}$ vanishes at some critical points inside Ω, under the form:

$$\mathcal{J}(\Omega_{\mathbf{z},\epsilon}) = \mathcal{J}(\Omega) + \sum_{i=1}^{I} f_i(\epsilon)\delta^i\mathcal{J}(\mathbf{z}) + o(f_I(\epsilon)),$$

where f_i, $1\le i\le I$ are scalar positives functions verifying $f_{i+1}(\epsilon) = o(f_i(\epsilon))$ and vanish with ϵ. $\delta^i\mathcal{J}$ is the i^{th} topological derivative of the Kohn-Vogelius functional $\mathcal{J}$.

To derive the expected expansion, we establish in the next section some preliminary results. The main results of this analysis will be presented in section 5.

4. Some preliminary results

The aim of this section is to present an asymptotic formula describing the variation of the solutions $\psi_{n,\epsilon}$ and $\psi_{d,\epsilon}$ caused by the perturbation of Ω by $B_{\mathbf{z},\epsilon}$.

In conductivity imperfections identification context, an asymptotic expansion describing the variation of the solutions for $I = 1$ was derived in [14, 19] for the Laplace equation. Another application was studied using Stokes system [1] for the detection of obstacles in a flow via the asymptotic expansion of the velocity filed.

In this work, to derive the desired formula, we need to find an asymptotic expansion of the exterior problem solution for the Laplace equation defined in $\mathbb{R}^3\setminus\overline{B}$. Let $\Phi\in H^{1/2}(\partial B)$, denoting by H the solution to

$$
\begin{aligned}
-\Delta H &= 0 \quad \text{in } \mathbb{R}^3\setminus\overline{B},\\
H &\to 0 \quad \text{at } \infty,
\end{aligned}
$$

$$H = \Phi \quad \text{on } \partial B,$$

Resorting to the simple layer potential representation [16, 28], H can be written as

$$H(\mathbf{y}) = \int_{\partial\omega} E(\mathbf{y}-t)\, q(t) ds(t), \quad \forall \mathbf{y} \in \mathbb{R}^3 \setminus \overline{B}, \tag{4.1}$$

where E is the Laplace equation fundamental solution in $\mathbb{R}^3$:

$$E(\mathbf{y}) = \frac{1}{4\pi\|\mathbf{y}\|},$$

and q is the boundary integral equation unique solution

$$\int_{\partial B} E(\mathbf{y}-t) q(t) ds(t) = \Phi(\mathbf{y}), \quad \forall \mathbf{y} \in \partial B. \tag{4.2}$$

By the change of variable $\mathbf{x} = \mathbf{z} + \epsilon\mathbf{y}$ and using the perturbation $B_{\mathbf{z},\epsilon}$ is not close to the boundary $\partial\Omega$, we have

$$H((\mathbf{x}-\mathbf{z})/\epsilon) = \epsilon \int_{\partial B} E(\mathbf{x}-\mathbf{z}-\epsilon\, t)\, q(t) ds(t), \quad \forall \mathbf{x} \in \mathbb{R}^3 \setminus \overline{B_{\mathbf{z},\epsilon}}.$$

Denoting by $\varphi_{\mathbf{x}-\mathbf{z},t}$ the function

$$\varphi_{\mathbf{x}-\mathbf{z},t} : \epsilon \longmapsto \varphi_{\mathbf{x}-\mathbf{z},t}(\epsilon) = \epsilon E((\mathbf{x}-\mathbf{z}) - \epsilon t), \quad \forall \epsilon > 0.$$

Using the fact that $\varphi_{\mathbf{x}-\mathbf{z},t}$ is smooth regarding ϵ and satisfies the following behavior

$$\varphi_{\mathbf{x}-\mathbf{z},t}(\epsilon) = \sum_{p=1}^{I} \frac{\epsilon^p}{p!} \varphi^{(p)}_{\mathbf{x}-\mathbf{z},t}(0) + O(\epsilon^{I+1}),$$

where $\varphi^{(p)}_{\mathbf{x}-\mathbf{z},t}(0)$ is the p^{th} derivative of $\varphi_{\mathbf{x}-\mathbf{z},t}$ at $\epsilon = 0$. Then the following lemma gives an asymptotic expansion of the function

$$\mathbf{x} \mapsto H(\frac{\mathbf{x}-\mathbf{z}}{\epsilon}).$$

Lemma 4.1. *For any $I \geq 0$, we have*

$$H((\mathbf{x}-\mathbf{z})/\epsilon) = \sum_{p=1}^{I} \epsilon^p\, H^{(p)}(\mathbf{x}-\mathbf{z}) + O(\epsilon^{I+1}), \quad \forall \mathbf{x} \in \mathbb{R}^3 \setminus \overline{B_{\mathbf{z},\epsilon}},$$

where $H^{(p)}$ is the smooth function defined by

$$H^{(p)}(\mathbf{x}-\mathbf{z}) = \frac{1}{p!} \int_{\partial B} \varphi^{(p)}_{\mathbf{x}-\mathbf{z},t}(0) q(t) ds(t).$$

Remark 4.2. The first-order asymptotic expansion of the exterior problem solution for the Laplace equation is proved by Guilaume and Sid Idris [21, page 1049].

4.1. **Asymptotic formula of the Neumann problem solution.** To present an asymptotic formula describing the variation of the Neumann Problem solution $\psi_{n,\epsilon}$, we define the sequences functions $(\Psi_{n,i})_{0\leq i\leq I}$ and $(W_{n,i})_{0\leq i\leq I}$, where for all $0 \leq i \leq I$, $\Psi_{n,i}$ are smooth function defined in the initial domain Ω, obtained as the solution to a interior problem with Neumann boundary condition on Γ_1 and $W_{n,i}$ are smooth function defined in $\mathbb{R}^3 \setminus \overline{B}$, obtained as the solution to a exterior problems. More precisely:

For $i = 0$: $\Psi_{n,0} = \psi_{n,0}$ and $W_{n,0}$ is the solution to

$$\begin{gathered} -\Delta W_{n,0} = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{B}, \\ W_{n,0} \to 0 \quad \text{at } \infty \\ W_{n,0} = -\psi_{n,0}(\mathbf{z}) \quad \text{on } \partial B. \end{gathered} \tag{4.3}$$

For $i = 1$: $\Psi_{n,1}$ is the solution to

$$\begin{gathered} -\Delta \Psi_{n,1} = 0 \quad \text{in } \Omega, \\ \nabla \Psi_{n,1} \cdot \mathbf{n} = -\nabla W_{n,0}^{(1)}(\mathbf{x} - \mathbf{z}) \cdot \mathbf{n} \quad \text{on } \Gamma_1, \\ \Psi_{n,1} = -W_{n,0}^{(1)}(\mathbf{x} - \mathbf{z}) \quad \text{on } \Gamma_2, \end{gathered} \tag{4.4}$$

with $W_{n,0}^{(1)}$ is defined by Lemma 4.1 in the particular case $i = 0$, $\phi = -\psi_{n,0}(\mathbf{z})$ and $p = 1$.

