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CHARACTERIZATION OF CONSTANT SIGN GREEN'S FUNCTION FOR A TWO-POINT BOUNDARY-VALUE PROBLEM BY MEANS OF SPECTRAL THEORY

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Dedicated to Prof. Stepan Tersian on his 65-th birthday

Abstract. This article is devoted to the study of the parameter's set where the Green's function related to a general linear n^{th}-order operator, depending on a real parameter, $T_n[M]$, coupled with many different two point boundary value conditions, is of constant sign. This constant sign is equivalent to the strongly inverse positive (negative) character of the related operator on suitable spaces related to the boundary conditions.

This characterization is based on spectral theory, in fact the extremes of the obtained interval are given by suitable eigenvalues of the differential operator with different boundary conditions. Also, we obtain a characterization of the strongly inverse positive (negative) character on some sets, where non homogeneous boundary conditions are considered. To show the applicability of the results, we give some examples. Note that this method avoids the explicit calculation of the related Green's function.

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1. Introduction

The study of the qualitative properties of the solution of a nonlinear two-point boundary-value problems for differential equations has been widely developed in the literature.

This work is devoted to the study of the strongly inverse positive (negative) character of the general n^{th}- order operator coupled with different boundary conditions. In order to do that, we characterize the parameter's set where the related Green's function is of constant sign. Our characterization is based on the spectral theory, actually, the extremes of the parameter's interval, where the Green's function is of constant sign, are just the eigenvalues of the given operator with suitable related boundary conditions.

This result avoids the requirement of obtaining the explicit expression of the Green's function, which can be very complicate to work with. In a wider class of situations, specially in the non constant coefficients case, it is not possible to obtain such an expression. Moreover, a slight change on the operator or in the boundary conditions may produce a big change on the expression of the related Green's function and its behavior. So, it is very useful to give a direct and easy way to characterize its sign.

It is well-known that the constant sign of the Green's function related to the linear part of a nonlinear problem is equivalent to the validity of the method of lower and upper solutions, coupled with monotone iterative techniques, that allows to deduce the existence of solution of such a problem, see for instance [2, 3, 17, 21].

Moreover, by using the constant sign of the related Green's function, nonexistence, existence and multiplicity results for nonlinear boundary-value problems are derived, by means of the well-known Kranosel'skiĭ contraction/expansion fixed point theorem [20], from the construction of suitable cones on Banach spaces [1, 4, 11, 19, 24]. The combination of these two methods has also been proved as a useful tool to ensure the existence of solution [5, 6, 16, 18, 23].

It is important to point out that the study of the constant sign of the related Green's function has been widely developed along the literature, by means of studying its expression [8, 9, 10, 22]. In all of them the expression of the Green's function has been obtained in order to prove the optimality of the previously obtained bounds. This work generalizes the ones given in [12] for the problems with the so-called $(k, n-k)$ boundary conditions and in [14] for a fourth order problem with the simply supported beam boundary conditions.

In this article, we study a huge number of different boundary conditions including the previously mentioned.

First, we introduce two sets of indices which describe the boundary conditions in each case. Let $k \in \{1, \dots, n-1\}$ and consider the following sets of indices

$\{\sigma_1,\dots,\sigma_k\}\subset\{0,\dots,n-1\}$ and $\{\varepsilon_1,\dots,\varepsilon_{n-k}\}\subset\{0,\dots,n-1\}$, such that

$$0\le\sigma_1<\sigma_2<\dots<\sigma_k\le n-1\,,\quad 0\le\varepsilon_1<\varepsilon_2<\dots<\varepsilon_{n-k}\le n-1\,.$$

Definition 1.1. Let us say that $\{\sigma_1,\dots,\sigma_k\}-\{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ satisfy property (N_a) if

$$\sum_{\sigma_j<h}1+\sum_{\varepsilon_j<h}1\ge h\,,\quad\forall h\in\{1,\dots,n-1\}\,. \tag{1.1}$$

Notation 1.2. Let us denote α, $\beta\in\{0,\dots,n-1\}$, such that

$$\alpha\notin\{\sigma_1,\dots,\sigma_k\}\,,\quad\text{and if }\alpha\ne0\,,\quad\{0,\dots,\alpha-1\}\subset\{\sigma_1,\dots,\sigma_k\}\,, \tag{1.2}$$

$$\beta\notin\{\varepsilon_1,\dots,\varepsilon_{n-k}\}\,,\quad\text{and if }\beta\ne0\,,\quad\{0,\dots,\beta-1\}\subset\{\varepsilon_1,\dots,\varepsilon_{n-k}\}\,. \tag{1.3}$$

Note that $\alpha\le k$ and $\beta\le n-k$. Let us define the following family of n^{th}-order linear differential equations

$$T_n[M]u(t)=u^{(n)}(t)+p_1(t)u^{(n-1)}(t)+\dots+(p_n(t)+M)u(t)=0\,,\quad t\in I\,, \tag{1.4}$$

where $I=[a,b]$ is a real fixed interval, $M\in\mathbb{R}$ a parameter and $p_j\in C^{n-j}(I)$ are given functions.

Note that this equation represents a general n order equation. In fact, we could think of

$$u^{(n)}(t)+p_1(t)u^{(n-1)}(t)+\dots+\tilde p_n(t)u(t)=0\,,\ t\in I\,,$$

where

$$\tilde p_n(t)=p_n(t)+\frac{1}{b-a}\int_a^b\tilde p_n(s)\,ds\equiv p_n(t)+M,\quad t\in I.$$

So, if p_n is a function of average equals to zero, the parameter M represents the average of the coefficient of u and, as a consequence, the problem of finding the values of M for which the Green's function has constant sign is equivalent to look for the values of the average of such a coefficient.

We study (1.4), coupled with the boundary conditions:

$$u^{(\sigma_1)}(a)=\dots=u^{(\sigma_k)}(a)=0\,, \tag{1.5}$$

$$u^{(\varepsilon_1)}(b)=\dots=u^{(\varepsilon_{n-k})}(b)=0\,. \tag{1.6}$$

This boundary conditions cover many different problems. As an example, we can consider $n=4$, $\{\sigma_1,\sigma_2\}=\{0,2\}$ and $\{\varepsilon_1,\varepsilon_2\}=\{0,2\}$ which correspond to the simply supported beam boundary conditions.

Note that, in the second order case, the Neumman conditions do not satisfy property (N_a). However, Dirichlet and Mixed conditions are included.

In this article, we will illustrate the obtained results with an example based on the choice of $\{\sigma_1,\sigma_2\}=\{0,2\}$ and $\{\varepsilon_1,\varepsilon_2\}=\{1,2\}$.

We consider the following definitions related to the boundary conditions (1.5)-(1.6):

$$\begin{aligned}X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}=\big\{u\in C^n(I):u^{(\sigma_1)}(a)=\dots=u^{(\sigma_k)}(a)=u^{(\varepsilon_1)}(b)\\ =\dots=u^{(\varepsilon_{n-k})}(b)=0\big\}.\end{aligned} \tag{1.7}$$

Remark 1.3. In this article we consider different choices of boundary conditions. Sometimes, we do not know the relative position of the given indices which define the spaces of definition. In particular, if we consider the following boundary conditions

$$u^{(\sigma_1)}(a)=\dots=u^{(\sigma_{k-1})}(a)=0\,,$$

$$\begin{gathered} u^{(\alpha)}(a)=0\,,\\ u^{(\varepsilon_1)}(b)=\dots=u^{(\varepsilon_{n-k})}(b)=0\,, \end{gathered}$$

with α defined by (1.2).

To point out this setting of the indices we use the following notation:

$$\begin{aligned} X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_{k-1}|\alpha\}} = \big\{ & u\in C^n(I): u^{(\sigma_1)}(a)=\dots=u^{(\sigma_{k-1})}(a)=0\,,\\ & u^{(\alpha)}(a)=0\,, u^{(\varepsilon_1)}(b)=\dots=u^{(\varepsilon_{n-k})}(b)=0\big\}. \end{aligned}$$

Analogously, we denote the following sets of functions:

$$\begin{aligned} X^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}_{\{\sigma_1,\dots,\sigma_{k}|\alpha\}} = \big\{ & u\in C^n(I): u^{(\sigma_1)}(a)=\dots=u^{(\sigma_{k})}(a)=0\,,\\ & u^{(\alpha)}(a)=0\,, u^{(\varepsilon_1)}(b)=\dots=u^{(\varepsilon_{n-k-1})}(b)=0\big\},\\ X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}_{\{\sigma_1,\dots,\sigma_{k-1}\}} = \big\{ & u\in C^n(I): u^{(\sigma_1)}(a)=\dots=u^{(\sigma_{k-1})}(a)=0\,,\\ & u^{(\varepsilon_1)}(b)=\dots=u^{(\varepsilon_{n-k})}(b)=0\,,\ u^{(\beta)}(b)=0\big\},\\ X^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}}_{\{\sigma_1,\dots,\sigma_{k}\}} = \big\{ & u\in C^n(I): u^{(\sigma_1)}(a)=\dots=u^{(\sigma_{k})}(a)=0\,,\\ & u^{(\varepsilon_1)}(b)=\dots=u^{(\varepsilon_{n-k-1})}(b)=0\,, u^{(\beta)}(b)=0\big\}. \end{aligned}$$

For instance, if $n=4$, $\sigma_1=0$, $\sigma_2=2$, $\varepsilon_1=0$ and $\varepsilon_2=1$, then $X^{\{\varepsilon_1,\varepsilon_2\}}_{\{\sigma_1|\alpha\}}=X^{\{0,1\}}_{\{0,1\}}$, where $\sigma_1=0<\alpha=1$. On another hand, if $\sigma_1=2$, $\sigma_2=3$, $\varepsilon_1=0$ and $\varepsilon_2=1$, then $X^{\{\varepsilon_1,\varepsilon_2\}}_{\{\sigma_1|\alpha\}}=X^{\{0,1\}}_{\{0,2\}}$ where $\alpha=0<\sigma_1=2$.

So, we are interested into characterize the parameter's set for which the operator $T_n[M]$ is either strongly inverse positive or negative on $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$.

Moreover, once we have obtained such a characterization with the homogeneous boundary conditions (1.5)-(1.6), we study its strongly inverse positive (negative) character on related spaces with non homogeneous boundary conditions.

The work is structured as follows, at first, in order to make the paper more readable, we introduce a preliminary section where some previous known results are shown. After that, in the next section, we introduce the hypotheses that both the operator $T_n[M]$ and the boundary conditions should satisfy to our results be applied. In Section 4 we obtain the expression of the related adjoint operator and boundary conditions. We deduce suitable properties of them. Next section is devoted to the study of operator $T_n[M]$ for a given $M=\bar{M}$ that satisfies some suitable previously introduced hypotheses. After that, in the two next sections, we study the existence and properties of the related eigenvalues of the operator and its adjoint, respectively, together to additional properties of the associated eigenfunctions. Section 8 is devoted to prove the main result of the work, where the characterization of the interval of parameters, where the Green's function has constant sign, is attained. At the end of such a section, some examples are shown. In Section 10, we obtain a necessary condition that M should verify, in order to allow $T_n[M]$ to be strongly inverse negative (positive) on the non considered cases on Section 8. At the end of such a section, we prove that this necessary condition can give an optimal interval in some cases. Once we have worked with the homogeneous boundary conditions, we obtain a characterization for a particular case of non homogeneous boundary conditions. Finally, we study a class of operators that satisfy the imposed hypotheses. Moreover, for this type of operators, we obtain a characterization for more general non homogeneous boundary conditions. The section finishes by showing some examples of this class of operators.

2. Preliminaries

In this section, for the convenience of the reader, we introduce the fundamental tools in the theory of disconjugacy and Green's functions that will be used in the development of further sections.

Definition 2.1. Let $p_k \in C^{n-k}(I)$ for $k = 1, \dots, n$. The n^{th}-order linear differential equation (1.4) is said to be disconjugate on I if every non trivial solution has less than n zeros on I, multiple zeros being counted according to their multiplicity.

Definition 2.2. The functions $u_1, \dots, u_n \in C^n(I)$ are said to form a Markov system on the interval I if the n Wronskians

$$W(u_1, \dots, u_k) = \begin{vmatrix} u_1 & \cdots & u_k \\ \vdots & \dots & \vdots \\ u_1^{(k-1)} & \cdots & u_k^{(k-1)} \end{vmatrix}, \quad k = 1, \dots, n, \tag{2.1}$$

are positive throughout I.

The following results about this concept are collected on [15, Chapter 3].

Theorem 2.3. *The linear differential equation* (1.4) *has a Markov fundamental system of solutions on the compact interval I if and only if it is disconjugate on I.*

Theorem 2.4. *The linear differential equation* (1.4) *has a Markov system of solutions if and only if the operator $T_n[M]$ has a representation*

$$T_n[M]\, y \equiv v_1 v_2 \dots v_n \frac{d}{dt}\Big(\frac{1}{v_n}\frac{d}{dt}\Big(\cdots \frac{d}{dt}\big(\frac{1}{v_2}\frac{d}{dt}\big(\frac{1}{v_1}\, y\big)\big)\Big)\Big), \tag{2.2}$$

where $v_k > 0$ on I and $v_k \in C^{n-k+1}(I)$ for $k = 1, \dots, n$.

To introduce the concept of Green's function related to the n^{th}-order scalar problem (1.4)–(1.6), we consider the following equivalent first-order vectorial problem:

$$x'(t) = A(t)\, x(t)\,,\ t \in I\,, \quad B\, x(a) + C\, x(b) = 0, \tag{2.3}$$

with $x(t) \in \mathbb{R}^n$, $A(t), B, C \in \mathcal{M}_{n\times n}$, defined by

$$x(t) = \begin{pmatrix} u(t) \\ u'(t) \\ \vdots \\ u^{(n-1)}(t) \end{pmatrix}, \quad A(t) = \left(\begin{array}{c|c} 0 & I_{n-1} \\ \hline -(p_n(t) + M) & -p_{n-1}(t) \cdots - p_1(t) \end{array}\right),$$

$$B = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{pmatrix}, \quad C = \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix}, \tag{2.4}$$

where $b_{j,1+\sigma_j} = 1$ for $j = 1, \dots, k$ and $c_{j+k,1+\varepsilon_j} = 1$ for $j = 1, \dots, n-k$; otherwise, $b_{ij} = 0$ and $c_{ij} = 0$.

Definition 2.5. We say that G is a Green's function for problem (2.3) if it satisfies the following properties:

(1) $G \equiv (G_{i,j})_{i,j \in \{1,\dots,n\}} \colon (I \times I) \backslash \{(t,t)\,,\ t \in I\} \to \mathcal{M}_{n\times n}$.

(2) G is a C^1 function on the triangles $\{(t,s)\in\mathbb{R}^2\,,\ a\le s<t\le b\}$ and $\{(t,s)\in\mathbb{R}^2\,,\ a\le t<s\le b\}$.
(3) For all $i\ne j$ the scalar functions $G_{i,j}$ have a continuous extension to $I\times I$.
(4) For all $s\in(a,b)$, the following equality holds:

$$\frac{\partial}{\partial t}G(t,s)=A(t)\,G(t,s),\quad \text{for all } t\in I\setminus\{s\}\,.$$

(5) For all $s\in(a,b)$ and $i\in\{1,\dots,n\}$, the following equalities are fulfilled:

$$\lim_{t\to s^+}G_{i,i}(t,s)=\lim_{t\to s^-}G_{i,i}(s,t)=1+\lim_{t\to s^+}G_{i,i}(s,t)=1+\lim_{t\to s^-}G_{i,i}(t,s).$$

(6) For all $s\in(a,b)$, the function $t\to G(t,s)$ satisfies the boundary conditions

$$B\,G(a,s)+C\,G(b,s)=0.$$

Remark 2.6. On the previous definition, item (G5) can be modified to obtain the characterization of the lateral limits for $s=a$ and $s=b$ as follows:

$$\lim_{t\to a^+}G_{i,i}(t,a)=1+\lim_{t\to a^+}G_{i,i}(a,t),\quad \text{and}\quad \lim_{t\to b^-}G_{i,i}(b,t)=1+\lim_{t\to b^-}G_{i,i}(t,b).$$

It is well known that Green's function related to this problem is given by the following expression [2, Section 1.4]

$$G(t,s)=\begin{pmatrix} g_1(t,s) & g_2(t,s) & \cdots & g_{n-1}(t,s) & g_M(t,s)\\ \frac{\partial}{\partial t}g_1(t,s) & \frac{\partial}{\partial t}g_2(t,s) & \cdots & \frac{\partial}{\partial t}g_{n-1}(t,s) & \frac{\partial}{\partial t}g_M(t,s)\\ \vdots & \vdots & \cdots & \vdots & \vdots\\ \frac{\partial^{n-1}}{\partial t^{n-1}}g_1(t,s) & \frac{\partial^{n-1}}{\partial t^{n-1}}g_2(t,s) & \cdots & \frac{\partial^{n-1}}{\partial t^{n-1}}g_{n-1}(t,s) & \frac{\partial^{n-1}}{\partial t^{n-1}}g_M(t,s)\end{pmatrix}, \tag{2.5}$$

where $g_M(t,s)$ is the scalar Green's function related to $T_n[M]$ on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.

Using Definition 2.5 we can deduce the properties fulfilled by $g_M(t,s)$. In particular, $g_M\in C^{n-2}(I\times I)$ and it is a C^n function on the triangles $a\le s<t\le b$ and $a\le t<s\le b$. Moreover it satisfies, as a function of t, the two-point boundary-value conditions (1.5)-(1.6) and solves equation (1.4) whenever $t\ne s$.

In [12] $g_{n-j}(t,s)$ are expressed as functions of $g_M(t,s)$ for all $j=1,\dots,n-1$ as follows:

$$g_{n-j}(t,s)=(-1)^j\frac{\partial^j}{\partial s^j}g_M(t,s)+\sum_{i=0}^{j-1}\alpha_i^j(s)\frac{\partial^i}{\partial s^i}g_M(t,s)\,, \tag{2.6}$$

where $\alpha_i^j(s)$ are functions of $p_1(s),\dots,\ p_j(s)$ and of its derivatives up to order $(j-1)$ and follow the recurrence formula

$$\alpha_0^0(s)=0\,, \tag{2.7}$$

$$\alpha_i^{j+1}(s)=0\,,\quad i\ge j+1\ge 1, \tag{2.8}$$

$$\alpha_0^{j+1}(s)=p_{j+1}(s)-\big(\alpha_0^j\big)'(s),\quad j\ge 0, \tag{2.9}$$

$$\alpha_i^{j+1}(s)=-\Big(\alpha_{i-1}^j(s)+\big(\alpha_i^j\big)'(s)\Big),\quad 1\le i\le j. \tag{2.10}$$

The adjoint of the operator $T_n[M]$ is given by the following expression, see for details [2, Section 1.4] or [15, Chapter 3, Section 5],

$$T_n^*[M]v(t) \equiv (-1)^n v^{(n)}(t) + \sum_{j=1}^{n-1} (-1)^j \left(p_{n-j} v\right)^{(j)}(t) + (p_n(t) + M)v(t)\,, \quad (2.11)$$

and its domain of definition is

$$\begin{aligned} D(T_n^*[M]) = \Big\{ v \in C^n(I) : & \sum_{j=1}^{n} \sum_{i=0}^{j-1} (-1)^{j-1-i} (p_{n-j} v)^{(j-1-i)}(b) u^{(i)}(b) \\ &= \sum_{j=1}^{n} \sum_{i=0}^{j-1} (-1)^{j-1-i} (p_{n-j} v)^{(j-1-i)}(a) u^{(i)}(a) \\ &(\text{with } p_0 = 1)\,, \forall u \in D(T_n[M]) \Big\}. \end{aligned} \quad (2.12)$$

Next result appears in [15, Chapter 3, Theorem 9].

Theorem 2.7. *Equation* (1.4) *is disconjugate on an interval I if and only if the adjoint equation, $T_n^*[M]\, y(t) = 0$ is disconjugate on I.*

We denote $g_M^*(t,s)$ as the Green's function related to the adjoint operator, $T_n^*[M]$. In [2, Section 1.4] it is proved the following relationship

$$g_M^*(t,s) = g_M(s,t)\,. \quad (2.13)$$

Now, let us define the operator

$$\widehat{T}_n[(-1)^n\, M] := (-1)^n T_n^*[M]\,, \quad (2.14)$$

we deduce, from the previous expressions, that

$$\widehat{g}_{(-1)^n\, M}(t,s) = (-1)^n g_M^*(t,s) = (-1)^n g_M(s,t)\,, \quad (2.15)$$

where $\widehat{g}_{(-1)^n M}(t,s)$ is the scalar Green's function related to operator $\widehat{T}_n[(-1)^n M]$ in $D\,(T_n^*[M])$.

Obviously, Theorem 2.7 remains true for operator $\widehat{T}_n[(-1)^n\, M]$.

Definition 2.8. Operator $T_n[M]$ is inverse positive (negative) on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ if every function $u \in X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ such that $T_n[M]\, u \ge 0$ on I, satisfies $u \ge 0$ ($u \le 0$) on I.

Next results are consequence of the ones proved on [2, Section 1.6, Section 1.8] for several two-point n-order operators.

Theorem 2.9. *Operator $T_n[M]$ is inverse positive (negative) on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ if and only if Green's function related to problem* (1.4)–(1.6) *is non-negative (non-positive) on its square of definition.*

Theorem 2.10. *Let M_1, $M_2 \in \mathbb{R}$ and suppose that operators $T_n[M_j]$, $j = 1, 2$, are invertible on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$. Let g_j, $j = 1, 2$, be Green's functions related to operators $T_n[M_j]$ and suppose that both functions have the same constant sign on $I \times I$. Then, if $M_1 < M_2$, it is satisfied that $g_2 \le g_1$ on $I \times I$.*

Theorem 2.11. *Let $M_1 < \bar{M} < M_2$ be three real constants. Suppose that operator $T_n[M]$ is invertible on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ for $M = M_j$, $j = 1, 2$ and that the corresponding Green's function satisfies $g_2 \leq g_1 \leq 0$ (resp. $0 \leq g_2 \leq g_1$) on $I \times I$. Then the operator $T_n[\bar{M}]$ is invertible on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ and the related Green's function $\bar{g}$ satisfies $g_2 \leq \bar{g} \leq g_1 \leq 0$ ($0 \leq g_2 \leq \bar{g} \leq g_1$) on $I \times I$.*

Now, we introduce a stronger concept of inverse positive (negative) character.

Definition 2.12. Operator $T_n[M]$ is strongly inverse positive on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ if every function $u \in X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ such that $T_n[M]\, u \gneqq 0$ on I, must verify $u > 0$ on (a, b) and, moreover, $u^{(\alpha)}(a) > 0$ and $u^{(\beta)}(b) > 0$ if β is even, $u^{(\beta)}(b) < 0$ if β is odd, where α and β are defined in (1.2) and (1.3), respectively.

Definition 2.13. Operator $T_n[M]$ is strongly inverse negative on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ if every function $u \in X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ such that $T_n[M]\, u \gneqq 0$ on I, must verify $u < 0$ on (a, b) and, moreover, $u^{(\alpha)}(a) < 0$ and $u^{(\beta)}(b) < 0$ if β is even, $u^{(\beta)}(b) > 0$ if β is odd, where α and β are defined in (1.2) and (1.3), respectively.

Analogously to Theorem 2.9, the following ones can be shown.

Theorem 2.14. *Operator $T_n[M]$ is strongly inverse positive on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ if and only if Green's function related to problem* (1.4)–(1.6)*, $g_M(t, s)$, satisfies the following properties:*

- *$g_M(t, s) > 0$ a.e. on $(a, b) \times (a, b)$.*
- *$\frac{\partial^\alpha}{\partial t^\alpha} g_M(t, s)\big|_{t=a} > 0$ for a.e. $s \in (a, b)$.*
- *$\frac{\partial^\beta}{\partial t^\beta} g_M(t, s)\big|_{t=b} > 0$ if β is even and $\frac{\partial^\beta}{\partial t^\beta} g_M(t, s)\big|_{t=b} < 0$ if β is odd for a.e. $s \in (a, b)$.*

Theorem 2.15. *Operator $T_n[M]$ is strongly inverse negative on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ if and only if Green's function related to problem* (1.4)–(1.6)*, $g_M(t, s)$, satisfies the following properties:*

- *$g_M(t, s) < 0$ a.e. on $(a, b) \times (a, b)$.*
- *$\frac{\partial^\alpha}{\partial t^\alpha} g_M(t, s)\big|_{t=a} < 0$ for a.e. $s \in (a, b)$.*
- *$\frac{\partial^\beta}{\partial t^\beta} g_M(t, s)\big|_{t=b} < 0$ if β is even and $\frac{\partial^\beta}{\partial t^\beta} g_M(t, s)\big|_{t=b} > 0$ if β is odd for a.e. $s \in (a, b)$.*

Next, we introduce two conditions on $g_M(t, s)$ that will be used in this paper.

(A2.1) Suppose that there is a continuous function $\phi(t) > 0$ for all $t \in (a, b)$ and $k_1, k_2 \in \mathcal{L}^1(I)$, such that $0 < k_1(s) < k_2(s)$ for a.e. $s \in I$, satisfying

$$\phi(t)\, k_1(s) \leq g_M(t, s) \leq \phi(t)\, k_2(s)\,, \quad \text{a.e. } (t, s) \in I \times I\,.$$

(A2.2) Suppose that there is a continuous function $\phi(t) > 0$ for all $t \in (a, b)$ and $k_1, k_2 \in \mathcal{L}^1(I)$, such that $k_1(s) < k_2(s) < 0$ a.e. $s \in I$, satisfying

$$\phi(t)\, k_1(s) \leq g_M(t, s) \leq \phi(t)\, k_2(s)\,, \quad \text{a.e. } (t, s) \in I \times I\,.$$

Finally, we introduce the following sets that characterize where the Green's function is of constant sign,

$$P_T = \{M \in \mathbb{R} : g_M(t, s) \geq 0 \;\forall (t, s) \in I \times I\}\,, \tag{2.16}$$

$$N_T = \{M \in \mathbb{R} : g_M(t,s) \le 0 \;\forall (t,s) \in I \times I\}\,. \tag{2.17}$$

Note that, using Theorem 2.10, we can affirm that the two previous sets are real intervals (which can be empty in some situations).

The next results describe one of the extremes of the two previous intervals (see [2, Theorems 1.8.31 and 1.8.23]).

Theorem 2.16. *Let $\bar{M} \in \mathbb{R}$ be fixed. If $T_n[\bar{M}]$ is an invertible operator on the set $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ and its related Green's function satisfies condition (A2.1), then the following statements hold:*

- *There is $\lambda_1 > 0$, the least eigenvalue in absolute value of operator $T_n[\bar{M}]$ on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$. Moreover, there exists a nontrivial constant sign eigenfunction corresponding to the eigenvalue λ_1.*
- *Green's function related to operator $T_n[M]$ is nonnegative on $I \times I$ for all $M \in (\bar{M} - \lambda_1, \bar{M}]$.*
- *Green's function related to operator $T_n[M]$ cannot be nonnegative on $I \times I$ for all $M < \bar{M} - \lambda_1$.*
- *If there is $M \in \mathbb{R}$ for which Green's function related to operator $T_n[M]$ is non-positive on $I \times I$, then $M < \bar{M} - \lambda_1$.*

Theorem 2.17. *Let $\bar{M} \in \mathbb{R}$ be fixed. If $T_n[\bar{M}]$ is an invertible operator on the set $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ and its related Green's function satisfies condition* (A2.2), *then the following statements hold:*

- *There is $\lambda_2 < 0$, the least eigenvalue in absolute value of operator $T_n[\bar{M}]$ on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$. Moreover, there exists a nontrivial constant sign eigenfunction corresponding to the eigenvalue λ_2.*
- *Green's function related to operator $T_n[M]$ is non-positive on $I \times I$ for all $M \in [\bar{M}, \bar{M} - \lambda_2)$.*
- *Green's function related to operator $T_n[M]$ cannot be non-positive on $I \times I$ for all $M > \bar{M} - \lambda_2$.*
- *If there is $M \in \mathbb{R}$ for which Green's function related to operator $T_n[M]$ is nonnegative on $I \times I$, then $M > \bar{M} - \lambda_2$.*

Next results give some relevant properties of the intervals N_T and P_T.

Theorem 2.18. *Let $\bar{M} \in \mathbb{R}$ be fixed. If $T_n[\bar{M}]$ is an invertible operator on the set $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ and its related Green's function satisfies condition (A2.1); then if the interval $N_T \neq \emptyset$, then* $\sup(N_T) = \inf(P_T)$.

Theorem 2.19. *Let $\bar{M} \in \mathbb{R}$ be fixed. If $T_n[\bar{M}]$ is an invertible operator on the set $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ and its related Green's function satisfies condition (A2.2); then if the interval $P_T \neq \emptyset$, then* $\sup(N_T) = \inf(P_T)$.

By using these results, we know that one of the extremes of the interval of constant sign of the Green's function, if it is not empty, is characterized by its first eigenvalue. So, the rest of the paper is devoted to characterize the other extreme of the interval, provided that it is bounded.

3. Hypotheses on the operator $T_n[M]$

As we have mentioned at the introduction, the aim of this work is to generalize the results given in [12] and [14].

In [12], the problems studied are the so-called $(k, n-k)$ boundary conditions which correspond to $\{\sigma_1, \dots, \sigma_k\} = \{0, \dots, k-1\}$ and $\{\varepsilon_1, \dots, \varepsilon_{n-k}\} = \{0, \dots, n-k-1\}$. We will characterize the parameter's set where the Green's function has constant sign, by assuming that the boundary conditions satisfy (N_a), which, clearly, holds for $(k, n-k)$.

By using Theorems 2.3 and 2.4, under the hypothesis that (1.4) is disconjugate on $[a,b]$, it is proved in [12] the existence of a decomposition as follows:

$$T_0 u(t) = u(t)\,, \quad T_k u(t) = \frac{d}{dt}\Big(\frac{T_{k-1}u(t)}{v_k(t)}\Big)\,, \quad k = 1, \dots, n\,,$$

where $v_k > 0$, $v_k \in C^n(I)$ such that

$$T_n[M]u(t) = v_1(t)\,\dots v_n(t)\,T_n u(t)\,,\ t \in I\,,$$

and, moreover, this decomposition satisfies, for every $u \in X_{\{0,\dots,k-1\}}^{\{0,\dots,n-k-1\}}$:

$$\begin{gathered} T_0 u(a) = \dots = T_{k-1}u(a) = 0\,,\\ T_0 u(b) = \dots = T_{n-k-1}u(b) = 0\,. \end{gathered}$$

In [14], it is studied a fourth order problem coupled with the simply supported beam boundary conditions, that is, $\{\sigma_1, \sigma_2\} = \{\varepsilon_1, \varepsilon_2\} = \{0, 2\}$. It is also obtained a decomposition as follows:

$$T_0 u(t) = u(t)\,, \quad T_k u(t) = \frac{d}{dt}\Big(\frac{T_{k-1}u(t)}{v_k(t)}\Big)\,, \quad k = 1, \dots, 4\,,$$

where $v_k > 0$, $v_k \in C^4(I)$ such that

$$T_4[M]u(t) = v_1(t)\,\dots v_4(t)\,T_4 u(t)\,, \quad t \in I\,,$$

and, moreover, this decomposition satisfies, for every $u \in X_{\{0,2\}}^{\{0,2\}}$:

$$\begin{gathered} T_0 u(a) = T_2 u(a) = 0\,,\\ T_0 u(b) = T_2 u(b) = 0\,. \end{gathered}$$

Furthermore, the simplest n^{th}-order operator which we can study is $T_n[0]u(t) = u^{(n)}(t)$. It is obvious that such an operator satisfies

$$T_0 u(t) = u(t)\,, \quad T_k u(t) = \frac{d}{dt}\Big(\frac{T_{k-1}u(t)}{v_k(t)}\Big)\,, \quad k = 1, \dots, n\,,$$

where $v_k \equiv 1$ on I and

$$T_n[0]u(t) = v_1(t)\,\dots v_n(t)\,T_n u(t)\,, \quad t \in I\,,$$

and, moreover, this decomposition satisfies, for every $u \in X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$:

$$\begin{gathered} T_{\sigma_1}u(a) = u^{(\sigma_1)}(a) = 0\,, \dots\,, T_{\sigma_k}u(a) = u^{(\sigma_k)}(a) = 0\,,\\ T_{\varepsilon_1}u(b) = u^{(\varepsilon_1)}(b) = 0\,, \dots\,, T_{\varepsilon_{n-k}}u(b) = u^{(\varepsilon_{n-k})}(b) = 0\,. \end{gathered}$$

Thus, it is natural to impose that the operator $T_n[M]$ satisfies the following property.

Definition 3.1. We say that the operator $T_n[M]$ satisfies the property (T_d) on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ if and only if there exists the following decomposition:

$$T_0 u(t) = u(t)\,, \quad T_k u(t) = \frac{d}{dt}\Big(\frac{T_{k-1}u(t)}{v_k(t)}\Big)\,, \quad k = 1, \dots, n\,, \tag{3.1}$$

where $v_k > 0$, $v_k \in C^n(I)$ such that

$$T_n[M]u(t) = v_1(t) \dots v_n(t)\, T_n u(t)\,, \quad t \in I\,, \tag{3.2}$$

and, moreover, such a decomposition satisfies, for every $u \in X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$:

$$\begin{gathered} T_{\sigma_1}u(a) = \dots = T_{\sigma_k}u(a) = 0\,,\\ T_{\varepsilon_1}u(b) = \dots = T_{\varepsilon_{n-k}}u(b) = 0\,. \end{gathered}$$

As we have shown above, the operator $T_n[M]u(t) \equiv u^{(n)}(t) + Mu(t)$ satisfies property (T_d) for $M = 0$. Indeed, the existence of such a decomposition for $M = \bar{M}$ allows to express the operator $T_n[\bar{M}]$ as a composition of operators of order 1 verifying the boundary conditions given on (1.5)-(1.6). That is, in order to study the oscillation, we can think that the operator $T_n[\bar{M}]u(t)$ has an analogous behavior to $u^{(n)}(t)$.

Remark 3.2. By Theorems 2.3 and 2.4, the disconjugacy of the linear differential equation (1.4) on I is a necessary condition for the operator $T_n[M]$ to satisfy property (T_d) on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$. Furthermore, as it has been proved in [12], the disconjugacy hypothesis is also a sufficient condition for the operator $T_n[M]$ to satisfy property (T_d) on $X_{\{0,\dots,k-1\}}^{\{0,\dots,n-k-1\}}$.

Remark 3.3. There may exist different decompositions (3.1) depending on the choice of v_k for $k = 1,\dots,n$. Moreover, even if we are not able to obtain such a decomposition, we cannot ensure that it does not exist, unless we prove that the linear differential equation (1.4) is not disconjugate.

In [12], it is shown that

$$T_\ell u(t) = \frac{1}{v_1(t)\dots v_\ell(t)} u^{(\ell)}(t) + p_{\ell_1}(t)u^{(\ell-1)}(t) + \cdots p_{\ell_\ell}(t)u(t)\,, \tag{3.3}$$

where $p_{\ell_i} \in C^{n-\ell}(I)$, for every $i = 1,\dots,\ell$, and $\ell = 0,\dots,n$. Now, let us see that

$$p_{\ell_1}(t) = a_1^1(v_1(t),\dots,v_\ell(t))v_1'(t) + \dots + a_1^\ell(v_1(t),\dots,v_\ell(t))v_\ell'(t)\,, \tag{3.4}$$

$$\begin{aligned} p_{\ell_2}(t) = {} & a_2^1(v_1(t),\dots,v_\ell(t))v_1''(t) + \dots + a_2^{\ell-1}(v_1(t),\dots,v_\ell(t))v_{\ell-1}''(t)\\ & + f_2(v_1(t),\dots,v_\ell(t),v_1'(t),\dots,v_\ell'(t))\,, \end{aligned} \tag{3.5}$$

$$\begin{aligned} p_{\ell_3}(t) = {} & a_3^1(v_1(t),\dots,v_\ell(t))v_1'''(t) + \dots + a_3^{\ell-2}(v_1(t),\dots,v_\ell(t))v_{\ell-2}'''(t)\\ & + f_3(v_1(t),\dots,v_\ell(t),v_1'(t),\dots,v_\ell'(t),v_1''(t),\dots,v_{\ell-1}''(t))\,, \end{aligned} \tag{3.6}$$

$$\dots$$

$$\begin{aligned} p_{\ell_\ell}(t) = {} & a_\ell^1(v_1(t),\dots,v_\ell(t))v_1^{(\ell)}(t)\\ & + f_\ell(v_1(t),\dots,v_\ell(t),v_1'(t),\dots,v_\ell'(t),\dots,v_1^{(\ell-1)}(t),v_2^{(\ell-1)}(t))\,, \end{aligned} \tag{3.7}$$

where $a_i^j \in C^\infty\big((0,+\infty)^\ell\big)$, $f_i \in C^\infty\big((0,\infty)^\ell \times \mathbb{R}^{(i-1)\frac{2\ell-i+2}{2}}\big)$ for all $\ell = 0,\dots,n$, $i = 1,\dots,\ell$ and $j = 1,\dots,\ell-i+1$.

