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A GENERALIZATION OF SCHAUDER'S THEOREM AND ITS APPLICATION TO CAUCHY-KOVALEVSKAYA PROBLEM

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Abstract. We extend the classical majorant functions method to a PDE system which right hand side is a mapping of one functional space to another. This extension is based on some generalization of the Schauder fixed point theorem.

1. Introduction

Kovalevskaya proved that the analytic Cauchy problem has an unique analytic solution in 1842. She used the method of majorant functions developed by Cauchy and Weierstrass. In this article, we consider the classical method of majorant functions from an abstract viewpoint and extend this method to a PDE system which right hand side is a mapping of one functional space to another. This mapping can be non-analytic in the evolution variable. Then this result is used for obtaining estimates for the evolution variable interval on which the solution of the problem exists and also to obtain majorant estimates for this solution. The estimated obtained can be used in some problems of perturbation theory [3].

Our version of the majorant functions method is based on some generalization of Schauder's fixed point theorem to the case of seminormed spaces. Our results do not follow from the abstract Cauchy-Kovalevskaya theorems in [2] and [4].

Preliminaries in topology. Following [5] we introduce some definitions.

Let M be a semimetric space with a collection of semimetrics $\{\rho_\omega\}_{\omega\in\Omega}$. Recall that a function $\rho : M\times M \to \mathbb{R}$ is referred as semimetric if it satisfies all the metric axioms except the axiom of non-degenerateness; i. e., it is possibly that $\rho(x, y) = 0$ for some $x, y \in M$ such that $x \neq y$.

We assume that for any finite set $Q \subset \Omega$ there exists $\omega \in \Omega$ such that

$$\rho_q(\cdot,\cdot) \le \rho_\omega(\cdot,\cdot), \quad q \in Q.$$

This assumption allows us to consider M as a topological space. A basis of the topology in this space is given by the balls

$$B_\omega(r, y) = \{x \in M : \rho_\omega(x, y) < r\}.$$

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Definition 1.1. We say that a set $U \subset M$ is bounded if for every $\omega \in \Omega$ there exists r and y such that $U \subseteq B_\omega(r, y)$.

Definition 1.2. We say that a space M satisfies Montel's axiom if any closed and bounded subset of M is compact.

In this article, we assume that all spaces satisfy the first axiom of countability: For any $y \in M$ there exists a countable collection of the balls $\{B_\tau(r_\tau, y)\}_{\tau\in\mathbb{N}}$ such that if G is a neighborhood of y then $B_\tau(r_\tau, y) \subseteq G$ for some τ. This assumption enables to prove topological assertions in terms of sequences instead of using neighborhoods.

Definition 1.3. We say that a sequence $\{x_k\}_{k\in\mathbb{N}}$ converges to x as $k \to \infty$ if for every $\varepsilon > 0$ and $\omega \in \Omega$ there exists N such that for all $n > N$, $\rho_\omega(x_n, x) < \varepsilon$.

Thus a set $K \subset M$ is called compact if any sequence $\{x_k\} \in K$ contains a subsequence $\{x'_k\}$ such that $x'_k \to \hat{x} \in K$ as $k \to \infty$.

In similar way, we introduce a seminormed linear space E with a collection of seminorms $\{\|\cdot\|_\omega\}_{\omega\in\Omega}$. Consider the following examples:

Let $\{(E_\omega, \|\cdot\|_\omega)\}_{0<\omega<1}$ be a scale of normed spaces over the field $\mathbb{R}$ or $\mathbb{C}$:

$$E_{\omega+\delta} \subseteq E_\omega, \quad \|\cdot\|_\omega \le \|\cdot\|_{\omega+\delta}, \quad \delta > 0.$$

We construct a seminormed space $E = \bigcap_{0<\omega<1} E_\omega$ with the collection of norms $\{\|\cdot\|_\omega\}_{0<\omega<1}$. (We use the term 'seminormed space' even if all seminorms are norms.)