The function $W_{n,1}$ depends on $\Psi_{n,0}$ and $\Psi_{n,1}$, it is solution of the exterior problem

$$\begin{gathered} -\Delta W_{n,1} = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{B}, \\ W_{n,1} \to 0 \quad \text{at } \infty \\ W_{n,1} = -\Psi_{n,1}(\mathbf{z}) - D\Psi_{n,0}(\mathbf{z})(\mathbf{y}) \quad \text{on } \partial B. \end{gathered} \tag{4.5}$$

For $1 \le i \le I$: The function $\Psi_{n,i}$ depends on $W_{n,j}$ for $0 \le j \le i-1$ and is solution of the interior problem

$$\begin{gathered} -\Delta \Psi_{n,i} = 0 \quad \text{in } \Omega, \\ \nabla \Psi_{n,i} \cdot \mathbf{n} = -\sum_{p=1}^{i} \nabla W_{n,i-p}^{(p)}(\mathbf{x} - \mathbf{z}) \cdot \mathbf{n} \quad \text{on } \Gamma_1, \\ \Psi_{n,i} = -\sum_{p=1}^{i} W_{n,i-p}^{(p)}(\mathbf{x} - \mathbf{z}) \quad \text{on } \Gamma_2, \end{gathered} \tag{4.6}$$

with $W_{n,j}^{(p)}$ is defined by Lemma 4.1.

The function $W_{n,i}$ depends on $\Psi_{n,j}$ for $0 \le j \le i$ and is solution of the exterior problem

$$\begin{gathered} -\Delta W_{n,i} = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{B}, \\ W_{n,i} \to 0 \quad \text{at } \infty \\ W_{n,i} = -\Psi_{n,i}(\mathbf{z}) - \sum_{p=1}^{i} \frac{1}{p!} D^p \Psi_{n,i-p}(\mathbf{z})(\mathbf{y}^p) \quad \text{on } \partial B, \end{gathered} \tag{4.7}$$

where $D^p \Psi_{n,i-p}(\mathbf{z})$ is the p^{th} derivative of $\Psi_{n,i-p}$ (the harmonic function) at $\mathbf{z} \in \Omega$ and $\mathbf{y}^p = (\mathbf{y}, \dots, \mathbf{y}) \in (\mathbb{R}^3)^p$.

We are now ready to present an asymptotic formula describing the variation of the solution $\psi_{n,\epsilon}$ raised from the perturbation of Ω by $B_{\mathbf{z},\epsilon}.\Omega$.

Theorem 4.3. *In the perturbed domain $\Omega_{\mathbf{z},\epsilon}$, the solution ψ_n^ϵ of the Neumann Laplace equation has the asymptotic expansion*

$$\psi_{n,\epsilon}(\mathbf{x}) = \sum_{i=0}^{I} \epsilon^i [\Psi_{n,i}(\mathbf{x}) + W_{n,i}(\frac{\mathbf{x} - \mathbf{z}}{\epsilon})] + O(\epsilon^{I+1}) \quad \textit{in } \Omega_{\mathbf{z},\epsilon}.$$

4.2. **Asymptotic formula of the Dirichlet problem solution.** Similarly to the asymptotic of the Neumann solution, to present an asymptotic formula describing the variation of the Dirichlet Problem solution $\psi_{d,\epsilon}$, we define the sequences functions $(\Psi_{d,i})_{0\le i\le I}$ and $(W_{d,i})_{0\le i\le I}$, where for all $0\le i\le I$, $\Psi_{d,i}$ are smooth function defined in the initial domain Ω, obtained as the solution to a interior problem with Dirichlet boundary condition on Γ_1 and $W_{d,i}$ are smooth function defined in $\mathbb{R}^3\setminus\overline{B}$, obtained as the solution to a exterior problems. More precisely, the sequences functions $(\Psi_{d,i})_{0\le i\le I}$ and $(W_{d,i})_{0\le i\le I}$, are defined as follow:

For $i=0$: $\Psi_{d,0}=\psi_{d,0}$ and $W_{d,0}$ is the solution to

$$\begin{gathered} -\Delta W_{d,0}=0 \quad \text{in } \mathbb{R}^3\setminus\overline{B},\\ W_{d,0}\to 0 \quad \text{at } \infty\\ W_{d,0}=-\psi_{d,0}(\mathbf{z}) \quad \text{on } \partial B. \end{gathered} \tag{4.8}$$

For $i=1$: $\Psi_{d,1}$ is the solution to

$$\begin{gathered} -\Delta\Psi_{d,1}=0 \quad \text{in } \Omega,\\ \Psi_{d,1}=-W^{(1)}_{d,0}(\mathbf{x}-\mathbf{z}) \quad \text{on } \partial\Omega, \end{gathered} \tag{4.9}$$

with $W^{(1)}_{d,0}$ is defined by Lemma 4.1 in the particular case $i=0$, $\phi=-\psi_{d,0}(\mathbf{z})$ and $p=1$.

The function $W_{d,1}$ depends on $\Psi_{d,0}$ and $\Psi_{d,1}$, and is solution of the exterior problem

$$\begin{gathered} -\Delta W_{d,1}=0 \quad \text{in } \mathbb{R}^3\setminus\overline{B},\\ W_{d,1}\to 0 \quad \text{at } \infty\\ W_{d,1}=-\Psi_{d,1}(\mathbf{z})-D\Psi_{d,0}(\mathbf{z})(\mathbf{y}) \quad \text{on } \partial B. \end{gathered} \tag{4.10}$$

For $1\le i\le I$: The function $\Psi_{d,i}$ depends on $W_{d,j}$ for $0\le j\le i-1$ and is solution of the interior problem

$$\begin{gathered} -\Delta\Psi_{d,i}=0 \quad \text{in } \Omega,\\ \Psi_{d,i}=-\sum_{p=1}^{i} W^{(p)}_{d,i-p}(\mathbf{x}-\mathbf{z}) \quad \text{on } \partial\Omega, \end{gathered} \tag{4.11}$$

with $W^{(p)}_{d,j}$ defined in Lemma 4.1.