We can see that for $\ell = 1$ the result is true:

$$T_1u(t) = \frac{d}{dt}\Big(\frac{u(t)}{v_1(t)}\Big) = \frac{u'(t)}{v_1(t)} - \frac{v_1'(t)}{v_1^2(t)}u(t)\,, \tag{3.8}$$

hence $a_1^1(x) = -\frac{1}{x^2}$.

Suppose, by induction hypothesis, that the result is true for a given $\ell \geq 1$. Then, let us see what happens for $\ell + 1$.

$$T_{\ell+1}u(t) = \frac{d}{dt}\Big(\frac{1}{v_1(t)\dots v_{\ell+1}(t)}u^{(\ell)}(t) + \frac{p_{\ell_1}(t)}{v_{\ell+1}(t)}u^{(\ell-1)}(t) + \dots + \frac{p_{\ell_\ell}(t)}{v_{\ell+1}(t)}u(t)\Big)\,,$$

or, which is the same,

$$T_{\ell+1}u(t) = \frac{1}{v_1(t)\dots v_{\ell+1}(t)}u^{(\ell+1)}(t) + p_{\ell+1_1}(t)u^{(\ell)}(t) + \dots + p_{\ell+1_{\ell+1}}(t)u(t)\,,$$

where

$$p_{\ell+1_1}(t) = \frac{d}{dt}\Big(\frac{1}{v_1(t)\dots v_{\ell+1}(t)}\Big) + \frac{p_{\ell_1}(t)}{v_{\ell+1}(t)}\,,$$

$$p_{\ell+1_j}(t) = \frac{d}{dt}\Big(\frac{p_{\ell_{j-1}}(t)}{v_{\ell+1}(t)}\Big) + \frac{p_{\ell_j}(t)}{v_{\ell+1}(t)}\,, \quad 2 \leq j \leq \ell\,,$$

$$p_{\ell+1_{\ell+1}}(t) = \frac{d}{dt}\Big(\frac{p_{\ell_\ell}(t)}{v_{\ell+1}(t)}\Big)\,,$$

which clearly satisfy (3.4)–(3.7) for $\ell + 1$.

Example 3.4. Now, let us show, as an example, the expression of $T_2u(t)$:

$$\begin{aligned} T_2u(t) &= \frac{d}{dt}\Big(\frac{T_1u(t)}{v_2(t)}\Big) \\ &= \frac{u''(t)}{v_1(t)v_2(t)} - u'(t)\frac{2v_2(t)v_1'(t) + v_1(t)v_2'(t)}{v_1^2(t)v_2^2(t)} \\ &\quad + u(t)\frac{v_1(t)v_1'(t)v_2'(t) + v_2(t)\big(2{v_1'}^2(t) - v_1(t)v_1''(t)\big)}{v_1^3(t)v_2^2(t)}\,. \end{aligned}$$

In this case, $a_1^1(x,y) = -\frac{2}{x^2\,y}$, $a_1^2(x,y) = -\frac{1}{x\,y^2}$, $a_2^1(x,y) = -\frac{1}{x^2\,y}$ and $f(x,y,z,t) = \frac{x\,z\,t + 2\,y\,z^2}{x^3\,y^2}$.

Remark 3.5. From the arbitrariness of the choice of $u \in X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, if the operator $T_n[\bar M]$ satisfies the property (T_d) on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ then, for each ℓ which belongs to $\{\sigma_1,\dots,\sigma_k\}$, we have that

$$T_\ell u(a) = \frac{1}{v_1(a)\dots v_\ell(a)}u^{(\ell)}(a) + p_{\ell_1}(a)u^{(\ell-1)}(a) + \cdots p_{\ell_\ell}(a)u(a) = 0\,,$$

implies that $p_{\ell_h}(a) = 0$ for each $h \in \{1,\dots,\ell\}$ such that $\ell - h \notin \{\sigma_1,\dots,\sigma_k\}$.

Analogously, for each $\ell \in \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$

$$T_\ell u(b) = \frac{1}{v_1(b)\dots v_\ell(b)}u^{(\ell)}(b) + p_{\ell_1}(b)u^{(\ell-1)}(b) + \cdots p_{\ell_\ell}(b)u(b) = 0\,,$$

implies that $p_{\ell_h}(b) = 0$ for each $h \in \{1,\dots,\ell\}$ such that $\ell - h \notin \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$.

Now, we deduce two results which are a straight consequence of property (T_d) and previous Remark.

Lemma 3.6. *Let $\bar M \in \mathbb{R}$ be such that $T_n[\bar M]$ satisfies property (T_d) on the set $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$. If $u \in C^n([a,c))$, where $c > a$, is a function that satisfies $u^{(\sigma_1)}(a) = \cdots = u^{(\sigma_{\ell-1})}(a) = 0$, for $\ell = 1,\dots,k$, then*

$$T_{\sigma_1}u(a) = \cdots = T_{\sigma_{\ell-1}}u(a) = 0\,,$$

$$T_{\sigma_\ell} u(a) = f(a) u^{(\sigma_\ell)}(a)\,,$$

where

$$f(t) = \frac{1}{v_1(t)\,\dots v_{\sigma_\ell}(t)} > 0\,, \quad t \in I\,.$$

In particular, $u^{(\sigma_\ell)}(a) = 0$ *if and only if* $T_{\sigma_\ell} u(a) = 0$.

Proof. We only have to take into account expression (3.3) and Remark 3.5 to deduce the result directly. □

We have an analogous result for $t = b$.

Lemma 3.7. *Let* $\bar{M} \in \mathbb{R}$ *be such that* $T_n[\bar{M}]$ *satisfies property* (T_d) *on the set* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$. *If* $u \in C^n((c,b])$, *where* $c < b$, *is a function that satisfies* $u^{(\varepsilon_1)}(b) = \dots = u^{(\varepsilon_{\ell-1})}(b) = 0$, *for* $\ell \in \{1,\dots,n-k\}$, *then*

$$T_{\varepsilon_1} u(b) = \dots = T_{\varepsilon_{\ell-1}} u(b) = 0\,,$$
$$T_{\varepsilon_\ell} u(b) = g(b) u^{(\varepsilon_\ell)}(b)\,,$$

where

$$g(t) = \frac{1}{v_1(t)\dots v_{\varepsilon_\ell}(t)} > 0\,,\ t \in I\,.$$

In particular, if $u^{(\varepsilon_\ell)}(b) = 0$, *then* $T_{\varepsilon_\ell} u(b) = 0$.

As in Lemma 3.6, the proof follows from (3.3) and Remark 3.5. Now, we prove a preliminary result, which ensures that Green's function is well-defined for the operator $T_n[M]$ on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, provided that it satisfies the property (T_d) on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.

Lemma 3.8. *Let* $\bar{M} \in \mathbb{R}$ *be such that* $T_n[\bar{M}]$ *satisfies property* (T_d) *on the set* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$. *Then* $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ *satisfy* (N_a) *if and only if* $M = 0$ *is not an eigenvalue of* $T_n[\bar{M}]$ *on* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.

Proof. To prove the sufficient condition, let us consider $u \in X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, such that

$$T_n[\bar{M}] u(t) = 0\,, t \in I\,.$$

We will see that necessarily $u \equiv 0$ in I. Since the operator $T_n[\bar{M}]$ satisfies the property (T_d) on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, we can use the decomposition given in (3.1); so, we have

$$0 = T_n[\bar{M}] u(t) = v_1(t)\,\dots v_n(t)\, T_n u(t)\,, \quad t \in I\,,$$

which, since $v_1\,,\dots,v_n > 0$, implies that

$$T_n u(t) = \frac{d}{dt}\Big(\frac{T_{n-1} u(t)}{v_n(t)}\Big) = 0\,, \quad t \in I\,,$$

hence $\frac{T_{n-1} u(t)}{v_n(t)}$ is a constant function on I. So, since $v_n > 0$ on I, $T_{n-1} u(t)$ is of constant sign on I. Hence $\frac{T_{n-2} u(t)}{v_{n-2}(t)}$ is a monotone function, with at most one zero on I. As before, since $v_{n-2}(t) > 0$ on I, we can conclude that $T_{n-2} u(t)$ can have at most one zero on I. Proceeding analogously, we conclude that u can have at most $n-1$ zeros on I.

If $T_\ell u \neq 0$ for all $\ell = 1, \dots, n-1$, then each time that $T_\ell u(a) = 0$ or $T_\ell u(b) = 0$ a possible oscillation is lost. Indeed, if the maximum number of zeros for $T_\ell u$ on I is h and one of them is found in either $t = a$ or $t = b$, then $T_\ell u$ can have at most $h-1$ sign changes on I ($h-2$ if both $T_\ell u(a) = T_\ell u(b) = 0$). Since $T_n[\bar{M}]$ satisfies property (T_d) on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, $T_\ell u(a) = 0$ or $T_\ell u(b) = 0$, at least n times.

If $T_\ell u(t) \neq 0$ for all $\ell = 1, \dots, n-1$, n possible oscillations are lost, since u can have $n-1$ zeros with maximal oscillation, this implies that necessarily $u \equiv 0$.

If there exists some $\ell \in \{1, \dots, n-1\}$, such that $T_\ell u(t) \equiv 0$ on I, let us choose the least ℓ that satisfy this property. With the same arguments as before, we can conclude that the maximum number of zeros which u can have is $\ell - 1$.

Using the fact that $\{\sigma_1, \dots, \sigma_k\} - \{\varepsilon_1, \dots, \varepsilon_{n-k}\}$ satisfy (N_a), we know that $T_h u(a) = 0$ or $T_h u(b) = 0$ at least ℓ times from $h = 0$ to ℓ. Therefore, we lose ℓ possible oscillations, hence $u \equiv 0$. And, we can conclude that 0 is not an eigenvalue of $T_n[\bar{M}]$ on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.

Reciprocally, to prove the necessary condition, let us assume that $\{\sigma_1, \dots, \sigma_k\} - \{\varepsilon_1, \dots, \varepsilon_{n-k}\}$ do not satisfy (N_a). Then, there exists $h_0 \in \{1, \dots, n-1\}$ such that

$$\sum_{\sigma_j < h_0} 1 + \sum_{\varepsilon_j < h_0} 1 < h_0 \,.$$

Thus, there always exists a nontrivial function verifying the boundary conditions (1.5)-(1.6) for $\sigma_\ell < h_0$ and $\varepsilon_\ell < h_0$ such that $T_h u(t) = 0$.

Trivially, $T_{\sigma_\ell} u(a) = 0$ and $T_{\varepsilon_\ell} u(b) = 0$ for either $\sigma_\ell > h_0$ or $\varepsilon_\ell > h_0$. Thus, by applying Lemmas 3.6 and 3.7 inductively, we conclude that $u \in X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$. As a consequence, it is obvious that $M = 0$ is an eigenvalue of $T_n[\bar{M}]$ on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$. □

4. Study of the adjoint operator, $T_n^*[M]$

To obtain the characterization of the Green's function sign, as it has been done in [12] and [14], it is necessary to study the adjoint operator, $T_n^*[M]$, defined in (2.11). So, this section is devoted to make an analysis of such an operator and some of its properties in relation with the hypotheses on operator $T_n[M]$ given in the previous section.

So, we describe the space $D(T_n^*[M])$, defined in (2.12) by taking into account that, in our case, $D(T_n[M]) = X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.

Let us denote $D(T_n^*[M]) = X_{\{\sigma_1,\dots,\sigma_k\}}^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, and consider the sets

$$\{\delta_1, \dots, \delta_k\}\,, \{\tau_1, \dots, \tau_{n-k}\} \subset \{0, \dots, n-1\},$$

such that $\delta_i < \delta_{i+1}$ and $\tau_j < \tau_{j+1}$, for $i = 1, \dots, k-1$ and $j = 1, \dots, n-k-1$, satisfying:

$$\{\sigma_1, \dots, \sigma_k, n-1-\tau_1, \dots, n-1-\tau_{n-k}\} = \{0, \dots, n-1\}$$
$$\{\varepsilon_1, \dots, \varepsilon_{n-k}, n-1-\delta_1, \dots, n-1-\delta_k\} = \{0, \dots, n-1\}\,.$$

Remark 4.1. *By the definition of* α *and* β *given in* (1.2) *and* (1.3), *respectively, we have* $\alpha = n-1-\tau_{n-k}$ *and* $\beta = n-1-\delta_k$.

Hence we choose $u \in X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, such that

$$u^{(n-1-\tau_1)}(a) = 1\,,$$

$$u^{(i)}(a)=0\,,\quad \forall i=0,\dots,n-1\,,\ i\neq n-1-\tau_1\,,$$
$$u^{(i)}(b)=0\,,\quad \forall i=0,\dots,n-1\,.$$

Thus, from (2.12) we can conclude that every $v\in X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$, satisfies

$$v^{(\tau_1)}(a)+\sum_{j=n-\tau_1}^{n-1}(-1)^{n-j}(p_{n-j}v)^{(\tau_1+j-n)}(a)=0\,.$$

Proceeding analogously for $\tau_2,\dots,\tau_{n-k}$, we can obtain the boundary conditions for the adjoint operator at $t=a$, and working at $t=b$ for $\delta_1,\dots,\delta_k$ we are able to complete the boundary conditions related to the adjoint operator. So, we conclude that every $v\in X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$ is a $C^n(I)$ function that satisfies the following conditions

$$v^{(\tau_1)}(a)+\sum_{j=n-\tau_1}^{n-1}(-1)^{n-j}(p_{n-j}v)^{(\tau_1+j-n)}(a)=0\,,\tag{4.1}$$
$$\dots$$
$$v^{(\tau_{n-k-1})}(a)+\sum_{j=n-\tau_{n-k-1}}^{n-1}(-1)^{n-j}(p_{n-j}v)^{(\tau_{n-k-1}+j-n)}(a)=0\,,\tag{4.2}$$
$$v^{(\tau_{n-k})}(a)+\sum_{j=n-\tau_{n-k}}^{n-1}(-1)^{n-j}(p_{n-j}v)^{(\tau_{n-k}+j-n)}(a)=0\,,\tag{4.3}$$
$$v^{(\delta_1)}(b)+\sum_{j=n-\delta_1}^{n-1}(-1)^{n-j}(p_{n-j}v)^{(\delta_1+j-n)}(b)=0\,,\tag{4.4}$$
$$\dots$$
$$v^{(\delta_{k-1})}(b)+\sum_{j=n-\delta_{k-1}}^{n-1}(-1)^{n-j}(p_{n-j}v)^{(\delta_{k-1}+j-n)}(b)=0\,,\tag{4.5}$$
$$v^{(\delta_k)}(b)+\sum_{j=n-\delta_k}^{n-1}(-1)^{n-j}(p_{n-j}v)^{(\delta_k+j-n)}(b)=0\,.\tag{4.6}$$

Let us denote η, $\gamma\in\{0,\dots,n-1\}$ as follows

$$\eta\notin\{\tau_1,\dots,\tau_{n-k}\}\,,\ \text{and if } \eta\neq 0\,,\ \{0,\dots,\eta-1\}\subset\{\tau_1,\dots,\tau_{n-k}\}\,,\tag{4.7}$$
$$\gamma\notin\{\delta_1,\dots,\delta_k\}\,,\ \text{and if } \gamma\neq 0\,,\ \{0,\dots,\gamma-1\}\subset\{\delta_1,\dots,\delta_k\}\,.\tag{4.8}$$

Remark 4.2. As in Remark 4.1, we have that $\eta=n-1-\sigma_k$ and $\gamma=n-1-\varepsilon_{n-k}$.

From the boundary conditions (4.1)–(4.6), since $p_j\in C^{n-j}(I)$, the following assertions are fulfilled:

- If $\eta\neq 0$, for all $v\in X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$ it is satisfied $v(a)=\dots=v^{(\eta-1)}(a)=0$.
- If $\gamma\neq 0$, for all $v\in X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$ it is satisfied $v(b)=\dots=v^{(\gamma-1)}(b)=0$.

Example 4.3. Let us consider the fourth order operator $T_4[M]$ coupled with the boundary conditions

$$u(a)=u''(a)=u'(b)=u''(b)=0\,.\tag{4.9}$$

Now, we describe the domain of definition of the adjoint operator, $T_4^*[M]$. In this case, $\{\tau_1,\tau_2\}=\{0,2\}$ and $\{\delta_1,\delta_2\}=\{0,3\}$. Thus, from (4.1)–(4.6), we deduce that:

$$\begin{aligned} X_{\{0,2\}}^{*\{1,2\}} = \big\{v\in C^4(I) : v(a)=v''(a)-p_1(a)v'(a)=v(b)=0\,,\\ v^{(3)}(b)-p_1(b)v''(b)+(p_2(b)-2p_1'(b))v'(b)=0\big\}. \end{aligned} \tag{4.10}$$

Definition 4.4. We say that operator $T_n^*[M]$ satisfies property (T_d^*) on the set $X_{\{\sigma_1,\dots,\sigma_k\}}^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ if, there exists a decomposition:

$$T_0^*v(t)=w_0(t)v(t)\,,\quad T_k^*v(t)=\frac{-1}{w_k(t)}\frac{d}{dt}\left(T_{k-1}^*v(t)\right)\,,\quad k=1,\dots,n\,, \tag{4.11}$$

where $w_k>0$, $w_k\in C^n(I)$ and

$$T_n^*[M]v(t)=T_n^*v(t)\,,\ t\in I\,.$$

Moreover, this decomposition satisfies that for every $v\in X_{\{\sigma_1,\dots,\sigma_k\}}^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$:

$$T_{\tau_1}^*v(a)=\dots=T_{\tau_{n-k}}^*v(a)=0\,, \tag{4.12}$$

$$T_{\delta_1}^*v(b)=\dots=T_{\delta_k}^*v(b)=0\,. \tag{4.13}$$

Lemma 4.5. *Let $\bar M\in\mathbb{R}$ be such that $T_n[\bar M]$ satisfies property (T_d) on the set $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$. Then the adjoint operator $T_n^*[\bar M]$ also satisfies property (T_d^*) on the set $X_{\{\sigma_1,\dots,\sigma_k\}}^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*

Proof. From [15, Chapter 3, Theorem 10], it is fulfilled that if $T_n[\bar M]v$ satisfies (3.2), then $T_n^*[\bar M]$ can be decomposed as:

$$T_n^*[\bar M]v(t)=\frac{(-1)^n}{v_1(t)}\frac{d}{dt}\Big(\frac{1}{v_2(t)}\frac{d}{dt}\Big(\cdots\frac{d}{dt}\big(\frac{1}{v_n(t)}\frac{d}{dt}\big(v_1(t)\dots v_n(t)v(t)\big)\big)\Big)\Big)\,. \tag{4.14}$$

Hence,

$$T_0^*v(t)=v_1(t)\dots v_n(t)v(t)\,,\quad\text{and}\quad T_k^*v(t)=\frac{-1}{v_{n+1-k}(t)}\frac{d}{dt}\left(T_{k-1}^*v(t)\right)\,;$$

so, the existence of the decomposition given in (4.11) is proved by taking $w_0(t)=v_1(t)\dots v_n(t)$ and $w_k(t)=v_{n+1-k}(t)$ for $k=1,\dots,n$.

Let us see that for every $v\in X_{\{\sigma_1,\dots,\sigma_k\}}^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, the boundary conditions (4.12)-(4.13) are satisfied.

Obviously, the expression of the $n^{\rm th}$- order scalar problem (1.4)–(1.6) as a first order vectorial problem, given in (2.3), does not depend on the property (T_d) of $T_n[\bar M]$. In our case, using the decomposition given by (T_d), we can transform the $n^{\rm th}$-order problem $T_n[\bar M]u(t)=0$ into a first-order vectorial problem in an alternative way as follows

$$U_u'(t)=A_1(t)\,U_u(t)\,,\quad t\in I\,,\quad B\,U_u(a)+C\,U_u(b)=0\,, \tag{4.15}$$

with B, $C\in\mathcal{M}_{n\times n}$ defined in (2.4) and $U_u(t)\in\mathbb{R}^n$, $A_1(t)\in\mathcal{M}_{n\times n}$, defined by

$$U_u(t)=\begin{pmatrix}u_{1u}(t)\\u_{2u}(t)\\\vdots\\u_{nu}(t)\end{pmatrix}\,,\quad A_1(t)=\begin{pmatrix}0&v_2(t)&0&\dots&0\\0&0&v_3(t)&\dots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\dots&v_n(t)\\0&0&0&\dots&0\end{pmatrix}\,, \tag{4.16}$$

where ${u_\ell}_u(t) := \frac{T_{\ell-1}u(t)}{v_\ell(t)}$ for $\ell = 1,\dots,n$ and $u \in X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$. Indeed, if $1 \le \ell \le n-1$:

$$ {u_\ell}_u'(t) = \frac{d}{dt}\left(\frac{T_{\ell-1}u(t)}{v_\ell(t)}\right) = \frac{T_\ell u(t)}{v_{\ell+1}(t)} v_{\ell+1}(t) = v_{\ell+1}(t)\, {u_{\ell+1}}_u(t)\,, $$

and, if $\ell = n$

$$ u_n'(t) = \frac{d}{dt}\left(\frac{T_{n-1}u(t)}{v_{n-1}(t)}\right) = T_n u(t) = \frac{T_n[\bar M]u(t)}{v_1(t)\dots v_n(t)} = 0\,. $$

Taking into account that $T_n[\bar M]$ satisfies property (T_d) on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, we have

$$ {u_{\sigma_1+1}}_u(a) = \dots = {u_{\sigma_k+1}}_u(a) = {u_{\varepsilon_1+1}}_u(b) = \dots = {u_{\varepsilon_{n-k}+1}}_u(b) = 0\,. \tag{4.17} $$

Moreover, using similar arguments, by means of the decomposition (4.14), we can transform the n^{th}-order scalar problem

$$ T_n^*[\bar M]v(t) = 0\,, \quad t \in I\,, \tag{4.18} $$

coupled with the boundary conditions (4.1)–(4.6) into the equivalent first-order vectorial problem

$$ Z_v'(t) = -A_1^T(t)\, Z_v(t)\,, \quad t \in I\,, \tag{4.19} $$

where $A_1(t) \in \mathcal{M}_{n\times n}$ is defined in (4.16) and $Z_v(t) \in \mathbb{R}^n$ is given by

$$ Z_v(t) = \begin{pmatrix} {z_1}_v(t) \\ {z_2}_v(t) \\ \dots \\ {z_n}_v(t) \end{pmatrix}\,, $$

with ${z_\ell}_v(t) := T_{n-\ell}^* v(t)$ for $\ell = 0,\dots,n-1$ and $v \in X_{\{\sigma_1,\dots,\sigma_k\}}^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.

Indeed, if $2 \le \ell \le n$:

$$ {z_\ell}_v'(t) = \frac{d}{dt}\left(T_{n-\ell}^* v(t)\right) = T_{n-\ell+1}^* v(t)(-v_{n+1-(n-\ell+1)}(t)) = -v_\ell(t)\, {z_{\ell-1}}_v(t)\,, $$

and, if $\ell = 1$:

$$ {z_1}_v'(t) = \frac{d}{dt}\left(T_{n-1}^* v(t)\right) = -v_1(t)\, T_n^* v(t) = -v_1(t)\, T_n^*[\bar M]v(t) = 0\,. $$

Let us consider the n^{th}-order linear differential operators $T_n[\bar M]$ and $T_n^*[\bar M]$ in a vectorial way as follows:

$$ {T_n}^v[\bar M]\, U_u(t) = U_u'(t) - A_1(t)\, U_u(t)\,, $$
$$ T_n^{*v}[\bar M]\, Z_v(t) = -Z_v'(t) - A_1^T(t)\, Z_v(t)\,, $$

with $U_u(t)$, $Z_v(t) \in \mathbb{R}$ and $A_1(t) \in \mathcal{M}_{n\times n}$ previously defined.

As it can be seen in [2, Section 1.3], $T_n^{*v}[\bar M]$ is the adjoint operator of ${T_n}^v[\bar M]$ and vice-versa. As consequence, by definition of adjoint operator, we have that for every $u \in X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ and $v \in X_{\{\sigma_1,\dots,\sigma_k\}}^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, the following equality is fulfilled

$$ \langle {T_n}^v[\bar M]\, U_u(t), Z_v(t)\rangle = \langle U_u(t), T_n^{*v}[\bar M]\, Z_v(t)\rangle\,, $$

where $\langle\cdot,\cdot\rangle$ is the scalar product in $\mathcal{L}^2(I,\mathbb{R}^n)$. Moreover, from [2, Section 1.3], we have

$$ \langle U_u(a), Z_v(a)\rangle = \langle U_u(b), Z_v(b)\rangle\,, \quad \forall u \in X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}, \quad v \in X_{\{\sigma_1,\dots,\sigma_k\}}^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}\,. $$

Taking into account the boundary conditions (4.17), we conclude that for every $v \in X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$ it is satisfied:

$$z_{n-\tau_1\,v}(a) = \dots = z_{n-\tau_n\,v}(a) = z_{n-\delta_1\,v}(b) = \dots = z_{n-\delta_k\,v}(b) = 0\,,$$

which implies that

$$T^*_{\tau_1} v(a) = \dots = T^*_{\tau_{n-k}} v(a) = T^*_{\delta_1} v(b) = \dots = T^*_{\delta_k} v(b) = 0.$$

Or, which is the same, $T_n^*[\bar{M}]$ satisfies the property (T_d^*) on $X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$. □

Example 4.6. Let us consider the fourth-order operator $T_4[M]$. Moreover, let us assume that $T_4[M]$ satisfies property (T_d) on $X^{\{1,2\}}_{\{0,2\}}$. That is, $T_0u(a) = T_2u(a) = T_1u(b) = T_2u(b) = 0$ for all $u \in X^{\{1,2\}}_{\{0,2\}}$.

From (3.8) and Example 4.3, taking into account the boundary conditions (4.9), we obtain that the following equalities are fulfilled for every $u \in X^{\{1,2\}}_{\{0,2\}}$.

$$\begin{gathered} T_0u(a) = 0\,, \\ T_2u(a) = -u'(a)\frac{2v_2(a)v_1'(a) + v_1(a)v_2'(a)}{v_1^2(a)v_2^2(a)}\,, \\ T_1u(b) = -u(b)\frac{v_1'(b)}{v_1^2(b)}\,, \\ T_2u(b) = u(b)\frac{v_1(b)v_1'(b)v_2'(b) + v_2(b)\big(2{v_1'}^2(b) - v_1(b)v_1''(b)\big)}{v_1^3(b)v_2^2(b)}\,. \end{gathered}$$

So, $T_4[M]$ satisfies property (T_d) on $X^{\{1,2\}}_{\{0,2\}}$ if and only if, there exists a decomposition as (3.1)-(3.2), where $v_1\,,\ v_2 \in C^4(I)$ are such that:

$$\frac{2v_1'(a)}{v_1(a)} = -\frac{v_2'(a)}{v_2(a)}\,, \tag{4.20}$$

$$v_1'(b) = v_1''(b) = 0\,. \tag{4.21}$$

Let us verify that in such a case, the operator $T_4^*[M]$ satisfies property (T_d^*) on $X^{*\{1,2\}}_{\{0,2\}}$. To do that, we express p_1 and p_2 as functions of v_1, v_2, v_3 and v_4. Expanding the related expression (2.2) for $n = 4$, we obtain that

$$p_1 \equiv -\frac{4v_1'}{v_1} - \frac{3v_2'}{v_2} - \frac{2v_3'}{v_3} - \frac{v_4'}{v_4}\,,$$

and

$$\begin{aligned} p_2 \equiv{}& \frac{12{v_1'}^2}{v_1^2} + \frac{6{v_2'}^2}{v_2} + \frac{2{v_3'}^2}{v_3^2} + \frac{9v_1'v_2'}{v_1v_2} + \frac{6v_1'v_3'}{v_1v_3} + \frac{4v_2'v_3'}{v_2v_3} + \frac{3v_1'v_4'}{v_1v_4} + \frac{2v_2'v_4'}{v_2v_4} \\ &+ \frac{v_3'v_4'}{v_3v_4} - \frac{6v_1''}{v_1} - \frac{3v_2''}{v_2} - \frac{v_3''}{v_3}\,. \end{aligned}$$

Moreover,

$$p_1' \equiv \frac{4{v_1'}^2}{v_1^2} + \frac{3{v_2'}^2}{v_2^2} + \frac{2{v_3'}^2}{v_3^2} + \frac{{v_4'}^2}{v_4^2} - \frac{4v_1''}{v_1} - \frac{3v_2''}{v_2} - \frac{2v_3''}{v_3} - \frac{v_4''}{v_4}\,.$$

Taking into account (4.20)-(4.21), the boundary conditions for the adjoint operator, given in Example 4.3, can be expressed in terms of v_1, v_2, v_3 and v_4 as follows:

$$v(a) = v''(a) + \Big(\frac{v_2'(a)}{v_2(a)} + \frac{2v_3'(a)}{v_3(a)} + \frac{v_4'(a)}{v_4(a)}\Big) v'(a) = v(b) = 0, \tag{4.22}$$

$$\begin{aligned} &v^{(3)}(b) + \Big(\frac{3v_2'(b)}{v_2(b)} + \frac{2v_3'(b)}{v_3(b)} + \frac{v_4'(b)}{v_4(b)}\Big) v''(b) \\ &+ \Big(\frac{4v_2'(b)v_3'(b)}{v_2(b)v_3(b)} - \frac{2{v_3'}^2(b)}{v_3^2(b)} + \frac{2v_2'(b)v_4'(b)}{v_2(b)v_4(b)} + \frac{v_3'(b)v_4'(b)}{v_3(b)v_4(b)} - \frac{2{v_4'}^2(b)}{v_4^2(b)} \\ &+ \frac{3v_2''(b)}{v_2(b)} + \frac{3v_3''(b)}{v_3(b)} + \frac{2v_4''(b)}{v_4(b)}\Big) v'(b) = 0\,. \end{aligned} \tag{4.23}$$

Now, let us see that $T_0^*v(a) = T_2^*v(a) = T_0^*v(b) = T_3^*v(b) = 0$ for all $v \in X_{\{0,2\}}^{*\{1,2\}}$. Trivially, $T_0^*v(a) = v(a) = 0$ and $T_0^*v(b) = v(b) = 0$. Using the decomposition (4.11), we have

$$T_2^*v(t) = -\frac{1}{v_3(t)} \frac{d}{dt}\Big(\frac{-1}{v_4(t)} \frac{d}{dt}\left(v_1(t)v_2(t)v_3(t)v_4(t)v(t)\right)\Big),$$

from which, considering (4.20) and (4.22), we obtain

$$T_2^*v(a) = v_1(a)v_2(a)\Big(v''(a) + \Big(\frac{v_2'(a)}{v_2(a)} + \frac{2v_3'(a)}{v_3(a)} + \frac{v_4'(a)}{v_4(a)}\Big) v'(a)\Big) = 0\,.$$

Finally,

$$T_3^*v(t) = -\frac{1}{v_2(t)} \frac{d}{dt}\Big(\frac{-1}{v_3(t)} \frac{d}{dt}\Big(\frac{-1}{v_4(t)} \frac{d}{dt}\big(v_1(t)v_2(t)v_3(t)v_4(t)v(t)\big)\Big)\Big)\,.$$

Combining the previous expression with (4.21) –(4.23), we obtain

$$\begin{aligned} T_3^*v(b) = -v_1(b)\Big(&v^{(3)}(b) + \Big(\frac{3v_2'(b)}{v_2(b)} + \frac{2v_3'(b)}{v_3(b)} + \frac{v_4'(b)}{v_4(b)}\Big) v''(b) \\ &+ \Big(\frac{4v_2'(b)v_3'(b)}{v_2(b)v_3(b)} - \frac{2{v_3'}^2(b)}{v_3^2(b)} + \frac{2v_2'(b)v_4'(b)}{v_2(b)v_4(b)} + \frac{v_3'(b)v_4'(b)}{v_3(b)v_4(b)} \\ &- \frac{2{v_4'}^2(b)}{v_4^2(b)} + \frac{3v_2''(b)}{v_2(b)} + \frac{3v_3''(b)}{v_3(b)} + \frac{2v_4''(b)}{v_4(b)}\Big) v'(b)\Big) = 0\,. \end{aligned}$$

As a particular case of Lemma 4.5, we have proved that if $T_4[M]$ satisfies property (T_d) on $X_{\{0,2\}}^{\{1,2\}}$, then $T_4^*[M]$ satisfies property (T_d^*) on $X_{\{0,2\}}^{*\{1,2\}}$.

It is obvious that we can enunciate an analogous result to Lemma 4.5 referring to operator $\widehat{T}_n[(-1)^n \bar{M}]$ defined in (2.14).

Lemma 4.7. *Let $\bar{M} \in \mathbb{R}$ be such that $T_n[\bar{M}]$ satisfies property (T_d) on the set $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, then $\widehat{T}_n[(-1)^n \bar{M}]$ also satisfies property (T_d^*) on $X_{\{\sigma_1,\dots,\sigma_k\}}^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*

Proof. We only have to consider $\widehat{T}_\ell v(t) = (-1)^\ell T_\ell^* v(t)$, $\ell = 0, \dots, 1$ and the result follows directly from Lemma 4.5. □

Arguing as in [12] to deduce the equality (3.3), let us see that the expression of $\widehat{T}_\ell v(t)$ is given by

$$\widehat{T}_\ell v(t) = v_1(t) \dots v_{n-\ell} v^{(\ell)}(t) + \widehat{p}_{\ell_1}(t) v^{(\ell-1)}(t) + \dots + \widehat{p}_{\ell_\ell}(t) v(t)\,, \tag{4.24}$$

where $\widehat{p}_{\ell_i} \in C^{n-\ell}(I)$.

For $\ell = 0$, we have that $\widehat{T}_0 v(t) = v_1(t) \dots v_n(t) v(t)$. Let us assume that (4.24) is true for a given $\ell \geq 0$, then, by (4.11), we have

$$\widehat{T}_{\ell+1} v(t) = \frac{1}{v_{n-\ell}(t)} \frac{d}{dt} \big(\widehat{T}_\ell v(t) \big) .$$

Thus, using the induction hypothesis,

$$\widehat{T}_{\ell+1} v(t) = \frac{1}{v_{n-\ell}(t)} \frac{d}{dt} \Big(v_1(t) \, \dots v_{n-\ell} v^{(\ell)}(t) + \widehat{p}_{\ell_1}(t) v^{(\ell-1)}(t) + \dots + \widehat{p}_{\ell_\ell}(t) v(t) \Big) \, ,$$

from which follows the expression (4.24) for $\ell + 1$.

As a consequence of the previous results, we are able to obtain analogous results to Lemmas 3.6 and 3.7 for $\widehat{T}_n[(-1)^n M]$.

Lemma 4.8. *Let $\bar{M} \in \mathbb{R}$ be such that $T_n[(-1)^n \bar{M}]$ satisfies the property (T_d^*) on $X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$. If $v \in C^n([a,c))$, with $c > a$, is a function that satisfies (4.1)–(4.2), then*

$$\begin{gathered} \widehat{T}_{\tau_1} v(a) = \dots = \widehat{T}_{\tau_{n-k-1}} v(a) = 0 \, , \\ \widehat{T}_{\tau_{n-k}} v(a) = \widehat{f}(a) \Big(v^{(\tau_{n-k})}(a) + \sum_{j=n-\tau_{n-k}}^{n-1} (-1)^{n-j} (p_{n-j} v)^{(\tau_{n-k}+j-n)}(a) \Big) \, , \end{gathered}$$

where $\widehat{f}(t) = v_1(t) \dots v_{n-\tau_{n-k}}(t) > 0$ on I. In particular, if v satisfies (4.3), then $\widehat{T}_{\tau_{n-k}} v(a) = 0$.

Proof. The proof is analogous to the one given in Lemma 3.6, but in this case we have

$$\begin{aligned} \widehat{T}_{\tau_{n-k}} v(a) = {} & v_1(a) \, \dots v_{n-\tau_{n-k}}(a) v^{(\tau_{n-k})}(a) \\ & + \widehat{p}_{\tau_{n-k_1}}(a) v^{(\tau_{n-k}-1)}(a) + \dots + \widehat{p}_{\tau_{n-k_{\tau_{n-k}}}}(a) v(a) \, . \end{aligned}$$

If (4.3) is satisfied, then $\widehat{T}_{\tau_{n-k} v(a)} = 0$, and the result follows. ☐

Lemma 4.9. *Let $\bar{M} \in \mathbb{R}$ be such that $T_n[(-1)^n \bar{M}]$ satisfies property (T_d^*) on $X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$. If $v \in C^n((c,b])$, with $c < b$, is a function that satisfies (4.4)–(4.5), then*

$$\begin{gathered} \widehat{T}_{\delta_1} v(b) = \dots = \widehat{T}_{\delta_{k-1}} v(b) = 0 \, , \\ \widehat{T}_{\delta_k} v(b) = \widehat{g}(b) \Big(v^{(\delta_k)}(b) + \sum_{j=n-\delta_k}^{n-1} (-1)^{n-j} (p_{n-j} v)^{(\delta_k+j-n)}(b) \Big) \, , \end{gathered}$$

where $\widehat{g}(t) = v_1(t) \dots v_{n-\delta_k}(t) > 0$ on I. In particular, if v satisfies (4.6), then $\widehat{T}_{\delta_k} v(b) = 0$.