Let $U_r^n = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : |z| = \max_k |z_k| < r\}$ be a polycircle. Consider a space $\mathcal{H}_n$ of a functions $f : U_R^n \to \mathbb{C}$ that are analytic in U_R^n. The space $\mathcal{H}_n$ is seminormed with a collection of norms

$$\|f\|_r = \max_{|z|\le r} |f(z)|, \quad 0 < r < R.$$

Theorem 1.1 (Montel's theorem [5])**.** *The space $\mathcal{H}_n$ satisfies Montel's axiom.*

Consider the linear operator $D : \mathcal{H}_n \to \mathcal{H}_n$ defined as

$$Df = \frac{\partial f}{\partial z_1}.$$

This operator is continuous with respect to definition 1.3. Indeed, let $u_k \to u, \quad u_k, u \in \mathcal{H}_n$ as $k \to \infty$. According to the Cauchy inequality we get

$$\|Du_k - Du\|_r \le \frac{K}{\delta}\|u_k - u\|_{r+\delta} \to 0,$$

where K is a positive constant and $r + \delta < R$. Nevertheless, it is well known that this operator is not continuous with respect to any fixed norm $\|\cdot\|_r$.

2. Main theorem

Let $(L, \{\|\cdot\|_\omega\}_{\omega\in\Omega})$ be a seminormed space and $\omega' \in \Omega$ be such that $\|\cdot\|_{\omega'}$ is a norm. Then a compact set $K \subset L$ is convex.

Now, we consider a continuous map $f : K \to K$.

Theorem 2.1 (Generalized Schauder's theorem)**.** *There exists a point $\hat{x} \in K$ such that $f(\hat{x}) = \hat{x}$.*

Recall the original formulation of Schauder's theorem. Let $(L, \|\cdot\|)$ be a Banach space and $K \subset L$ be a convex compact set. Then a continuous map $f : K \to K$ has a fixed point $\hat{x} \in K$.

Note that though this formulation includes completeness of the space, actually this condition is not necessary. The point is that the proof of this theorem (see [1]) considers the map f only on the compact K but any compact set is complete and can be embedded to a completion of the space L.

Proof of Theorem 2.1. Let $(E, \{\rho_\omega\}_{\omega\in\Omega})$ and $(F, \{d_\sigma\}_{\sigma\in\Sigma})$ be semimetric spaces. and there exist ω', σ' such that the semimetrics $\rho_{\omega'}$ and $d_{\sigma'}$ are metrics.

Consider a compact set (with respect to the semimetric topology) $K \subset (E, \{\rho_\omega\}_{\omega\in\Omega})$ and a map $f : E \to F$.

Lemma 2.2. *If the map $f : E \to F$ is continuous on K, with respect to the semimetric topology, then it is continuous on K as a map of the metric space $(E, \rho_{\omega'})$ to the metric space $(F, d_{\sigma'})$.*

Proof. Let $\{x_n\} \subset K$ be a sequence such that $\rho_{\omega'}(x_n, a) \to 0$ as $n \to \infty$ where $a \in K$ and we put $y_n = f(x_n)$. So we must prove that $d_{\sigma'}(y_n, b) \to 0$ where $b = f(a)$.

Assume the converse. Then there exists a subsequence $\{y'_n\} \subseteq \{y_n\}$ such that $d_{\sigma'}(y'_n, b) \geq c > 0$. A set $\hat{K} = f(K)$ is compact as an image of a compact set under a continuous map and $\{y'_n\} \subset \hat{K}$. Thus, there exists a subsequence $\{y''_n\} \subseteq \{y'_n\}$ such that

$$d_\sigma(y''_n, \beta) \to 0, \quad \sigma \in \Sigma, \quad \beta \neq b. \tag{2.1}$$

Let $\{x''_n\} \subseteq \{x_n\}$ be a sequence such that $y''_n = f(x''_n)$. Consider a subsequence $\{x'''_n\} \subseteq \{x''_n\}$ that converges with respect to the semimetric topology: $\rho_\omega(x'''_n, a) \to 0$ for all $\omega \in \Omega$ and let $y'''_n = f(x'''_n)$. Note that $\{y'''_n\} \subseteq \{y''_n\}$.

Since f is continuous we have $d_\sigma(y'''_n, b) \to 0$ for all $\sigma \in \Sigma$. On other hand we have (2.1). This contradiction proves the Lemma. □

Theorem 2.1 follows, almost directly, from original Schauder's theorem and Lemma 2.2. Indeed, by Lemma 2.2 the map f is continuous on K with respect to the norm $\|\cdot\|_{\omega'}$. By $\overline{L}$ denote a completion of L with respect to the same norm.