The function $W_{d,i}$ depends on $\Psi_{d,j}$ for $0\le j\le i$ and is solution of the exterior problem

$$\begin{gathered} -\Delta W_{d,i}=0 \quad \text{in } \mathbb{R}^3\setminus\overline{B},\\ W_{d,i}\to 0 \quad \text{at } \infty\\ W_{d,i}=-\Psi_{d,i}(\mathbf{z})-\sum_{p=1}^{i}\frac{1}{p!}D^p\Psi_{d,i-p}(\mathbf{z})(\mathbf{y}^p) \quad \text{on } \partial B. \end{gathered} \tag{4.12}$$

We are now ready to present an asymptotic formula giving the variation of $\psi_{d,\epsilon}$ raised from the perturbation of Ω by $B_{\mathbf{z},\epsilon}$.

Theorem 4.4. *In the perturbed domain $\Omega_{\mathbf{z},\epsilon}$, the solution $\psi_{d,\epsilon}$ of the Dirichlet Laplace equation has the asymptotic expansion*

$$\psi_{d,\epsilon}(\mathbf{x}) = \sum_{i=0}^{I} \epsilon^i [\Psi_{d,i}(\mathbf{x}) + W_{d,i}(\frac{\mathbf{x}-\mathbf{z}}{\epsilon})] + O(\epsilon^{I+1}) \quad \text{in } \Omega_{\mathbf{z},\epsilon}.$$

5. Asymptotic formula

The main result is presented in this section. A high-order topological asymptotic expansion is derived for the semi-norm Kohn-Vogelius functional $\mathcal{J}$, when a Dirichlet perturbation is introduced in the initial domain. The functional $\mathcal{J}$ can be decomposed as

$$\begin{aligned}\mathcal{J}(\Omega_{\mathbf{z},\epsilon}) &= \int_{\Omega_{\mathbf{z},\epsilon}} |\nabla \psi_{d,\epsilon}|^2 \mathrm{dx} + \int_{\Omega_{\mathbf{z},\epsilon}} |\nabla \psi_{n,\epsilon}|^2 \mathrm{dx} - 2\int_{\Omega_{\mathbf{z},\epsilon}} \nabla \psi_{d,\epsilon} . \nabla \psi_{n,\epsilon} \mathrm{dx} \\ &= \mathcal{J}_d(\Omega_{\mathbf{z},\epsilon}) + \mathcal{J}_n(\Omega_{\mathbf{z},\epsilon}) + \mathcal{J}_{dn}(\Omega_{\mathbf{z},\epsilon}),\end{aligned}$$

where

$$\begin{aligned}\mathcal{J}_d(\Omega_{\mathbf{z},\epsilon}) &= \int_{\Omega_{\mathbf{z},\epsilon}} |\nabla \psi_{d,\epsilon}|^2 \mathrm{dx}, \\ \mathcal{J}_n(\Omega_{\mathbf{z},\epsilon}) &= \int_{\Omega_{\mathbf{z},\epsilon}} |\nabla \psi_{n,\epsilon}|^2 \mathrm{dx}, \\ \mathcal{J}_{d,n}(\Omega_{\mathbf{z},\epsilon}) &= -2\int_{\Omega_{\mathbf{z},\epsilon}} \nabla \psi_{d,\epsilon} . \nabla \psi_{n,\epsilon} \mathrm{dx}.\end{aligned}$$

The Dirichlet term $\mathcal{J}_d$ has the following variation

$$\begin{aligned}\mathcal{J}_d(\Omega_{\mathbf{z},\epsilon}) - \mathcal{J}_d(\Omega) &= \int_{\Omega_{\mathbf{z},\epsilon}} |\nabla \psi_{d,\epsilon}|^2 \mathrm{dx} - \int_{\Omega} |\nabla \psi_{d,0}|^2 \mathrm{dx} \\ &= \int_{\Omega_{\mathbf{z},\epsilon}} \nabla(\psi_{d,\epsilon} + \psi_{d,0}) \cdot \nabla(\psi_{d,\epsilon} - \psi_{d,0}) \mathrm{dx} - \int_{B_{\mathbf{z},\epsilon}} |\nabla \psi_{d,0}|^2 \mathrm{dx}.\end{aligned}$$

By the Green formula, from the problems (3.3) and (3.3) with $\epsilon = 0$, we deduce

$$\begin{aligned}&\int_{\Omega_{\mathbf{z},\epsilon}} \nabla(\psi_{d,\epsilon} + u_{d,0}) \cdot \nabla(\psi_{d,\epsilon} - \psi_{d,0}) \mathrm{dx} \\ &= -\int_{\partial B_{\mathbf{z},\epsilon}} \nabla(\psi_{d,\epsilon} + u_{d,0}) \cdot \mathbf{n} \psi_{d,0} \mathrm{ds} + 2\int_{\Omega_{\mathbf{z},\epsilon}} F(\psi_{d,\epsilon} - \psi_{d,0}) \mathrm{dx}.\end{aligned}$$

From problem (3.3) with $\epsilon = 0$ we derive

$$\int_{B_{\mathbf{z},\epsilon}} |\nabla \psi_{d,0}|^2 \mathrm{dx} = -\int_{\partial B_{\mathbf{z},\epsilon}} \nabla \psi_{d,0} \cdot \mathbf{n} \psi_{d,0} \ \mathrm{ds} + \int_{B_{\mathbf{z},\epsilon}} F \ \psi_{d,0} \ \mathrm{dx} \ , \tag{5.1}$$

then, we obtain

$$\begin{aligned}&\mathcal{J}_d(\Omega_{\mathbf{z},\epsilon}) - \mathcal{J}_d(\Omega) \\ &= -\int_{\partial B_{\mathbf{z},\epsilon}} \nabla \psi_{d,\epsilon} \cdot \mathbf{n} \psi_{d,0} \ \mathrm{ds} \ + 2\int_{\Omega_{\mathbf{z},\epsilon}} F\left(\psi_{d,\epsilon} - \psi_{d,0}\right) \ \mathrm{dx} \ - \int_{B_{\mathbf{z},\epsilon}} F \psi_{d,0} \ \mathrm{dx} \\ &= -\int_{\partial B_{\mathbf{z},\epsilon}} \nabla(\psi_{d,\epsilon} - \psi_{d,0}) \cdot \mathbf{n} \psi_{d,0} \ \mathrm{ds} - \int_{\partial B_{\mathbf{z},\epsilon}} \nabla \psi_{d,0} \cdot \mathbf{n} \psi_{d,0} \ \mathrm{ds} \\ &\quad + 2\int_{\Omega_{\mathbf{z},\epsilon}} F(\psi_{d,\epsilon} - \psi_{d,0}) \ \mathrm{dx} \ - \int_{B_{\mathbf{z},\epsilon}} F \psi_{d,0} \ \mathrm{dx} \ .\end{aligned}$$