The proof of the above lemma is analogous to the one of Lemma 4.8, and is omitted here.

5. Strongly inverse positive (negative) character of operator $T_n[\bar{M}]$

In this section we prove that if the operator $T_n[\bar{M}]$ satisfies property (T_d), then it is a strongly inverse positive (negative) operator on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$. Moreover, its related Green's function satisfies a suitable condition, which allows us to apply either Theorem 2.16 or Theorem 2.17 and obtain one of the extremes of the interval where the related Green's function is of constant sign. The result is the following.

Theorem 5.1. *Let $\bar{M} \in \mathbb{R}$ be such that $T_n[\bar{M}]$ satisfies (T_d) on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ and $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ satisfy condition (N_a). Then the following properties are fulfilled:*

- *If $n-k$ is even, then $T_n[\bar{M}]$ is strongly inverse positive on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ and, moreover, the related Green's function, $g_{\bar{M}}(t,s)$, satisfies* (A2.1).
- *If $n-k$ is odd, then $T_n[\bar{M}]$ is strongly inverse negative on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ and, moreover, the related Green's function, $g_{\bar{M}}(t,s)$, satisfies* (A2.2).

Proof. Firstly, let us verify the strongly inverse positive (negative) character. To this end, we use the decomposition of $T_n[\bar{M}]$ given on (3.1). Since $v_1(t)\dots v_n(t) > 0$; if $T_n[\bar{M}]u \gneqq 0$ on I, from (3.2), we conclude that $T_n u \gneqq 0$ on I. Hence, from (3.1) we know that $\frac{T_{n-1}u}{v_n}$ is a nontrivial nondecreasing function, with at most a sign change on I. Therefore, since $v_n > 0$, we can affirm that $T_{n-1}u$ can have at most a sign change, being negative at $t=a$ and positive at $t=b$.

Repeating this process for $T_{n-\ell}u$, with $\ell = 1,\dots n$, we can affirm that $T_0 u = u$ can have at most n zeros on (a,b), whenever the following inequalities are satisfied for every $\ell = 1,\dots,n$:

$$T_{n-\ell}u(a) \begin{cases} > 0\,, & \text{if } \ell \text{ is even,} \\ < 0\,, & \text{if } \ell \text{ is odd,} \end{cases} \quad \text{and} \quad T_{n-\ell}u(b) > 0\,. \tag{5.1}$$

Repeating the same argument as in Lemma 3.8, we can affirm that each time that $T_{n-\ell}u(a) = 0$ or $T_{n-\ell}u(b) = 0$, we lose a possible oscillation and, therefore, a possible zero of u in (a,b).

From property (T_d), we know that for all $u \in X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$

$$T_{\sigma_1}u(a) = \dots = T_{\sigma_k}u(a) = T_{\varepsilon_1}u(b) = \dots = T_{\varepsilon_{n-k}}u(b) = 0\,, \tag{5.2}$$

i.e, we lose the n possible zeros which u could ever have. Thus, we can conclude that u cannot have any zero on (a,b).

Let us see how is the sign of $u^{(\alpha)}(a)$ and $u^{(\beta)}(b)$ which gives the sign of u. Since $u(a) = \dots = u^{(\alpha-1)}(a) = 0$ and $u(b) = \dots = u^{(\beta-1)}(b) = 0$, from (3.3) we have

$$T_\alpha u(a) = \frac{u^{(\alpha)}(a)}{v_1(a)\dots v_\alpha(a)}\,, \quad T_\beta u(b) = \frac{u^{(\beta)}(b)}{v_1(b)\dots v_\beta(b)}\,, \tag{5.3}$$

hence, $u^{(\alpha)}(a)$ and $T_\alpha u(a)$, and $u^{(\beta)}(b)$ and $T_\beta u(b)$ have the same sign, respectively.

If either, $T_\ell u(a) = 0$ for any $\ell \notin \{\sigma_1,\dots,\sigma_k\}$, or $T_\ell u(b) = 0$ for any $\ell \notin \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$, then we lose another possible oscillation and, necessarily, $u \equiv 0$ on I which is a contradiction with $T_n[\bar{M}]u \gneqq 0$.

Moreover, taking into account (5.2), the sign of $T_\ell u(a)$ must allow the maximum number of oscillations for $T_\ell u$. Otherwise $u \equiv 0$ on I which is again a contradiction with $T_n[\bar{M}]u \gneqq 0$.

Notation 5.2. In this work, we understand for conditions of maximal oscillation those which allow u to have the maximum number of zeros depending on the fixed boundary conditions without being a trivial solution.

Hence $T_{n-\ell}$ must verify the conditions for maximal oscillation. That is, $T_{n-\ell}u(a)$ must change its sign each time that it is not null, i.e., if $T_{n-\ell}u(a) > 0$ for a given $\ell = 1, \dots, n$, then $T_{n-\ell-1}u(a) \le 0$ and if $T_{n-\ell-1}u(a) = 0$, we consider $\tilde{\ell} \in \{\ell+1, \dots, n\}$ such that $T_{n-\tilde{\ell}}u(a) \neq 0$ and $T_{n-h}u(a) = 0$ for $h \in \{\ell+1, \dots, \tilde{\ell}-1\}$, then $T_{n-\tilde{\ell}}u(a) < 0$.

From property (T_d), we know that $T_{n-\ell}u(a)$ vanishes $k-\alpha$ times for $\ell \in \{1, \dots, n-\alpha\}$. Hence, taking into account the previous argument and the conditions given in (5.1), we have

$$T_\alpha u(a) \begin{cases} > 0\,, & \text{if } n-\alpha-(k-\alpha) = n-k \text{ is even,} \\ < 0\,, & \text{if } n-k \text{ is odd.} \end{cases}$$

To obtain the previous inequalities, there are considered as many sign changes for $T_h u(a)$ as times that it is non null from $h = \alpha$ to $h = n-1$. That is, the $n-\alpha$ steps minus the $k-\alpha$ zeros that are found. Thus, from (5.3)

$$u^{(\alpha)}(a) \begin{cases} > 0\,, & \text{if } n-k \text{ is even,} \\ < 0\,, & \text{if } n-k \text{ is odd.} \end{cases} \tag{5.4}$$

From this, since $u \neq 0$ on (a,b), we conclude that

$$u(t) \begin{cases} > 0 \;\; t \in (a,b)\,, & \text{if } n-k \text{ is even,} \\ < 0 \;\; t \in (a,b)\,, & \text{if } n-k \text{ is odd.} \end{cases} \tag{5.5}$$

Taking into account that necessarily $T_\beta u(b) \neq 0$, since $\beta \notin \{\varepsilon_1, \dots, \varepsilon_{n-k}\}$, from (5.3) and (5.5) we have:

- If $n-k$ is even

$$u^{(\beta)}(b) \begin{cases} > 0\,, & \text{if } \beta \text{ is even,} \\ < 0\,, & \text{if } \beta \text{ is odd.} \end{cases} \tag{5.6}$$

- If $n-k$ is odd

$$u^{(\beta)}(b) \begin{cases} < 0\,, & \text{if } \beta \text{ is even,} \\ > 0\,, & \text{if } \beta \text{ is odd.} \end{cases} \tag{5.7}$$

Hence, from (5.4)–(5.7), we conclude that if $n-k$ is even, then the operator $T_n[\bar{M}]$ is a strongly inverse positive operator on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ and if $n-k$ is odd, then the operator $T_n[\bar{M}]$ is a strongly inverse negative operator on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.

Let us see that $g_{\bar{M}}(t,s)$ satisfies condition (A2.1) or (A2.2), respectively. Using Theorems 2.14 and 2.15, it is known that $(-1)^{n-k} g_{\bar{M}}(t,s) > 0$ for a.e. $(t,s) \in (a,b) \times (a,b)$. Let us see that, in fact, this inequality holds for all $(t,s) \in (a,b) \times (a,b)$.

For each fixed $s \in (a,b)$, let us define $u_s(t) = (-1)^{n-k} g_{\bar{M}}(t,s)$, $u_s \in C^{n-2}(I)$ and $u_s \in C^n([a,s) \cup (s,b])$. It is known that $u_s(t) \ge 0$ on I, and that it satisfies the boundary conditions (1.5)-(1.6). Moreover, since $g_{\bar{M}}(t,s)$ is the Green's function related to the operator $T_n[\bar{M}]$ on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, we have

$$T_n[\bar{M}]u_s(t) = v_1(t)\dots v_n(t)\, T_n u_s(t) = 0\,, \quad t \neq s\,.$$

Since $v_1 \dots v_n > 0$ on I, $T_n u_s(t) = 0$ if $t \neq s$. Hence,

$$\begin{aligned} \frac{1}{v_n(t)}\, T_{n-1} u_s(t) = c_1\,, \quad t < s\,, \\ \frac{1}{v_n(t)}\, T_{n-1} u_s(t) = c_2\,, \quad t > s\,, \end{aligned} \tag{5.8}$$

where c_1, $c_2 \in \mathbb{R}$ are of different sign to allow the maximal oscillation.

Since $v_n > 0$, $T_{n-1}u_s$ has the same sign as c_1 or c_2, if $t < s$ or $t > s$, respectively, i.e., in order to have maximal number of oscillations, it has two components of constant different sign. Then, since $\frac{1}{v_{n-1}} T_{n-2} u_s$ is a continuous function, it can have at most two sign changes and the same happens with $T_{n-2}u_s$.

Proceeding in a similar way, we conclude that with maximal oscillation $T_{n-\ell}u_s$ can have at most ℓ zeros, for $\ell = 2, \dots, n$. In particular, u_s has at most n sign changes on I.

Arguing as before, each time that $T_{n-\ell}u_s(a) = 0$ or $T_{n-\ell}u_s(b) = 0$ a possible oscillation is lost. Taking into account that $T_n[\bar{M}]$ satisfies (T_d) on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, we use Lemmas 3.6 and 3.7 to affirm that u_s satisfies (5.2). Thus, $T_{n-\ell}u_s(a)$ or $T_{n-\ell}u_s(b)$ vanish n times for $\ell = 1, \dots, n$. So, we have lost the n possible zeros and we can affirm that $u_s > 0$ on (a, b). Or, which is the same, $(-1)^{n-k} g_{\bar{M}}(t, s) > 0$ for all $(t, s) \in (a, b) \times (a, b)$.

Moreover, for each $s \in (a, b)$, we obtain the following limits:

$$\ell_1(s) = \lim_{t \to a^+} \frac{(-1)^{n-k} g_{\bar{M}}(t, s)}{(t-a)^\alpha (b-t)^\beta} = \frac{(-1)^{n-k} \frac{\partial^\alpha}{\partial t^\alpha} g_{\bar{M}}(t, s)\big|_{t=a}}{\alpha!\,(b-a)^\beta}\,,$$

$$\ell_2(s) = \lim_{t \to b^-} \frac{(-1)^{n-k} g_{\bar{M}}(t, s)}{(t-a)^\alpha (b-t)^\beta} = \frac{(-1)^{n-k-\beta} \frac{\partial^\beta}{\partial t^\beta} g_{\bar{M}}(t, s)\big|_{t=b}}{\beta!\,(b-a)^\alpha}\,.$$

For each $s \in (a, b)$, let us construct the continuous extension on I of u_s, as follows

$$\tilde{u}_s(t) = \frac{(-1)^{n-k} g_{\bar{M}}(t, s)}{(t-a)^\alpha (b-t)^\beta}\,.$$

Since $u_s > 0$ and $(t-a)^\alpha (b-t)^\beta > 0$ on (a, b), we have that $\tilde{u}_s > 0$ on (a, b). Moreover, using Theorems 2.14 and 2.15, we can affirm that $\ell_1(s) > 0$ and $\ell_2(s) > 0$ for a.e. $s \in (a, b)$. Hence, for a.e. $s \in (a, b)$, $\tilde{u}_s(a) > 0$ and $\tilde{u}_s(b) > 0$.

Furthermore, since $g_{\bar{M}}(t, s)$ is the related Green's function of $T_n[\bar{M}]$ on the set $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, we also can affirm that there exists $K > 0$ such that $\tilde{u}_s \leq K$ for every $(t, s) \in I \times (a, b)$. Hence, we construct the following functions:

$$\begin{aligned} \tilde{k}_1(s) &= \min_{t \in I} \tilde{u}_s(t)\,, \quad s \in (a, b)\,, \\ \tilde{k}_2(s) &= \max_{t \in I} \tilde{u}_s(t)\,, \quad s \in (a, b)\,, \end{aligned}$$

which are continuous on (a, b) and they are positive a.e. in (a, b).

Taking $\phi(t) = (t-a)^\alpha (b-t)^\beta > 0$ on (a, b), condition (A2.1) is trivially satisfied if $n - k$ is even with $k_1(s) = \tilde{k}_1(s)$ and $k_2(s) = \tilde{k}_2(s)$ and condition (A2.2) if $n - k$ is odd with $k_1(s) = -\tilde{k}_2(s)$ and $k_2(s) = -\tilde{k}_1(s)$. □

Remark 5.3. From Theorem 5.1, if $T_n[\bar{M}]$ satisfies property (T_d) on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon n-k\}}$, then either Theorem 2.16, if $n - k$ is even, or Theorem 2.17, if $n - k$ is odd, can be applied to operator $T_n[\bar{M}]$ on such a space.

Example 5.4. Let us continue the study of the fourth order operator given in Example 4.3. From Example 4.6, we can affirm that $T_4[M]$ satisfies condition (T_d) if and only if there exists a decomposition (3.1)-(3.2) such that (4.20)-(4.21) are satisfied. These equalities are true, in particular, if we choose $v_1(t) = v_2(t) = v_3(t) = v_4(t) = 1$ for all $t \in I$. That is, they are valid for the particular case of operator $T_4^0[0]u(t) = u^{(4)}(t)$. Such a choice has been done in order to simplify the calculations, the applicability of the results can be extended to a more complicated class of operators.

Now, let us check directly that this operator satisfies the thesis of Theorem 5.1. To do that, let us consider $I \equiv [0,1]$. In this case, $n-k=2$ is even, so let us study the strongly inverse positive character. If $u^{(4)} \gneqq 0$, then u'' is a convex function. Since $u''(0) = u''(1) = 0$, we have that $u'' \lneqq 0$ (if $u'' \equiv 0$, then $u^{(4)} \equiv 0$ which is a contradiction). Hence, u' is a decreasing function on I verifying $u'(1) = 0$, so $u' \gneqq 0$. In particular, $u'(0) > 0$.

Finally, taking into account that $u(0) = 0$, u is an increasing function on I and it cannot have infinite zeros without being a trivial solution of $T_4^0[0]u(t) = 0$, we have that $u(t) > 0$ for all $t \in (0,1]$.

Now, let us study the related Green's function, given by the expression:

$$g_0(t,s) = \begin{cases} \frac{1}{6}s\left(t\left(t^2-3t+3\right)-s^2\right), & 0 \le s \le t \le 1\,, \\ \frac{1}{6}(s-1)t\left(t^2-3s\right), & 0 < t < s \le 1\,. \end{cases}$$

Let us see that it satisfies the condition (A2.1). First, it is obvious that $g_0(1,s) = \frac{1}{6}\,s\,\left(1-s^2\right) > 0$ for all $s \in (0,1)$. Moreover,

$$\frac{\partial}{\partial t} g_0(t,s) = \begin{cases} \frac{1}{6}s\left(t^2+(2t-3)t-3t+3\right), & 0 \le s \le t \le 1\,, \\ \frac{1}{3}(s-1)t^2 + \frac{1}{6}(s-1)\left(t^2-3s\right), & 0 < t < s \le 1\,, \end{cases}$$

in particular, $\frac{\partial}{\partial t} g_0(t,s)\big|_{t=0} = \frac{1}{2}\left(s-s^2\right) > 0$ for all $s \in (0,1)$.

Now, let us verify that $g_0(t,s) > 0$ on $(0,1)\times(0,1)$. If $t < s$, we have that $s-1 < 0$ and $t^2-3s < -3s+s^2 < 0$ for all $s \in (0,1)$. If $t \ge s$, we have $t\left(t^2-3t+3\right)-s^2 \ge s\left(s^2-3s+3\right)-s^2 = 3s-4s^2+s^3 > 0$ for all $s \in (0,1)$. Hence, $g_0(t,s) > 0$ on $(0,1)\times(0,1)$.

On the other hand,

$$\tilde{u}_s(t) = \frac{g_0(t,s)}{t} = \begin{cases} \frac{1}{6}\frac{s}{t}\left(t\left(t^2-3t+3\right)-s^2\right), & 0 \le s \le t \le 1\,, \\ \frac{1}{6}(s-1)\left(t^2-3s\right), & 0 < t < s \le 1\,. \end{cases}$$

Thus, condition (A2.1) is satisfied for the following functions:

$$\begin{gathered} \phi(t) = t\,, \\ k_1(s) = \tilde{k}_1(s) = \min_{t\in I} \tilde{u}_s(t) = \frac{1}{6}\,s(1-s^2)\,, \\ k_2(s) = \tilde{k}_2(s) = \max_{t\in I} \tilde{u}_s(t) = \frac{s}{2}(1-s)\,, \end{gathered}$$

and

$$t\frac{1}{6}\,s(1-s^2) \le g_0(t,s) \le t\frac{s}{2}(1-s)\,, \quad \text{for all } (t,s) \in I\times I\,.$$

6. Existence and study of the eigenvalues of operator $T_n[\bar{M}]$ in different spaces

In [12] and [14], the characterization of the parameters set for which the related Green's function is of constant sign has been done by means of spectral theory. In fact, the extremes of the interval are characterized by suitable eigenvalues of the operator associated to different boundary conditions.

The characterization here obtained follows the same structure. Thus, in this section we study the existence of eigenvalues of operator $T_n[\bar{M}]$ in the spaces

$$X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}, \quad X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}, \quad X_{\{\sigma_1,\dots,\sigma_{k-1}\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}},$$
$$X_{\{\sigma_1,\dots,\sigma_{k-1}|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}, \quad X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}}.$$

Moreover, we study the constant sign of several solutions of the linear differential equation (1.4) coupled with different $n-1$ additional boundary conditions.

Firstly, let us see a result which allows us to affirm that, under the hypothesis that the property (T_d) is fulfilled on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, the operator $T_n[\bar{M}]$ satisfies such a property in all these spaces.

Lemma 6.1. *Let $\bar{M} \in \mathbb{R}$ be such that $T_n[\bar{M}]$ satisfies property (T_d) on the set $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$. Then the following properties are fulfilled:*

- *$T_n[\bar{M}]$ satisfies the property (T_d) on $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}$.*
- *$T_n[\bar{M}]$ satisfies the property (T_d) on $X_{\{\sigma_1,\dots,\sigma_{k-1}\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$.*
- *If $\sigma_k \neq k-1$, $T_n[\bar{M}]$ satisfies property (T_d) on $X_{\{\sigma_1,\dots,\sigma_{k-1}|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*
- *If $\varepsilon_{n-k} \neq n-k-1$, $T_n[\bar{M}]$ satisfies the property (T_d) on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}}$.*

Proof. The proof follows trivially from Lemmas 3.6 and 3.7, taking into account that under our hypotheses, from (3.3), we have

$$T_\alpha u(a) = \frac{u^{(\alpha)}(a)}{v_1(a)\dots v_\alpha(a)}, \quad T_\beta u(b) = \frac{u^{(\beta)}(b)}{v_1(b)\dots v_\beta(b)}. \tag{6.1}$$

□

Remark 6.2. If $\sigma_k = k-1$ or $\varepsilon_{n-k} = n-k-1$, then $\alpha = k$ or $\beta = n-k$, respectively. So, if either $u \in X_{\{\sigma_1,\dots,\sigma_{k-1}|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ or $u \in X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}}$, then (6.1) can be not true.

Example 6.3. Let us consider the fourth-order operator $T_4[M]$. In Example 4.6, we have seen that if $T_4[M]$ satisfies (T_d) on $X_{\{0,2\}}^{\{1,2\}}$, then (4.20)-(4.21) are fulfilled. Let us see that, in such a case, (T_d) also holds on $X_{\{0,1,2\}}^{\{1\}}$, $X_{\{0\}}^{\{0,1,2\}}$, $X_{\{0,1\}}^{\{1,2\}}$ and $X_{\{0,2\}}^{\{0,1\}}$.

- $X_{\{0,1,2\}}^{\{1\}}$: Trivially, since $T_\ell u(t)$ is a linear combination of $u(t),\dots,u^{(\ell)}(t)$, $T_0u(a) = T_1u(a) = T_2u(a) = 0$. Moreover, from (3.8), it follows that $T_1u(b) = -\frac{v_1'(b)}{v_1^2(b)}u(b) = 0$.
- $X_{\{0\}}^{\{0,1,2\}}$: Obviously, $T_0u(a) = T_0u(b) = T_1u(b) = T_2u(b) = 0$.

- $X_{\{0,1\}}^{\{1,2\}}$: Directly, $T_0u(a) = T_1u(a) = 0$. From (3.8) and Example 3.4, $T_1u(b) = \frac{-v_1'(b)}{v_1^2(b)}u(b) = 0$ and

$$T_2u(b) = \frac{v_1(b)v_1'(b)v_2'(b) + v_2(b)\left(2{v_1'}^2(b) - v_1(b)v_1''(b)\right)}{v_1^3(b)v_2^2(b)}u(b) = 0\,.$$

- $X_{\{0,2\}}^{\{0,1\}}$: Trivially, $T_0u(a) = T_0u(b) = T_1u(b) = 0$. Finally, from Example 3.4, $T_2u(a) = -\frac{2v_2(a)v_1'(a)+v_1(a)v_2'(a)}{v_1^2(a)v_2^2(a)}u'(a) = 0$.

As a consequence we can prove the following corollary.

Corollary 6.4. *Let $\bar M \in \mathbb{R}$ be such that $T_n[\bar M]$ satisfies property (T_d) on the set $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, and $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ satisfy (N_a). Then*

- *If $n-k$ is even:*
  - * *$T_n[\bar M]$ is strongly inverse positive and satisfies condition (A2.1) on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ and $X_{\{\sigma_1,\dots,\sigma_{k-1}|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$. If, in addition, $\varepsilon_{n-k} \neq n-k-1$, then this property is also fulfilled on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}}$.*
  - * *$T_n[\bar M]$ is strongly inverse negative and satisfies condition (A2.2) on $X_{\{\sigma_1,\dots,\sigma_{k-1}\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$. If, in addition, $\sigma_k \neq k-1$, then this property is also fulfilled on $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}$.*
- *If $n-k$ is odd:*
  - * *$T_n[\bar M]$ is strongly inverse negative and satisfies condition (A2.2) on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ and $X_{\{\sigma_1,\dots,\sigma_{k-1}|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$. If, in addition, $\varepsilon_{n-k} \neq n-k-1$, then this property is also fulfilled on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}}$.*
  - * *$T_n[\bar M]$ is strongly inverse positive and satisfies condition (A2.1) on on $X_{\{\sigma_1,\dots,\sigma_{k-1}\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$. If, in addition, $\sigma_k \neq k-1$, then this property is also fulfilled on $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}$.*

Proof. It is obvious that if $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ satisfy (N_a), then the sets $\{\sigma_1,\dots,\sigma_k|\alpha\} - \{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}$, $\{\sigma_1,\dots,\sigma_{k-1}\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}$ also do.

Moreover, if $\sigma_k \neq k-1$, then $\alpha < \sigma_k$ and if $\varepsilon_{n-k} \neq n-k-1$, then $\beta < \varepsilon_{n-k}$. So, if $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ satisfy (N_a), then $\{\sigma_1,\dots,\sigma_{k-1}|\alpha\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ and $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}$ also do. Thus, using Theorem 5.1 and Lemma 6.1, the result follows. □

Now, from the previous Corollary and the first assertion on Theorems 2.16 and 2.17, we obtain, as a direct consequence, the following result.

Corollary 6.5. *Let $\bar M \in \mathbb{R}$ be such that $T_n[\bar M]$ satisfies the property (T_d) on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, and $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ satisfy (N_a). Then*

- *If $n-k$ is even:*
  - *There is $\lambda_1 > 0$, the least positive eigenvalue of $T_n[\bar M]$ on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$. Moreover, there exists a nontrivial constant sign eigenfunction corresponding to the eigenvalue λ_1.*

  - *If $k > 1$, there is $\lambda_2' < 0$, the largest negative eigenvalue of $T_n[\bar{M}]$ on $X_{\{\sigma_1,\dots,\sigma_{k-1}\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$. Moreover, there exists a nontrivial constant sign eigenfunction corresponding to the eigenvalue λ_2'.*
  - *If $k < n-1$, there is $\lambda_2'' < 0$, the largest negative eigenvalue of $T_n[\bar{M}]$ on $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}$. Moreover, there exists a nontrivial constant sign eigenfunction corresponding to the eigenvalue λ_2''.*
  - *If $\sigma_k \neq k-1$, there is $\lambda_3' > 0$, the least positive eigenvalue of $T_n[\bar{M}]$ on $X_{\{\sigma_1,\dots,\sigma_{k-1}|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$. Moreover, there exists a nontrivial constant sign eigenfunction corresponding to the eigenvalue λ_3'.*
  - *If $\varepsilon_{n-k} \neq n-k-1$, there is $\lambda_3'' > 0$, the least positive eigenvalue of $T_n[\bar{M}]$ on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}}$. Moreover, there exists a nontrivial constant sign eigenfunction corresponding to the eigenvalue λ_3''.*
- *If $n-k$ is odd:*
  - *There exists $\lambda_1 < 0$, the largest negative eigenvalue of $T_n[\bar{M}]$ on the set $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$. Moreover, there is a nontrivial constant sign eigenfunction corresponding to the eigenvalue λ_1.*
  - *If $k > 1$, there is $\lambda_2' > 0$, the least positive eigenvalue of $T_n[\bar{M}]$ on $X_{\{\sigma_1,\dots,\sigma_{k-1}\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$. Moreover, there exists a nontrivial constant sign eigenfunction corresponding to the eigenvalue λ_2'.*
  - *If $k < n-1$, there is $\lambda_2'' > 0$, the least positive eigenvalue of $T_n[\bar{M}]$ on $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}$. Moreover, there exists a nontrivial constant sign eigenfunction corresponding to the eigenvalue λ_2''.*
  - *If $\sigma_k \neq k-1$, there is $\lambda_3' < 0$, the largest negative eigenvalue of $T_n[\bar{M}]$ on $X_{\{\sigma_1,\dots,\sigma_{k-1}|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$. Moreover, there exists a nontrivial constant sign eigenfunction corresponding to the eigenvalue λ_3'.*
  - *If $\varepsilon_{n-k} \neq n-k-1$, there is $\lambda_3'' < 0$, the largest negative eigenvalue of $T_n[\bar{M}]$ on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}}$. Moreover, there exists a nontrivial constant sign eigenfunction corresponding to the eigenvalue λ_3''.*

Example 6.6. Continuing the study of the operator $T_4^0[M]u(t) = u^{(4)}(t) + Mu(t)$ introduced in Example 5.4, we can affirm the existence of the eigenvalues of $T_4^0[0]$ in the different spaces introduced in Example 6.3 and the related constant sign eigenfunctions.

Next, we obtain those eigenvalues and related eigenfunctions.

• The eigenvalues of $T_4^0[0]$ on $X_{\{0,2\}}^{\{1,2\}}$ are given by $\lambda = m^4$, where m is a positive solution of the following equation:

$$\tan(m) + \tanh(m) = 0\,. \tag{6.2}$$

- The least positive eigenvalue is $\lambda_1 = m_1^4 \approxeq 2,36502^4$, where m_1 is the least positive solution of (6.2). The related constant sign eigenfunctions are given by:

$$u(t) = K\Big(\frac{\sinh(m_1\,t)}{\cosh(m_1)} - \frac{\sin(m_1\,t)}{\cos(m_1)}\Big)\,,$$

  where $K \in \mathbb{R}$.

- The largest negative eigenvalue of $T_4^0[0]$ on $X_{\{0,1,2\}}^{\{1\}}$ is $\lambda_2'' = -4\pi^4$. The related constant sign eigenfunctions are given by:
$$u(t) = K\ (\cosh(\pi\, t)\sin(\pi\, t) - \cos(\pi\, t)\sinh(\pi\, t))\ ,$$
where $K \in \mathbb{R}$.

• The eigenvalues of $T_4^0[0]$ on $X_{\{0\}}^{\{0,1,2\}}$ are given by $\lambda = -m^4$, where m is a positive solution of the following equation:
$$\tan\left(\frac{m}{\sqrt{2}}\right) - \tanh\left(\frac{m}{\sqrt{2}}\right) = 0\,. \tag{6.3}$$
The largest negative eigenvalue is $\lambda_2' = -m_2^4 \approxeq -5,55305^4$, where m_2 is the least positive solution of (6.3). The related constant sign eigenfunctions are given by:
$$\begin{aligned} u(t) = K\Big(&\cosh\left(\frac{m_2}{\sqrt{2}}(1-t)\right)\sin\left(\frac{m_2}{\sqrt{2}}(1-t)\right)\\ &-\cos\left(\frac{m_2}{\sqrt{2}}(1-t)\right)\sinh\left(\frac{m_2}{\sqrt{2}}(1-t)\right)\Big)\,, \end{aligned}$$
where $K \in \mathbb{R}$.

• The eigenvalues of $T_4^0[0]$ on $X_{\{0,2\}}^{\{0,1\}}$ are given by $\lambda = m^4$, where m is a positive solution of the following equation:
$$\tan(m) - \tanh(m) = 0\,. \tag{6.4}$$
The least positive eigenvalue is $\lambda_3'' = m_3^4 \approxeq 3,9266^4$, where m_3 is the least positive solution of (6.4). The related constant sign eigenfunctions are given by:
$$u(t) = K\ \left(\frac{\sinh(m_3\, t)}{\cosh(m_3)} - \frac{\sin(m_3\, t)}{\cos(m_3)}\right)\ ,$$
where $K \in \mathbb{R}$.

The least positive eigenvalue of $T_4^0[0]$ on $X_{\{0,1\}}^{\{1,2\}}$ is $\lambda_3' = \pi^4$. The related constant sign eigenfunctions are given by:
$$u(t) = K\, e^{-\pi(t+1)}\left(e^{2\pi t} + e^{\pi t}\left((e^{\pi}-1)\sin(\pi t) + (-1-e^{\pi})\cos(\pi t)\right) + e^{\pi}\right)\ ,$$
where $K \in \mathbb{R}$.

Now, we introduce some results that provide sufficient conditions to ensure that suitable solutions of (1.4) are of constant sign.

Proposition 6.7. *Let $\bar{M} \in \mathbb{R}$ be such that $T_n[\bar{M}]$ satisfies property (T_d) on the set $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ and $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ satisfy (N_a). If $u \in C^n(I)$ is a solution of (1.4) on (a,b), satisfying the boundary conditions:*
$$u^{(\sigma_1)}(a) = \cdots = u^{(\sigma_{k-1})}(a) = 0\,, \tag{6.5}$$
$$u^{(\varepsilon_1)}(b) = \cdots = u^{(\varepsilon_{n-k})}(b) = 0\,, \tag{6.6}$$
then it does not have any zero on (a,b) provided that one of the following assertions is satisfied:

- *Let $n-k$ be even:*
  - *If $k>1$, $\sigma_k \neq k-1$ and $M \in [\bar{M}-\lambda_3', \bar{M}-\lambda_2']$, where:*
    - *$\lambda_3' > 0$ is the least positive eigenvalue of $T_n[\bar{M}]$ on the set $X_{\{\sigma_1,\dots,\sigma_{k-1}|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*

        * $\lambda_2' < 0$ *is the largest negative eigenvalue of* $T_n[\bar M]$ *on the set* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}_{\{\sigma_1,\dots,\sigma_{k-1}\}}$.
    - *If* $k = 1$, $\sigma_1 \neq 0$ *and* $M \in [\bar M - \lambda_3', +\infty)$, *where:*
        * $\lambda_3' > 0$ *is the least positive eigenvalue of* $T_n[\bar M]$ *on* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\alpha\}}$, *where* $\alpha = 0$.
    - *If* $k > 1$, $\sigma_k = k - 1$ *and* $M \in [\bar M - \lambda_1, \bar M - \lambda_2']$, *where:*
        * $\lambda_1 > 0$ *is the least positive eigenvalue of* $T_n[\bar M]$ *on* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{0,\dots,k-1\}}$.
        * $\lambda_2' < 0$ *is the largest negative eigenvalue of* $T_n[\bar M]$ *on the set* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}_{\{0,\dots,k-2\}}$.
    - *If* $k = 1$ *and* $\sigma_1 = 0$ *and* $M \in [\bar M - \lambda_1, +\infty)$, *where:*
        * $\lambda_1 > 0$ *is the least positive eigenvalue of* $T_n[\bar M]$ *on* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-1}\}}_{\{0\}}$.
- *Let* $n - k$ *be odd:*
    - *If* $k > 1$, $\sigma_k \neq k - 1$ *and* $M \in [\bar M - \lambda_2', \bar M - \lambda_3']$, *where:*
        * $\lambda_3' < 0$ *is the largest negative eigenvalue of* $T_n[\bar M]$ *on the set* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_{k-1}|\alpha\}}$.
        * $\lambda_2' > 0$ *is the least positive eigenvalue of* $T_n[\bar M]$ *on* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}_{\{\sigma_1,\dots,\sigma_{k-1}\}}$.
    - *If* $k = 1$, $\sigma_1 \neq 0$ *and* $M \in (-\infty, \bar M - \lambda_3']$, *where:*
        * $\lambda_3' < 0$ *is the largest negative eigenvalue of* $T_n[\bar M]$ *on the set* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-1}\}}_{\{\alpha\}}$, *where* $\alpha = 0$.
    - *If* $k > 1$, $\sigma_k = k - 1$ *and* $M \in [\bar M - \lambda_2', \bar M - \lambda_1]$, *where:*
        * $\lambda_1 < 0$ *is the largest negative eigenvalue of* $T_n[\bar M]$ *on the set* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{0,\dots,k-1\}}$.
        * $\lambda_2' > 0$ *is the least positive eigenvalue of* $T_n[\bar M]$ *on* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}_{\{0,\dots,k-2\}}$.
    - *If* $k = 1$ *and* $\sigma_1 = 0$ *and* $M \in (-\infty, \bar M - \lambda_1]$, *where:*
        * $\lambda_1 < 0$ *is the largest negative eigenvalue of* $T_n[\bar M]$ *on the set* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{0\}}$.

Proof. Firstly, let us see that for $M = \bar M$ every solution of (1.4) on (a, b) satisfying the boundary conditions (6.5)-(6.6) does not have any zero on (a, b). On the proof of Lemma 3.8 we have seen that, without taking into account the boundary conditions, every solution of (1.4) for $M = \bar M$ has at most $n-1$ zeros on (a, b). Let us prove that this $n-1$ possible oscillations are not attained because of the boundary conditions.

Let us denote, $u_M \in C^n(I)$ a solution of (1.4) verifying the boundary conditions (6.5)-(6.6). Each time that $T_{n-\ell}u_M(a) = 0$ or $T_{n-\ell}u_M(b) = 0$ for $\ell = 1, \dots n$ a possible oscillation is lost.

Since $T_n[\bar M]$ satisfies property (T_d) on $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$, by applying Lemmas 3.6 and 3.7 we conclude that for every $M \in \mathbb{R}$

$$T_{\sigma_1}u_M(a) = \cdots = T_{\sigma_{k-1}}u_M(a) = 0\,, \tag{6.7}$$
$$T_{\varepsilon_1}u_M(b) = \cdots = T_{\varepsilon_{n-k}}u_M(b) = 0\,. \tag{6.8}$$

In particular, this property holds for $M = \bar M$. Hence, we lose the $n-1$ possible oscillations and we can affirm that $u_{\bar M}$ does not have any zero on (a, b).

Now, to prove the result, let us move u_M in a continuous way with M in a neighborhood of $\bar{M}$. We have that u_M is a solution of (1.4) on (a,b), hence

$$T_n[\bar{M}]u_M(t) = (\bar{M} - M)u_M(t)\,, \quad t \in (a,b)\,. \tag{6.9}$$

First, let us see that while u_M is of constant sign it cannot have any double zero on (a,b). Let us assume that $u_{\bar{M}} > 0$ on (a,b) (if $u_{\bar{M}} < 0$ on (a,b) the arguments are valid by multiplying by -1). Thus, in equation (6.9) we have

$$T_n[\bar{M}]u_M(t) \begin{cases} \geq 0\,,\ t \in (a,b)\,, & \text{if } M < \bar{M}, \\ \leq 0\,,\ t \in (a,b)\,, & \text{if } M > \bar{M}. \end{cases} \tag{6.10}$$

In both cases, $T_n[\bar{M}]u_M$ is a constant sign function. Then, since $v_1 \dots v_n > 0$, $T_{n-1}u_M$ is a monotone function with, at most, one zero.