It is easy to check that the compactness of the set K with respect to the seminormed topology involves the compactness of K with respect to the norm $\|\cdot\|_{\omega'}$. So we obtain the continuous map $f : K \to K$ where K is a convex compact set in the Banach space $\overline{L}$.

By the original Schauder's theorem we get the fixed point $\hat{x}$. Then Theorem 2.1 is proved.

3. Application: majorant method for Cauchy-Kovalevskaya problem

Now we study an existence of Cauchy-Kovalevskaya problem's solutions for a single partial differential equation. Extension of this theory to the case of countable PDE system contains in [6]. Consider the problem

$$u_t = f(u), \quad u\big|_{t=0} = u_0(z) \in \mathcal{H}_n. \tag{3.1}$$

By a subscript we denote a derivative. For example u_t is the derivative of the function u with respect to the variable t.

Let I_T be the interval $[0, T]$. Denote by $C(I_T, \mathcal{H}_n)$ the seminormed space of continues maps $v : I_T \to \mathcal{H}_n$ with a collection of seminorms:

$$\|v\|_r^c = \max_{t \in I_T} \|v(z,t)\|_r.$$

We imply that the space $\mathcal{H}_{n+1}$ consists of such a type functions: $u(z,t) \in \mathcal{H}_{n+1}$.

We consider problem (3.1) in the following two setups. Complex-time setup: f is a continues map of the set $\mathcal{H}_{n+1}$ to itself. Real-time setup: f is a continues map of the set $C(I_T, \mathcal{H}_n)$ to itself.

Note that we consider continuity of the map f with respect to the seminormed topology of the space $\mathcal{H}_n$. For example f can contain derivatives such as

$$\frac{\partial^{j_1+\dots+j_n}}{\partial {z_1}^{j_1} \dots \partial {z_n}^{j_n}}.$$

Now we give the following definition. An analytic function

$$G(z) = \sum_{k_1,\dots k_n \geq 0} G_{k_1,\dots k_n} z_1^{k_1} \cdot \dots \cdot z_n^{k_n}$$

is said to be a majorant function (or majorant) for another analytic function

$$g(z) = \sum_{k_1,\dots k_n \geq 0} g_{k_1,\dots k_n} z_1^{k_1} \cdot \dots \cdot z_n^{k_n}$$

if $|g_{k_1,\dots k_n}| \leq G_{k_1,\dots k_n}$ for all integer $k_1, \dots k_n \geq 0$. This condition is denoted by $g \ll G$.

If functions $g, G \in C(I_T, \mathcal{H}_n)$, then their Taylor coefficients depend on t and the relation $g \ll G$ implies that $|g_{k_1,\dots k_n}(t)| \leq G_{k_1,\dots k_n}(t)$ for all $t \in I_T$.

Define a relation '$\ll$' for maps as follows:

Real-time setup: A map $Q : C(I_T, \mathcal{H}_n) \to C(I_T, \mathcal{H}_n)$ is said to be majorant for a map $q : C(I_T, \mathcal{H}_n) \to C(I_T, \mathcal{H}_n)$ if for all $v, V \in C(I_T, \mathcal{H}_n)$ such that $v \ll V$ we have $q(v) \ll Q(V)$.

Complex-time setup: A map $Q : \mathcal{H}_{n+1} \to \mathcal{H}_{n+1}$ is said to be majorant for a map $q : \mathcal{H}_{n+1} \to \mathcal{H}_{n+1}$ if for all $v, V \in \mathcal{H}_{n+1}$ such that $v \ll V$ we have $q(v) \ll Q(V)$.

Define the following majorant pair $(U(z,t), F(U))$ for problem (3.1).

Real-time setup:

$$U \in C(I_T, \mathcal{H}_n), \quad F : C(I_T, \mathcal{H}_n) \to C(I_T, \mathcal{H}_n).$$

The function F is majorant for the function f and the following conditions hold:

$$\begin{aligned} &U(z,0) \gg u_0(z), \\ &U(z,t) \gg U(z,0) + \int_0^t F(U)\, ds, \end{aligned} \tag{3.2}$$

where $t \in I_T$.