Then, from (5.1) it follows that

$$\begin{aligned}\mathcal{J}_d(\Omega_{\mathbf{z},\epsilon}) - \mathcal{J}_d(\Omega) = &- \int_{\partial B_{\mathbf{z},\epsilon}} (\nabla\psi_{d,\epsilon} - \nabla\psi_{d,0}).\mathbf{n}\psi_{d,0} \text{ ds } + \int_{B_{\mathbf{z},\epsilon}} |\nabla\psi_{d,0}|^2 \text{ dx} \\ &+ 2\int_{\Omega_{\mathbf{z},\epsilon}} F\,(\psi_{d,\epsilon} - \psi_{d,0}) \text{ dx } - 2\int_{B_{\mathbf{z},\epsilon}} F\,\psi_{d,0} \text{ dx}\,.\end{aligned}$$

Similarly, the Neumann term $\mathcal{J}_n$ has the variation

$$\begin{aligned}\mathcal{J}_n(\Omega_{\mathbf{z},\epsilon}) - \mathcal{J}_n(\Omega) &= \int_{\Omega_{\mathbf{z},\epsilon}} |\nabla\psi_{n,\epsilon}|^2 \text{dx} - \int_{\Omega} |\nabla\psi_{n,0}|^2 \text{dx} \\ &= \int_{\partial B_{\mathbf{z},\epsilon}} (\nabla\psi_{n,\epsilon} - \nabla\psi_{n,0}).\mathbf{n}\,\psi_{n,0} \text{ ds } - \int_{B_{\mathbf{z},\epsilon}} |\nabla\psi_{n,0}|^2 \text{ dx}.\end{aligned}$$

The Dirichlet/Neumann term $\mathcal{J}_{d,n}$ has the variation

$$\begin{aligned}\mathcal{J}_{d,n}(\Omega_{\mathbf{z},\epsilon}) - \mathcal{J}_{d,n}(\Omega) &= \int_{\Omega_{\mathbf{z},\epsilon}} \nabla\psi_{d,\epsilon}.\nabla\psi_{n,\epsilon} \text{ dx } - \int_{\Omega} \nabla\psi_{d,0}.\nabla\psi_{n,0} \text{ dx} \\ &= \int_{\Omega_{\mathbf{z},\epsilon}} F(\psi_{d,\epsilon} - \psi_{d,0}) \text{ dx } - \int_{B_{\mathbf{z},\epsilon}} F\,\psi_{d,0} \text{ dx}\,.\end{aligned}$$

Then the functional $\mathcal{J}$ has the variation

$$\begin{aligned}\mathcal{J}(\Omega_{\mathbf{z},\epsilon}) - \mathcal{J}(\Omega) = &\int_{B_{\mathbf{z},\epsilon}} |\nabla\psi_{d,0}|^2 d\mathbf{x} - \int_{B_{\mathbf{z},\epsilon}} |\nabla\psi_{n,0}|^2 d\mathbf{x} \\ &- \int_{\partial B_{\mathbf{z},\epsilon}} (\nabla\psi_{d,\epsilon} - \nabla\psi_{d,0}).\mathbf{n}\,\psi_{d,0} ds \\ &+ \int_{\partial B_{\mathbf{z},\epsilon}} (\nabla\psi_{n,\epsilon} - \nabla\psi_{n,0}).\mathbf{n}\psi_{n,0} ds.\end{aligned}$$

From Theorem 4.3, we have

$$\begin{aligned}&\int_{\partial B_{\mathbf{z},\epsilon}} (\nabla\psi_{n,\epsilon} - \nabla\psi_{n,0}).\mathbf{n}\psi_{n,0} ds \\ &= \sum_{i=1}^{I} \epsilon^i \int_{\partial B_{\mathbf{z},\epsilon}} \nabla\Psi_{n,i}(\mathbf{x}).\mathbf{n}(\mathbf{x})\,\psi_{n,0}(\mathbf{x}) ds \\ &\quad + \sum_{i=0}^{I} \epsilon^i \int_{\partial B_{\mathbf{z},\epsilon}} \nabla_{\mathbf{x}} W_{n,i}((\mathbf{x} - \mathbf{z})/\epsilon)) \cdot \mathbf{n}\,\psi_{n,0} ds + O(\epsilon^{I+1}),\end{aligned}$$

and using Theorem 4.4, we have

$$\begin{aligned}&\int_{\partial B_{\mathbf{z},\epsilon}} (\nabla\psi_{d,\epsilon} - \nabla\psi_{d,0}).\mathbf{n}\,\psi_{d,0} ds \\ &= \sum_{i=1}^{I} \epsilon^i \int_{\partial B_{\mathbf{z},\epsilon}} \nabla\Psi_{d,i}(\mathbf{x}).\mathbf{n}(\mathbf{x})\,\psi_{d,0}(\mathbf{x}) ds \\ &\quad + \sum_{i=0}^{I} \epsilon^i \int_{\partial B_{\mathbf{z},\epsilon}} \nabla_{\mathbf{x}} W_{d,i}(\frac{\mathbf{x} - \mathbf{z}}{\epsilon}) \cdot \mathbf{n}\psi_{d,0} ds + O(\epsilon^{I+1}).\end{aligned}$$

Consequently, the functional $\mathcal{J}$ has the following variation

$$
\begin{aligned}
\mathcal{J}(\Omega_{\mathbf{z},\epsilon}) - \mathcal{J}(\Omega) = &\sum_{i=0}^{I} \epsilon^i \int_{\partial B_{\mathbf{z},\epsilon}} \nabla_{\mathbf{x}} W_{n,i}(\frac{\mathbf{x}-\mathbf{z}}{\epsilon}) \cdot \mathbf{n}\, \psi_{n,0} ds \\
&- \sum_{i=0}^{I} \epsilon^i \int_{\partial B_{\mathbf{z},\epsilon}} \nabla_{\mathbf{x}} W_{d,i}(\frac{\mathbf{x}-\mathbf{z}}{\epsilon}) \cdot \mathbf{n}\, \psi_{d,0} ds \\
&+ \sum_{i=1}^{I} \epsilon^i \int_{\partial B_{\mathbf{z},\epsilon}} \nabla \Psi_{n,i}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})\, \psi_{n,0}(\mathbf{x}) ds \\
&- \sum_{i=1}^{I} \epsilon^i \int_{\partial B_{\mathbf{z},\epsilon}} \nabla \Psi_{d,i}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})\, \psi_{d,0}(\mathbf{x}) ds \\
&+ \int_{B_{\mathbf{z},\epsilon}} |\nabla \psi_{d,0}|^2 d\mathbf{x} - \int_{B_{\mathbf{z},\epsilon}} |\nabla \psi_{n,0}|^2 d\mathbf{x} + O(\epsilon^{I+1}).
\end{aligned} \tag{5.2}
$$