Under analogous arguments, we conclude that $T_{n-\ell}u_M$ has at most ℓ zeros, for $\ell = 1, \dots, n$. In particular, u_M can have n zeros at most. But, u_M satisfies (6.7)-(6.8), i.e., $n-1$ possible oscillations are lost. Thus u_M is only allowed to have a simple zero on (a,b), but this is not possible while it is of constant sign.

Let us assume that $k > 1$ and $\sigma_k \neq k-1$. In such a case, we can affirm that u_M is of constant sign up to one of the two following boundary conditions is satisfied:

$$u_M^{(\alpha)}(a) = 0 \quad \text{or} \quad u_M^{(\beta)}(b) = 0\,.$$

Now, to see when the sign change begins, let us study the problem with different signs of M. Since we are considering $u_M \geq 0$, it is obvious that

$$u_M^{(\alpha)}(a) \geq 0\,, \quad \text{and} \quad u_M^{(\beta)}(b) \begin{cases} \geq 0\,, & \text{if } \beta \text{ is even}, \\ \leq 0\,, & \text{if } \beta \text{ is odd}. \end{cases} \tag{6.11}$$

Let us study the behavior of $u_M^{(\alpha)}(a)$ and $u_M^{(\beta)}(b)$, to keep the maximal oscillation, considered as in Notation 5.2, in each case. In this case, the maximum number of zeros which u can have, taking into account the boundary conditions (6.5)-(6.6) is 1. Then, a zero on the boundary is allowed without implying that $u \equiv 0$. If $T_{n-\ell}u_M(a) = 0$ for $\ell \neq n-\alpha$ and $n-\ell \notin \{\sigma_1, \dots, \sigma_{k-1}\}$ or $T_{n-\ell}u_M(b) = 0$ for $\ell \neq n-\beta$ and $n-\ell \notin \{\varepsilon_1, \dots, \varepsilon_{n-k}\}$, then the maximum number of zeros which u can have is 0 and we cannot have more zeros on the boundary for any nontrivial solution of (1.4). Therefore, let us assume that the only zero which is allowed is found at $T_\alpha u_M(a)$ or $T_\beta u_M(b)$.

At first, consider $M < \bar{M}$, we have that $T_n[\bar{M}]u_M \geq 0$, hence, with maximal oscillation, if $T_{n-\ell}u_M(a) \neq 0$ and $T_{n-\ell}u_M(b) \neq 0$ for all $\ell = 1, \dots n$ (5.1) is satisfied.

However each time that $T_{n-\ell}u_M(a) = 0$, the sign change come on the next $\tilde{\ell}$ for which $T_{n-\tilde{\ell}}u_M(a) \neq 0$. And, if $T_{n-\ell}u_M(b) = 0$, it changes its sign on the next $\tilde{\ell}$ for which $T_{n-\tilde{\ell}}u_M(b) \neq 0$ many times as it has vanished. From $\ell = 1$ to $n-\alpha$ there are $k-1-\alpha$ zeros for $T_{n-\ell}u_M(a)$ and from $\ell = 1$ to $n-\beta$ there are $n-k-\beta$ zeros for $T_{n-\ell}u_M(b)$. Hence, to allow the maximal oscillation it is necessary that

$$T_\alpha u_M(a) \begin{cases} \geq 0\,, & \text{if } n-\alpha-(k-\alpha-1) = n-k+1 \text{ is even}, \\ \leq 0\,, & \text{if } n-k+1 \text{ is odd}, \end{cases} \tag{6.12}$$

and

$$T_\beta u_M(b) \begin{cases} \geq 0\,, & \text{if } n-k-\beta \text{ is even}, \\ \leq 0\,, & \text{if } n-k-\beta \text{ is odd}. \end{cases} \tag{6.13}$$

From (6.1), we can affirm that with maximal oscillation

$$u_M^{(\alpha)}(a)\begin{cases}\geq 0\,, & \text{if } n-k \text{ is odd,}\\ \leq 0\,, & \text{if } n-k \text{ is even,}\end{cases}$$

and,

- If $n-k$ is even
$$u_M^{(\beta)}(b)\begin{cases}\geq 0\,, & \text{if } \beta \text{ is even,}\\ \leq 0\,, & \text{if } \beta \text{ is odd.}\end{cases}$$
- If $n-k$ is odd
$$u_M^{(\beta)}(b)\begin{cases}\leq 0\,, & \text{if } \beta \text{ is even,}\\ \geq 0\,, & \text{if } \beta \text{ is odd.}\end{cases}$$

Hence, we arrive at the following conclusions, taking into account (6.11):

- If $n-k$ is even, the maximal oscillation is not allowed for u_M if $u_N^{(\alpha)}(a)\neq 0$ for all N between $\bar M$ and M; which implies that $u_M>0$ on (a,b) for $M\in[\bar M-\lambda_3',\bar M]$.
- If $n-k$ is odd, the maximal oscillation is not allowed for u_M if $u_N^{(\beta)}(b)\neq 0$ for all N between $\bar M$ and M; which implies that $u_M>0$ on (a,b) for $M\in[\bar M-\lambda_2',\bar M]$.

Now, considering $M>\bar M$, we have that $T_n[\bar M]u_M\leq 0$, hence with maximal oscillation, if $T_{n-\ell}u_M(a)\neq 0$ and $T_{n-\ell}u_M(b)\neq 0$, for all $\ell=1,\dots n$, the following inequalities are satisfied:

$$T_{n-\ell}u_M(a)\begin{cases}<0\,, & \text{if } \ell \text{ is even,}\\ >0\,, & \text{if } \ell \text{ is odd,}\end{cases}\qquad T_{n-\ell}u_M(b)<0\,. \tag{6.14}$$

In this case, since we have contrary signs from the previous case where $M<\bar M$, to allow the maximal oscillation, the following inequalities must be satisfied:

$$T_\alpha u_M(a)\begin{cases}\leq 0\,, & \text{if } n-k-1 \text{ is even,}\\ \geq 0\,, & \text{if } n-k-1 \text{ is odd,}\end{cases} \tag{6.15}$$

and

$$T_\beta u_M(b)\begin{cases}\leq 0\,, & \text{if } n-k-\beta \text{ is even,}\\ \geq 0\,, & \text{if } n-k-\beta \text{ is odd.}\end{cases} \tag{6.16}$$

Hence, from (6.1), we can affirm that with maximal oscillation

$$u_M^{(\alpha)}(a)\begin{cases}\leq 0\,, & \text{if } n-k \text{ is odd,}\\ \geq 0\,, & \text{if } n-k \text{ is even,}\end{cases}$$

and,

- If $n-k$ is even
$$u_M^{(\beta)}(b)\begin{cases}\leq 0\,, & \text{if } \beta \text{ is even,}\\ \geq 0\,, & \text{if } \beta \text{ is odd.}\end{cases}$$
- If $n-k$ is odd
$$u_M^{(\beta)}(b)\begin{cases}\geq 0\,, & \text{if } \beta \text{ is even,}\\ \leq 0\,, & \text{if } \beta \text{ is odd.}\end{cases}$$

Hence, we arrive at the following conclusions, taking into account (6.11):

- If $n-k$ is even, the maximal oscillation is not allowed for u_M if $u_N^{(\beta)}(b) \neq 0$ for all N between $\bar{M}$ and M; which implies that $u_M > 0$ on (a,b) for $M \in [\bar{M}, \bar{M} - \lambda_2']$.
- If $n-k$ is odd, the maximal oscillation is not allowed for u_M if $u_N^{(\alpha)}(a) \neq 0$ for all N between $\bar{M}$ and M; which implies that $u_M > 0$ on (a,b) for $M \in [\bar{M}, \bar{M} - \lambda_3']$.

The proof is complete since if $k = 1$, $u_M^{(\beta)}(b) \neq 0$ for every $M \neq \bar{M}$, because the contrary will imply that u_M is a nontrivial solution of the linear differential equation (1.4) with a zero of multiplicity n at $t = b$ and this is not possible.

And, if $\sigma_k = k-1$, consider $u_M^{(k-1)}(a)$ instead of $u_M^{(\alpha)}(a) = u^{(k)}(a)$, since it is the first non null derivative at $t = a$. Since $u_M \geq 0$, then $u_M^{(k-1)}(a) \geq 0$. But, with maximal oscillation, $T_{k-1} u_M(a)$ follows (5.1) if $M < \bar{M}$ and (6.14) if $M > \bar{M}$ for $\ell = n-k-1$. Hence, from (6.1), we can affirm that, with maximal oscillation, the following inequalities must be fulfilled:

- If $M < \bar{M}$

$$u_M^{(k-1)}(a) \begin{cases} \geq 0\,, & \text{if } n-k \text{ is odd,} \\ \leq 0\,, & \text{if } n-k \text{ is even.} \end{cases}$$

- If $M > \bar{M}$

$$u_M^{(k-1)}(a) \begin{cases} \leq 0\,, & \text{if } n-k \text{ is odd,} \\ \geq 0\,, & \text{if } n-k \text{ is even.} \end{cases}$$

Then, we can conclude the proof:

- If $n-k$ is even and $M < \bar{M}$, the maximal oscillation is not allowed for u_M if $u_N^{(k-1)}(a) \neq 0$ for all N between $\bar{M}$ and M; which implies that $u_M > 0$ on (a,b) for $M \in [\bar{M} - \lambda_1, \bar{M}]$.
- If $n-k$ is odd and $M > \bar{M}$, the maximal oscillation is not allowed for u_M if $u_N^{(n-k-1)}(b) \neq 0$ for all N between $\bar{M}$ and M; which implies that $u_M > 0$ on (a,b) for $M \in [\bar{M}, \bar{M} - \lambda_1]$.

□

Example 6.8. From Proposition 6.7 and Example 6.6, we can affirm that any nontrivial solution of $T_4^0[M] \equiv u^{(4)}(t) + M u(t) = 0$ on $[0,1]$, verifying the boundary conditions:

$$u(0) = u'(1) = u''(1) = 0\,,$$

does not have any zero on $(0,1)$ for $M \in [-\pi^4, m_2^4]$, where $m_2^4 = -\lambda_1$ with λ_1 the first negative eigenvalue of $T_4^0[0]$ on $X_{\{0\}}^{\{0,1,2\}}$ and m_2 has been introduced in Example 6.6 as the least positive solution of (6.3).

Such functions are given as multiples of the following expression:

$$\begin{cases} \cos(m-mt)(\sin(m)-\sinh(m))+\sin(m-mt)(-\cos(m)-\cosh(m)) \\ +\sinh(m-mt)(\cos(m)+\cosh(m))+\cosh(m-mt)(\sin(m)-\sinh(m))\,, \\ \quad \text{if } M=-m^4<0\,, \\ t^3-3t^2+3t\,, \quad \text{if } M=0\,, \\ e^{-\frac{mt}{\sqrt{2}}}\Big(-\big(e^{\sqrt{2}m(t-1)}+e^{\sqrt{2}mt}+e^{\sqrt{2}m}+1\big)\sin\big(\frac{mt}{\sqrt{2}}\big) \\ +\big(e^{\sqrt{2}mt}-1\big)\cos\big(\frac{m(t-2)}{\sqrt{2}}\big)+\big(e^{\sqrt{2}mt}-1\big)\cos\big(\frac{mt}{\sqrt{2}}\big)\Big)\,, \\ \quad \text{if } M=m^4>0\,. \end{cases}$$

Now, we enunciate a similar result, which refers to the eigenvalues on the sets $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}$ and $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}}$.

Proposition 6.9. *Let $\bar{M}\in\mathbb{R}$ be such that $T_n[\bar{M}]$ satisfies property (T_d) on the set $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ and $\{\sigma_1,\dots,\sigma_k\}-\{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ satisfy (N_a). If $u\in C^n(I)$ is a solution of (1.4) on (a,b) satisfying the boundary conditions:*

$$u^{(\sigma_1)}(a)=\dots=u^{(\sigma_k)}(a)=0\,, \tag{6.17}$$

$$u^{(\varepsilon_1)}(b)=\dots=u^{(\varepsilon_{n-k-1})}(b)=0\,, \tag{6.18}$$

then it does not have any zero on (a,b) provided that one of the following assertions is satisfied:

- *Let $n-k$ be even:*
  - *If $\varepsilon_{n-k}\neq n-k-1$ and $M\in[\bar{M}-\lambda_3'',\bar{M}-\lambda_2'']$, where:*
    - *$\lambda_3''>0$ is the least positive eigenvalue of $T_n[\bar{M}]$ in the set $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}}$.*
    - *$\lambda_2''<0$ is the largest negative eigenvalue of $T_n[\bar{M}]$ on the set $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}$.*
  - *If $\varepsilon_{n-k}=n-k-1$ and $M\in[\bar{M}-\lambda_1,\bar{M}-\lambda_2'']$, where:*
    - *$\lambda_1>0$ is the least positive eigenvalue of $T_n[\bar{M}]$ on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{0,\dots,n-k-1\}}$.*
    - *$\lambda_2''<0$ is the largest negative eigenvalue of $T_n[\bar{M}]$ on the set $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{0,\dots,k-2\}}$.*
- *Let $n-k$ be odd:*
  - *If $k<n-1$, $\varepsilon_{n-k}\neq n-k-1$ and $M\in[\bar{M}-\lambda_2'',\bar{M}-\lambda_3'']$, where:*
    - *$\lambda_3''<0$ is the largest negative eigenvalue of $T_n[\bar{M}]$ on the set $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}}$.*
    - *$\lambda_2''>0$ is the least positive eigenvalue of $T_n[\bar{M}]$ in the set $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}$.*
  - *If $k=n-1$, $\varepsilon_1\neq 0$ and $M\in(-\infty,\bar{M}-\lambda_3'']$, where:*
    - *$\lambda_3''<0$ is the largest negative eigenvalue of $T_n[\bar{M}]$ on the set $X_{\{\sigma_1,\dots,\sigma_{n-1}\}}^{\{\beta\}}$, where $\beta=0$.*
  - *If $k<n-1$, $\varepsilon_{n-k}=n-k-1$ and $M\in[\bar{M}-\lambda_2'',\bar{M}-\lambda_1]$, where:*
    - *$\lambda_1<0$ is the largest negative eigenvalue of $T_n[\bar{M}]$ on the set $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{0,1,\dots,n-k-1\}}$.*

  * $\lambda_2'' > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *on the set* $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}$.
- *If* $k = n-1$ *and* $\varepsilon_{n-k} = 0$ *and* $M \in (-\infty, \bar{M} - \lambda_1]$, *where:*
  * $\lambda_1 < 0$ *is the largest negative eigenvalue of* $T_n[\bar{M}]$ *on the set* $X_{\{\sigma_1,\dots,\sigma_{n-1}\}}^{\{0\}}$.

The proof of the above proposition is analogous to the one of Proposition 6.7, and is omitted here.

Example 6.10. Consider the fourth order differential equation $u^{(4)}(t)+Mu(t)=0$ coupled with the boundary conditions $u(0)=u''(0)=u'(1)=0$. Using Proposition 6.9 and Example 6.6, we conclude that such functions do not have any zero on $(0,1)$ if $M \in [-m_3^4, 4\pi^4]$, where $m_3^3 = -\lambda_1$, with λ_1 the first negative eigenvalue of $T_4^0[0]$ on $X_{0,1,2}^1$ and m_3 has been introduced in Example 6.6 as the least positive solution of (6.4).

It is not difficult to verify that the solutions of this problem are given as multiples of the following expression:

$$\begin{cases} \frac{\sin(m\,t)}{\cos(m)} - \frac{\sinh(m\,t)}{\cosh(m)}\,, \quad \text{if } M = -m^4 < 0\,, \\ t^3 - 3t\,, \quad \text{if } M = 0\,, \\ e^{-\frac{mt}{\sqrt{2}}}\Big(\big(e^{\sqrt{2}m(t+1)}+1\big)\sin\big(\frac{m(t-1)}{\sqrt{2}}\big) + \big(e^{\sqrt{2}mt}+e^{\sqrt{2}m}\big)\sin\big(\frac{m(t+1)}{\sqrt{2}}\big) \\ +\big(1-e^{\sqrt{2}m(t+1)}\big)\cos\big(\frac{m(t-1)}{\sqrt{2}}\big) + \big(e^{\sqrt{2}m}-e^{\sqrt{2}mt}\big)\cos\big(\frac{m(t+1)}{\sqrt{2}}\big)\Big)\,, \\ \quad \text{if } M = m^4 > 0\,. \end{cases}$$

To complete this section, we show a result which gives an order on the previously obtained eigenvalues λ_1, λ_3' and λ_3''. First, let us introduce some notation.

Notation 6.11. Let us denote $\alpha_1 \in \{1,\dots,n-1\}$ such that $\alpha_1 \notin \{\sigma_1,\dots,\sigma_{k-1}|\alpha\}$ and $\{0,\dots,\alpha_1-1\} \subset \{\sigma_1,\dots,\sigma_{k-1}|\alpha\}$. Let $\beta_1 \in \{1,\dots,n-1\}$ be such that $\beta_1 \notin \{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}$ and $\{0,\dots,\beta_1-1\} \subset \{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}$.

Proposition 6.12. *Let* $\bar{M} \in \mathbb{R}$ *be such that* $T_n[\bar{M}]$ *satisfies property* (T_d) *on* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ *and* $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ *satisfy* (N_a). *Then the following assertions are fulfilled:*

- *Let* $n-k$ *be even, we have:*
  - *If* $\sigma_k \neq k-1$, *then* $\lambda_3' > \lambda_1 > 0$, *where*
    * $\lambda_3' > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *on* $X_{\{\sigma_1,\dots,\sigma_{k-1}|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.
    * $\lambda_1 > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *on* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.

    Moreover if there exists $\lambda_1' > \lambda_1$ *another eigenvalue of* $T_n[\bar{M}]$ *on* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, *then* $\lambda_1' > \lambda_3'$.
  - *If* $\varepsilon_{n-k} \neq n-k-1$, *then* $\lambda_3'' > \lambda_1 > 0$, *where*
    * $\lambda_3'' > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *in the set* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}}$.
    * $\lambda_1 > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *on* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.

    Moreover if there exists $\lambda_1' > \lambda_1$ *another eigenvalue of of* $T_n[\bar{M}]$ *on* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, *then* $\lambda_1' > \lambda_3''$.

- *Let $n-k$ be odd, we have:*
  - *If $\sigma_k \neq k-1$, then $\lambda_3' < \lambda_1 < 0$, where*
    - *$\lambda_3' < 0$ is the largest negative eigenvalue of $T_n[\bar{M}]$ in the set $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_{k-1}|\alpha\}}$.*
    - *$\lambda_1 < 0$ is the largest negative eigenvalue of $T_n[\bar{M}]$ on the set $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$.*

    Moreover if there exists $\lambda_1' < \lambda_1$ another eigenvalue of $T_n[\bar{M}]$ on $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$, then $\lambda_1' < \lambda_3'$.
  - *If $\varepsilon_{n-k} \neq n-k-1$, then $\lambda_3'' < \lambda_1 < 0$, where*
    - *$\lambda_3'' < 0$ is the largest negative eigenvalue of $T_n[\bar{M}]$ on the set $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}}_{\{\sigma_1,\dots,\sigma_k\}}$.*
    - *$\lambda_1 < 0$ is the largest negative eigenvalue of $T_n[\bar{M}]$ on the set $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$.*

    Moreover if there exists $\lambda_1' < \lambda_1$ another eigenvalue of $T_n[\bar{M}]$ on $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$, then $\lambda_1' < \lambda_3''$.

Proof. At the beginning, we focus on the relation between λ_1 and λ_3'. We have seen in Proposition 6.7 that a function u_M, solution of (1.4), satisfying the boundary conditions (6.5)-(6.6) cannot have any zero on (a,b) for $M \in [\bar{M}-\lambda_3', \bar{M}]$ if $n-k$ is even and for $M \in [\bar{M}, \bar{M}-\lambda_3']$ if $n-k$ is odd.

Moreover, it is proved that for $M = \bar{M}$, without taking into account the boundary conditions, $u_{\bar{M}}$ has at most $n-1$ zeros, moreover, conditions (6.7)-(6.8) are satisfied by $u_{\bar{M}}$. Hence, we lose the $n-1$ possible oscillations. So, for $M = \bar{M}$ with the given boundary conditions, the maximal oscillation is achieved for the boundary conditions (6.5)-(6.6).

Let us assume that $u_{\bar{M}} \geq 0$ (if $u_{\bar{M}} \leq 0$ the arguments are valid by multiplying by -1), hence $T_\alpha u_{\bar{M}}(a) = \frac{u^{(\alpha)}_{\bar{M}}(a)}{v_1(a)\dots v_\alpha(a)} > 0$.

As we have said before, $T_h u(a)$ changes its sign for every $h = 0,\dots,n-1$ if it is non null. From $h=\alpha$ to σ_k, taking into account (6.7), $k-1-\alpha$ zeros for $T_h u(a)$ are found. Hence, with maximal oscillation:

$$T_{\sigma_k} u_{\bar{M}}(a) \begin{cases} >0\,, & \text{if } (\sigma_k-\alpha)-(k-1-\alpha) = \sigma_k-k+1 \text{ is even},\\ <0\,, & \text{if } \sigma_k-k+1 \text{ is odd}. \end{cases}$$

On the other hand, by means of Lemma 3.6, we have

$$u^{(\sigma_k)}_{\bar{M}}(a) \begin{cases} >0\,, & \text{if } \sigma_k-k \text{ is odd},\\ <0\,, & \text{if } \sigma_k-k \text{ is even}. \end{cases} \tag{6.19}$$

Let us move u_M continuously on M up to $M = \bar{M}-\lambda_3'$. On Proposition 6.7 we have proved that u_M has at most n zeros for every $M \in [\bar{M}-\lambda_3', \bar{M}]$ ($[\bar{M}, \bar{M}-\lambda_3]$ if $n-k$ is odd) if $u_M \geq 0$, without taking into account the boundary conditions.

Since λ_3' is an eigenvalue of $T_n[\bar{M}]$ on $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_{k-1}|\alpha\}}$, we have that $u^{(\alpha)}_{\bar{M}-\lambda_3'}(a) = 0$. Thus, $T_\alpha u_{\bar{M}-\lambda_3'}(a) = 0$. This fact, coupled with the boundary conditions (6.7)-(6.8), allows us to affirm that $u_{\bar{M}-\lambda_3'}$ cannot lose more oscillations if it is a nontrivial solution. Hence, the maximal oscillation is verified.

Since we have moved continuously from $\bar{M}$ to $\bar{M}-\lambda_3'$ and it was assumed $u_{\bar{M}} \geq 0$ on I, we conclude that $u_{\bar{M}-\lambda_3'} \geq 0$, hence

$$T_{\alpha_1} u_{\bar{M}}(a) = \frac{u_{\bar{M}}^{(\alpha_1)}(a)}{v_1(a)\dots v_{\alpha_1}(a)} > 0,$$

where α_1 has been introduced in Notation 6.11.

As for $M = \bar{M}$, provided it is non null, $T_h u(a)$ changes its sign for every $h = 0, \dots, n-1$. From $h = \alpha_1$ to σ_k, taking into account (6.7), $k-\alpha_1$ zeros are found. Hence, with maximal oscillation

$$T_{\sigma_k} u_{\bar{M}-\lambda_3'}(a) \begin{cases} > 0\,, & \text{if } (\sigma_k-\alpha_1)-(k-\alpha_1) = \sigma_k - k \text{ is even,} \\ < 0\,, & \text{if } \sigma_k - k \text{ is odd.} \end{cases}$$

From Lemma 3.6 again, we have

$$u_{\bar{M}-\lambda_3'}^{(\sigma_k)}(a) \begin{cases} > 0\,, & \text{if } \sigma_k - k \text{ is even,} \\ < 0\,, & \text{if } \sigma_k - k \text{ is odd.} \end{cases} \tag{6.20}$$

Hence, since we have been moving with continuity, from (6.19) and (6.20), we can ensure the existence of a $\tilde{M}$ between $\bar{M}$ and $\bar{M}-\lambda_3'$ such that $u_{\tilde{M}}^{(\sigma_k)}(a) = 0$. As consequence:

- If $n-k$ is even, $0 < \lambda_1 = \bar{M} - \tilde{M} < \lambda_3'$.
- If $n-k$ is odd, $0 > \lambda_1 = \bar{M} - \tilde{M} > \lambda_3'$.

The relation between λ_1 and λ_3'' is proved analogously by using Proposition 6.9.

The assertion referring to λ_1' is due to the fact that, if $0 < \lambda_1 < \lambda_1' < \lambda_3'$ on the case where $n-k$ is even, then, by Proposition 6.7, the eigenfunctions related to λ_1 and λ_1' are of constant sign and this is not possible for an strongly inverse positive (negative) operator (see [25, Corollary 7.27] and [2, Section 1.8]). The same happens when $n-k$ is odd and $0 > \lambda_1 > \lambda_1' > \lambda_3$.

Similarly, if either $n-k$ is even and $0 < \lambda_1 < \lambda_1' < \lambda_3''$ or $n-k$ is odd and $0 > \lambda_1 > \lambda_1' > \lambda_3''$, then, by Proposition 6.9, the eigenfunctions related to λ_1 and λ_1' are of constant sign. Thus, the result is proved. □

Example 6.13. Let us return to Example 6.6, where we have obtained the different eigenvalues for the operator $T_4^0[0]$. Let us see that the Assumptions of Proposition 6.12 are fulfilled.

- $\lambda_1 = m_1^4 \approxeq 2.36502^4 < \lambda_3' = \pi^4$.
- $\lambda_1 < \lambda_3'' = m_3^4 \approxeq 3.9266^4$.

Moreover, we have seen in Example 6.6 that the eigenvalues of $T_4^0[0]$ on $X_{\{0,2\}}^{\{1,2\}}$ are given as $\lambda = m^4$, where m is a positive solution of (6.2). So $\lambda_1' \approxeq 5.497^4 > \lambda_3'$ and $\lambda_1' > \lambda_3''$.

7. Study of the eigenvalues of the adjoint operator $T_n^*[\bar{M}]$ and of $\widehat{T}_n[(-1)^n \bar{M}]$ in different spaces

This section is devoted to the study of the eigenvalues of the adjoint operator $T_n^*[\bar{M}]$ in the different spaces $X_{\{\sigma_1,\dots,\sigma_k\}}^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{*\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}$, $X_{\{\sigma_1,\dots,\sigma_{k-1}\}}^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$, $X_{\{\sigma_1,\dots,\sigma_{k-1}|\alpha\}}^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ and $X_{\{\sigma_1,\dots,\sigma_k\}}^{*\{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}}$.

In Section 4 we have proved that the boundary conditions satisfied for every $v \in X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$ are given by (4.1)–(4.6). Proceeding analogously in the different spaces, taking into account that $\eta = n-1-\sigma_k$, $\gamma = n-1-\varepsilon_{n-k}$, $\alpha = n-1-\tau_{n-k}$ and $\beta = n-1-\delta_k$, we have the following assertions:

- If $v \in X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}_{\{\sigma_1,\dots,\sigma_{k-1}\}}$, then it satisfies (4.1)–(4.3) and (4.4)–(4.5) coupled with $v^{(\eta)}(a) = 0$.
- If $v \in X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_{k-1}|\alpha\}}$, then it satisfies (4.1)–(4.2) and (4.4)–(4.6) coupled with $v^{(\eta)}(a) = 0$.
- If $v \in X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}_{\{\sigma_1,\dots,\sigma_k|\alpha\}}$, then it satisfies (4.1)–(4.2) and (4.4)–(4.6) coupled with $v^{(\gamma)}(b) = 0$.
- If $v \in X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}}_{\{\sigma_1,\dots,\sigma_k\}}$, then it satisfies (4.1)–(4.3) and (4.4)–(4.5) coupled with $v^{(\gamma)}(b) = 0$.

Example 7.1. Arguing in an analogous way to Example 4.3, we obtain

$$\begin{aligned}
X^{*\{1\}}_{\{0,1,2\}} &= \left\{v \in C^4(I) : v(a) = v(b) = v'(b) = v^{(3)}(b) - p_1(b)v''(b) = 0\right\},\\
X^{*\{0,1,2\}}_{\{0\}} &= \left\{v \in C^4(I) : v(a) = v'(a) = v''(a) = v(b) = 0\right\} = X^{\{0\}}_{\{0,1,2\}},\\
X^{*\{0,1\}}_{\{0,2\}} &= \left\{v \in C^4(I) : v(a) = v''(a) - p_1(a)v'(a) = v(b) = v'(b) = 0\right\},\\
X^{*\{1,2\}}_{\{0,1\}} &= \big\{v \in C^4(I) : v^{(3)}(b) - p_1(b)v''(b) + (p_2(b) - 2p_1'(b))v'(b) = 0,\\
&\qquad v(a) = v'(a) = v(b) = 0\big\}.
\end{aligned}$$

Next, we prove analogous results to those of the previous section referring to functions defined in these spaces.

Remark 7.2. In this case, taking into account that the eigenvalues of one operator and those of its adjoint are the same, we do not need to prove the existence of the eigenvalues. Such existence follows from the one of the eigenvalues of $T_n[\bar M]$ in the correspondent spaces,

First, we prove two results which refer to the operator $T_n^*[M]$ and then we will be able to extrapolate them for $\widehat{T}_n[(-1)^n M]$.

Proposition 7.3. *Let $\bar M \in \mathbb{R}$ be such that $T_n[\bar M]$ satisfies property (T_d) on the set $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$ and $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ satisfy (N_a). Then every solution of $T_n^*[M]v(t) = 0$, for $t \in (a,b)$, satisfying the boundary conditions* (4.1)–(4.3) *and* (4.4)–(4.5) *does not have any zero on (a,b) provided that one of the following assertions is fulfilled:*

- *Let $n-k$ be even:*
  - *If $k > 1$, $\varepsilon_{n-k} \neq n-k-1$ and $M \in [\bar M - \lambda_3'', \bar M - \lambda_2']$, where:*
    - *$\lambda_3'' > 0$ is the least positive eigenvalue of $T_n[\bar M]$ on the set $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}}_{\{\sigma_1,\dots,\sigma_k\}}$.*
    - *$\lambda_2' < 0$ is the largest negative eigenvalue of $T_n[\bar M]$ on the set $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}_{\{\sigma_1,\dots,\sigma_{k-1}\}}$.*
  - *If $k = 1$, $\varepsilon_{n-1} \neq n-2$ and $M \in [\bar M - \lambda_3'', +\infty)$, where:*
    - *$\lambda_3'' > 0$ is the least positive eigenvalue of $T_n[\bar M]$ on $X^{\{\varepsilon_1,\dots,\varepsilon_{n-2}|\beta\}}_{\{\sigma_1\}}$.*

  - *If* $k > 1$, $\varepsilon_{n-k} = n-k-1$ *and* $M \in [\bar{M} - \lambda_1, \bar{M} - \lambda_2']$, *where:*
    * $\lambda_1 > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *on* $X^{\{0,\dots,n-k-1\}}_{\{\sigma_1,\dots,\sigma_k\}}$.
    * $\lambda_2' < 0$ *is the largest negative eigenvalue of* $T_n[\bar{M}]$ *on the set* $X^{\{0,\dots,n-k-1,n-k\}}_{\{\sigma_1,\dots,\sigma_{k-1}\}}$.
  - *If* $k = 1$, $\varepsilon_{n-1} = n-2$ *and* $M \in [\bar{M} - \lambda_1, +\infty)$, *where:*
    * $\lambda_1 > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *on* $X^{\{0,\dots,n-2\}}_{\{\sigma_1\}}$.
- *Let* $n-k$ *be odd:*
  - *If* $1 < k < n-1$, $\varepsilon_{n-k} \neq n-k-1$ *and* $M \in [\bar{M} - \lambda_2', \bar{M} - \lambda_3'']$, *where:*
    * $\lambda_3'' < 0$ *is the largest negative eigenvalue of* $T_n[\bar{M}]$ *on the set* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}}_{\{\sigma_1,\dots,\sigma_k\}}$.
    * $\lambda_2' > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *on* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}_{\{\sigma_1,\dots,\sigma_{k-1}\}}$.
  - *If* $1 < k = n-1$, $\varepsilon_1 \neq 0$ *and* $M \in [\bar{M} - \lambda_2', \bar{M} - \lambda_3'']$, *where:*
    * $\lambda_3'' < 0$ *is the least largest negative eigenvalue of* $T_n[\bar{M}]$ *on* $X^{\{\beta\}}_{\{\sigma_1,\dots,\sigma_{n-1}\}}$, *where* $\beta = 0$.
    * $\lambda_2' > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *on* $X^{\{\varepsilon_1|\beta\}}_{\{\sigma_1,\dots,\sigma_{n-2}\}}$.
  - *If* $k = 1 < n-1$, $\varepsilon_{n-1} \neq n-2$ *and* $M \in (-\infty, \bar{M} - \lambda_3'']$, *where:*
    * $\lambda_3'' < 0$ *is the largest negative eigenvalue of* $T_n[\bar{M}]$ *on the set* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-2}|\beta\}}_{\{\sigma_1\}}$.
  - *If* $k = 1$, $n = 2$, $\varepsilon_1 \neq 0$ *and* $M \in (-\infty, \bar{M} - \lambda_3'']$, *where:*
    * $\lambda_3'' < 0$ *is the largest negative eigenvalue of* $T_n[\bar{M}]$ *on* $X^{\{\beta\}}_{\{\sigma_1\}} = X^{\{0\}}_{\{0\}}$.
  - *If* $1 < k$, $\varepsilon_{n-k} = n-k-1$ *and* $M \in [\bar{M} - \lambda_2', \bar{M} - \lambda_1]$, *where:*
    * $\lambda_1 < 0$ *is the largest negative eigenvalue of* $T_n[\bar{M}]$ *on the set* $X^{\{0,\dots,n-k-1\}}_{\{\sigma_1,\dots,\sigma_k\}}$.
    * $\lambda_2' > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *on the set* $X^{\{0,\dots,n-k-1,n-k\}}_{\{\sigma_1,\dots,\sigma_{k-1}\}}$.
  - *If* $k = 1$, $\varepsilon_{n-1} = n-2$ *and* $M \in (-\infty, \bar{M} - \lambda_1]$, *where:*
    * $\lambda_1 < 0$ *is the largest negative eigenvalue of* $T_n[\bar{M}]$ *on* $X^{\{0,\dots,n-2\}}_{\{\sigma_1\}}$.

Proof. The proof of the above proposition follows the same steps as Proposition 6.7. Since working with the adjoint operator is slightly different, for convenience of the reader, we describe the steps below.

Let us denote by $v_M \in C^n(I)$ a solution of

$$T_n^*[M]v(t) = 0\,, \quad t \in (a,b)\,, \tag{7.1}$$

satisfying the boundary conditions (4.1)–(4.3) and (4.4)–(4.5). At the beginning, let us see that $v_{\bar{M}}$ does not have any zero on (a,b). In order to see that, we consider the decomposition (4.11) whose existence is guaranteed by Lemma 4.5.

Analogously to the proof of Lemma 3.8, since $w_0, \dots, w_n > 0$, we can conclude that, without taking into account the boundary conditions, a solution of (7.1) for $M = \bar{M}$ can have at most $n-1$ zeros. However, as we have said before, each time that either $T_\ell^* v_M(a) = 0$ or $T_\ell^* v_M(b) = 0$, a possible oscillation is lost. From the boundary conditions and Lemmas 4.8 and 4.9, taking into account that

$T^*_\ell v_M(t)=(-1)^\ell \widehat{T}_\ell v_M(t)$, we can affirm that for every $M\in\mathbb{R}$:

$$T^*_{\tau_1} v_M(a)=\dots=T^*_{\tau_{n-k}} v_M(a)=0\,, \tag{7.2}$$

$$T^*_{\delta_1} v_M(b)=\dots=T^*_{\delta_{k-1}} v_M(b)=0\,. \tag{7.3}$$

Thus, every nontrivial solution of (7.1) for $M=\bar{M}$ does not have zeros on (a,b).

Now, let us move v_M continuously as a function of M on a neighborhood of $M=\bar{M}$. We have that v_M is a solution of (7.1), hence:

$$T^*_n[\bar{M}]v_M(t)=(\bar{M}-M)v_M\,,\quad t\in(a,b)\,. \tag{7.4}$$

Analogously to the proof of Proposition 6.7, we will see that, while v_M is of constant sign, it cannot have any double zero on (a,b).

We can assume that $v_{\bar{M}}>0$ on I (if $v_{\bar{M}}<0$, then the arguments are valid by multiplying by -1). So, in equation (7.4) we have:

$$T^*_n[\bar{M}]v_M(t)\begin{cases}\geq 0\,,\ t\in I\,, & \text{if } M<\bar{M},\\ \leq 0\,,\ t\in I\,, & \text{if } M>\bar{M}.\end{cases} \tag{7.5}$$

In both cases, since $\frac{-1}{w_n}<0$, $T^*_{n-1}v_M$ is a monotone function, with at most one zero. Studying the maximal oscillation of $T^*_{n-\ell}v_M$ for $\ell=2,\dots,n$, we conclude that $T^*_{n-\ell}v_M$ has at most ℓ zeros.