Complex-time setup:

$$U \in \mathcal{H}_{n+1}, \quad F : \mathcal{H}_{n+1} \to \mathcal{H}_{n+1},$$

the function F is majorant for the function f and conditions (3.2) hold for $t \in U_R^n$.

The function F is continues on the respective sets. Particularly if the map F is majorant for the map f and $U(z,t)$ is a solution of the following problem:

$$U_t = F(U), \quad U\big|_{t=0} = U_0(z) \gg u_0(z)$$

then the pair $(U(z,t), F(U))$ is majorant for problem (3.1).

Theorem 3.1. *If problem (3.1) admits a majorant pair $(U(z,t), F(U))$ then it has solution $u(z,t)$ such that $u \in \mathcal{H}_{n+1}$ – for the complex-time setup, $u \in C(I_T, \mathcal{H}_n)$ – for the real-time setup and*

$$u(z,t) \ll U(z,t).$$

The technique of majorant pairs building was developed by D. Treschev. Nontrivial applications of this technique to perturbation theory are shown in [3].

Proof of Theorem 3.1. We will prove the theorem just in the real-time setup. The case of the complex-time can be considered in analogous way. Consider the following subset of $C(I_T, \mathcal{H}_n)$:

$$W = \{w(z,t) : w \ll U,$$
$$\|w(z,t') - w(z,t'')\|_r \le \|F(U)\|_r^c \cdot |t' - t''|, \quad r < R, \quad t', t'' \in I_T\}.$$

Lemma 3.2. *The set W is a convex compact.*

Proof. It is easy to check that W is a convex closed set. The set W is uniformly continues: there exist a set of constants $\{M_r\}$ such that for any $w \in W$ and for any $t', t'' \in I_T$ we have

$$\|w(z,t') - w(z,t'')\|_r \le M_r |t' - t''|.$$

Indeed, we can put $M_r = \|F(U)\|_r^c$.

The set W can be written as

$$W = \prod_{t \in I_T} W(t),$$

where $W(t) = \{w(z,t) \in W\} \subset \mathcal{H}_n$ and by $\prod$ we denote the cross product. The set $W(t)$ is bounded: if $w \in W(t)$ then $\|w(z,t)\|_r \le \|U\|_r^c$. By Montel's theorem it follows that $W(t)$ is compact in the space $\mathcal{H}_n$.

Then the proof will be complete when we apply the following theorem.

Theorem 3.3 ([5])**.** *If a closed set $W \subset C(I_T, \mathcal{H}_n)$ is uniformly continuous and for any $t \in I_T$ the set $W(t)$ is compact in $\mathcal{H}_n$, then W is compact in $C(I_T, \mathcal{H}_n)$.*

This complete the proof of Lemma 3.2 □

Let the map $P : C(I_T, \mathcal{H}_n) \to C(I_T, \mathcal{H}_n)$ be given by

$$P(w) = u_0(z) + \int_0^t f(w)\, ds.$$

Taking into account (3.2) one can check that $P(W) \subseteq W$. Then by Theorem 2.1 and Lemma 3.2 we obtain a fixed point $u(z,t) \in W$ for the map P:

$$P(u) = u.$$

This fixed point is a solution of the problem (3.1). which proves Theorem 3.1.

References

[1] L. Nirenberg, *Topics in nonlinear functional analisis*, Courant Institute of Math. Sciences New York University, 1974.

[2] T. Nishida, *A Note on a Theorem of Nirenberg*, J. Differential Geometry 12 (1977) 629-633

[3] A. Pronin, D. Treschev, *Continuous averaging in multi-frequency slow-fast systems*, Regular and Chaotic Dynamics, V.5, No.2, 2000, 157–170.

[4] M. Safonov, *The Abstract Cauchy-Kovalevskaya Theorem in a Weighted Banah Space*. Communications on Pure and Applied Mathmatics, 1995 vol.48, P. 629-643.

[5] L. Schwartz. *Analyse Mathématique*, Hermann. 1967.

[6] O. Zubelevich, *On the Magorant Method for Cauchy-Kovalevskaya Problem*, Mathematical Notes, 2001, 69(3), 363-374. (in Rusian)

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