To present the desired asymptotic expansion of the Kohn-Vogelius functional $\mathcal{J}$, for all $\mathbf{z} \in \Omega$ we consider the following notation:

$$
\begin{aligned}
\mathcal{T}_{n,1}^i(\mathbf{z}) &= \sum_{p=0}^{i} \frac{1}{p!} \int_{\partial B} \nabla_{\mathbf{y}} W_{n,i-p}(\mathbf{y}) \cdot \mathbf{n}(\mathbf{y}) [\nabla^{(p)} \psi_{n,0}(\mathbf{z})(\mathbf{y}^p)] ds(\mathbf{y}), \\
\mathcal{T}_{d,1}^i(\mathbf{z}) &= -\sum_{p=0}^{i} \frac{1}{p!} \int_{\partial B} \nabla_{\mathbf{y}} W_{d,i-p}(\mathbf{y}) \cdot \mathbf{n}(\mathbf{y}) [\nabla^{(p)} \psi_{d,0}(\mathbf{z})(\mathbf{y}^p)] ds(\mathbf{y}), \\
\mathcal{T}_{n,2}^i(\mathbf{z}) &= \sum_{p=0}^{i} \sum_{q=0}^{p} \frac{1}{q!(p-q)!} \int_{\partial B} [\nabla^{(q+1)} \Psi_{n,i-p+1}(\mathbf{z})(\mathbf{y}^q)] \cdot \mathbf{n}(\mathbf{y}) \\
&\quad \times [\nabla^{(p-q)} \psi_{n,0}(\mathbf{z})(\mathbf{y}^{p-q})] ds(\mathbf{y}), \\
\mathcal{T}_{d,2}^i(\mathbf{z}) &= -\sum_{p=0}^{i} \sum_{q=0}^{p} \frac{1}{q!(p-q)!} \int_{\partial B} [\nabla^{(q+1)} \Psi_{d,i-p+1}(\mathbf{z})(\mathbf{y}^q)] \cdot \mathbf{n}(\mathbf{y}) \\
&\quad \times [\nabla^{(p-q)} \psi_{d,0}(\mathbf{z})(\mathbf{y}^{p-q})] ds(\mathbf{y}), \\
\mathcal{T}_{d,3}^i(\mathbf{z}) &= \sum_{p=0}^{i} \frac{1}{p!(i-p)!} \int_{B} \nabla^{(p+1)} \psi_{d,0}(\mathbf{z})(\mathbf{y}^p) \cdot \nabla^{(i-p+1)} \psi_{d,0}(\mathbf{z})(\mathbf{y}^{i-p}) d\mathbf{y}, \\
\mathcal{T}_{n,3}^i(\mathbf{z}) &= -\sum_{p=0}^{i} \frac{1}{p!(i-p)!} \int_{B} \nabla^{(p+1)} \psi_{n,0}(\mathbf{z})(\mathbf{y}^p) \cdot \nabla^{(i-p+1)} \psi_{n,0}(\mathbf{z})(\mathbf{y}^{i-p}) d\mathbf{y}.
\end{aligned}
$$

We can know derive the topological asymptotic expansion of the Kohn-Vogelius cost functional $\mathcal{J}$ by giving the variation $\mathcal{J}(\Omega_{\mathbf{z},\epsilon}) - \mathcal{J}(\Omega)$ regarding to the geometric perturbation of the domain at any point. The main result is described by the following theorem.

Theorem 5.1. *The topological asymptotic expansion of the Kohn-Vogelius functional $\mathcal{J}$ is given by*

$$
\mathcal{J}(\Omega_{\mathbf{z},\epsilon}) - \mathcal{J}(\Omega) = \sum_{i=1}^{I} \epsilon^i \delta^i \mathcal{J}(\mathbf{z}) + O(\epsilon^{I+1}),
$$

where

$$\delta^i \mathcal{J}(\mathbf{z}) = \begin{cases} \mathcal{T}_{n,1}^{i-1}(\mathbf{z}) - T_{d,1}^{i-1}(\mathbf{z}) & \text{if } i \le 2, \\ \mathcal{T}_{n,1}^{i-1}(\mathbf{z}) - T_{d,1}^{i-1}(\mathbf{z}) + T_{n,2}^{i-3}(\mathbf{z}) - T_{d,2}^{i-3} + T_{n,3}^{i-3} - T_{d,3}^{i-3}(\mathbf{z}) & \text{if } 3 \le i \le I. \end{cases}$$

6. Proofs

The aim of this section is to prove Theorems 4.3 and 4.4, and the main result described in Theorem 5.1.

Proofs of Theorems 4.3 and 4.4. To prove Theorem 4.3, in $\Omega_{\mathbf{z},\epsilon}$ we define the function

$$\begin{aligned} R_{n,I}^{\epsilon} &= \Psi_{n,0}(\mathbf{x}) + W_{n,0}(\frac{\mathbf{x}-\mathbf{z}}{\epsilon}) + \epsilon\,(\Psi_{n,1}(\mathbf{x}) + W_{n,1}(\frac{\mathbf{x}-\mathbf{z}}{\epsilon}) \\ &\quad + \dots + \epsilon^N(\Psi_{n,I}(\mathbf{x}) + W_{n,I}(\frac{\mathbf{x}-\mathbf{z}}{\epsilon}) - \psi_n^{\epsilon}(\mathbf{x}). \end{aligned}$$

We can easily show that $R_{n,I}^{\epsilon}$ is harmonic in $\Omega_{\mathbf{z},\epsilon}$.