In particular, $T^*_0 v_M$ has no more than n zeros. Since $w_0>0$, we can affirm that v_M has at most n zeros. However, v_M satisfies (7.2)-(7.3), hence $n-1$ possible oscillation are lost. Thus, v_M can have at most a simple zero on (a,b) which is not possible if it is of constant sign.

Let us assume that $k\neq 1$ and that $\delta_k\neq k-1$ (this is equivalent to $\varepsilon_{n-k}\neq n-k-1$). Under these assumptions, we can affirm that v_M is of constant sign up to one of the following boundary conditions is fulfilled:

$$v_M^{(\eta)}(a)=0\quad\text{or}\quad v_M^{(\gamma)}(b)=0\,.$$

Let us study what happens by moving M. Since we are considering $v_M\geq 0$, we have

$$v_M^{(\eta)}(a)\geq 0\,,\quad\text{and}\quad v_M^{(\gamma)}(b)\begin{cases}\geq 0\,, & \text{if } \gamma \text{ is even},\\ \leq 0\,, & \text{if } \gamma \text{ is odd}.\end{cases} \tag{7.6}$$

Now, let us see how $v_M^{(\eta)}(a)$ and $v_M^{(\gamma)}(b)$ are with maximal oscillation. As before, with maximal oscillation only one zero on the boundary is allowed. If $T^*_\ell v_M(a)=0$ for $\ell\notin\{\tau_1,\dots,\tau_{n-k},\eta\}$ or $T^*_\ell v_M(b)=0$ for $\ell\notin\{\delta_1,\dots,\delta_{k-1},\gamma\}$, we have that $T^*_\eta v_M(a)\neq 0$ and $T^*_\gamma v_M(b)\neq 0$. Because, otherwise, $v_M\equiv 0$ on I and we are looking for nontrivial solutions.

From (4.24), taking into account that $T^*_\ell v_M(t)=(-1)^\ell\widehat{T}_\ell v_M(t)$, we obtain

$$\begin{aligned}T^*_\eta v_M(a)&=(-1)^\eta v_1(a)\,\dots v_{n-\eta}(a)v^{(\eta)}(a)\,,\\ T^*_\gamma v_M(b)&=(-1)^\gamma v_1(b)\,\dots v_{n-\gamma}(b)v^{(\gamma)}(b)\,,\end{aligned} \tag{7.7}$$

where $v_1\,\dots,v_n>0$ are given in (3.1). Hence, if $T^*_\eta v_M(a)\neq 0$ and $T^*_\gamma v_M(b)\neq 0$, then $v_M^{(\eta)}(a)\neq 0$ and $v_M^{(\gamma)}(b)\neq 0$, thus the function v_M remains of constant sign. Thus, we can assume that the unique zero, which is allowed with maximal oscillation, is found either in $T^*_\eta v_M(a)$ or $T^*_\gamma v_M(b)$.

In this case, since $T^*_k v_M=\frac{-1}{w_k}\frac{d}{dt}\left(T^*_{k-1}v_M\right)$ with $w_k>0$, to allow the maximal oscillation, $T^*_{n-\ell}v_M(a)$ remains of constant sign, each time that it does not vanish and, if it vanishes, then it changes its sign the number of times that it has vanished on the next ℓ where it is non null. And $T^*_{n-\ell}v_M(b)$ changes its sign each time that it is non null.

At first, let us focus on the case $M < \bar{M}$, we have that $T^*_n[\bar{M}]v_M = T^*_n v_M \geq 0$ on I. In particular, $T^*_n v_M(a) \geq 0$ and $T^*_n v_M(b) \geq 0$. Using (7.2)-(7.3), from $\ell = 0$ to $n-\eta$, we have that $T_{n-\ell}v_M(a)$ vanishes $n-k-\eta$ times and from $\ell = 0$ to $n-\gamma$, $T_{n-\ell}v_M(b) = 0$ $k-1-\gamma$ times. Hence, to allow the maximal oscillation:

$$T^*_\eta v_M(a) \begin{cases} \geq 0\,, & \text{if } n-k-\eta \text{ is even,} \\ \leq 0\,, & \text{if } n-k-\eta \text{ is odd,} \end{cases} \tag{7.8}$$

and

$$T^*_\gamma v_M(b) \begin{cases} \geq 0\,, & \text{if } n-\gamma-(k-1-\gamma) = n-k+1 \text{ is even,} \\ \leq 0\,, & \text{if } n-k+1 \text{ is odd.} \end{cases} \tag{7.9}$$

Using (7.7) and (7.8)-(7.9), we can affirm that to set maximal oscillation:

$$v_M^{(\eta)}(a) \begin{cases} \geq 0\,, & \text{if } n-k \text{ is even,} \\ \leq 0\,, & \text{if } n-k \text{ is odd,} \end{cases} \tag{7.10}$$

and

- If $n-k$ is even
$$v_M^{(\gamma)}(b) \begin{cases} \geq 0\,, & \text{if } \gamma \text{ is odd,} \\ \leq 0\,, & \text{if } \gamma \text{ is even.} \end{cases} \tag{7.11}$$
- If $n-k$ is odd
$$v_M^{(\gamma)}(b) \begin{cases} \leq 0\,, & \text{if } \gamma \text{ is odd,} \\ \geq 0\,, & \text{if } \gamma \text{ is even.} \end{cases} \tag{7.12}$$

Hence, taking into account (7.6), we arrive at the following conclusions:

- If $n-k$ is even, the maximal oscillation is not allowed for v_M if $v_N^{(\gamma)}(b) \neq 0$ for all N between $\bar{M}$ and M; which implies that $v_M > 0$ on (a,b) for $M \in [\bar{M} - \lambda_3^{*\prime\prime}, \bar{M}]$, where $\lambda_3^{*\prime\prime} > 0$ is the least positive eigenvalue of $T^*_n[\bar{M}]$ on $X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\bar{\beta}\}}_{\{\sigma_1,\dots,\sigma_k\}}$.
- If $n-k$ is odd, the maximal oscillation is not allowed for v_M if $v_N^{(\eta)}(a) \neq 0$ for all N between $\bar{M}$ and M; which implies that $v_M > 0$ on (a,b) for $M \in [\bar{M} - \lambda_2^{*\prime}, \bar{M}]$, where $\lambda_2^{*\prime} > 0$ is the least positive eigenvalue of $T^*_n[\bar{M}]$ on $X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\bar{\beta}\}}_{\{\sigma_1,\dots,\sigma_{k-1}\}}$.

Moreover, since the eigenvalues of an operator and its adjoint are the same, we can affirm that $\lambda_3'' = \lambda_3^{*\prime\prime}$ and $\lambda_2' = \lambda_2^{*\prime}$.

Consider now the other case, i.e. $M > \bar{M}$. From (7.5), we have that $T^*_n v_M \leq 0$. Thus, to obtain the maximal oscillation, the inequalities (7.8)-(7.9) must be reversed. So, taking into account (7.7), we can affirm that to get maximal oscillation:

$$v_M^{(\eta)}(a) \begin{cases} \leq 0\,, & \text{if } n-k \text{ is even,} \\ \geq 0\,, & \text{if } n-k \text{ is odd,} \end{cases} \tag{7.13}$$

and

- If $n-k$ is even
$$v_M^{(\gamma)}(b)\begin{cases}\le 0\,, & \text{if } \gamma \text{ is odd,}\\ \ge 0\,, & \text{if } \gamma \text{ is even.}\end{cases} \tag{7.14}$$
- If $n-k$ is odd
$$v_M^{(\gamma)}(b)\begin{cases}\ge 0\,, & \text{if } \gamma \text{ is odd,}\\ \le 0\,, & \text{if } \gamma \text{ is even.}\end{cases} \tag{7.15}$$

Hence, we arrive at the following conclusions, taking into account (7.6):

- If $n-k$ is even, the maximal oscillation is not allowed for v_M if $v_N^{(\eta)}(a)\neq 0$ for all N between $\bar M$ and M; which implies that $v_M>0$ on (a,b) for $M\in[\bar M,\bar M-\lambda_2^{*\prime}]=[\bar M,\bar M-\lambda_2']$.
- If $n-k$ is odd, the maximal oscillation is not allowed for v_M if $v_N^{(\gamma)}(b)\neq 0$ for all N between $\bar M$ and M; which implies that $v_M>0$ on (a,b) for $M\in[\bar M,\bar M-\lambda_3^{*\prime\prime}]=[\bar M,\bar M-\lambda_3']$.

Now, we realize that if $k=1$, $v_M^{(\eta)}(a)\neq 0$ for all $M\in\mathbb{R}$, since the contrary implies that a nontrivial solution of the homogeneous linear differential equation (7.1) has a zero at $t=a$ of multiplicity n, which is not possible.

Finally, if $\varepsilon_{n-k}=n-k-1$ or, which is the same, $\delta_k=k-1$, we consider $v_M^{(k-1)}(b)$ instead of $v_M^{(\gamma)}(b)=v_M^{(k)}(b)$ and, taking into account that, from $\ell=0$ to $n-(k-1)$, $T_{n-\ell}v_M(b)\neq 0$, we obtain that to allow maximal oscillation the following properties hold:

- If $M<\bar M$
$$T_{k-1}^{*}v_M(b)\begin{cases}\ge 0\,, & \text{if } n-k \text{ is odd,}\\ \le 0\,, & \text{if } n-k \text{ is even.}\end{cases} \tag{7.16}$$
- If $M>\bar M$
$$T_{k-1}^{*}v_M(b)\begin{cases}\le 0\,, & \text{if } n-k \text{ is odd,}\\ \ge 0\,, & \text{if } n-k \text{ is even.}\end{cases} \tag{7.17}$$

From (4.24), since $T_\ell^{*}v_M(t)=(-1)^\ell\widehat{T}_\ell v_M(t)$, we have that
$$T_{k-1}^{*}v_M(b)=(-1)^{k-1}v_1(b)\cdots v_{n-k-1}(b)v_M^{(k-1)}(b)\,.$$
So, we obtain

- If $M<\bar M$
  - If $n-k$ is even
$$v_M^{(k-1)}(b)\begin{cases}\ge 0\,, & \text{if } k-1 \text{ is odd,}\\ \le 0\,, & \text{if } k-1 \text{ is even.}\end{cases} \tag{7.18}$$
  - If $n-k$ is odd
$$v_M^{(k-1)}(b)\begin{cases}\le 0\,, & \text{if } k-1 \text{ is odd,}\\ \ge 0\,, & \text{if } k-1 \text{ is even.}\end{cases} \tag{7.19}$$
- If $M>\bar M$
  - If $n-k$ is even
$$v_M^{(k-1)}(b)\begin{cases}\le 0\,, & \text{if } k-1 \text{ is odd,}\\ \ge 0\,, & \text{if } k-1 \text{ is even.}\end{cases} \tag{7.20}$$

– If $n-k$ is odd

$$v_M^{(k-1)}(b) \begin{cases} \geq 0\,, & \text{if } k-1 \text{ is odd,} \\ \leq 0\,, & \text{if } k-1 \text{ is even.} \end{cases} \tag{7.21}$$

From (7.6) we are able to complete the proof:

- If $n-k$ is even and $M < \bar{M}$, the maximal oscillation is not allowed for v_M if $v_N^{(k-1)}(b) \neq 0$ for all N between $\bar{M}$ and M; which implies that $v_M > 0$ on (a,b) for $M \in [\bar{M} - \lambda_1^*, \bar{M}]$, where $\lambda_1^* > 0$ is the least positive eigenvalue of $T_n^*[\bar{M}]$ on $X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$.
- If $n-k$ is odd and $M > \bar{M}$, the maximal oscillation is not allowed for v_M if $v_N^{(n-k-1)}(a) \neq 0$ for all N between $\bar{M}$ and M; which implies that $v_M > 0$ on (a,b) for $M \in [\bar{M}, \bar{M} - \lambda_1^*]$.

Because of the coincidence of the eigenvalues of an operator and the ones of its adjoint, we can affirm that $\lambda_1 = \lambda_1^*$ and the proof is complete. □

Now, we obtain an analogous result for different boundary conditions.

Proposition 7.4. *Let $\bar{M} \in \mathbb{R}$ be such that $T_n[\bar{M}]$ satisfies property (T_d) on the set $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$ and $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ satisfy (N_a). Then every solution of $T_n^*[M]v(t) = 0$ for $t \in (a,b)$, satisfying the boundary conditions* (4.1)–(4.2) *and* (4.4)–(4.6)*, does not have any zero on (a,b) provided that one of the following assertions is fulfilled:*

- *Let $n-k$ be even:*
  - *If $k > 1$, $\sigma_k \neq k-1$ and $M \in [\bar{M} - \lambda_3', \bar{M} - \lambda_2'']$, where:*
    - *$\lambda_3' > 0$ is the least positive eigenvalue of $T_n[\bar{M}]$ on the set $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_{k-1}|\alpha\}}$.*
    - *$\lambda_2'' < 0$ is the largest negative eigenvalue of $T_n[\bar{M}]$ on the set $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}_{\{\sigma_1,\dots,\sigma_k|\alpha\}}$.*
  - *If $k = 1$, $\sigma_1 \neq 0$ and $M \in [\bar{M} - \lambda_3', \bar{M} - \lambda_2'']$, where:*
    - *$\lambda_3' > 0$ is the least positive eigenvalue of $T_n[\bar{M}]$ on the set $X^{\{\varepsilon_1,\dots,\varepsilon_{n-1}\}}_{\{\alpha\}}$, where $\alpha = 0$.*
    - *$\lambda_2'' < 0$ is the largest negative eigenvalue of $T_n[\bar{M}]$ on the set $X^{\{\varepsilon_1,\dots,\varepsilon_{n-2}\}}_{\{\sigma_1|\alpha\}}$, where $\alpha = 0$.*
  - *If $\sigma_k = k-1$ and $M \in [\bar{M} - \lambda_1, \bar{M} - \lambda_2'']$, where:*
    - *$\lambda_1 > 0$ is the least positive eigenvalue of $T_n[\bar{M}]$ on $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{1,\dots,k-1\}}$.*
    - *$\lambda_2'' < 0$ is the largest negative eigenvalue of $T_n[\bar{M}]$ on the set $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}_{\{0,\dots,k-1,k\}}$.*
- *Let $n-k$ be odd:*
  - *If $1 < k < n-1$, $\sigma_k \neq k-1$ and $M \in [\bar{M} - \lambda_2'', \bar{M} - \lambda_3']$, where:*
    - *$\lambda_3' < 0$ is the largest negative eigenvalue of $T_n[\bar{M}]$ on the set $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_{k-1}|\alpha\}}$.*
    - *$\lambda_2'' > 0$ is the least positive eigenvalue of $T_n[\bar{M}]$ on the set $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}_{\{\sigma_1,\dots,\sigma_k|\alpha\}}$.*
  - *If $1 = k < n-1$, $\sigma_1 \neq 0$ and $M \in [\bar{M} - \lambda_2'', \bar{M} - \lambda_3']$, where:*

  - $*$ *$\lambda_3' < 0$ is the largest negative eigenvalue of $T_n[\bar{M}]$ on the set $X_{\{\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-1}\}}$, where $\alpha = 0$.*
  - $*$ *$\lambda_2'' > 0$ is the least positive eigenvalue of $T_n[\bar{M}]$ on the set $X_{\{\sigma_1|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-2}\}}$, where $\alpha = 0$.*
- *If $1 < k = n-1$, $\sigma_k \neq n-2$ and $M \in (-\infty, \bar{M} - \lambda_3']$, where:*
  - $*$ *$\lambda_3' < 0$ is the largest negative eigenvalue of $T_n[\bar{M}]$ on the set $X_{\{\sigma_1,\dots,\sigma_{n-2}|\alpha\}}^{\{\varepsilon_1\}}$.*
- *If $k = 1$, $n = 2$, $\sigma_1 \neq 0$ and $M \in (-\infty, \bar{M} - \lambda_3']$, where:*
  - $*$ *$\lambda_3' < 0$ is the largest negative eigenvalue of $T_n[\bar{M}]$ on $X_{\{\alpha\}}^{\{\varepsilon_1\}} = X_{\{0\}}^{\{0\}}$.*
- *If $k < n-1$, $\sigma_k = k-1$ and $M \in [\bar{M} - \lambda_2'', \bar{M} - \lambda_1]$, where:*
  - $*$ *$\lambda_1 < 0$ is the largest negative eigenvalue of $T_n[\bar{M}]$ on the set $X_{\{0,\dots,k-1\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*
  - $*$ *$\lambda_2'' > 0$ is the least positive eigenvalue of $T_n[\bar{M}]$ on the set $X_{\{0,\dots,k-1,k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}$.*
- *If $k = n-1$ and $\sigma_{n-1} = n-2$ and $M \in (-\infty, \bar{M} - \lambda_1]$, where:*
  - $*$ *$\lambda_1 < 0$ is the largest negative eigenvalue of $T_n[\bar{M}]$ on the set $X_{\{0,\dots,n-2\}}^{\{\varepsilon_1\}}$.*

The proof of the above proposition is analogous to Proposition 7.3, and is omitted here.

Example 7.5. Returning to our problem, introduced in Example 6.6, we have that operator $T_4^{0*}[M]v(t) = v^{(4)}(t) + Mv(t) = T_4^0[M]v(t)$ is defined on

$$X_{\{0,2\}}^{*\{1,2\}} = \left\{v \in C^4([0,1]) : v(0) = v''(0) = v(1) = v^{(3)}(1) = 0\right\},$$

as it is proved in (4.10) because, in this case, $p_1(t) = p_2(t) = p_3(t) = p_4(t) = 0$ for all $t \in [0,1]$.

From Proposition 7.3, we conclude that each solution of $v^{(4)}(t) + Mv(t) = 0$ on $[0,1]$ satisfying the boundary conditions $v(0) = v''(0) = v(1) = 0$ does not have any zero on $(0,1)$ for $M \in [-m_3^4, m_2^4]$, where m_2 and m_3 have been introduced in Example 6.6.

We note that such functions have the expressions:

$$\begin{cases} K\left(\sin(m\,t)\sinh(m) - \sinh(m\,t)\sin(m)\right), & M = -m^4 < 0, \\ K(t - t^3), & M = 0, \\ Ke^{-\frac{mt}{\sqrt{2}}}\Big(\left(e^{\sqrt{2}m(t+1)} - 1\right)\sin\left(\frac{m(t-1)}{\sqrt{2}}\right) & \\ + \left(e^{\sqrt{2}m} - e^{\sqrt{2}mt}\right)\sin\left(\frac{m(t+1)}{\sqrt{2}}\right)\Big), & M = m^4 > 0, \end{cases}$$

where $K \in \mathbb{R}$.

Moreover, from Proposition 7.4, we can affirm that any solution of $v^{(4)}(t) + Mv(t) = 0$ on $[0,1]$, satisfying the boundary conditions $v(0) = v(1) = v^{(3)}(1) = 0$, does not have any zero on $(0,1)$ for $M \in [-\pi^4, 4\pi^4]$. One can show that such solutions are given as multiples of:

$$\begin{cases} \cos(m-mt)(\sin(m)+\sinh(m))+\sin(m-mt)(\cosh(m)-\cos(m)) \\ +\sinh(m-mt)(\cosh(m)-\cos(m))-\cosh(m-mt)(\sin(m)+\sinh(m))\,, \\ \quad \text{if } M=-m^4<0\,, \\ t-t^2\,, \quad \text{if } M=0\,, \\ e^{-\frac{mt}{\sqrt{2}}}\Big(-\Big(e^{\sqrt{2}m(t-1)}-e^{\sqrt{2}mt}+e^{\sqrt{2}m}-1\Big)\sin\Big(\frac{mt}{\sqrt{2}}\Big) \\ +\Big(e^{\sqrt{2}mt}-1\Big)\cos\Big(\frac{m(t-2)}{\sqrt{2}}\Big)-\Big(e^{\sqrt{2}mt}-1\Big)\cos\Big(\frac{mt}{\sqrt{2}}\Big)\Big)\,, \\ \quad \text{if } M=m^4>0\,, \end{cases}$$

Taking into account that if v_M is a solution of (7.1), then $(-1)^n v_M$ is a solution of $\widehat{T}_n[(-1)^n M]v(t)=0$ for all $t\in I$, we obtain the analogous results for $\widehat{T}_n$.

Proposition 7.6. *Let $\bar{M}\in\mathbb{R}$ be such that $T_n[\bar{M}]$ satisfies property (T_d) on the set $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ and $\{\sigma_1,\dots,\sigma_k\}-\{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ satisfy (N_a). Then every solution of $\widehat{T}_n[(-1)^n M]v(t)=0$ for $t\in(a,b)$, satisfying the boundary conditions* (4.1)–(4.3) *and* (4.4)–(4.5)*, does not have any zero on (a,b) provided that one of the following assertions is fulfilled:*

- *Let k be even:*
  - *If $k<1$, $\varepsilon_{n-k}\neq n-k-1$ and $M\in[\bar{M}-\lambda_3'',\bar{M}-\lambda_2']$, where:*
    - *$\lambda_3''>0$ is the least positive eigenvalue of $T_n[\bar{M}]$ on the set $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}}$.*
    - *$\lambda_2'<0$ is the largest negative eigenvalue of $T_n[\bar{M}]$ on the set $X_{\{\sigma_1,\dots,\sigma_{k-1}\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$.*
  - *If $k=1$, $\varepsilon_{n-1}\neq n-2$ and $M\in[\bar{M}-\lambda_3'',+\infty)$, where:*
    - *$\lambda_3''>0$ is the least positive eigenvalue of $T_n[\bar{M}]$ on the set $X_{\{\sigma_1\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-2}|\beta\}}$.*
  - *If $1<k$, $\varepsilon_{n-k}=n-k-1$ and $M\in[\bar{M}-\lambda_1,\bar{M}-\lambda_2']$, where:*
    - *$\lambda_1>0$ is the least positive eigenvalue of $T_n[\bar{M}]$ on the set $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{0,\dots,n-k-1\}}$.*
    - *$\lambda_2'<0$ is the largest negative eigenvalue of $T_n[\bar{M}]$ on the set $X_{\{\sigma_1,\dots,\sigma_{k-1}\}}^{\{0,\dots,n-k-1,n-k\}}$.*
  - *If $k=1$ and $\varepsilon_{n-1}=n-2$ and $M\in[\bar{M}-\lambda_1,+\infty)$, where:*
    - *$\lambda_1>0$ is the least positive eigenvalue of $T_n[\bar{M}]$ on $X_{\{\sigma_1\}}^{\{0,\dots,n-2\}}$.*
- *Let k be odd:*
  - *If $1<k<n-1$, $\varepsilon_{n-k}\neq n-k-1$ and $M\in[\bar{M}-\lambda_2',\bar{M}-\lambda_3'']$, where:*
    - *$\lambda_3''<0$ is the largest negative eigenvalue of $T_n[\bar{M}]$ on the set $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}}$.*
    - *$\lambda_2'>0$ is the least positive eigenvalue of $T_n[\bar{M}]$ on $X_{\{\sigma_1,\dots,\sigma_{k-1}\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$.*
  - *If $1<k=n-1$, $\varepsilon_1\neq 0$ and $M\in[\bar{M}-\lambda_2',\bar{M}-\lambda_3'']$, where:*
    - *$\lambda_3''<0$ is the least largest negative eigenvalue of $T_n[\bar{M}]$ on $X_{\{\sigma_1,\dots,\sigma_{n-1}\}}^{\{\beta\}}$, where $\beta=0$.*
    - *$\lambda_2'>0$ is the least positive eigenvalue of $T_n[\bar{M}]$ on $X_{\{\sigma_1,\dots,\sigma_{n-2}\}}^{\{\varepsilon_1|\beta\}}$.*

- *If* $k = 1 < n-1$, $\varepsilon_{n-1} \neq n-2$ *and* $M \in (-\infty, \bar{M} - \lambda_3'']$, *where:*
  - $\lambda_3'' < 0$ *is the largest negative eigenvalue of* $T_n[\bar{M}]$ *on the set* $X_{\{\sigma_1\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-2}|\beta\}}$.
- *If* $k = 1$, $n = 2$, $\varepsilon_1 \neq 0$ *and* $M \in (-\infty, \bar{M} - \lambda_3'']$, *where:*
  - $\lambda_3'' < 0$ *is the largest negative eigenvalue of* $T_n[\bar{M}]$ *on the set* $X_{\{\sigma_1\}}^{\{\beta\}} = X_{\{0\}}^{\{0\}}$.
- *If* $1 < k$, $\varepsilon_{n-k} = n-k-1$ *and* $M \in [\bar{M} - \lambda_2', \bar{M} - \lambda_1]$, *where:*
  - $\lambda_1 < 0$ *is the largest negative eigenvalue of* $T_n[\bar{M}]$ *on the set* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{0,\dots,n-k-1\}}$.
  - $\lambda_2' > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *on the set* $X_{\{\sigma_1,\dots,\sigma_{k-1}\}}^{\{0,\dots,n-k-1,n-k\}}$.
- *If* $k = 1$ *and* $\varepsilon_{n-1} = n-2$ *and* $M \in (-\infty, \bar{M} - \lambda_1]$, *where:*
  - $\lambda_1 < 0$ *is the largest negative eigenvalue of* $T_n[\bar{M}]$ *on* $X_{\{\sigma_1\}}^{\{0,\dots,n-2\}}$.

Proposition 7.7. *Let* $\bar{M} \in \mathbb{R}$ *be such that* $T_n[\bar{M}]$ *satisfies property* (T_d) *on the set* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ *and* $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ *satisfy* (N_a). *Then every solution of* $\widehat{T}_n[(-1)^n M]v(t) = 0$ *for* $t \in (a,b)$, *satisfying the boundary conditions* (4.1)–(4.2) *and* (4.4)–(4.6), *does not have any zero on* (a,b) *provided that one of the following assertions is fulfilled:*

- *Let* k *be even:*
  - *If* $k > 1$, $\sigma_k \neq k-1$ *and* $M \in [\bar{M} - \lambda_3', \bar{M} - \lambda_2'']$, *where:*
    - $\lambda_3' > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *on the set* $X_{\{\sigma_1,\dots,\sigma_{k-1}|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.
    - $\lambda_2'' < 0$ *is the largest negative eigenvalue of* $T_n[\bar{M}]$ *on the set* $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}$.
  - *If* $k = 1$, $\sigma_1 \neq 0$ *and* $M \in [\bar{M} - \lambda_3', \bar{M} - \lambda_2'']$, *where:*
    - $\lambda_3' > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *on the set* $X_{\{\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-1}\}}$, *where* $\alpha = 0$.
    - $\lambda_2'' < 0$ *is the largest negative eigenvalue of* $T_n[\bar{M}]$ *on the set* $X_{\{\sigma_1|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-2}\}}$, *where* $\alpha = 0$.
  - *If* $\sigma_k = k-1$ *and* $M \in [\bar{M} - \lambda_1, \bar{M} - \lambda_2'']$, *where:*
    - $\lambda_1 > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *on* $X_{\{1,\dots,k-1\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.
    - $\lambda_2'' < 0$ *is the largest negative eigenvalue of* $T_n[\bar{M}]$ *on the set* $X_{\{0,\dots,k-1,k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}$.
- *Let* k *be odd:*
  - *If* $1 < k < n-1$, $\sigma_k \neq k-1$ *and* $M \in [\bar{M} - \lambda_2'', \bar{M} - \lambda_3']$, *where:*
    - $\lambda_3' < 0$ *is the largest negative eigenvalue of* $T_n[\bar{M}]$ *on the set* $X_{\{\sigma_1,\dots,\sigma_{k-1}|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.
    - $\lambda_2'' > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *on the set* $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}$.
  - *If* $1 = k < n-1$, $\sigma_1 \neq 0$ *and* $M \in [\bar{M} - \lambda_2'', \bar{M} - \lambda_3']$, *where:*
    - $\lambda_3' < 0$ *is the largest negative eigenvalue of* $T_n[\bar{M}]$ *on the set* $X_{\{\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-1}\}}$, *where* $\alpha = 0$.

  * $\lambda_2'' > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *on* $X_{\{\sigma_1|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-2}\}}$, *where* $\alpha = 0$.
- *If* $1 < k = n-1$, $\sigma_{n-1} \neq n-2$ *and* $M \in (-\infty, \bar{M} - \lambda_3']$, *where:*
  * $\lambda_3' < 0$ *is the largest negative eigenvalue of* $T_n[\bar{M}]$ *on the set* $X_{\{\sigma_1,\dots,\sigma_{n-2}|\alpha\}}^{\{\varepsilon_1\}}$.
- *If* $k = 1$, $n = 2$, $\sigma_1 \neq 0$ *and* $M \in (-\infty, \bar{M} - \lambda_3']$, *where:*
  * $\lambda_3' < 0$ *is the largest negative eigenvalue of* $T_n[\bar{M}]$ *on* $X_{\{\alpha\}}^{\{\varepsilon_1\}} = X_{\{0\}}^{\{0\}}$.
- *If* $k < n-1$, $\sigma_k = k-1$ *and* $M \in [\bar{M} - \lambda_2'', \bar{M} - \lambda_1]$, *where:*
  * $\lambda_1 < 0$ *is the largest negative eigenvalue of* $T_n[\bar{M}]$ *on the set* $X_{\{0,\dots,k-1\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.
  * $\lambda_2'' > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *on the set* $X_{\{0,\dots,k-1,k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}$.
- *If* $k = n-1$ *and* $\sigma_{n-1} = n-2$ *and* $M \in (-\infty, \bar{M} - \lambda_1]$, *where:*
  * $\lambda_1 < 0$ *is the largest negative eigenvalue of* $T_n[\bar{M}]$ *on* $X_{\{0,\dots,n-2\}}^{\{\varepsilon_1\}}$.

Remark 7.8. In this example, we have $n = 4$ which is even, so $\tilde{T}_4[(-1)^4 M] \equiv T_4^*[M]$. Then, Example 7.5 is also valid to illustrate Propositions 7.6 and 7.7.

8. Characterization of the strongly inverse positive (negative) character of $T_n[M]$ on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$

This section is devoted to obtaining the main result of this work, such a result gives the characterization of the parameter's set where $T_n[M]$ is either strongly inverse positive or strongly inverse negative on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$. Such a characterization is obtained under the hypotheses that there exists $\bar{M} \in \mathbb{R}$ such that the operator $T_n[\bar{M}]$ satisfies property (T_d) and, moreover, $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ satisfy (N_a). In such a case, from Theorem 5.1, it is known that if $n-k$ is even, then $T_n[\bar{M}]$ is strongly inverse positive on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ and, if $n-k$ is odd, then $T_n[\bar{M}]$ is strongly inverse negative on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.

The characterization here obtained is related to the parameter's set which contains $\bar{M}$. That is, if $n-k$ is even we characterize the parameter's set where $T_n[M]$ is strongly inverse positive on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ and, if $n-k$ is odd we characterize the parameter's set where $T_n[M]$ is strongly inverse negative on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$. In particular, $\bar{M}$ belongs to those intervals.

Theorem 8.1. *Let* $\bar{M} \in \mathbb{R}$ *be such that* $T_n[\bar{M}]$ *satisfies property* (T_d) *on the set* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ *and* $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ *satisfy* (N_a). *The following properties are fulfilled:*

- *If* $n-k$ *is even and* $2 \le k \le n-1$, *then* $T_n[M]$ *is strongly inverse positive on* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ *if and only if* $M \in (\bar{M} - \lambda_1, \bar{M} - \lambda_2]$, *where*
  * $\lambda_1 > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *on* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.
  * $\lambda_2 < 0$ *is the maximum between:*

    $\lambda_2' < 0$, *the largest negative eigenvalue of* $T_n[\bar{M}]$ *on the set* $X_{\{\sigma_1,\dots,\sigma_{k-1}\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$.

$\lambda_2'' < 0$ *is the largest negative eigenvalue of* $T_n[\bar{M}]$ *on the set* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}_{\{\sigma_1,\dots,\sigma_k|\alpha\}}$.

- *If* $k = 1$ *and* n *is odd, then* $T_n[M]$ *is strongly inverse positive on* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-1}\}}_{\{\sigma_1\}}$ *if and only if* $M \in (\bar{M} - \lambda_1, \bar{M} - \lambda_2]$, *where*
  * $\lambda_1 > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *on* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-1}\}}_{\{\sigma_1\}}$.
  * $\lambda_2 < 0$ *is the largest negative eigenvalue of* $T_n[\bar{M}]$ *on* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-2}\}}_{\{\sigma_1|\alpha\}}$.
- *If* $n - k$ *is odd and* $2 \le k \le n - 2$, *then* $T_n[M]$ *is strongly inverse negative on* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$ *if and only if* $M \in [\bar{M} - \lambda_2, \bar{M} - \lambda_1)$, *where*
  * $\lambda_1 < 0$ *is the largest negative eigenvalue of* $T_n[\bar{M}]$ *on* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$.
  * $\lambda_2 > 0$ *is the minimum between:*
    · $\lambda_2' > 0$, *the least positive eigenvalue of* $T_n[\bar{M}]$ *on the set* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}_{\{\sigma_1,\dots,\sigma_{k-1}\}}$.
    · $\lambda_2'' > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *on the set* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}_{\{\sigma_1,\dots,\sigma_k|\alpha\}}$.
- *If* $k = 1$ *and* $n > 2$ *is even, then* $T_n[M]$ *is strongly inverse negative on* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-1}\}}_{\{\sigma_1\}}$ *if and only if* $M \in [\bar{M} - \lambda_2, \bar{M} - \lambda_1)$, *where*
  * $\lambda_1 < 0$ *is the largest negative eigenvalue of* $T_n[\bar{M}]$ *on* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-1}\}}_{\{\sigma_1\}}$.
  * $\lambda_2 > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *on* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-2}\}}_{\{\sigma_1|\alpha\}}$.
- *If* $k = n - 1$ *and* $n > 2$, *then* $T_n[M]$ *is strongly inverse negative on* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-1}\}}_{\{\sigma_1\}}$ *if and only if* $M \in [\bar{M} - \lambda_2, \bar{M} - \lambda_1)$, *where*
  * $\lambda_1 < 0$ *is the largest negative eigenvalue of* $T_n[\bar{M}]$ *on* $X^{\{\varepsilon_1\}}_{\{\sigma_1,\dots,\sigma_{n-1}\}}$.
  * $\lambda_2 > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *on* $X^{\{\varepsilon_1|\beta\}}_{\{\sigma_1,\dots,\sigma_{n-2}\}}$.
- *If* $n = 2$, *then* $T_n[M]$ *is strongly inverse negative on* $X^{\{\varepsilon_1\}}_{\{\sigma_1\}}$ *if and only if* $M \in (-\infty, \bar{M} - \lambda_1)$, *where*
  * $\lambda_1 < 0$ *is the largest negative eigenvalue of* $T_n[\bar{M}]$ *on* $X^{\{\varepsilon_1\}}_{\{\sigma_1\}}$.

Proof. From Lemma 5.1, we know that operator $T_n[\bar{M}]$ satisfies property (A2.1) and is strongly inverse positive on $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$ if $n-k$ is even. Moreover, it satisfies (A2.2) and is strongly inverse negative on $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$ if $n - k$ is odd.

Then, using Theorems 2.10, 2.14, 2.16 and 2.17, we conclude that

- If $n - k$ is even and $M \le \bar{M}$, then $T_n[M]$ is strongly inverse positive on $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$ if and only if $M \in (\bar{M} - \lambda_1, \bar{M}]$.
- If $n - k$ is odd and $M \ge \bar{M}$, then $T_n[M]$ is strongly inverse negative on $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$ if and only if $M \in [\bar{M}, \bar{M} - \lambda_1)$.

To obtain the other extreme of the interval we use the characterization of the strongly inverse positive (negative) character given in Theorems 2.14 and 2.15.

The proof follows several steps. To make the paper more readable, we indicate the different steps for the case with $n - k$ even. For the case with $n - k$ odd the proof is analogous.

Step 1. Study of the related Green's function at $s = a$.
Step 2. Study of the related Green's function at $s = b$.

Step 3. Study of the related Green's function at $t=a$.
Step 4. Study of the related Green's function at $t=b$.
Step 5. Study of the related Green's function on $(a,b)\times(a,b)$.

Let us denote

$$g_M(t,s)=\begin{cases} g_M^1(t,s)\,, & a\le s\le t\le b\,,\\ g_M^2(t,s)\,, & a<t<s<b\,,\end{cases}$$

as the related Green's function of $T_n[M]$ on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.

Step 1. Study of the related Green's function at $s=a$. Let us consider $w_M(t)=\frac{\partial^\eta}{\partial s^\eta}g_M^1(t,s)\big|_{s=a}$, where η has been defined in (4.7). Using (2.13) and the boundary conditions of the adjoint operator given in (4.1)–(4.6), if $\eta>0$, we obtain

$$g_M^1(t,a)=\frac{\partial}{\partial s}g_M^1(t,s)\big|_{s=a}=\dots=\frac{\partial^{\eta-1}}{\partial s^{\eta-1}}g_M^1(t,s)\big|_{s=a}=0\,.$$

Note that a necessary condition to ensure the inverse positive character is that $w_M\ge 0$. Indeed, if there exists $t^*\in[a,b]$, such that $w_M(t^*)<0$, then there exists $\rho(t^*)>0$ such that $g_M(t^*,s)<0$ for all $s\in(0,\rho(t^*))$, which contradicts the inverse positive character. Hence from Lemma 5.1, we have $w_{\bar M}\ge 0$ if $n-k$ is even.

Moreover, since $g_M(t,s)$ is the related Green's function of $T_n[M]$ on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, we have that $T_n[M]g_M(t,a)=0$ for all $t\in(a,b]$. Hence

$$\frac{\partial^\eta}{\partial s^\eta}\left(T_n[M]g_M(t,s)\right)\big|_{s=a}=T_n[M]\,w_M(t)=0\,,\quad t\in(a,b]\,.$$

Now, let us see which boundary conditions are satisfied by w_M. To this end, we use the Green's matrix for the vectorial problem (2.3)-(2.4), introduced in (2.5), where the expression of $g_{n-j}(t,s)$ is given in (2.6) for $j=1,\dots,n-1$. If $k>1$, considering the first row of (2.4), we have

$$\begin{gathered}\frac{\partial^{\sigma_1}}{\partial t^{\sigma_1}}g_M^2(t,s)\big|_{t=a}=0\,,\\ -\frac{\partial^{\sigma_1+1}}{\partial t^{\sigma_1}\partial s}g_M^2(t,s)\big|_{t=a}+\alpha_0^1(s)\frac{\partial^{\sigma_1}}{\partial t^{\sigma_1}}g_M^2(t,s)\big|_{t=a}=0\,,\\ \dots\\ (-1)^\eta\frac{\partial^{\sigma_1+\eta}}{\partial t^{\sigma_1}\partial s^\eta}g_M^2(t,s)\big|_{t=a}+\sum_{i=0}^{\eta-1}\alpha_i^\eta(s)\frac{\partial^{i+\sigma_1}}{\partial t^{\sigma_1}\partial s^i}g_M^2(t,s)\big|_{t=a}=0\,.\end{gathered}$$

This system is satisfied in particular for $s=a$. Since $\eta+\sigma_1<n-1$ we do not reach any diagonal element of $G(t,s)$, hence we obtain by continuity:

$$\begin{gathered}\frac{\partial^{\sigma_1}}{\partial t^{\sigma_1}}g_M^1(t,s)\big|_{(t,s)=(a,a)}=0\,,\\ -\frac{\partial^{\sigma_1+1}}{\partial t^{\sigma_1}\partial s}g_M^1(t,s)\big|_{(t,s)=(a,a)}+\alpha_0^1(a)\frac{\partial^{\sigma_1}}{\partial t^{\sigma_1}}g_M^1(t,s)\big|_{(t,s)=(a,a)}=0\,,\\ \dots\\ (-1)^\eta\frac{\partial^{\sigma_1+\eta}}{\partial t^{\sigma_1}\partial s^\eta}g_M^1(t,s)\big|_{(t,s)=(a,a)}+\sum_{i=0}^{\eta-1}\alpha_i^\eta(a)\frac{\partial^{i+\sigma_1}}{\partial t^{\sigma_1}\partial s^i}g_M^1(t,s)\big|_{(t,s)=(a,a)}=0\,.\end{gathered}$$

Taking into account that $\alpha_i^j \in C(I)$, we have

$$w_M^{(\sigma_1)}(a) = \frac{\partial^{\sigma_1+\eta}}{\partial t^{\sigma_1}\partial s^{\eta}} g_M^1(t,s)\big|_{(t,s)=(a,a)} = 0\,.$$

Proceeding analogously for $\sigma_2,\dots,\sigma_{k-1}$, we obtain

$$w_M^{(\sigma_2)}(a) = \dots = w_M^{(\sigma_{k-1})}(a) = 0\,.$$

Now, let us choose the row σ_k of $G(t,s)$. From (2.4), we have

$$\begin{gathered}
\frac{\partial^{\sigma_k}}{\partial t^{\sigma_k}} g_M^2(t,s)\big|_{t=a} = 0\,,\\
-\frac{\partial^{\sigma_k+1}}{\partial t^{\sigma_k}\partial s} g_M^2(t,s)\big|_{t=a} + \alpha_0^1(s)\frac{\partial^{\sigma_k}}{\partial t^{\sigma_k}} g_M^2(t,s)\big|_{t=a} = 0\,,\\
\dots\\
(-1)^{\eta}\frac{\partial^{\sigma_k+\eta}}{\partial t^{\sigma_k}\partial s^{\eta}} g_M^2(t,s)\big|_{t=a} + \sum_{i=0}^{\eta-1}\alpha_i^{\eta}(s)\frac{\partial^{i+\sigma_k}}{\partial t^{\sigma_k}\partial s^{i}} g_M^2(t,s)\big|_{t=a} = 0\,.
\end{gathered}$$

This system is satisfied in particular for $s=a$. However, since $\sigma_k+\eta = n-1$, we reach a diagonal element of $G(t,s)$. Hence, to express the previous system by means of $g_M^1(t,s)$, we have to take into account Remark 2.6 to obtain

$$\begin{gathered}
\frac{\partial^{\sigma_k}}{\partial t^{\sigma_k}} g_M^1(t,s)\big|_{(t,s)=(a,a)} = 0\,,\\
-\frac{\partial^{\sigma_k+1}}{\partial t^{\sigma_k}\partial s} g_M^1(t,s)\big|_{(t,s)=(a,a)} + \alpha_0^1(a)\frac{\partial^{\sigma_k}}{\partial t^{\sigma_k}} g_M^1(t,s)\big|_{(t,s)=(a,a)} = 0\,,\\
\dots\\
(-1)^{\eta}\frac{\partial^{\sigma_k+\eta}}{\partial t^{\sigma_k}\partial s^{\eta}} g_M^1(t,s)\big|_{(t,s)=(a,a)} + \sum_{i=0}^{\eta-1}\alpha_i^{\eta}(a)\frac{\partial^{i+\sigma_k}}{\partial t^{\sigma_k}\partial s^{i}} g_M^1(t,s)\big|_{(t,s)=(a,a)} = 1\,.
\end{gathered}$$

So, since $\alpha_i^j \in C(I)$, we have

$$w_M^{(\sigma_k)}(a) = \frac{\partial^{\sigma_k+\eta}}{\partial t^{\sigma_k}\partial s^{\eta}} g_M^1(t,s)\big|_{(t,s)=(a,a)} = (-1)^{\eta} = (-1)^{(n-1-\sigma_k)}\,.$$

Analogously, if $k=1$, then $w_M^{(\sigma_1)}(a) = (-1)^{n-1-\sigma_1}$.

Now, let us see what happens at $t=b$. If we consider the $(k+1)^{\text{th}}$ row of (2.4), we have

$$\begin{gathered}
\frac{\partial^{\varepsilon_1}}{\partial t^{\varepsilon_1}} g_M^1(t,s)\big|_{t=b} = 0\,,\\
-\frac{\partial^{\varepsilon_1+1}}{\partial t^{\varepsilon_1}\partial s} g_M^1(t,s)\big|_{t=b} + \alpha_0^1(s)\frac{\partial^{\varepsilon_1}}{\partial t^{\varepsilon_1}} g_M^1(t,s)\big|_{t=b} = 0\,,\\
\dots\\
(-1)^{\eta}\frac{\partial^{\varepsilon_1+\eta}}{\partial t^{\varepsilon_1}\partial s^{\eta}} g_M^1(t,s)\big|_{t=b} + \sum_{i=0}^{\eta-1}\alpha_i^{\eta}(s)\frac{\partial^{i+\varepsilon_1}}{\partial t^{\varepsilon_1}\partial s^{i}} g_M^1(t,s)\big|_{t=b} = 0\,.
\end{gathered}$$

Since $b \neq a$, this system is satisfied in particular at $s=a$. Thus, using that $\alpha_i^j \in C(I)$, we conclude:

$$w_M^{(\varepsilon_1)}(b) = \frac{\partial^{\varepsilon_1+\eta}}{\partial t^{\varepsilon_1}\partial s^{\eta}} g_M^1(t,s)\big|_{(t,s)=(b,a)} = 0\,.$$

Proceeding analogously we obtain:

$$w_M^{(\varepsilon_2)}(b) = \cdots = w_M^{(\varepsilon_{n-k})}(b) = 0\,.$$

Hence, w_M satisfies the boundary conditions (6.5)-(6.6), so we can apply Proposition 6.7 to affirm that

- If $n-k$ is even and $k>1$, then $w_M>0$ on (a,b) for all $M\in[\bar{M},\bar{M}-\lambda_2']$.
- If $k=1$ and n is odd, then $w_M>0$ on (a,b) for all $M\geq\bar{M}$.

To complete this Step, let us see that if $n-k$ is even and $k>1$, then $T_n[M]$ cannot be inverse positive for $M>\bar{M}-\lambda_2'$.

Suppose that there exists $\widehat{M}>\bar{M}-\lambda_2'$ such that $T_n[\widehat{M}]$ is inverse positive on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, thus from Theorems 2.10 and 2.11, we can affirm that for every $M\in[\bar{M}-\lambda_2',\widehat{M}]$ operator $T_n[M]$ is inverse positive on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ and, moreover, $w_{\bar{M}-\lambda_2'}\geq w_M\geq w_{\widehat{M}}$.

In particular, $0=w_{\bar{M}-\lambda_2'}^{(\beta)}(b)\leq w_M^{(\beta)}(b)\leq w_{\widehat{M}}^{(\beta)}(b)$ if β is even and $0=w_{\bar{M}-\lambda_2'}^{(\beta)}(b)\geq w_M^{(\beta)}(b)\geq w_{\widehat{M}}^{(\beta)}(b)$ if β is odd.

If $w_{\widehat{M}}^{(\beta)}(b)\neq 0$, then there exists $\rho>0$ such that $w_{\widehat{M}}(t)<0$ for all $t\in(b-\rho,b)$, which contradicts our assumption. So

$$0=w_{\bar{M}-\lambda_2'}^{(\beta)}(b)=w_M^{(\beta)}(b)=w_{\widehat{M}}^{(\beta)}(b)\,,\quad \forall M\in[\bar{M}-\lambda_2',\widehat{M}]\,,$$

and this fact contradicts the discrete character of the spectrum of the operator $T_n[\bar{M}]$ on $X_{\{\sigma_1,\dots,\sigma_{k-1}\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$. From this Step we obtain the following conclusions:

- If $n-k$ is even, $k>1$ and $M\in[\bar{M},\bar{M}-\lambda_2']$: for each $t\in(a,b)$ there exists $\rho(t)>0$ such that
$$g_M(t,s)>0\ \forall s\in(a,a+\rho(t))\,.$$
Moreover, if $M>\bar{M}-\lambda_2'$; then $T_n[M]$ is not inverse positive on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.
- If $k=1$, n is odd and $M\geq\bar{M}$: for each $t\in(a,b)$ there exists $\rho(t)>0$ such that
$$g_M(t,s)>0\ \forall s\in(a,a+\rho(t))\,.$$

Step 2. Study of the related Green's function at $s=b$. Analogously to Step 1, we consider the function

$$y_M(t)=\frac{\partial^\gamma}{\partial s^\gamma}g_M^2(t,s)\big|_{s=b}\,.$$

In this case, from (2.13) and the boundary conditions (4.1)–(4.6), we obtain that if $\gamma>0$, then

$$g_M^2(t,b)=\frac{\partial}{\partial s}g_M^2(t,s)\big|_{s=b}=\cdots=\frac{\partial^{\gamma-1}}{\partial s^{\gamma-1}}g_M^2(t,s)\big|_{s=b}=0\,.$$

We have the following assertions:

- If γ is even and there exist $t^*\in(a,b)$ such that $y_M(t^*)<0$, then $T_n[M]$ cannot be inverse positive on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.
- If γ is odd and there exist $t^*\in(a,b)$ such that $y_M(t^*)>0$, then $T_n[M]$ cannot be inverse positive on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.

The proof of these assertions is analogous to the proof in Step 1 for w_M.

From Lemma 5.1, we obtain that if γ is even, then $y_{\bar M} \geq 0$ and if γ is odd, then $y_{\bar M} \leq 0$. As in Step 1, it can be shown that

$$T_n[M]\, y_M(t) = 0\,, \quad \forall t \in [a,b)\,.$$

Moreover, we can obtain the boundary conditions which y_M satisfies. Studying the Green's function related to the first order vectorial problem given in (2.5) and the boundary conditions (2.4), we obtain that y_M satisfies:

$$\begin{gathered} y_M^{(\sigma_1)}(a) = \cdots = y_M^{(\sigma_k)}(a) = 0\,, \\ y_M^{(\varepsilon_1)}(b) = \cdots = y_M^{(\varepsilon_{n-k-1})}(b) = 0\,, \\ y_M^{(\varepsilon_{n-k})}(b) = (-1)^{(n-\varepsilon_{n-k})}\,. \end{gathered}$$

So, y_M satisfies the boundary conditions (6.17)-(6.18), then we can apply Proposition 6.9 to conclude that

- If $n-k$ is even and $k < n-1$, then $y_M > 0$ if γ is even and $y_M < 0$ if γ is odd on (a,b) for all $M \in [\bar M, \bar M - \lambda_2'']$.

Analogously to Step 1, it can be seen that if $n-k$ is even, then $T_n[M]$ cannot be inverse positive for $M > \bar M - \lambda_2''$.

So from Step 2, we obtain the following conclusions:

- If $n-k$ is even and $M \in [\bar M, \bar M - \lambda_2'']$: for each $t \in (a,b)$ there exists $\rho(t) > 0$ such that

$$g_M(t,s) > 0 \quad \forall s \in (b-\rho(t), b)\,.$$

  Moreover, if $M > \bar M - \lambda_2''$; then $T_n[M]$ is not inverse positive on the set $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.

From the two steps above we can conclude that the intervals where $T_n[M]$ is strongly inverse positive cannot be increased.

The rest of the proof is focused into see that these intervals are the optimal ones.

Step 3. Study of the related Green's function at $t = a$. Let us denote

$$\widehat{g}_{(-1)^n M}(t,s) = \begin{cases} \widehat{g}^1_{(-1)^n M}(t,s)\,, & a \leq s \leq t \leq b\,, \\ \widehat{g}^2_{(-1)^n M}(t,s)\,, & a < t < s < b\,, \end{cases}$$

as the related Green's function of $\widehat{T}_n[(-1)^n M]$ on $X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$. To study the behavior at $t = a$, we consider the function

$$\widehat{w}_M(t) = (-1)^n \frac{\partial^\alpha}{\partial s^\alpha} \widehat{g}^1_{(-1)^n M}(t,s)\big|_{s=a}\,, \quad t \in I\,.$$

From (2.15), it is satisfied that

$$\widehat{w}_M(s) = \frac{\partial^\alpha}{\partial t^\alpha} g^2_M(t,s)\big|_{t=a}\,, \quad s \in I\,, \tag{8.1}$$

moreover, from the boundary conditions (1.5)-(1.6), if $\alpha > 0$ we obtain

$$g_M(a,s) = \frac{\partial}{\partial s} g_M(t,s)\big|_{t=a} = \cdots = \frac{\partial^{\alpha-1}}{\partial s^{\alpha-1}} g_M(t,s)\big|_{t=a} = 0\,, \quad \forall s \in (a,b).$$

Using the arguments of Step 1, we can affirm that if there exists $t^* \in (a,b)$ such that $\widehat{w}_M(t^*) < 0$, then $T_n[M]$ is not inverse positive on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$. Moreover, from Lemma 5.1, if $n-k$ is even, then $\widehat{w}_{\bar{M}} \geq 0$.

From the expression of $\widehat{T}_n[(-1)^n M]$ given in (2.14) and (2.11), we construct the associated vectorial problem (2.3) tacking, in this case

$$\begin{aligned}\widehat{p}_{n-j}(t) &= (-1)^{n+j} p_{n-j}(t) + (-1)^{(n+j+1)}(j+1)\,p'_{n-j-1}(t) + \cdots \\ &\quad - \binom{n-1}{j} p_1^{(n-j-1)}(t)\,, \quad j = 1,\dots,n-1\,, \\ \widehat{p}_n(t) &= (-1)^n p_n(t) + (-1)^{n+1} p'_{n-1}(t) + (-1)^{n+2}\binom{2}{0} p''_{n-2}(t) \\ &\quad + \cdots - p_1^{(n-1)}(t)\,.\end{aligned}$$

Now, the related Green's function is

$$\widehat{G}(t,s) = \begin{pmatrix} \widehat{g}_1(t,s) & \cdots & \widehat{g}_{n-1}(t,s) & \widehat{g}_{(-1)^n M}(t,s) \\ \frac{\partial}{\partial t}\,\widehat{g}_1(t,s) & \cdots & \frac{\partial}{\partial t}\,\widehat{g}_{n-1}(t,s) & \frac{\partial}{\partial t}\,\widehat{g}_{(-1)^n M}(t,s) \\ \vdots & \cdots & \vdots & \vdots \\ \frac{\partial^{n-1}}{\partial t^{n-1}}\,\widehat{g}_1(t,s) & \cdots & \frac{\partial^{n-1}}{\partial t^{n-1}}\,\widehat{g}_{n-1}(t,s) & \frac{\partial^{n-1}}{\partial t^{n-1}}\,\widehat{g}_{(-1)^n M}(t,s) \end{pmatrix}. \tag{8.2}$$

Repeating the arguments done with $T_n[M]$, we obtain

$$\widehat{g}_{n-j}(t,s) = (-1)^j \frac{\partial^j}{\partial s^j}\widehat{g}_{(-1)^n M}(t,s) + \sum_{i=0}^{j-1} \widehat{\alpha}_i^j(s)\frac{\partial^i}{\partial s^i}\widehat{g}_{(-1)^n M}(t,s)\,, \tag{8.3}$$

where $\widehat{\alpha}_i^j(s)$ follow the recurrence formula (2.7)–(2.10) for this problem with the obvious notation.

The correspondent boundary conditions (2.4) are given by the matrices $\widehat{B}$, $\widehat{C} \in \mathcal{M}_{n\times n}$, defined as follows:

$$\begin{gathered}\big(\widehat{B}\big)_{i\,\tau_i+1} = 1\,, \quad i = n-k+1,\dots,n\,, \\ \big(\widehat{B}\big)_{i\,j} = 0\,, \quad \tau_i + 1 < j \leq n\,, \quad i = n-k+1,\dots,n\,, \\ \big(\widehat{B}\big)_{i\,\tau_i-h} = \widehat{p}_{h+1}(a)\,, \quad h = 0,\dots\tau_i - 1\,, \quad i = n-k+1,\dots,n\,, \\ \big(\widehat{B}\big)_{i\,j} = 0\,, \quad j = 0,\dots,n\,,\ i = n-k+1,\dots n \\ \big(\widehat{C}\big)_{i\,j} = 0\,, \quad j = 0,\dots,n\,,\ i = 0,\dots,n-k\,, \\ \big(\widehat{C}\big)_{i\,\delta_{i-(n-k)}+1} = 1\,, \quad i = n-k+1,\dots,n\,, \\ \big(\widehat{C}\big)_{i\,j} = 0\,, \quad \delta_{i-(n-k)} + 1 < j \leq n\,, \quad i = n-k+1,\dots,n\,, \\ \big(\widehat{C}\big)_{i\,\delta_{i-(n-k)}-h} = \widehat{p}_{h+1}(b)\,, \quad h = 0,\dots\delta_{i-(n-k)} - 1\,,\ i = n-k+1,\dots,n\,;\end{gathered}$$

that is, for every $v \in C^n(I)$, we have

$$\widehat{B}\begin{pmatrix} v(a) \\ \vdots \\ v^{(n-1)}(a)\end{pmatrix} + \widehat{C}\begin{pmatrix} v(b) \\ \vdots \\ v^{(n-1)}(b)\end{pmatrix} = \begin{pmatrix} W_1 \\ \vdots \\ W_n\end{pmatrix}\,,$$

where,

$$W_1 = v^{(\tau_1)}(a) + \sum_{j=n-\tau_1}^{n-1} (-1)^{n-j}(p_{n-j}v)^{(\tau_1+j-n)}(a)\,,$$
$$\dots$$
$$W_{n-k} = v^{(\tau_{n-k})}(a) + \sum_{j=n-\tau_{n-k}}^{n-1} (-1)^{n-j}(p_{n-j}v)^{(\tau_{n-k}+j-n)}(a)\,,$$
$$W_{n-k+1} = v^{(\delta_1)}(b) + \sum_{j=n-\delta_1}^{n-1} (-1)^{n-j}(p_{n-j}v)^{(\delta_1+j-n)}(b)\,,$$
$$\dots$$
$$W_n = v^{(\delta_k)}(b) + \sum_{j=n-\delta_k}^{n-1} (-1)^{n-j}(p_{n-j}v)^{(\delta_k+j-n)}(b)\,.$$

As in Steps 1 and 2, we can conclude that

$$\widehat{T}_n[(-1)^n M]\,\widehat{w}_M(t) = 0\,, \quad t \in (a,b]\,.$$

Next, we obtain the boundary conditions for $\widehat{w}_M$. The used arguments are similar to the two previous steps. By definition, $\widehat{G}(t,s)$ satisfies:

$$\widehat{B}\,\widehat{G}(a,s) + \widehat{C}\,\widehat{G}(b,s) = 0\,, \quad \forall s \in (a,b)\,. \tag{8.4}$$

If $k < n-1$, we consider the first row of (8.4) to deduce:

$$\frac{\partial^{\tau_1}}{\partial t^{\tau_1}}\widehat{g}^2_{(-1)^n M}(t,s)\big|_{t=a} + \sum_{j=n-\tau_1}^{n-1}(-1)^{n-j}\frac{\partial^{\tau_1+j-n}}{\partial t^{\tau_1+j-n}}\Big(p_{n-j}(t)\widehat{g}^2_{(-1)^n M}(t,s)\Big)\big|_{t=a} = 0\,,$$
$$-\Big(\sum_{j=n-\tau_1}^{n-1}(-1)^{n-j}\frac{\partial^{\tau_1+j-n+1}}{\partial t^{\tau_1+j-n}\partial s}\big(p_{n-j}(t)\widehat{g}^2_{(-1)^n M}(t,s)\big)\big|_{t=a}$$
$$+\frac{\partial^{\tau_1+1}}{\partial t^{\tau_1}\partial s}\widehat{g}^2_{(-1)^n M}(t,s)\big|_{t=a}\Big) + \widehat{\alpha}^1_0(s)\Big(\frac{\partial^{\tau_1}}{\partial t^{\tau_1}}\widehat{g}^2_{(-1)^n M}(t,s)\big|_{t=a}$$
$$+\sum_{j=n-\tau_1}^{n-1}(-1)^{n-j}\frac{\partial^{\tau_1+j-n}}{\partial t^{\tau_1+j-n}}\Big(p_{n-j}(t)\widehat{g}^2_{(-1)^n M}(t,s)\Big)\big|_{t=a}\Big) = 0\,,$$
$$\dots$$
$$(-1)^\alpha\sum_{j=n-\tau_1}^{n-1}(-1)^{n-j}\frac{\partial^{\tau_1+j-n+\alpha}}{\partial t^{\tau_1+j-n}\partial s^\alpha}\Big(p_{n-j}(t)\widehat{g}^1_{(-1)^n M}(t,s)\Big)\big|_{t=a}$$
$$+(-1)^\alpha\left(\frac{\partial^{\tau_1+\alpha}}{\partial t^{\tau_1}\partial s^\alpha}\widehat{g}^2_{(-1)^n M}(t,s)\big|_{t=a}\right) + \sum_{i=0}^{\alpha-1}\widehat{\alpha}^\alpha_i(s)\Big(\frac{\partial^{\tau_1+i}}{\partial t^{\tau_1}\partial s^i}\widehat{g}^2_{(-1)^n M}(t,s)\big|_{t=a}$$
$$+\sum_{j=n-\tau_1}^{n-1}(-1)^{n-j}\frac{\partial^{\tau_1+j-n+i}}{\partial t^{\tau_1+j-n}\partial s^i}\Big(p_{n-j}(t)\widehat{g}^2_{(-1)^n M}(t,s)\Big)\big|_{t=a}\Big) = 0\,.$$

Since $\tau_1+\alpha<n-1$, we do not reach any diagonal element, hence the previous system is satisfied for $s=a$, and we obtain

$$
\begin{aligned}
&\frac{\partial^{\tau_1}}{\partial t^{\tau_1}}\widehat{g}^1_{(-1)^nM}(t,s)\big|_{(t,s)=(a,a)}\\
&\quad+\sum_{j=n-\tau_1}^{n-1}(-1)^{n-j}\frac{\partial^{\tau_1+j-n}}{\partial t^{\tau_1+j-n}}\Big(p_{n-j}(t)\widehat{g}^1_{(-1)^nM}(t,s)\Big)\,|_{(t,s)=(a,a)}=0\,,\\
&-\sum_{j=n-\tau_1}^{n-1}(-1)^{n-j}\frac{\partial^{\tau_1+j-n+1}}{\partial t^{\tau_1+j-n}\partial s}\Big(p_{n-j}(t)\widehat{g}^1_{(-1)^nM}(t,s)\Big)|_{(t,s)=(a,a)}\\
&-\frac{\partial^{\tau_1+1}}{\partial t^{\tau_1}\partial s}\widehat{g}^1_{(-1)^nM}(t,s)\big|_{(t,s)=(a,a)}+\widehat{\alpha}^1_0(a)\Big(\frac{\partial^{\tau_1}}{\partial t^{\tau_1}}\widehat{g}^1_{(-1)^nM}(t,s)\big|_{(t,s)=(a,a)}\\
&+\sum_{j=n-\tau_1}^{n-1}(-1)^{n-j}\frac{\partial^{\tau_1+j-n}}{\partial t^{\tau_1+j-n}}\Big(p_{n-j}(t)\widehat{g}^1_{(-1)^nM}(t,s)\Big)\,|_{(t,s)=(a,a)}\Big)=0\,,\\
&\qquad\qquad\qquad\dots\\
&(-1)^{\alpha}\sum_{j=n-\tau_1}^{n-1}(-1)^{n-j}\frac{\partial^{\tau_1+j-n+\alpha}}{\partial t^{\tau_1+j-n}\partial s^{\alpha}}\Big(p_{n-j}(t)\widehat{g}^1_{(-1)^nM}(t,s)\Big)\,|_{(t,s)=(a,a)}\\
&+(-1)^{\alpha}\Big(\frac{\partial^{\tau_1+\alpha}}{\partial t^{\tau_1}\partial s^{\alpha}}\widehat{g}^1_{(-1)^nM}(t,s)\big|_{(t,s)=(a,a)}\Big)\\
&+\sum_{i=0}^{\alpha-1}\widehat{\alpha}^{\alpha}_i(a)\Big(\frac{\partial^{\tau_1+i}}{\partial t^{\tau_1}\partial s^{i}}\widehat{g}^1_{(-1)^nM}(t,s)\big|_{(t,s)=(a,a)}\\
&+\sum_{j=n-\tau_1}^{n-1}(-1)^{n-j}\frac{\partial^{\tau_1+j-n+i}}{\partial t^{\tau_1+j-n}\partial s^{i}}\Big(p_{n-j}(t)\widehat{g}^1_{(-1)^nM}(t,s)\Big)\,|_{(t,s)=(a,a)}\Big)=0\,.
\end{aligned}
$$

Since $\widehat{\alpha}^j_i\in C(I)$, we conclude that

$$
\widehat{w}_M^{(\tau_1)}(a)+\sum_{j=n-\tau_1}^{n-1}(-1)^{n-j}\left(p_{n-j}\widehat{w}_M\right)^{(\tau_1+j-n)}(a)=0\,.
$$

Proceeding analogously with $\tau_2,\dots,\tau_{n-k-1}$, we can ensure that $\widehat{w}_M$ satisfies the boundary conditions (4.1)–(4.2).

Now, let us see what happens for τ_{n-k}. From (8.4), we obtain that for all $s\in(a,b)$ the following equalities hold:

$$
\begin{aligned}
&\frac{\partial^{\tau_{n-k}}}{\partial t^{\tau_{n-k}}}\widehat{g}^2_{(-1)^nM}(t,s)\big|_{t=a}\\
&\quad+\sum_{j=n-\tau_{n-k}}^{n-1}(-1)^{n-j}\frac{\partial^{\tau_{n-k}+j-n}}{\partial t^{\tau_{n-k}+j-n}}\Big(p_{n-j}(t)\widehat{g}^2_{(-1)^nM}(t,s)\Big)\,|_{t=a}\quad=0\,,
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{j=n-\tau_{n-k}}^{n-1} (-1)^{n-j} \frac{\partial^{\tau_{n-k}+j-n+1}}{\partial t^{\tau_{n-k}+j-n} \partial s} \Big(p_{n-j}(t) \widehat{g}^2_{(-1)^n M}(t,s) \Big) \big|_{t=a} \\
&- \frac{\partial^{\tau_{n-k}+1}}{\partial t^{\tau_{n-k}} \partial s} \widehat{g}^2_{(-1)^n M}(t,s)\big|_{t=a} + \widehat{\alpha}^1_0(s) \Big(\frac{\partial^{\tau_{n-k}}}{\partial t^{\tau_{n-k}}} \widehat{g}^2_{(-1)^n M}(t,s)\big|_{t=a} \\
&+ \sum_{j=n-\tau_{n-k}}^{n-1} (-1)^{n-j} \frac{\partial^{\tau_{n-k}+j-n}}{\partial t^{\tau_{n-k}+j-n}} \Big(p_{n-j}(t) \widehat{g}^2_{(-1)^n M}(t,s) \Big) \big|_{t=a} \Big) = 0\,, \\
&\qquad\qquad\dots \\
&(-1)^\alpha \sum_{j=n-\tau_{n-k}}^{n-1} (-1)^{n-j} \frac{\partial^{\tau_{n-k}+j-n+\alpha}}{\partial t^{\tau_{n-k}+j-n} \partial s^\alpha} \Big(p_{n-j}(t) \widehat{g}^1_{(-1)^n M}(t,s) \Big) \big|_{t=a} \\
&+ (-1)^\alpha \frac{\partial^{\tau_{n-k}+\alpha}}{\partial t^{\tau_{n-k}} \partial s^\alpha} \widehat{g}^2_{(-1)^n M}(t,s)\big|_{t=a} + \sum_{i=0}^{\alpha-1} \widehat{\alpha}^\alpha_i(s) \Big(\frac{\partial^{\tau_{n-k}+i}}{\partial t^{\tau_{n-k}} \partial s^i} \widehat{g}^2_{(-1)^n M}(t,s)\big|_{t=a} \\
&+ \sum_{j=n-\tau_{n-k}}^{n-1} (-1)^{n-j} \frac{\partial^{\tau_{n-k}+j-n+i}}{\partial t^{\tau_{n-k}+j-n} \partial s^i} \Big(p_{n-j}(t) \widehat{g}^2_{(-1)^n M}(t,s) \Big) \big|_{t=a} \Big) = 0\,.
\end{aligned}
$$

In this case, since $\tau_{n-k}+\alpha = n-1$, we reach a diagonal element of $\widehat{G}(t,s)$, hence by Remark 2.6, we obtain the following system for $s=a$:

$$
\begin{aligned}
&\frac{\partial^{\tau_{n-k}}}{\partial t^{\tau_{n-k}}} \widehat{g}^1_{(-1)^n M}(t,s)\big|_{(t,s)=(a,a)} \\
&\quad + \sum_{j=n-\tau_{n-k}}^{n-1} (-1)^{n-j} \frac{\partial^{\tau_{n-k}+j-n}}{\partial t^{\tau_{n-k}+j-n}} \Big(p_{n-j}(t) \widehat{g}^1_{(-1)^n M}(t,s) \Big) \big|_{(t,s)=(a,a)} = 0\,, \\
&- \sum_{j=n-\tau_{n-k}}^{n-1} (-1)^{n-j} \frac{\partial^{\tau_1+j-n+1}}{\partial t^{\tau_{n-k}+j-n} \partial s} \Big(p_{n-j}(t) \widehat{g}^1_{(-1)^n M}(t,s) \Big) \big|_{(t,s)=(a,a)} \\
&- \frac{\partial^{\tau_{n-k}+1}}{\partial t^{\tau_{n-k}} \partial s} \widehat{g}^1_{(-1)^n M}(t,s)\big|_{(t,s)=(a,a)} + \widehat{\alpha}^1_0(a) \Big(\frac{\partial^{\tau_{n-k}}}{\partial t^{\tau_{n-k}}} \widehat{g}^1_{(-1)^n M}(t,s)\big|_{(t,s)=(a,a)} \\
&+ \sum_{j=n-\tau_{n-k}}^{n-1} (-1)^{n-j} \frac{\partial^{\tau_{n-k}+j-n}}{\partial t^{\tau_{n-k}+j-n}} \Big(p_{n-j}(t) \widehat{g}^1_{(-1)^n M}(t,s) \Big) \big|_{(t,s)=(a,a)} \Big) = 0\,, \\
&\qquad\qquad\dots \\
&(-1)^\alpha \sum_{j=n-\tau_{n-k}}^{n-1} (-1)^{n-j} \frac{\partial^{\tau_{n-k}+j-n+\alpha}}{\partial t^{\tau_{n-k}+j-n} \partial s^\alpha} \Big(p_{n-j}(t) \widehat{g}^1_{(-1)^n M}(t,s) \Big) \big|_{(t,s)=(a,a)} \\
&(-1)^\alpha \frac{\partial^{\tau_{n-k}+\alpha}}{\partial t^{\tau_{n-k}} \partial s^\alpha} \widehat{g}^1_{(-1)^n M}(t,s)\big|_{(t,s)=(a,a)} \\
&+ \sum_{i=0}^{\alpha-1} \widehat{\alpha}^\alpha_i(s) \Big(\frac{\partial^{\tau_{n-k}+i}}{\partial t^{\tau_{n-k}} \partial s^i} \widehat{g}^1_{(-1)^n M}(t,s)\big|_{(t,s)=(a,a)} \\
&+ \sum_{j=n-\tau_{n-k}}^{n-1} (-1)^{n-j} \frac{\partial^{\tau_{n-k}+j-n+i}}{\partial t^{\tau_{n-k}+j-n} \partial s^i} \Big(p_{n-j}(t) \widehat{g}^1_{(-1)^n M}(t,s) \Big) \big|_{(t,s)=(a,a)} \Big) \qquad = 1\,.
\end{aligned}
$$

Since $\widehat{\alpha}_i^j \in C(I)$, from the definition of $\widehat{w}_M$, we deduce that

$$\widehat{w}_M^{(\tau_{n-k})}(a) + \sum_{j=n-\tau_{n-k}}^{n-1} (-1)^{n-j} \left(p_{n-j}\widehat{w}_M\right)^{(\tau_{n-k}+j-n)}(a) = (-1)^{n-\alpha} = (-1)^{1+\tau_{n-k}}.$$

Now, let us study the behavior of $\widehat{w}_M$ at $t=b$. Studying the $(n-k+1)^{\text{th}}$ row of (8.4), we have for all $s \in (a,b)$:

$$\frac{\partial^{\delta_1}}{\partial t^{\delta_1}}\widehat{g}^1_{(-1)^n M}(t,s)\big|_{t=b} + \sum_{j=n-\delta_1}^{n-1} (-1)^{n-j}\frac{\partial^{\delta_1+j-n}}{\partial t^{\delta_1+j-n}}\Big(p_{n-j}(t)\widehat{g}^1_{(-1)^n M}(t,s)\Big)\big|_{t=b} = 0,$$

$$\begin{aligned}
&-\Big(\sum_{j=n-\delta_1}^{n-1} (-1)^{n-j}\frac{\partial^{\delta_1+j-n+1}}{\partial t^{\delta_1+j-n}\partial s}\Big(p_{n-j}(t)\widehat{g}^1_{(-1)^n M}(t,s)\Big)\big|_{t=b}\\
&+\frac{\partial^{\delta_1+1}}{\partial t^{\delta_1}\partial s}\widehat{g}^1_{(-1)^n M}(t,s)\big|_{t=b}\Big) + \widehat{\alpha}_0^1(s)\Big(\frac{\partial^{\delta_1}}{\partial t^{\delta_1}}\widehat{g}^1_{(-1)^n M}(t,s)\big|_{t=b}\\
&+\sum_{j=n-\delta_1}^{n-1} (-1)^{n-j}\frac{\partial^{\delta_1+j-n}}{\partial t^{\delta_1+j-n}}\Big(p_{n-j}(t)\widehat{g}^1_{(-1)^n M}(t,s)\Big)\big|_{t=b}\Big) = 0,
\end{aligned}$$

$$\dots$$

$$\begin{aligned}
&(-1)^{\alpha}\sum_{j=n-\delta_1}^{n-1} (-1)^{n-j}\frac{\partial^{\delta_1+j-n+\alpha}}{\partial t^{\delta_1+j-n}\partial s^{\alpha}}\Big(p_{n-j}(t)\widehat{g}^1_{(-1)^n M}(t,s)\Big)\big|_{t=b}\\
&(-1)^{\alpha}\frac{\partial^{\delta_1+\alpha}}{\partial t^{\delta_1}\partial s^{\alpha}}\widehat{g}^1_{(-1)^n M}(t,s)\big|_{t=b} + \sum_{i=0}^{\alpha-1}\widehat{\alpha}_i^{\alpha}(s)\Big(\frac{\partial^{\delta_1+i}}{\partial t^{\delta_1}\partial s^{i}}\widehat{g}^1_{(-1)^n M}(t,s)\big|_{t=b}\\
&+\sum_{j=n-\delta_1}^{n-1} (-1)^{n-j}\frac{\partial^{\delta_1+j-n+i}}{\partial t^{\delta_1+j-n}\partial s^{i}}\Big(p_{n-j}(t)\widehat{g}^1_{(-1)^n M}(t,s)\Big)\big|_{t=b}\Big) = 0.
\end{aligned}$$

Since $b \neq a$, this system is satisfied for $s=a$. Taking into account that $\widehat{\alpha}_i^j \in C(I)$, we conclude that

$$\widehat{w}_M^{(\delta_1)}(b) + \sum_{j=n-\delta_1}^{n-1} (-1)^{n-j}\left(p_{n-j}\widehat{w}_M\right)^{(\delta_1+j-n)}(b) = 0.$$

Proceeding analogously with $\delta_2,\dots,\delta_k$, we can affirm that $\widehat{w}_M$ satisfies the boundary conditions (4.4)–(4.6).