On $\partial B_{\mathbf{z},\epsilon}$ we have

$$\begin{aligned} R_{n,I}^{\epsilon}(\mathbf{x}) &= \Psi_{n,0}(\mathbf{x}) + W_{n,0}(\frac{\mathbf{x}-\mathbf{z}}{\epsilon}) + \sum_{i=1}^{I} \epsilon^i [\Psi_{n,i}(\mathbf{x}) + W_{n,i}(\frac{\mathbf{x}-\mathbf{z}}{\epsilon})] \\ &= \sum_{i=0}^{I} \epsilon^i \Psi_{n,i}(\mathbf{x}) - \sum_{i=0}^{I} \epsilon^i \Big[\sum_{p=0}^{i} \frac{1}{p!} D^p \Psi_{n,i-p}(\mathbf{z})((\frac{\mathbf{x}-\mathbf{z}}{\epsilon})^p)\Big]. \end{aligned} \tag{6.1}$$

From the multilinearity of $D^p \Psi_{n,i-p}(\mathbf{z})$, it follows that

$$\begin{aligned} \sum_{i=1}^{I} \epsilon^i \Big[\sum_{p=0}^{i} \frac{1}{p!} D^p \Psi_{n,i-p}(\mathbf{z})((\frac{\mathbf{x}-\mathbf{z}}{\epsilon})^p)\Big] &= \sum_{i=0}^{I} \sum_{p=0}^{i} \frac{\epsilon^{i-p}}{p!} D^p \Psi_{n,i-p}(\mathbf{z})((\mathbf{x}-\mathbf{z})^p) \\ &= \sum_{i=0}^{I} \epsilon^i \sum_{p=0}^{N-i} \frac{1}{p!} D^p \Psi_{n,i}(\mathbf{z})((\mathbf{x}-\mathbf{z})^p). \end{aligned}$$

Then, one can deduce

$$R_{n,I}^{\epsilon} = \sum_{i=0}^{I} \epsilon^i \Big[\Psi_{n,i}(\mathbf{x}) - \sum_{p=0}^{I-i} \frac{1}{p!} D^p \Psi_{n,i}(\mathbf{z})((\mathbf{x}-\mathbf{z})^p)\Big]. \tag{6.2}$$

Using that $\|\mathbf{x}-\mathbf{z}\| = O(\epsilon)$ on $\partial B_{\mathbf{z},\epsilon}$ and Taylor's Theorem [29], we have

$$R_{n,I}^{\epsilon}(\mathbf{x}) = O(\epsilon^{I+1}), \quad \text{on } \partial B_{\mathbf{z},\epsilon}.$$

On Γ_2 we have

$$\begin{aligned} R_{n,I}^{\epsilon}(\mathbf{x}) &= \sum_{i=0}^{I} \epsilon^i W_{n,i}(\frac{\mathbf{x}-\mathbf{z}}{\epsilon}) - \sum_{i=1}^{I} \epsilon^i \big[\sum_{p=1}^{i} W_{n,i-p}^{(p)}(\mathbf{x}-\mathbf{z})\big] \\ &= \sum_{i=0}^{I} \epsilon^i W_{n,i}(\frac{\mathbf{x}-\mathbf{z}}{\epsilon}) - \sum_{i=0}^{I-1} \epsilon^i \big[\sum_{p=1}^{I-i} \epsilon^p W_{n,i}^{(p)}(\mathbf{x}-\mathbf{z})\big]. \end{aligned}$$

This equality can be written as

$$R_{n,I}^{\epsilon}(\mathbf{x}) = \epsilon^I W_{n,I}(\frac{\mathbf{x}-\mathbf{z}}{\epsilon}) + \sum_{i=0}^{I-1} \epsilon^i \big[W_{n,i}(\frac{\mathbf{x}-\mathbf{z}}{\epsilon}) - \sum_{p=1}^{I-i} \epsilon^p W_{n,i}^{(p)}(\mathbf{x}-\mathbf{z})\big].$$

Then, by Lemma 4.1 we obtain

$$R^{\epsilon}_{n,I} = O(\epsilon^{I+1}) \quad \text{on } \Gamma_2.$$

On Γ_n, using the same analysis we obtain

$$\nabla R^{\epsilon}_{n,I} \cdot \mathbf{n} = O(\epsilon^{I+1}) \quad \text{on } \Gamma_1.$$

Similarly, to prove Theorem 4.4, we define the function $R^{\epsilon}_{d,I}$ in $\Omega_{\mathbf{z},\epsilon}$ by

$$\begin{aligned} R^{\epsilon}_{d,I} &= \Psi_{d,0}(\mathbf{x}) + W_{d,0}(\frac{\mathbf{x}-\mathbf{z}}{\epsilon}) + \epsilon\,(\Psi_{d,1}(\mathbf{x}) + W_{d,1}(\frac{\mathbf{x}-\mathbf{z}}{\epsilon}) \\ &\quad + \dots + \epsilon^I(\Psi_{d,I}(\mathbf{x}) + W_{d,I}(\frac{\mathbf{x}-\mathbf{z}}{\epsilon}) - \psi^{\epsilon}_d(\mathbf{x}) \end{aligned}$$

and using the same analysis in the proof of Theorem 4.3 we derive $R^{\epsilon}_{d,I} = O(\epsilon^{I+1})$.

6.1. **Proofs of the main results in Theorem 5.1.** To prove Theorem 5.1, we have to estimate each term of the equality (5.2).

Estimate for the first and the second terms. By changing $\mathbf{x} = \mathbf{z} + \epsilon\mathbf{y}$, we have

$$\int_{\partial B_{\mathbf{z},\epsilon}} \nabla_{\mathbf{x}} W_{n,i}(\frac{\mathbf{x}-\mathbf{z}}{\epsilon}) \cdot \mathbf{n}(\mathbf{x})\psi_{n,0}(\mathbf{x})ds = \epsilon \int_{\partial B} \nabla_{\mathbf{y}} W_{n,i}(\mathbf{y}) \cdot \mathbf{n}(\mathbf{y})\, \psi_{n,0}(\mathbf{z}+\epsilon\mathbf{y})ds(\mathbf{y}).$$

Since $\psi_{n,0}$ is smooth in a neighborhood of $\mathbf{z}$, one obtains

$$\begin{aligned} \psi_{n,0}(\mathbf{z}+\epsilon\mathbf{y}) &= \psi_{n,0}(\mathbf{z}) + \sum_{p=1}^{I-1} \frac{\epsilon^p}{p!} \nabla^{(p)}\psi_{n,0}(\mathbf{z})(\mathbf{y}^p) + O(\epsilon^I) \\ &= \sum_{p=0}^{I-1} \frac{\epsilon^p}{p!} \nabla^{(p)}\psi_{n,0}(\mathbf{z})(\mathbf{y}^p) + O(\epsilon^I). \end{aligned}$$

Then, we have

$$\begin{aligned} &\int_{\partial B_{\mathbf{z},\epsilon}} \nabla_{\mathbf{x}} W_{n,i}(\frac{\mathbf{x}-\mathbf{z}}{\epsilon}) \cdot \mathbf{n}(\mathbf{x})\psi_{n,0}(\mathbf{x})ds \\ &= \sum_{p=0}^{I-1} \frac{\epsilon^{p+1}}{p!} \int_{\partial B} \nabla_{\mathbf{y}} W_{n,i}(\mathbf{y}) \cdot \mathbf{n}(\mathbf{y})[\nabla^{(p)}\psi_{n,0}(\mathbf{z})(\mathbf{y}^p)]ds(\mathbf{y}) + O(\epsilon^{I+1}). \end{aligned}$$