We have proved that $\widehat{w}_M$ satisfies the boundary conditions (4.1)–(4.2) and (4.4)–(4.6), thus we can apply Proposition 7.4 to conclude that

- If $n-k$ is even and $k<n-1$, then $\widehat{w}_M > 0$ on

From (8.1), we obtain the following conclusion:

- If $n-k$ is even and $M \in [\bar{M}, \bar{M}-\lambda_2'']$: for each $s \in (a,b)$ there exists $\rho(s)>0$ such that $g_M(t,s)>0$ for all $t \in (a, a+\rho(s))$.

Step 4. Study of the related Green's function at $t=b$. To study the behavior at $t=b$, we consider the function

$$\widehat{y}_M(t) = (-1)^n\frac{\partial^{\beta}}{\partial s^{\beta}}\widehat{g}^2_{(-1)^n M}(t,s)\big|_{s=b}.$$

From (2.15), it is satisfied that

$$\widehat{y}_M(s) = \frac{\partial^\beta}{\partial t^\beta} g^1_M(t,s)\big|_{t=b}\,, \tag{8.5}$$

moreover, from the boundary conditions (1.5)-(1.6), if $\beta > 0$ we obtain

$$g_M(b,t) = \frac{\partial}{\partial s} g_M(t,s)\big|_{t=b} = \dots = \frac{\partial^{\beta-1}}{\partial s^{\beta-1}} g_M(t,s)\big|_{t=b} = 0\,.$$

As in the previous Steps, we can affirm that if there exists $t^* \in (a,b)$ such that either $\widehat{y}_M(t^*) < 0$ and β even or $\widehat{y}_M(t^*) > 0$ and β odd, then $T_n[M]$ is not inverse positive on $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$.

From Lemma 5.1, if $n-k$ is even, then $\widehat{y}_{\bar M} \ge 0$ if β is even and $\widehat{y}_{\bar M} \le 0$ if β is odd. Furthermore, we have

$$\widehat{T}_n[(-1)^n M]\,\widehat{y}_M(t) = 0\,, \quad t \in [a,b)\,.$$

Now, using similar arguments as before, we obtain that $\widehat{y}_M$ satisfies the boundary conditions (4.1)–(4.3) and (4.4)–(4.5). Moreover, it satisfies:

$$\widehat{y}_M^{(\delta_k)}(b) + \sum_{j=n-\delta_k}^{n-1} (-1)^{n-j} \left(p_{n-j}\widehat{y}_M\right)^{(\delta_k+j-n)}(b) = (-1)^{n-\beta+1} = (-1)^{\delta_k}\,.$$

Thus, we can apply Proposition 7.3 to conclude that

- If $n-k$ is even and $k > 1$, then $\widehat{y}_M > 0$ on (a,b) if β is even and $\widehat{y}_M < 0$ on (a,b) if β is odd for all $M \in [\bar M, \bar M - \lambda_2']$.
- If $k = 1$ and n is odd, then $\widehat{y}_M > 0$ on (a,b) if β is even and $\widehat{y}_M < 0$ on (a,b) if β is odd for all $M \ge \bar M$.

So, from this Step, we obtain the following conclusions:

- If $n-k$ is even, $k > 1$ and $M \in [\bar M, \bar M - \lambda_2']$: for each $s \in (a,b)$ there exists $\rho(s) > 0$ such that $g_M(t,s) > 0$ $\forall t \in (b-\rho(s), b)$.
- If $k = 1$, n is odd and $M \ge \bar M$: for each $s \in (a,b)$ there exists $\rho(s) > 0$ such that $g_M(t,s) > 0$ for all $t \in (b-\rho(s), b)$.

Step 5 Study of the related Green's function on $(a,b)\times(a,b)$. To finish the proof we only need to verify that $(-1)^{n-k} g_M(t,s) > 0$ for a.e. $(t,s) \in I \times I$ if M belongs to the given intervals. In fact, we will prove that $(-1)^{n-k} g_M(t,s) > 0$ on $(a,b)\times(a,b)$ for those M. To this end, for all $s \in (a,b)$, let us denote $u^s_M(t) = g_M(t,s)$.

By the definition of a Green's function it is known that for all $s \in (a,b)$:

$$T_n[\bar M] u^s_M(t) = \left(\bar M - M\right) u^s_M(t)\,, \quad \forall t \ne s\,,\ t \in I\,. \tag{8.6}$$

Moreover, $u^s_M \in C^{n-2}(I)$ and it satisfies the boundary conditions (1.5)-(1.6).

From Lemma 5.1, it is known that $(-1)^{n-k} u^s_{\bar M} \ge 0$ on I. Now, moving continuously with M, we will verify that while u^s_M is of constant sign on I, it cannot have a double zero on (a,b), which implies that the sign change must be either at $t = a$ or $t = b$ and then the result is proved. We study separately the cases where $n-k$ is even or odd.

First, let us assume that $n-k$ is even. In this case, from Theorem 2.16, we only need to study the behavior for $M > \bar M$ and $u^s_M \ge 0$. From (8.6), we have that $T_n[\bar M] u^s_M \le 0$; hence, since $v_1 \dots v_n > 0$, $\frac{1}{v_n} T_{n-1} u^s_M$ is a decreasing function, with two continuous components. Then, it has at most two zeros on I (see Figure 1).

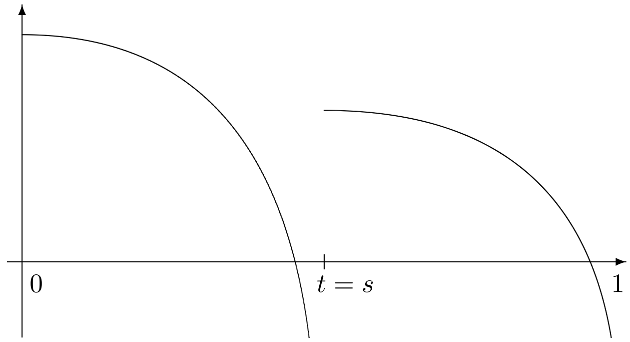

FIGURE 1. $\frac{1}{v_n(t)} T_{n-1} u_M^s(t)$, maximal oscillation with $t \in I = [0, 1]$

Although we cannot know the increasing or decreasing intervals of $T_{n-1} u_M^s$; since $v_n > 0$, it has the same sign as $\frac{1}{v_n} T_{n-1} u_M^s$. Thus, $T_{n-1} u_M^s$ has at most two zeros on I. So, $\frac{1}{v_{n-1}} T_{n-2} u_M^s$ is a continuous function, with at most four zeros on I (see Figure 2).

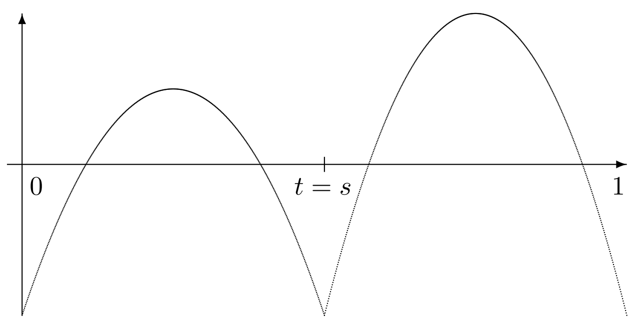

FIGURE 2. $\frac{1}{v_{n-1}(t)} T_{n-2} u_M^s(t)$, maximal oscillation with $t \in I = [0, 1]$

Again, since $v_{n-1} > 0$, it follows that $T_{n-2} u_M^s$ has the same sign as $\frac{1}{v_{n-1}} T_{n-2} u_M^s$. So, $T_{n-2} u_M^s$ has at most fourth zeros on I. Hence, $\frac{1}{v_{n-2}} T_{n-3} u_M^s$ has at most five zeros on I, the same as $T_{n-3} u_M^s$.

By recurrence, we conclude that $T_{n-\ell} u_M^s$ has, with maximal oscillation, at most $\ell + 2$ zeros on I. However, each time that either $T_{n-\ell} u_M^s(a) = 0$ or $T_{n-\ell} u_M^s(b) = 0$, a possible oscillation on (a, b) is lost. From the boundary conditions (1.5)-(1.6), coupled with Lemmas 3.6 and 3.7, we can affirm that n possible oscillations are lost. Hence, u_M^s can have at most two zeros on (a, b).

Let us see that, despite this fact does not inhibit that, with maximal oscillation, u_M^s has a double zero on (a, b), this double zero is not possible. If $T_\ell u(a) = 0$ for $\ell \notin \{\sigma_1, \dots, \sigma_k\}$, then u_s can have only a simple zero and this is not possible while it is of constant sign.

Now, to allow this possible double zero, let us study which should be the sign of $u^{(\alpha)}(a)$. We have already said that $T_{n-\ell} u_M^s(a)$ changes its sign for two consecutive

$\ell \in \{0, \dots, n\}$ if it does not vanish. Moreover, at every time that $T_{n-\ell} u_M^s(a) = 0$ the sign change comes on the next $\tilde{\ell}$ for which $T_{n-\tilde{\ell}} u_M^s(a) \neq 0$. Since, from $\ell = 0$ to $n - \alpha$ there are $k - \alpha$ zeros of $T_{n-\ell} u_M^s(a)$, to allow the maximal oscillation it is necessary to have

$$T_\alpha u_M^s(a) \begin{cases} \leq 0\,, & \text{if } n - \alpha - (k - \alpha) = n - k \text{ is even,} \\ \geq 0\,, & \text{if } n - k \text{ is odd.} \end{cases}$$

As a direct consequence of (6.1), we can affirm that with maximal oscillation it must be verified

$$u_M^s{}^{(\alpha)}(a) \begin{cases} \leq 0\,, & \text{if } n - k \text{ is even,} \\ \geq 0\,, & \text{if } n - k \text{ is odd.} \end{cases} \tag{8.7}$$

On the other hand, since $u_M^s \geq 0$, it must be satisfied that $u_M^s{}^{(\alpha)}(a) \geq 0$. We can assume that $u_M^s{}^{(\alpha)}(a) > 0$. Because, in other case, i.e. if $u_M^s{}^{(\alpha)}(a) = 0$, then $T_\alpha u_M^s(a) = 0$ and another possible oscillation is lost, so it only remains the possibility of having a simple zero on (a, b), which is not possible when u_M^s is of constant sign.

If $u_M^s{}^{(\alpha)}(a) > 0$, from (8.7) the maximal oscillation is not allowed. So, we have again only the possibility of a simple zero on (a, b). Hence we conclude:

- If $n - k$ even and $M > \bar{M}$, if $u_M^s \geq 0$, then $u_M^s > 0$ on (a, b).

Thus, combining these assertions with the previous Steps the result is proved. □

Example 8.2. In Example 6.6 the eigenvalues related to operator $T_4^0[0]$ the different sets, $X_{\{0,2\}}^{\{1,2\}}$, $X_{\{0,2\}}^{\{0,1\}}$ and $X_{\{0,1\}}^{\{1,2\}}$, have been obtained. They are denoted by λ_1, λ_2' and λ_2'', respectively. We have that $\lambda_1 = m_1^4$ and

$$\lambda_2 = \max\{\lambda_2', \lambda_2''\} = \max\{-m_2^4, -4\pi^4\} = -4\pi^4\,,$$

where $m_1 \approxeq 2.36502$ and $m_2 \approxeq 5.550305$ have been introduced in Example 6.6 as the least positive solutions of (6.2) and (6.3), respectively.

So, we can affirm that $T_4^0[M]$ is a strongly inverse positive operator on $X_{\{0,2\}}^{\{1,2\}}$ if and only if $M \in (-m_1^4, 4\pi^4]$

Remark 8.3. In Steps 1 and 2, to obtain that w_M and y_M satisfy the boundary conditions (6.5)-(6.6) and (6.17)-(6.18), respectively, we do not need to impose that the operator $T_n[\bar{M}]$ satisfies property (T_d).

Taking into account the previous Remark, we obtain the following result.

Theorem 8.4. *If either $\sigma_k = k - 1$ or $\varepsilon_{n-k} = n - k - 1$, we have the following properties:*

- *If $n-k$ is even, then there is no $M \in \mathbb{R}$ such that $T_n[M]$ is inverse negative on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*
- *If $n - k$ is odd, then there is no $M \in \mathbb{R}$ such that $T_n[M]$ is inverse positive on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*

Proof. If $\sigma_k = k - 1$, then $\{\sigma_1, \dots, \sigma_k\} = \{0, \dots, k - 1\}$. We consider

$$w_M(t) = \frac{\partial^\eta}{\partial s^\eta} g_M^1(t, s)\big|_{s=a}\,,$$

defined in Step 1 of the proof of Theorem 8.1.

By the calculations done in the proof of the mentioned result, we conclude that for all $M \in \mathbb{R}$, w_M satisfies the following boundary conditions:

$$w_M(a) = \cdots = w_M^{(k-2)}(a) = 0\,, \quad w_M^{(k-1)}(a) = (-1)^{n-\sigma_k-1} = (-1)^{n-k}\,.$$

Hence, if $n-k$ is even, then there exists $\rho > 0$, such that $w_M(t) > 0$ for all $t \in (a, a+\rho)$. So, $T_n[M]$ cannot be inverse negative for any real M.

Now, if $n-k$ is odd, then there exists $\rho > 0$, such that $w_M(t) < 0$ for all $t \in (a, a+\rho)$. Thus, $T_n[M]$ cannot be inverse positive for any $M \in \mathbb{R}$. Analogously, if $\varepsilon_{n-k} = n-k-1$, then $\{\varepsilon_1, \dots, \varepsilon_{n-k}\} = \{0, \dots, n-k-1\}$ and $\gamma = n-\varepsilon_{n-k}-1 = k$. We consider now

$$y_M(t) = \frac{\partial^\gamma}{\partial s^\gamma} g_M^2(t,s)\big|_{s=b}\,,$$

defined in Step 2 of the proof of Theorem 8.1.

By the previous calculations, we conclude that for all $M \in \mathbb{R}$, y_M satisfies the following boundary conditions:

$$y_M(b) = \cdots = y_M^{(n-k-2)}(b) = 0\,, \quad y_M^{(n-k-1)}(b) = (-1)^{n-\varepsilon_{n-k}} = (-1)^{k+1}\,.$$

Hence, if $n-k$ and k are even, then there exists $\rho > 0$, such that $y_M(t) > 0$ for all $t \in (b-\rho, b)$. So, $T_n[M]$ cannot be inverse negative for any real M. Moreover, if $n-k$ is even and k odd, then there exists $\rho > 0$, such that $y_M(t) < 0$ for all $t \in (b-\rho, b)$. So, $T_n[M]$ cannot be inverse negative for any real M.

Now, if $n-k$ and k are odd, then there exists $\rho > 0$, such that $y_M(t) > 0$ for all $t \in (b-\rho, b)$. So, $T_n[M]$ cannot be inverse positive for any real M. Finally, if $n-k$ is odd and k even, then there exists $\rho > 0$, such that $y_M(t) < 0$ for all $t \in (b-\rho, b)$. As consequence, $T_n[M]$ cannot be inverse positive for any real M. $\square$

9. Particular Cases

This Section is devoted to show the applicability of Theorem 8.1 to particular situations. Let us consider $I = [0,1]$. We have showed every result for the particular case where $n = 4$, $\{\sigma_1, \sigma_2\} = \{0, 2\}$, $\{\varepsilon_1, \varepsilon_2\} = \{1, 2\}$ and $T_4^0[M]u(t) = u^{(4)}(t) + Mu(t)$, which satisfies the hypotheses of Theorem 8.1 for $\bar{M} = 0$.

If we wanted to study the strongly inverse positive character of $T_4^0[M]$ on $X_{\{0,2\}}^{\{1,2\}}$ without taking into account Theorem 8.1, we would have to study the related Green's function, which is given by the following expression for $M = m^4 > 0$, obtained by means of the Mathematica program developed in [7]:
If $0 \le s \le t \le 1$, then we have

$$\begin{aligned}
&e^{-\sqrt{2}m(s+t-2)}\Big(2e^{\frac{m(s+t-4)}{\sqrt{2}}}\sin\big(\frac{m(s-t)}{\sqrt{2}}\big) - e^{\frac{m(3s+t-2)}{\sqrt{2}}}\sin\big(\frac{m(s-t)}{\sqrt{2}}\big)\\
&+ e^{\frac{m(s+3t-6)}{\sqrt{2}}}\sin\big(\frac{m(s-t)}{\sqrt{2}}\big) - 2e^{\frac{m(3s+3t-4)}{\sqrt{2}}}\sin\big(\frac{m(s-t)}{\sqrt{2}}\big)\\
&+ e^{\frac{m(3s+t-4)}{\sqrt{2}}}\sin\big(\frac{m(s-t+2)}{\sqrt{2}}\big) - e^{\frac{m(s+3t-4)}{\sqrt{2}}}\sin\big(\frac{m(s-t+2)}{\sqrt{2}}\big)\\
&+ e^{\frac{m(s+t-4)}{\sqrt{2}}}\sin\big(\frac{m(s+t-2)}{\sqrt{2}}\big) - e^{\frac{m(3s+3t-4)}{\sqrt{2}}}\sin\big(\frac{m(s+t-2)}{\sqrt{2}}\big)\\
&+ \Big(-e^{\frac{m(s+t-2)}{\sqrt{2}}} + e^{\frac{3m(s+t-2)}{\sqrt{2}}} + 2e^{\frac{m(3s+t-4)}{\sqrt{2}}} - 2e^{\frac{m(s+3t-4)}{\sqrt{2}}}\Big)\sin\big(\frac{m(s+t)}{\sqrt{2}}\big)
\end{aligned}$$

$$
\begin{aligned}
&+\Big(e^{\frac{m(3s+t-2)}{\sqrt{2}}}+e^{\frac{m(s+3t-6)}{\sqrt{2}}}\Big)\cos\big(\frac{m(s-t)}{\sqrt{2}}\big)-e^{\frac{m(3s+3t-4)}{\sqrt{2}}}\cos\big(\frac{m(s+t-2)}{\sqrt{2}}\big)\\
&+\Big(e^{\frac{m(3s+t-4)}{\sqrt{2}}}+e^{\frac{m(s+3t-4)}{\sqrt{2}}}\Big)\cos\big(\frac{m(s-t+2)}{\sqrt{2}}\big)-e^{\frac{m(s+t-4)}{\sqrt{2}}}\cos\big(\frac{m(s+t-2)}{\sqrt{2}}\big)\\
&-e^{\frac{m(s+t-2)}{\sqrt{2}}}\Big(e^{\sqrt{2}m(s+t-2)}+1\Big)\cos\big(\frac{m(s+t)}{\sqrt{2}}\big)\Big)\\
&\div\Big(4\sqrt{2}m^3\Big(\sin(\sqrt{2}m)+\sinh(\sqrt{2}m)\Big)\Big)
\end{aligned}
$$

If $0<t<s\le 1$, then we have

$$
\begin{aligned}
&e^{-\frac{m(3s+t-6)}{\sqrt{2}}}\Big(e^{2\sqrt{2}m(s-1)}\sin\big(\frac{m(s-t-2)}{\sqrt{2}}\big)-e^{\sqrt{2}m(s+t-2)}\sin\big(\frac{m(s-t-2)}{\sqrt{2}}\big)\\
&+2e^{\sqrt{2}m(s-2)}\sin\big(\frac{m(s-t)}{\sqrt{2}}\big)-e^{\sqrt{2}m(2s-3)}\sin\big(\frac{m(s-t)}{\sqrt{2}}\big)+e^{\sqrt{2}m(s+t-1)}\\
&\times\sin\big(\frac{m(s-t)}{\sqrt{2}}\big)-2e^{\sqrt{2}m(2s+t-2)}\sin\big(\frac{m(s-t)}{\sqrt{2}}\big)+e^{\sqrt{2}m(s-2)}\sin\big(\frac{m(s+t-2)}{\sqrt{2}}\big)\\
&-e^{\sqrt{2}m(2s+t-2)}\sin\big(\frac{m(s+t-2)}{\sqrt{2}}\big)+\Big(e^{\sqrt{2}m(s+t-2)}+e^{2\sqrt{2}m(s-1)}\Big)\\
&\times\cos\big(\frac{m(s-t-2)}{\sqrt{2}}\big)+\Big(-2e^{\sqrt{2}m(s+t-2)}+e^{\sqrt{2}m(2s+t-3)}-e^{\sqrt{2}m(s-1)}\\
&+2e^{2\sqrt{2}m(s-1)}\Big)\sin\big(\frac{m(s+t)}{\sqrt{2}}\big)+\Big(e^{\sqrt{2}m(s+t-1)}+e^{\sqrt{2}m(2s-3)}\Big)\\
&\times\cos\big(\frac{m(s-t)}{\sqrt{2}}\big)-e^{\sqrt{2}m(2s+t-2)}\cos\big(\frac{m(s+t-2)}{\sqrt{2}}\big)\\
&-\Big(e^{\sqrt{2}m(2s+t-3)}+e^{\sqrt{2}m(s-1)}\Big)\cos\big(\frac{m(s+t)}{\sqrt{2}}\big)\\
&-e^{\sqrt{2}m(s-2)}\cos\big(\frac{m(s+t-2)}{\sqrt{2}}\big)\Big)\\
&\div\Big(2\sqrt{2}m^3\Big(e^{2\sqrt{2}m}+2e^{\sqrt{2}m}\sin(\sqrt{2}m)-1\Big)\Big)\,.
\end{aligned}
$$

With this example we can see the applicability of Theorem 8.1 to characterize the Green's function constant sign. The usefulness of the result increases in much more complicated problems, where its expression may be inapproachable. Moreover, in some cases, for instance in problems with non constant coefficients, we cannot even obtain its expression. So, Theorem 8.1 is very useful because it allows us to see which is the sign of the related Green's function without knowing its expression. We point out that to calculate the corresponding eigenvalues is very simple in the constant coefficient case and can be numerically approached in the non constant case.

Next, we see examples, where the applicability of Theorem 8.1 is shown.

In [12], there are studied the operators $T_n[M]$ in the spaces $X_{\{0,\dots,k-1\}}^{\{0,\dots,n-k-1\}}$ under the hypothesis that there exists $\bar{M}\in\mathbb{R}$ such that $T_n[\bar{M}]u(t)=0$ is a disconjugate equation on I.

In fact, it is proved there that in such a case $T_n[\bar{M}]$ satisfies the property (T_d). The result there obtained is a particular case of Theorem 8.1 when $\sigma_k=k-1$ and

$\varepsilon_{n-k} = n-k-1$. Moreover, since in such a case both $\sigma_k = k-1$ and $\varepsilon_{n-k} = n-k-1$, it is proved the correspondent to Theorem 8.4.

In [12], there are several examples, some of them with operators of non constant coefficients, such as

$$\bar{T}_4[M]u(t) = u^{(4)}(t) + e^{2t}u'(t) + Mu(t)\,, \quad t \in [0,1]\,,$$

on $X^{\{0,1\}}_{\{0,1\}}$, $X^{\{0\}}_{\{0,1,2\}}$ and $X^{\{0,1,2\}}_{\{0\}}$.

In [14], there is a particular type of fourth-order operators

$$T_4(p_1,p_2)[M]u(t) = u^{(4)}(t) + p_1(t)u^{(3)}(t) + p_2(t)u^{(2)}(t) + Mu(t)$$

on $X^{\{0,2\}}_{\{0,2\}}$, under the hypothesis that the second-order equation $u''(t)+p_1(t)u'(t)+p_2(t)u(t) = 0$ is disconjugate on I.

This allows us to prove that the operator $T_4(p_1,p_2)[0]$ satisfies property (T_d) on $X^{\{0,2\}}_{\{0,2\}}$. Hence, the result there obtained for the strongly inverse positive character is a particular case of Theorem 8.1.

In [14], there are also several examples of this type of operators coupled with the well-known simply supported beam boundary conditions. Again, some of the examples have non-constant coefficients, such as

$$\widehat{T}_4[M]u(t) = u^{(4)}(t) + 2tu^{(3)}(t) + 2u^{(2)}(t) + Mu(t)\,, \quad t \in [0,1]\,.$$

Before giving some results for this type of operator, we take into account the following remarks:

Remark 9.1. If we choose $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ satisfying (N_a), then the hypotheses of Theorem 8.1 are fulfilled for $\bar{M} = 0$ for the operator $T_n^0[\bar{M}]$ by choosing $v_1(t) = \dots = v_n(t) = 1$ for all $t \in I$.

Remark 9.2. For this type of operators the behavior on $X^{\{\sigma_1,\dots,\sigma_k\}}_{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ can be known by studying the behavior on $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$. This is so because the eigenvalues are the same if n is even or the opposed if n is odd.

Indeed, if u is a nontrivial solution of $u^{(n)}(t) + Mu(t) = 0$ on $X^{\{\sigma_1,\dots,\sigma_k\}}_{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, then $y(t) = u(1-t)$ is a solution of $u^{(n)}(t) + (-1)^n\, Mu(t) = 0$ on $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$.

So, we do not need to study all the cases to obtain conclusions about the strongly inverse positive (negative) character.

Next, we show different examples of this type of operators.

Second order. The only possibility in second order is to consider $k=1$. However, there are three options for the choice of $\{\sigma_1\} - \{\varepsilon_1\}$. First of them is $\sigma_1 = \varepsilon_1 = 0$ which correspond to the Dirichlet case, that is the boundary conditions $(1,1)$. This case has been considered in [12], where it is obtained that the operator $T_2^0[M]$ is strongly inverse negative on $X^{\{0\}}_{\{0\}}$ if, and only if $M \in (-\infty, \pi^2)$. The other two choices correspond to the mixed boundary conditions and are equivalent, $\sigma_1 = 0$ and $\varepsilon_1 = 1$ or $\sigma_1 = 1$ and $\varepsilon_1 = 0$.

The largest negative eigenvalue of $T_2^0[0]$ on $X^{\{1\}}_{\{0\}}$ is $\lambda_1 = -\frac{\pi^2}{4}$. So, using Theorem 8.1, we can affirm that $T_2^0[M]$ is a strongly inverse negative operator on $X^{\{1\}}_{\{0\}}$ or on $X^{\{0\}}_{\{1\}}$ if, and only if $M \in (-\infty, \frac{\pi^2}{4})$. Moreover, from Theorem 8.4, we can conclude that there is no $M \in \mathbb{R}$ such that $T_2^0[M]$ is strongly inverse positive either on $X_{\{0\}}^{\{1\}}$ or $X_{\{1\}}^{\{0\}}$.

Third order. In this case the number of possible cases increases to twelve, which can be reduced to six. The cases $\{\sigma_1, \sigma_2\} = \{0, 1\}$, $\{\varepsilon_1\} = \{0\}$ and $\{\sigma_1\} = \{0\}$, $\{\varepsilon_1, \varepsilon_2\} = \{0, 1\}$ have been considered in [12]. Let us see some of the rest.

First, let us consider $\{\sigma_1, \sigma_2\} = \{1, 2\}$ and $\{\varepsilon_1\} = \{0\}$. The largest negative eigenvalue of $T_3^0[0]$ on $X_{\{1,2\}}^{\{0\}}$ is $\lambda_1 = -m_4^3$, where $m_4 \approxeq 1.85$ is the least positive solution of

$$e^{-m} + 2\, e^{m/2} \cos\big(\frac{\sqrt{3}}{2} m\big) = 0\,. \tag{9.1}$$

To apply Theorem 8.1, we need to obtain the least positive eigenvalue of $T_3^0[0]$ on $X_{\{1\}}^{\{0,1\}}$, which is $\lambda_2 = m_5^3$, where $m_5 \approxeq 3.017$ is the least positive solution of

$$e^{-m} - e^{m/2}\Big(\cos\big(\frac{\sqrt{3}}{2} m\big) + \sqrt{3} \sin\big(\frac{\sqrt{3}}{2} m\big)\Big) = 0\,. \tag{9.2}$$

Since, $k = 2 = n - 1$, we can apply Theorem 8.1 to affirm that $T_3^0[M]$ is strongly inverse negative on $X_{\{1,2\}}^{\{0\}}$ if and only if $M \in [-m_5^3, m_4^3) \approxeq [-3.017^3, 1.85^3)$.

From Theorem 8.4, we can conclude that there is no $M \in \mathbb{R}$ such that $T_3^0[M]$ is strongly inverse positive on $X_{\{1,2\}}^{\{0\}}$. Now, from Remark 9.2, we can affirm that $T_3^0[M]$ is strongly inverse positive in $X_{\{0\}}^{\{1,2\}}$ if and only if $M \in (-m_4^3, m_5^3] \approxeq (-1.85^3, 3.017^3]$. Moreover, we can conclude that there is no $M \in \mathbb{R}$ such that $T_3^0[M]$ is strongly inverse negative in $X_{\{0\}}^{\{1,2\}}$.

Now, let us consider $\{\sigma_1, \sigma_2\} = \{0, 2\}$ and $\{\varepsilon_1\} = \{1\}$. The largest negative eigenvalue of $T_3^0[0]$ on $X_{\{0,2\}}^{\{1\}}$ is $\lambda_1 = -m_4^3$, where m_4 has been defined as the least positive solution of (9.1). Moreover, the least positive eigenvalue on $X_{\{0\}}^{\{0,1\}}$ is $\lambda_2 = m_6^3$, where $m_6 \approxeq 4.223$ is the least positive solution of

$$e^{-m} - e^{m/2}\Big(\cos\big(\frac{\sqrt{3}}{2} m\big) - \sqrt{3} \sin\big(\frac{\sqrt{3}}{2} m\big)\Big) = 0\,. \tag{9.3}$$

Thus, from Theorem 8.1, we conclude that $T_3^0[M]$ is strongly inverse negative on $X_{\{0,2\}}^{\{1\}}$ if and only if $M \in [-m_6^3, m_4^3) \approxeq [-4.223^3, 1.85^3)$. We note that in this case $\sigma_2 = 2 > 1$ and $\varepsilon_1 = 1 > 0$, thus we cannot apply Theorem 8.4 to obtain conclusions about the strongly inverse positive character of $T_3^0[M]$ on $X_{\{0,2\}}^{\{1\}}$. However, from Remark 9.2, we can affirm that $T_3^0[M]$ is strongly inverse positive on $X_{\{1\}}^{\{0,2\}}$ if and only if $M \in (-m_4^3, m_6^3] \approxeq (-1.85^3, 4.223^3]$.

Fourth order. There are forty possibilities, which can be decreased, by using Remark 9.2, to twenty one. There are three possibilities which have been studied in [12], they are represented on the sets $X_{\{0,1,2\}}^{\{0\}}$, $X_{\{0\}}^{\{0,1,2\}}$ and $X_{\{0,1\}}^{\{0,1\}}$. The characterization on $X_{\{0,2\}}^{\{0,2\}}$ has been obtained in [14]. Moreover, along the paper we have studied the case $X_{\{0,2\}}^{\{1,2\}}$. From Remark 9.2, the obtained characterization remains valid for the set $X_{\{1,2\}}^{\{0,2\}}$.

Next we see a pair of different cases. For instance, $X_{\{1,2,3\}}^{\{0\}}$ (which also gives the characterization on $X_{\{0\}}^{\{1,2,3\}}$) and $X_{\{1,3\}}^{\{0,2\}}$ ($X_{\{0,2\}}^{\{1,3\}}$).

Let us work on the space $X_{\{1,2,3\}}^{\{0\}}$. First, we obtain the necessary eigenvalues in order to apply Theorem 8.1 to this case: The largest negative eigenvalue of $T_4^0[0]$ on $X_{\{1,2,3\}}^{\{0\}}$ is $\lambda_1 = -\pi^4/4$. The least positive eigenvalue of $T_4^0[0]$ on $X_{\{1,2\}}^{\{0,1\}}$ is $\lambda_2 = \pi^4$. Then, $T_4^0[M]$ is strongly inverse negative on $X_{\{1,2,3\}}^{\{0\}}$ if and only if $M \in [-\pi^4, \frac{\pi^4}{4})$. Since $\varepsilon_1 = 0$, we can apply Theorem 8.4 to conclude that there is no $M \in \mathbb{R}$ such that $T_4^0[M]$ is strongly inverse positive on $X_{\{1,2,3\}}^{\{0\}}$.

Concerning to the space $X_{\{0,2\}}^{\{1,3\}}$, we have the following eigenvalues: The least positive eigenvalue of $T_4^0[0]$ on $X_{\{0,2\}}^{\{1,3\}}$ is $\lambda_1 = \pi^4/16$. The largest negative eigenvalue of $T_4^0[0]$ on $X_{\{0,1,2\}}^{\{1\}}$ is $\lambda_2' = -4\pi^4$. The largest negative eigenvalue of $T_4^0[0]$ on $X_{\{0\}}^{\{0,1,3\}}$ is $\lambda_2'' = -4\pi^4$. So, $\lambda_2 = \max\{-4\pi^4, -4\pi^4\} = -4\pi^4$. Hence, we conclude, from Theorem 8.1, that $T_4^0[M]$ is strongly inverse positive on $X_{\{0,2\}}^{\{1,3\}}$ if and only if $M \in (-\pi^4/16, 4\pi^4]$. Since $\sigma_2 = 2 > 1$ and $\varepsilon_2 = 3 > 1$, we cannot apply Theorem 8.4 to affirm that it cannot be strongly inverse negative for any $M \in \mathbb{R}$.

Higher order. If we increase the order of the problem, because the related Green's function gets more complex, the usefulness of Theorem 8.1 also increases. Even if we cannot obtain the eigenvalues analytically, we can obtain them numerically, using different methods.

Let us show an example of sixth order, where we can obtain the eigenvalues analytically. The largest negative eigenvalue of $T_6^0[0]$ on $X_{\{0,2,4\}}^{\{0,2,4\}}$ is $\lambda_1 = -\pi^6$. The least positive eigenvalue of $T_6^0[0]$ on $X_{\{0,1,2,4\}}^{\{0,2\}}$ and $X_{\{0,2\}}^{\{0,1,2,4\}}$ is $\lambda_2 = \lambda_2' = \lambda_2'' = m_7^6$, where $m_7 \approxeq 5.47916$ is the least positive solution of

$$\cos\big(\sqrt{3}m\big) - \cosh(m) + 8\cos\big(\frac{\sqrt{3}m}{2}\big)\sinh^2\big(\frac{m}{2}\big)\cosh\big(\frac{m}{2}\big) = 0\,.$$

Hence, from Theorem 8.1, we conclude that $T_6^0[M]$ is a strongly inverse negative operator on $X_{\{0,2,4\}}^{\{0,2,4\}}$ if and only if $M \in [-m_7^7, \pi^6) \approxeq [-5.47916^6, \pi^6)$. Since $\sigma_3 = 4 > 2$ and $\varepsilon_3 = 4 > 2$, we cannot apply Theorem 8.4 to obtain any conclusion about the strongly inverse positive character.

Now we consider the operator

$$T_4^N[M]u(t) = u^{(4)}(t) + Nu'(t) + Mu(t) \quad \text{on } X_{\{0,2\}}^{\{1,2\}}.$$

When $N = n^3$. the fourth order operator is $T_4^{n^3}[M]u(t) = u^{(4)}(t) + n^3u'(t) + Mu(t)$ on $X_{\{0,2\}}^{\{1,2\}}$. Note that for $n = 0$ this operator coincides with the example that we have been considering in the different examples in this article.