Consequently,

$$\begin{aligned} &\sum_{i=0}^{I} \epsilon^i \int_{\partial B_{\mathbf{z},\epsilon}} \nabla_{\mathbf{x}} W_{n,i}((\mathbf{x}-\mathbf{z})/\epsilon)) \cdot \mathbf{n}\, \psi_{n,0} ds \\ &= \sum_{i=0}^{I} \epsilon^i \sum_{p=0}^{I-1} \frac{\epsilon^{p+1}}{p!} \int_{\partial B} \nabla_{\mathbf{y}} W_{n,i}(\mathbf{y}) \cdot \mathbf{n}(\mathbf{y})[\nabla^{(p)}\psi_{n,0}(\mathbf{z})(\mathbf{y}^p)]ds(\mathbf{y}) + O(\epsilon^{I+1}) \\ &= \sum_{i=1}^{I} \epsilon^i \sum_{p=0}^{i-1} \frac{1}{p!} \int_{\partial\omega} \nabla_{\mathbf{y}} W_{n,i-p-1}(\mathbf{y}) \cdot \mathbf{n}(\mathbf{y})[\nabla^{(p)}\psi_{n,0}(\mathbf{z})(\mathbf{y}^p)]ds(\mathbf{y}) + O(\epsilon^{I+1}) \\ &= \sum_{i=1}^{I} \epsilon^i \mathcal{T}^{i-1}_{n,1}(\mathbf{z}) + O(\epsilon^{I+1}), \end{aligned}$$

Similarly, we obtain

$$\sum_{i=0}^{I} \epsilon^i \int_{\partial B_{\mathbf{z},\epsilon}} \nabla_{\mathbf{x}} W_{d,i}((\mathbf{x}-\mathbf{z})/\epsilon)) \cdot \mathbf{n}\, \psi_{d,0} ds = -\sum_{i=1}^{I} \epsilon^i \mathcal{T}_{d,1}^{i-1}(\mathbf{z}) + O(\epsilon^{I+1}).$$

Estimate for the third and the fourth terms. By changing $\mathbf{x} = \mathbf{z} + \epsilon\mathbf{y}$, we have

$$\int_{\partial B_{\mathbf{z},\epsilon}} \nabla\Psi_{n,i}(\mathbf{x})\cdot\mathbf{n}(\mathbf{x})\psi_{n,0}(\mathbf{x})\, ds = \epsilon^2 \int_{\partial B} \nabla\Psi_{n,i}(\mathbf{z}+\epsilon\mathbf{y})\cdot\mathbf{n}(\mathbf{z}+\epsilon\mathbf{y})\, u_{n,0}(\mathbf{z}+\epsilon\mathbf{y}) ds(\mathbf{y}).$$

Since $\psi_{n,0}$ is smooth in a neighborhood of $\mathbf{z}$, one obtains

$$\begin{aligned} \psi_{n,0}(\mathbf{z}+\epsilon\mathbf{y}) &= \psi_{n,0}(\mathbf{z}) + \sum_{p=1}^{I-1} \frac{\epsilon^p}{p!} \nabla^{(p)}\psi_{n,0}(\mathbf{z})(\mathbf{y}^p) + O(\epsilon^I) \\ &= \sum_{p=0}^{I-1} \frac{\epsilon^p}{p!} \nabla^{(p)}\psi_{n,0}(\mathbf{z})(\mathbf{y}^p) + O(\epsilon^I). \end{aligned}$$

Similarly, Ψ_i is smooth in a neighborhood of $\mathbf{z}$, then

$$\nabla\Psi_{n,i}(\mathbf{z}+\epsilon\mathbf{y}) = \sum_{q=0}^{I-1} \frac{\epsilon^q}{q!} \nabla^{(q+1)}\Psi_{n,i}(\mathbf{z})(\mathbf{y}^q) + O(\epsilon^I).$$

Then

$$\begin{aligned} &\int_{\partial B_{\mathbf{z},\epsilon}} \nabla U_{n,i}(\mathbf{x})\cdot\mathbf{n}(\mathbf{x}) u_{n,0}(\mathbf{x}) ds \\ &= \epsilon^2 \int_{\partial B} [\sum_{q=0}^{I-1} \frac{\epsilon^q}{q!} \nabla^{(q+1)} U_{n,i}(\mathbf{z})(\mathbf{y}^q)]\cdot\mathbf{n}(\mathbf{y}) [\sum_{p=0}^{I-1} \frac{\epsilon^p}{p!} \nabla^{(p)} u_{n,0}(\mathbf{z})(\mathbf{y}^p)] ds(\mathbf{y}) + O(\epsilon^{I+1}). \end{aligned}$$

Using the Cauchy product formula, we derive

$$\begin{aligned} &\int_{\partial B_{\mathbf{z},\epsilon}} \nabla\Psi_{n,i}(\mathbf{x})\cdot\mathbf{n}(\mathbf{x})\psi_{n,0}(\mathbf{x}) ds \\ &= \sum_{p=0}^{I-2} \epsilon^{p+2} \sum_{q=0}^{p} \frac{1}{q!(p-q)!} \int_{\partial B} [\nabla^{(q+1)}\Psi_{n,i}(\mathbf{z})(\mathbf{y}^q)]\cdot\mathbf{n}(\mathbf{y}) \\ &\quad \times [\nabla^{(p-q)}\psi_{n,0}(\mathbf{z})(\mathbf{y}^{p-q})] ds(\mathbf{y}) + O(\epsilon^{I+1}). \end{aligned}$$

Consequently,

$$\begin{aligned} &\sum_{i=1}^{I} \epsilon^i \int_{\partial B_{\mathbf{z},\epsilon}} \nabla\Psi_{n,i}(\mathbf{x})\cdot\mathbf{n}(\mathbf{x})\psi_{n,0}(\mathbf{x}) ds \\ &= \sum_{i=1}^{I}\sum_{p=0}^{I-2} \epsilon^{i+p+2} \sum_{q=0}^{p} \frac{1}{q!(p-q)!} \\ &\quad \times \int_{\partial B} [\nabla^{(q+1)}\Psi_{n,i}(\mathbf{z})(\mathbf{y}^q)]\cdot\mathbf{n}(\mathbf{y}) [\nabla^{(p-q)}\psi_{n,0}(\mathbf{z})(\mathbf{y}^{p-q})] ds(\mathbf{y}) + O(\epsilon^{I+1}) \\ &= \sum_{i=3}^{I} \epsilon^i \sum_{p=0}^{i-3}\sum_{q=0}^{p} \frac{1}{q!(p-q)!} \end{aligned}$$

$$
\begin{aligned}
&\times \int_{\partial B} [\nabla^{(q+1)} \Psi_{n,i-p-2}(\mathbf{z})(\mathbf{y}^q)] \cdot \mathbf{n}(\mathbf{y}) [\nabla^{p-q} \psi_{n,0}(\mathbf{z})(\mathbf{y}^{(p-q)})] ds(\mathbf{y}) + O(\epsilon^{I+1}) \\
&= \sum_{i=3}^{I} \epsilon^i \mathcal{T}_{n,2}^{i-3}(\mathbf{z}) + O(\epsilon^{I+1}).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\sum_{i=1}^{I} \epsilon^i \int_{\partial B_{\mathbf{z},\epsilon}} \nabla \Psi_{d,i}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) \psi_{d,0}(\mathbf{x}) ds \\
&= \sum_{i=3}^{I} \epsilon^i \sum_{p=0}^{i-3} \sum_{q=0}^{p} \frac{1}{q!(p-q)!} \\
&\quad \times \int_{\partial B} [\nabla^{(q+1)} \Psi_{d,i-p-2}(\mathbf{z})(\mathbf{y}^q)] \cdot \mathbf{n}(\mathbf{y}) [\nabla^{p-q} \psi_{d,0}(\mathbf{z})(\mathbf{y}^{(p-q)})] ds(\mathbf{y}) + O(\epsilon^{I+1}) \\
&= -\sum_{i=3}^{I} \epsilon^i \mathcal{T}_{d,2}^{i-3}(\mathbf{z}) + O(\epsilon^{I+1}).
\end{aligned}
$$

Estimate for the fifth and the sixth terms. Since $\psi_{d,0}$ and $\psi_{n,0}$ are sufficiently regular in $B_{\mathbf{z},\epsilon}$, we have

$$
\begin{aligned}
\nabla \psi_{d,0}(\mathbf{z} + \epsilon \mathbf{y}) &= \nabla \psi_{d,0}(\mathbf{z}) + \sum_{i=1}^{I-1} \frac{\epsilon^i}{i!} \nabla^{(i+1)} \psi_{d,0}(\mathbf{z})(\mathbf{y}^i) + O(\epsilon^I) \\
\nabla \psi_{n,0}(\mathbf{z} + \epsilon \mathbf{y}) &= \nabla \psi_{n,0}(\mathbf{z}) + \sum_{i=1}^{I-1} \frac{\epsilon^i}{i!} \nabla^{(i+1)} \psi_{n,0}(\mathbf{z})(\mathbf{y}^i) + O(\epsilon^I).
\end{aligned}
$$

By the change of variable $\mathbf{x} = \mathbf{z} + \epsilon \mathbf{y}$, we obtain

$$
\begin{aligned}
\int_{B_{\mathbf{z},\epsilon}} |\nabla \psi_{d,0}|^2 d\mathbf{x} &= \epsilon^3 \int_B |\nabla \psi_{d,0}(\mathbf{z} + \epsilon \mathbf{y})|^2 d\mathbf{y} \\
&= \epsilon^3 \int_B \Big(\sum_{i=0}^{I-1} \frac{\epsilon^i}{i!} |\nabla^{(i+1)} \psi_{d,0}(\mathbf{z})(\mathbf{y}^i) \Big) |^2 d\mathbf{y} + O(\epsilon^{I+1}).
\end{aligned}
$$

Using the Cauchy product formula, we obtain

$$
\begin{aligned}
&\int_{B_{\mathbf{z},\epsilon}} |\nabla \psi_{d,0}|^2 d\mathbf{x} \\
&= \sum_{i=0}^{I-3} \epsilon^{i+3} \Big(\sum_{p=0}^{i} \frac{1}{p!(i-p)!} \int_B \nabla^{(p+1)} \psi_{d,0}(\mathbf{z})(\mathbf{y}^p) \cdot \nabla^{(i-p+1)} \psi_{d,0}(\mathbf{z})(\mathbf{y}^{i-p}) d\mathbf{y} \Big) \\
&\quad + O(\epsilon^{I+1}) \\
&= \sum_{i=3}^{I} \epsilon^i \mathcal{T}_{d,3}^{i-3}(\mathbf{z}) + O(\epsilon^{I+1}).
\end{aligned}
$$

Similarly,

$$
\int_{B_{\mathbf{z},\epsilon}} |\nabla \psi_{n,0}|^2 d\mathbf{x}
$$

$$
\begin{aligned}
&= \sum_{i=0}^{I-3} \epsilon^{i+3} \Big(\sum_{p=0}^{i} \frac{1}{p!(i-p)!} \int_B \nabla^{(p+1)} \psi_{n,0}(\mathbf{z})(\mathbf{y}^p) \cdot \nabla^{(i-p+1)} \psi_{n,0}(\mathbf{z})(\mathbf{y}^{i-p}) d\mathbf{y} \Big) \\
&\quad + O(\epsilon^{I+1}) \\
&= - \sum_{i=3}^{I} \epsilon^i \, \mathcal{T}_{n,3}^{i-3}(\mathbf{z}) + O(\epsilon^{I+1}).
\end{aligned}
$$

Finally, the desired result is obtained by using the above estimates.

Concluding remarks. This work is concerned with a geometric inverse problem related to the Laplace operator in three-dimensional domain. More precisely, the topological sensitivity method is applied to calculate a high-order topological asymptotic expansion of the semi-norm Kohn-Vogelius functional, when a Dirichlet perturbation is introduced in the initial domain.

The obtained expansion of the semi-norm Kohn-Vogelius functional is of higher interest and improves the detection of objects with any size of perturbation. The other advantage is when the topological derivative of order one is equal to zero for some critical points in the initial domain.

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Khalifa Khelifi
Department of Mathematics, College of Sciences, Monastir University, Monastir, Tunisia
E-mail address: khalifakhelifi@hotmail.fr

Mohamed Abdelwahed (corresponding author)
Department of Mathematics, College of Sciences, King Saud University, Riyadh, Saudi Arabia
E-mail address: mabdelwahed@ksu.edu.sa

Nejmeddine Chorfi
Department of Mathematics, College of Sciences, King Saud University, Riyadh, Saudi Arabia
E-mail address: nchorfi@ksu.edu.sa

Maatoug Hassine
Department of Mathematics, College of Sciences, Monastir University, Monastir, Tunisia
E-mail address: Maatoug.Hassine@enit.rnu.tn