Let us see that for $n \in (-\frac{4\pi}{3\sqrt{3}}, \frac{2\pi}{3\sqrt{3}})$, $T_4^{n^3}[0]$ satisfies property (T_d) on $X_{\{0,2\}}^{\{1,2\}}$. To show that, we consider the fundamental system of solutions

$$y_1^n(t) = 1\,, \quad y_2^n(t) = \sqrt{3}\,e^{nt/2}\cos\big(\frac{\sqrt{3}}{2}n\,t\big) + e^{nt/2}\sin\big(\frac{\sqrt{3}}{2}n\,t\big)\,,$$

$$y_3^n(t) = e^{nt/2}\sin\big(\frac{\sqrt{3}}{2}n\,t\big)\,, \quad y_4^n(t) = e^{-nt}\,,$$

and the correspondent Wronskians:

$$W_1^n(t) = 1\,, \quad W_2^n(t) = \frac{1}{2}\, e^{nt/2}\, n^2 \Big(\cos\big(\frac{\sqrt{3}}{2} n\, t\big) - \sin\big(\frac{\sqrt{3}}{2} n\, t\big)\Big)\,,$$
$$W_3^n(t) = \frac{3}{2}\, n^3\, e^{n\,t}\,, \quad W_4^n(t) = -\frac{9\, n^6}{2}\,.$$

If, $n \neq 0$, then W_1^n, W_3^n and W_4^n are non null in $[0,1]$. Moreover, it can be seen that if $n \in (-\frac{4\pi}{3\sqrt{3}}, \frac{2\pi}{3\sqrt{3}})$, then $W_2^n(t) \neq 0$ for all $t \in [0,1]$. So, we can obtain the representation given in (3.1)-(3.2).

We construct $v_1, \dots, v_4$ following the proof of Theorem 2.4 ([15, Theorem 2, Chapter 3]):

$$v_1^n(t) = 1\,, \quad v_2^n(t) = W_2^n(t) = \frac{1}{2}\, e^{nt/2}\, n^2 \Big(\cos\big(\frac{\sqrt{3}}{2} n\, t\big) - \sin\big(\frac{\sqrt{3}}{2} n\, t\big)\Big)\,,$$
$$v_3^n(t) = \frac{W_3^n(t)}{{W_2^n}^2(t)}\,, \quad v_4^n(t) = \frac{W_2^n(t) W_4^n(t)}{{W_3^n}^2(t)}\,.$$

In Example 4.6, we have proved that a fourth-order operator satisfies property (T_d) on $X_{\{0,2\}}^{\{1,2\}}$ if and only if there exists the decomposition (3.1)-(3.2) and (4.20)-(4.21) are fulfilled. Let us check it. Obviously, (4.21) is satisfied. Now, since ${v_1^n}'(0) = 0$ and $v_2^n(0) \neq 0$, we have to verify that ${v_2^n}'(0) = 0$. But, from the fact that

$${v_2^n}'(t) = -2\, e^{nt/2} \sin\big(\frac{\sqrt{3}}{2} n\, t\big)\,,$$

we deduce that it is trivially satisfied that ${v_2^n}'(0) = 0$. So, as a consequence, (4.20) is fulfilled and we conclude that if $n \in (-\frac{4\pi}{3\sqrt{3}}, \frac{2\pi}{3\sqrt{3}})$, then $T_4^{n^3}[0]$ satisfies the property (T_d) on $X_{\{0,2\}}^{\{1,2\}}$.

Remark 9.3. The interval $(-\frac{4\pi}{3\sqrt{3}}, \frac{2\pi}{3\sqrt{3}})$ is not necessarily optimal. If we study the disconjugacy set of $T_4^{n^3}[0]u(t) = 0$ on $[0,1]$, we obtain that such equation is disconjugate if and only if $n \in (-n_1, n_1)$, where $n_1 \approxeq 5.55$ is the least positive solution of

$$-3 + e^{-n} + 2^{n/2} \cos\big(\frac{\sqrt{3}\, n}{2}\big) = 0\,.$$

Then, it is possible that we may find different values of $n \in (-n_1, n_1)$ such that $T_4^{n^3}[0]$ satisfies property (T_d) on $X_{\{0,2\}}^{\{1,2\}}$ with a suitable choice of the fundamental system of solutions.

For instance, repeating the previous arguments for the fundamental system of solutions

$$y_1^n(t) = 1\,, \quad y_2^n(t) = \frac{2}{\sqrt{3}}\, e^{nt/2} \sin\big(\frac{\sqrt{3}}{2} n\, t\big) - e^{-nt}\,,$$
$$y_3^n(t) = e^{-nt}\,, \quad y_4^n(t) = e^{nt/2} \sin\big(\frac{\sqrt{3}}{2} n\, t\big)\,,$$

we obtain a decomposition to ensure that $T_4^{n^3}[0]$ verify property (T_d) for $n \in (-\frac{\pi}{\sqrt{3}}, \frac{\pi}{\sqrt{3}})$. Thus, we can say that for

$$n \in \big(-\frac{4\pi}{3\sqrt{3}}, \frac{2\pi}{3\sqrt{3}}\big) \cup \big(-\frac{\pi}{\sqrt{3}}, \frac{\pi}{\sqrt{3}}\big) = \big(-\frac{4\pi}{3\sqrt{3}}, \frac{\pi}{\sqrt{3}}\big) \subset (-n_1, n_1),$$

then $T_4^{n^3}[0]$ satisfies property (T_d). However, we cannot even affirm that such an interval is the optimal one.

Let us choose, for instance, $n = -\frac{\pi}{\sqrt{3}} \in (-\frac{4\pi}{3\sqrt{3}}, \frac{\pi}{\sqrt{3}}) \subset (-n_1, n_1)$ and we obtain the different eigenvalues numerically, by using Mathematica.

The least positive eigenvalue of $T_4^{-\frac{\pi^3}{3\sqrt{3}}}[0]$ on $X_{\{0,2\}}^{\{1,2\}}$ is $\lambda_1 \approxeq 2.21152^4$. The largest negative eigenvalue of $T_4^{-\frac{\pi^3}{3\sqrt{3}}}[0]$ on $X_{\{0,1,2\}}^{\{1\}}$ is $\lambda_2' \approxeq -4.53073^4$. The largest negative eigenvalue of $T_4^{-\frac{\pi^3}{3\sqrt{3}}}[0]$ on $X_{\{0\}}^{\{0,1,2\}}$ is $\lambda_2'' \approxeq -5.5014^4$. So, $\lambda_2 = \max\{\lambda_2', \lambda_2''\} = \lambda_2' \approxeq -4.53073^4$. From Theorem 8.1, we conclude that $T_4^{-\frac{\pi^3}{3\sqrt{3}}}[M]$ is strongly inverse positive on $X_{\{0,2\}}^{\{1,2\}}$ if and only if $M \in (-\lambda_1, -\lambda_2] \approxeq (-2.21152^4, 4.53073^4]$. Since $\sigma_2 = \varepsilon_2 = 2 > 1$, we cannot obtain any conclusion about the strongly inverse positive character from Theorem 8.4.

10. Necessary condition for the strongly inverse negative (positive) character of $T_n[M]$ on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$

In the previous section we have obtained a characterization of the parameter's set where the operator $T_n[M]$ is either strongly inverse positive or negative on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ if $n-k$ is even or odd, respectively.

In some cases, we can ensure that if $n-k$ is even, then there is no $M \in \mathbb{R}$ such that $T_n[M]$ is strongly inverse negative on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ and if $n-k$ is odd, then there is no $M \in \mathbb{R}$ such that $T_n[M]$ is strongly inverse positive on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$. However, on the cases which do not fulfill the hypotheses of Theorem 8.4, we have not said anything about the strongly inverse negative character if $n-k$ is even or about the strongly inverse positive character if $n-k$ is odd.

From Theorems 2.16 and 2.18, if $n-k$ is even and there exists $\bar{M} \in \mathbb{R}$ such that $T_n[\bar{M}]$ is strongly inverse negative on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, then the parameter's set, for which such a property is fulfilled, is given by an interval whose supremum is given by $\bar{M} - \lambda_1$. Moreover, from Theorems 2.17 and 2.19, if $n-k$ is odd and there exists $\bar{M} \in \mathbb{R}$ such that $T_n[\bar{M}]$ is strongly inverse positive on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, then the parameter's set, for which such a property is fulfilled, is given by an interval whose infimum is given by $\bar{M} - \lambda_1$.

This section is devoted to obtain a bound of the other extreme of the interval. Furthermore, we will see that, in such an interval, the Green's function satisfies a suitable property which allows to prove that the obtained interval is optimal if we prove that the Green's function cannot have any zero on $(a,b) \times (a,b)$.

Theorem 10.1. *Let $\bar{M} \in \mathbb{R}$ be such that $T_n[\bar{M}]$ satisfies property (T_d) on the set $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ and $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ fulfill (N_a). Then, the following properties are satisfied:*

- *If $n-k$ is even and $T_n[M]$ is inverse negative on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, then $M \in [\bar{M} - \lambda_3, \bar{M} - \lambda_1)$, where*
  - * *$\lambda_1 > 0$ is the least positive eigenvalue of $T_n[\bar{M}]$ on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*
  - * *$\lambda_3 > 0$ is the minimum between:*

  - $\cdot$ $\lambda_3' > 0$, *the least positive eigenvalue of* $T_n[\bar{M}]$ *on* $X_{\{\sigma_1,\dots,\sigma_{k-1}|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.
  - $\cdot$ $\lambda_3'' > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *on the set* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}}$.

- *If* $n-k$ *is odd and* $T_n[M]$ *is inverse positive on* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, *then* $M \in (\bar{M}-\lambda_1, \bar{M}-\lambda_3]$, *where*
  - * $\lambda_1 < 0$ *is the largest negative eigenvalue of* $T_n[\bar{M}]$ *on* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.
  - * $\lambda_3 < 0$ *is the maximum between:*
    - $\cdot$ $\lambda_3' < 0$, *the largest negative eigenvalue of* $T_n[\bar{M}]$ *on the set* $X_{\{\sigma_1,\dots,\sigma_{k-1}|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.
    - $\cdot$ $\lambda_3'' < 0$ *is the largest negative eigenvalue of* $T_n[\bar{M}]$ *on the set* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}}$.

Proof. From Theorem 8.4, we can affirm that $\sigma_k \neq k-1$ and $\varepsilon_{n-k} \neq n-k-1$. Hence, by Corollary 6.5 the existence of λ_3' and λ_3'' is ensured.

First, let us focus on the case where $n-k$ is even. Let us assume that there exists $M^* \notin [\bar{M}-\lambda_3, \bar{M}-\lambda_1)$, such that $T_n[M^*]$ is inverse negative on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$. From Theorem 2.16, we know that $M^* < \bar{M}-\lambda_3$. Moreover, using Theorem 2.18, we can affirm that for all $M \in [M^*, \bar{M}-\lambda_1)$ the operator $T_n[M]$ is inverse negative on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ and, by Theorem 2.10, that $0 \geq g_{M^*}(t,s) \geq g_M(t,s) \geq g_{\bar{M}-\lambda_3}(t,s)$. So, in particular

$$0 \geq w_{M^*}(t) \geq w_M(t) \geq w_{\bar{M}-\lambda_3}(t)\,,$$

and

$$\begin{gathered} 0 \geq y_{M^*}(t) \geq y_M(t) \geq y_{\bar{M}-\lambda_3}(t)\,, \quad \text{if } \gamma \text{ is even}, \\ 0 \leq y_{M^*}(t) \leq y_M(t) \leq y_{\bar{M}-\lambda_3}(t)\,, \quad \text{if } \gamma \text{ is odd}. \end{gathered}$$

If $\lambda_3 = \lambda_3'$, then $w_{\bar{M}-\lambda_3}^{(\alpha)}(a) = 0$. So, we conclude that, for all $M \in [M^*, \bar{M}-\lambda_3)$, $w_M^{(\alpha)}(a) = 0$, which contradicts the discrete character of the spectrum of $T_n[\bar{M}]$ on $X_{\{\sigma_1,\dots,\sigma_{k-1}|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.

If $\lambda_3 = \lambda_3''$, then $y_{\bar{M}-\lambda_3}^{(\beta)}(b) = 0$. So, we conclude that, for all $M \in [M^*, \bar{M}-\lambda_3)$, $y_M^{(\beta)}(b) = 0$, which contradicts the discrete character of the spectrum of $T_n[\bar{M}]$ on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}}$.

Analogously, if $n-k$ is odd and we assume that there exists $M^* \notin (\bar{M}-\lambda_1, \bar{M}-\lambda_3]$, such that $T_n[M^*]$ is inverse positive on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$. From Theorem 2.17, we know that $M^* > \bar{M}-\lambda_3$.

Using Theorem 2.19, we can affirm that for all $M \in (\bar{M}-\lambda_1, M^*]$ $T_n[M]$ is inverse positive on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ and, by Theorem 2.10, that $g_{\bar{M}-\lambda_3}(t,s) \geq g_M(t,s) \geq g_{M^*}(t,s) \geq 0$. So, in particular

$$w_{\bar{M}-\lambda_3}(t) \geq w_M(t) \geq w_{M^*}(t) \geq 0\,,$$

and

$$\begin{gathered} y_{\bar{M}-\lambda 3}(t) \geq y_M(t) \geq y_{M^*}(t) \geq 0\,, \quad \text{if } \gamma \text{ is even}, \\ y_{\bar{M}-\lambda 3}(t) \leq y_M(t) \leq y_{M^*}(t) \leq 0\,, \quad \text{if } \gamma \text{ is odd}. \end{gathered}$$

If $\lambda_3=\lambda_3'$, then $w^{(\alpha)}_{\bar M-\lambda_3}(a)=0$. So, we conclude that, for all $M\in(\bar M-\lambda_3,M^*]$, $w^{(\alpha)}_M(a)=0$, which contradicts the discrete character of the spectrum of $T_n[\bar M]$ on $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_{k-1}|\alpha\}}$.

If $\lambda_3=\lambda_3''$, then $y^{(\beta)}_{\bar M-\lambda_3}(b)=0$. So, we conclude that, for all $M\in(\bar M-\lambda_3,M^*]$, $y^{(\beta)}_M(b)=0$, which contradicts the discrete character of the spectrum of $T_n[\bar M]$ on $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}}_{\{\sigma_1,\dots,\sigma_k\}}$.

In all cases we arrive to a contradiction, thus the result is proved. □

Example 10.2. Now, let us focus again on our recurrent example $T_4^0[M]u(t)=u^{(4)}(t)+Mu(t)$. In Example 6.6, the different eigenvalues of $T_4^0[0]$ on the sets $X^{\{1,2\}}_{\{0,2\}}$, $X^{\{1,2\}}_{\{0,1\}}$ and $X^{\{0,1\}}_{\{0,2\}}$ are obtained. In particular, $\lambda_1=m_1^4$ and

$$\lambda_3=\min\{m_3^4,\pi^4\}=\pi^4\,,$$

where $m_1\approxeq 2.36502$ and $m_3\approxeq 3.9266$ have been introduced in Example 6.6 as the least positive solutions of (6.2) and (6.4), respectively. So, using Theorem 10.1, we can affirm that if $T_4^0[M]$ is strongly inverse negative on $X^{\{1,2\}}_{\{0,2\}}$, then $M\in[-\pi^4,-m_1^4)$.

In Theorem 10.1, we have established a necessary condition on operator $T_n[M]$ to be either inverse positive or inverse negative on $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$. Next result shows that this condition also ensures that the related Green's function satisfies a suitable condition on the boundary of $I\times I$.

Theorem 10.3. *Let $\bar M\in\mathbb{R}$ be such that $T_n[\bar M]$ satisfies property (T_d) on the set $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$ and $\{\sigma_1,\dots,\sigma_k\}-\{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ fulfill (N_a). Moreover, $\sigma_k\neq k-1$ and $\varepsilon_{n-k}\neq n-k-1$. Then, the following properties are satisfied:*

- *If $n-k$ is even and $M\in[\bar M-\lambda_3,\bar M-\lambda_1)$, where λ_1 and λ_3 are given in Theorem 10.1. Then: for each $t\in(a,b)$ there exists $\rho_1(t)>0$ such that*
$$g_M(t,s)<0\quad\forall s\in(a,a+\rho_1(t))\cup(b-\rho_1(t),b)\,,$$
and for each $s\in(a,b)$ there exists $\rho_2(s)>0$ such that
$$g_M(t,s)<0\quad\forall t\in(a,a+\rho_2(s))\cup(b-\rho_2(s),b)\,.$$
- *If $n-k$ is odd and $M\in(\bar M-\lambda_1,\bar M-\lambda_3]$, where λ_1 and λ_3 are given in Theorem 10.1. Then: for each $t\in(a,b)$ there exists $\rho_1(t)>0$ such that*
$$g_M(t,s)>0\quad\forall s\in(a,a+\rho_1(t))\cup(b-\rho_1(t),b)\,,$$
and for each $s\in(a,b)$, there exists $\rho_2(s)>0$ such that
$$g_M(t,s)>0\quad\forall t\in(a,a+\rho_2(s))\cup(b-\rho_2(s),b)\,.$$

Proof. To prove this result we consider the following functions introduced in the proof of Theorem 8.1:

$$w_M(t)=\frac{\partial^\eta}{\partial s^\eta}g_M(t,s)\big|_{s=a}\,,$$
$$y_M(t)=\frac{\partial^\gamma}{\partial s^\gamma}g_M(t,s)\big|_{s=b}\,,$$
$$\widehat{w}_M(t)=(-1)^n\frac{\partial^\alpha}{\partial s^\alpha}\widehat{g}_M(t,s)\big|_{t=a}\,,$$

$$\widehat{y}_M(s) = (-1)^n \frac{\partial^\beta}{\partial s^\beta} \widehat{g}_M(t,s)\Big|_{t=b}\,.$$

For these functions we obtained the following conclusions:

- $T_n[M]\, w_M(t) = 0$ for all $t \in (a,b]$ and w_M satisfies boundary conditions (6.5)-(6.6).
- $T_n[M]\, y_M(t) = 0$ for all $t \in [a,b)$ and y_M satisfies boundary conditions (6.17)-(6.18).
- $\widehat{T}_n[(-1)^n M]\, \widehat{w}_M(s) = 0$ for all $s \in (a,b]$ and $\widehat{w}_M$ satisfies boundary conditions (4.1)–(4.2) and (4.4)–(4.6).
- $\widehat{T}_n[(-1)^n M]\, \widehat{y}_M(s) = 0$ for all $s \in [a,b)$ and $\widehat{y}_M$ satisfies boundary conditions (4.1)–(4.3) and (4.4)–(4.5).

Thus, by applying Propositions 6.7, 6.9, 7.6 and 7.7, we know that w_M, y_M, $\widehat{w}_M$ and $\widehat{y}_M$ do not have any zero in (a,b) for all $M \in [\bar{M}-\lambda_3, \bar{M}-\lambda_1)$ if $n-k$ is even and for all $M \in (\bar{M}-\lambda_1, \bar{M}-\lambda_3]$ when $n-k$ is odd.

Moreover, by Proposition 6.12, since we do not reach any eigenvalue of $T_n[\bar{M}]$ on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, we have that the related Green's function is well-defined for every M in those intervals. So, since we are moving continuously on M, we conclude that its sign is the same in all the interval.

Let us, now, study the sign of these functions at a given M. We consider w_M and $\widehat{w}_M$ at $M = \bar{M}-\lambda_3'$ and y_M and $\widehat{y}_M$ at $M = \bar{M}-\lambda_3''$. As we have proved before, at this values of the real parameter the functions are of constant sign and satisfy the maximal oscillation, which means that verify the conditions at $t=a$ and $t=b$ to give the maximum number of zeros with the related boundary conditions, otherwise the function would be equivalent to zero and this is not true. Moreover, we know that they satisfy for all $M \in \mathbb{R}$ the following properties:

$$w_M^{(\sigma_k)}(a) = (-1)^{(n-1-\sigma_k)}\,, \tag{10.1}$$

$$y_M^{(\varepsilon_{n-k})}(b) = (-1)^{(n-\varepsilon_{n-k})}\,, \tag{10.2}$$

$$\widehat{w}_M^{(\tau_{n-k})}(a) + \sum_{j=n-\tau_{n-k}}^{n-1} (-1)^{n-j} (p_{n-j}\widehat{w}_M)^{(\tau_{n-k}+j-n)}(a) = (-1)^{1+\tau_{n-k}}\,, \tag{10.3}$$

$$\widehat{y}_M^{(\delta_k)}(b) + \sum_{j=n-\delta_k}^{n-1} (-1)^{n-j} (p_{n-j}\widehat{y}_M)^{(\delta_k+j-n)}(b) = (-1)^{\delta_k}\,. \tag{10.4}$$

• Study of $w_{\bar{M}-\lambda_3'}$. Let us consider $\alpha_1 \in \{0,\dots,n-1\}$, previously introduced in Notation 6.11.

Let us study the sign of $w_{\bar{M}-\lambda_3'}^{(\alpha_1)}(a)$ to obtain the sign of $w_{\bar{M}-\lambda_3'}$. From Lemma 3.6, coupled with (10.1), we conclude that

$$T_{\sigma_k} w_{\bar{M}-\lambda_3'}(a) \begin{cases} >0\,, & \text{if } n-\sigma_k \text{ is odd,} \\ <0\,, & \text{if } n-\sigma_k \text{ is even.} \end{cases}$$

As we said before, to allow the maximal oscillation, $T_{n-\ell} w_{\bar{M}-\lambda_3'}$ must change its sign each time that it is different from zero. Thus, since from α_1 to σ_k we have $k-\alpha_1$ zeros, with maximal oscillation the following inequalities are fulfilled:

If $n-\sigma_k$ is odd:

$$T_{\alpha_1}\, w_{\bar{M}-\lambda_3'}(a) \begin{cases} > 0\,, & \text{if } \sigma_k-\alpha_1-(k-\alpha_1)=\sigma_k-k \text{ is even,}\\ < 0\,, & \text{if } \sigma_k-k \text{ is odd.}\end{cases}$$

If $n-\sigma_k$ is even:

$$T_{\alpha_1}\, w_{\bar{M}-\lambda_3'}(a) \begin{cases} < 0\,, & \text{if } \sigma_k-k \text{ is even,}\\ > 0\,, & \text{if } \sigma_k-k \text{ is odd.}\end{cases}$$

From (3.3), we conclude that

$$T_{\alpha_1}\, w_{\bar{M}-\lambda_3'}(a) = \frac{1}{v_1(a)\dots v_{\alpha_1}(a)} w^{(\alpha_1)}_{\bar{M}-\lambda_3'}(a)\,,$$

hence, we can affirm that

$$w^{(\alpha_1)}_{\bar{M}-\lambda_3'}(a) \begin{cases} > 0\,, & \text{if } n-k \text{ is odd,}\\ < 0\,, & \text{if } n-k \text{ is even.}\end{cases}$$

Thus, we have proved that

- If $n-k$ is even and $M\in[\bar{M}-\lambda_3',\bar{M}-\lambda_1)$, then for each $t\in(a,b)$ there exists $\rho_{11}(t)>0$ such that
$$g_M(t,s)<0 \quad \forall s\in(a,a+\rho_{11}(t))\,.$$
- If $n-k$ is odd and $M\in(\bar{M}-\lambda_1,\bar{M}-\lambda_3']$, then for each $t\in(a,b)$ there exists $\rho_{11}(t)>0$ such that
$$g_M(t,s)>0 \quad \forall s\in(a,a+\rho_{11}(t))\,.$$

• Study of $y_{\bar{M}-\lambda_3''}$. Now, let us consider $\beta_1\in\{0,\dots,n-1\}$, introduced in Notation 6.11. To obtain the sign of $y_{\bar{M}-\lambda_3''}$, let us study the sign of $y^{(\beta_1)}_{\bar{M}-\lambda_3''}(b)$.

From Lemma 3.7 coupled with (10.2), we conclude that

$$T_{\varepsilon_{n-k}} y_{\bar{M}-\lambda_3''}(b) \begin{cases} > 0\,, & \text{if } n-\varepsilon_{n-k} \text{ is even,}\\ < 0\,, & \text{if } n-\varepsilon_{n-k} \text{ is odd.}\end{cases}$$

In this case, as we said on the proof of Theorem 5.1, to allow the maximal oscillation, $T_{n-\ell} w_{\bar{M}-\lambda_3'}(b)$ must change its sign each time that it vanishes. Thus, since from β_1 to ε_{n-k} we have, with maximal oscillation, $n-k-\beta_1$ zeros, we deduce de following properties:

If $n-\varepsilon_{n-k}$ is even:

$$T_{\beta_1}\, y_{\bar{M}-\lambda_3''}(b) \begin{cases} > 0\,, & \text{if } n-k-\beta_1 \text{ is even,}\\ < 0\,, & \text{if } n-k-\beta_1 \text{ is odd.}\end{cases}$$

If $n-\varepsilon_{n-k}$ is odd:

$$T_{\beta_1}\, y_{\bar{M}-\lambda_3''}(b) \begin{cases} < 0\,, & \text{if } n-k-\beta_1 \text{ is even,}\\ > 0\,, & \text{if } n-k-\beta_1 \text{ is odd.}\end{cases}$$

From (3.3), we conclude that

$$T_{\beta_1}\, y_{\bar{M}-\lambda_3''}(b) = \frac{1}{v_1(b)\dots v_{\beta_1}(b)} y^{(\beta_1)}_{\bar{M}-\lambda_3''}(b)\,,$$

hence, we can affirm that

$$y^{(\beta_1)}_{\bar M-\lambda_3''}(b)\begin{cases}>0\,, & \text{if } \varepsilon_{n-k}-k-\beta_1 \text{ is even},\\ <0\,, & \text{if } \varepsilon_{n-k}-k-\beta_1 \text{ is odd}.\end{cases}$$

Thus, we have proved that

$$y_{\bar M-\lambda_3''}(t)\begin{cases}\geq 0\,,\ t\in I\,, & \text{if } \varepsilon_{n-k}-k \text{ is even},\\ \leq 0\,,\ t\in I\,, & \text{if } \varepsilon_{n-k}-k \text{ is odd}.\end{cases}$$

Hence, since $\gamma=n-\varepsilon_{n-k}-1$, taking into account that $y_{\bar M-\lambda_3''}$ cannot have any zero on (a,b), we conclude

- If $n-k$ is even and $M\in[\bar M-\lambda_3'',\bar M-\lambda_1)$, then for each $t\in(a,b)$ there exists $\rho_{12}(t)>0$ such that
$$g_M(t,s)<0\quad\forall s\in(b-\rho_{12}(t),b)\,.$$
- If $n-k$ is odd and $M\in(\bar M-\lambda_1,\bar M-\lambda_3'']$, then for each $t\in(a,b)$ there exists $\rho_{12}(t)>0$ such that
$$g_M(t,s)>0\quad\forall s\in(b-\rho_{12}(t),b)\,.$$

- Study of $\widehat w_{\bar M-\lambda_3'}$.

Notation 10.4. Let us define $\eta_1\in\{0,\dots,n-1\}$ such that $\eta_1\notin\{\tau_1,\dots,\tau_{n-k-1},\eta\}$ and $\{0,\dots,\eta_1-1\}\subset\{\tau_1,\dots,\tau_{n-k-1},\eta\}$.

To obtain the sign of $\widehat w_{\bar M-\lambda_3'}$, let us study the sign of $\widehat w^{(\eta_1)}_{\bar M-\lambda_3'}(a)$. From Lemma 4.8 and (10.3), we conclude that

$$\widehat T_{\tau_{n-k}}\widehat w_{\bar M-\lambda_3'}(a)\begin{cases}>0\,, & \text{if } \tau_{n-k} \text{ is odd},\\ <0\,, & \text{if } \tau_{n-k} \text{ is even}.\end{cases}$$

Analogously to T_k, to allow the maximal oscillation we conclude that $\widehat T_{n-\ell}\widehat w_{\bar M-\lambda_3'}$ must change its sign each time that it is non null. Thus, since from η_1 to τ_{n-k} we have $n-k-\eta_1$ zeros, with maximal oscillation, the following inequalities are fulfilled.

If τ_{n-k} is odd:

$$\widehat T_{\eta_1}\widehat w_{\bar M-\lambda_3'}(a)\begin{cases}>0\,, & \text{if } \tau_{n-k}-\eta_1-(n-k-\eta_1)=\tau_{n-k}-n+k \text{ is even},\\ <0\,, & \text{if } \tau_{n-k}-n+k \text{ is odd}.\end{cases}$$

If τ_{n-k} is even:

$$\widehat T_{\eta_1}\,\widehat w_{\bar M-\lambda_3'}(a)\begin{cases}<0\,, & \text{if } \tau_{n-k}-n+k \text{ is even},\\ >0\,, & \text{if } \tau_{n-k}-n+k \text{ is odd}.\end{cases}$$

From (4.24), we conclude that

$$\widehat T_{\eta_1}\,\widehat w_{\bar M-\lambda_3'}(a)=v_1(a)\dots v_{n-\eta_1}(a)w^{(\eta_1)}_{\bar M-\lambda_3'}(a)\,,$$

hence, we can affirm that

$$\widehat w^{(\eta_1)}_{\bar M-\lambda_3'}(a)\begin{cases}>0\,, & \text{if } n-k \text{ is odd},\\ <0\,, & \text{if } n-k \text{ is even}.\end{cases}$$

Thus, we have proved that

$$\widehat{w}_{\bar{M}-\lambda_3'} \begin{cases} \geq 0\,, & \text{on } I \text{ if } n-k \text{ is odd,} \\ \leq 0\,, & \text{on } I \text{ if } n-k \text{ is even.} \end{cases}$$

Hence, we conclude:

- If $n-k$ is even and $M \in [\bar{M}-\lambda_3', \bar{M}-\lambda_1)$, then for each $s \in (a,b)$ there exists $\rho_{21}(s) > 0$ such that
$$g_M(t,s) < 0 \quad \forall t \in (a, a+\rho_{21}(s))\,.$$
- If $n-k$ is odd and $M \in (\bar{M}-\lambda_1, \bar{M}-\lambda_3']$, then for each $s \in (a,b)$ there exists $\rho_{21}(t) > 0$ such that
$$g_M(t,s) > 0 \quad \forall t \in (a, a+\rho_{21}(s))\,.$$

- Study of $\widehat{y}_{\bar{M}-\lambda_3''}$.

Notation 10.5. Let us denote $\gamma_1 \in \{0,\dots,n-1\}$, such that $\gamma_1 \notin \{\delta_1,\dots,\delta_{k-1},\gamma\}$ and $\{0,\dots,\gamma_1-1\} \subset \{\delta_1,\dots,\delta_{k-1},\gamma\}$.

To obtain the sign of $\widehat{y}_{\bar{M}-\lambda_3''}$, we study the sign of $\widehat{y}^{(\beta_1)}_{\bar{M}-\lambda_3''}(b)$. From Lemma 4.9 coupled with (10.4), we conclude that

$$\widehat{T}_{\delta_k}\,\widehat{y}_{\bar{M}-\lambda_3''}(b) \begin{cases} > 0\,, & \text{if } \delta_k \text{ is even,} \\ < 0\,, & \text{if } \delta_k \text{ is odd.} \end{cases}$$

In this case, analogously to T_k, to allow the maximal oscillation $\widehat{T}_{n-\ell} w_{\bar{M}-\lambda_3'}(b)$ changes its sign each time that it vanishes and it remains of constant sign if it does not vanish. Thus, with maximal oscillation, since from γ_1 to δ_k we have $k-\gamma_1$ zeros

If δ_k is even:

$$\widehat{T}_{\gamma_1}\,\widehat{y}_{\bar{M}-\lambda_3''}(b) \begin{cases} > 0\,, & \text{if } k-\gamma_1 \text{ is even,} \\ < 0\,, & \text{if } k-\gamma_1 \text{ is odd.} \end{cases}$$

If δ_k is odd:

$$\widehat{T}_{\gamma_1}\,\widehat{y}_{\bar{M}-\lambda_3''}(b) \begin{cases} < 0\,, & \text{if } k-\gamma_1 \text{ is even,} \\ > 0\,, & \text{if } k-\gamma_1 \text{ is odd.} \end{cases}$$

From (4.24), we conclude that

$$\widehat{T}_{\gamma_1}\,\widehat{y}_{\bar{M}-\lambda_3''}(b) = v_1(b)\dots v_{n-\gamma_1}(b)\widehat{y}^{(\gamma_1)}_{\bar{M}-\lambda_3''}(b)\,,$$

hence, we can affirm that

$$\widehat{y}^{(\gamma_1)}_{\bar{M}-\lambda_3''}(b) \begin{cases} > 0\,, & \text{if } k-\delta_k-\gamma_1 \text{ is even,} \\ < 0\,, & \text{if } k-\delta_k-\gamma_1 \text{ is odd.} \end{cases}$$

Thus, we have proved that

$$\widehat{y}_{\bar{M}-\lambda_3''}(t) \begin{cases} \geq 0\,, & \text{if } k-\delta_k \text{ is even,} \\ \leq 0\,, & \text{if } k-\delta_k \text{ is odd.} \end{cases}$$

Hence, since $\beta = n-\delta_k-1$, we conclude

- If $n-k$ is even and $M \in [\bar{M}-\lambda_3'', \bar{M}-\lambda_1)$, then for each $s \in (a,b)$ there exists $\rho_{22}(s) > 0$ such that
$$g_M(t,s) < 0 \;\forall t \in (b-\rho_{22}(s), b)\,.$$
- If $n-k$ is odd and $M \in (\bar{M}-\lambda_1, \bar{M}-\lambda_3'']$, then for each $s \in (a,b)$ there exists $\rho_{22}(s) > 0$ such that
$$g_M(t,s) > 0 \quad \forall t \in (b-\rho_{22}(s), b)\,.$$

By taking $\rho_1(t) = \min\{\rho_{11}(t), \rho_{12}(t)\}$ and $\rho_2(s) = \min\{\rho_{21}(s), \rho_{22}(s)\}$, we complete the proof. □

Remark 10.6. From Theorems 10.1 and 10.3, if we are able to prove that the sign change of the related Green's function must begin on the boundary of $I \times I$, then the intervals obtained in Theorem 10.1 are optimal.

Example 10.7. Now, we apply the Remark 10.6 to our recurrent example, the operator $T_4^0[M]$. Let us assume that there exists $M^* \in [-\pi^4, m_1^4)$ such that $g_{M^*}(t,s)$ changes its sign. Then, from Theorem 10.3 it must exist $s^* \in (0,1)$ such that $u^*(t) = g_{M^*}(t,s^*)$ has at least two zeros, $0 < c_1 < c_2 < 1$.

By the definition of the Green's function $u^* \in C^2([0,1])$. So, there exists $c^* \in (c_1,c_2)$ such that ${u^*}'(c^*) = 0$. There are two possibilities:

• $c^* \le s^*$. In this case, u^* is a solution of $T_4^0[M^*]u^*(t) = 0$ on $[0,c^*]$ satisfying the boundary conditions $u^*(0) = {u^*}''(0) = {u^*}'(c^*) = 0$. Moreover, it satisfies $u^*(c_1) = 0$.

The function $y^*(t) = u^*(c^*t)$ satisfies $y^*(0) = {y^*}''(0) = {y^*}'(1) = 0$ and $y^*\left(\frac{c_1}{c^*}\right) = 0$. Moreover, it is a solution of $T_4^0[{c^*}^4M^*]y^*(t) = 0$ on $[0,1]$, with $0 > {c^*}^4M^* > M^* > -\pi^4 > -m_3^4$, where m_3^4 has been introduced in Example 6.6. But, this is is a contradiction with Proposition 6.9.

• $c^* > s^*$. In this case, u^* is a solution of $T_4^0[M^*]u^*(t) = 0$ on $[c^*,1]$ satisfying the boundary conditions ${u^*}'(c^*) = {u^*}'(1) = {u^*}''(1) = 0$. Moreover, it satisfies $u^*(c_2) = 0$.

The function $y^*(t) = u^*((1-c^*)t + c^*)$ satisfies
$${y^*}'(0) = {y^*}'(1) = {y^*}''(1) = 0\,. \tag{10.5}$$
Moreover, it is a solution of $T_4^0[(1-c^*)^4M^*]y^*(t) = 0$ on $[0,1]$ and $y^*(\frac{c_2-c^*}{1-c^*}) = 0$, with $0 > (1-c^*)^4M^* > M^* > -\pi^4$. It can be seen that π^4 is the least positive eigenvalue of $T_4^0[0]$ on $X_{\{0,1\}}^{\{1,2\}}$.

Now, let us see that every solution of $u^{(4)}(t) + Mu(t) = 0$, satisfying the given boundary conditions (10.5), cannot have any zero on $(0,1)$ whenever $M \in (-\pi^4, 0)$. Which is a contradiction of supposing that there is a sign change on the Green's function.

First, let us choose $M = -\big(\pi/2\big)^4$, the solution is given as a multiple of
$$u(t) = f_1(1-t) + f_2(1-t)\,,$$
where, for $t \in [0,1]$,
$$f_1(t) = \Big(1 - \sinh\big(\frac{\pi}{2}\big)\Big)\Big(\sinh\big(\frac{\pi t}{2}\big) - \sin\big(\frac{\pi t}{2}\big)\Big) \ge f_1(1) = -\Big(\sinh\big(\frac{\pi}{2}\big) - 1\Big)^2,$$
$$f_2(t) = \cosh\big(\frac{\pi}{2}\big)\Big(\cos\big(\frac{\pi t}{2}\big) + \cosh\big(\frac{\pi t}{2}\big)\Big) \ge f_2(0) = 2\cosh\big(\frac{\pi}{2}\big)\,.$$

So, $u(t) \geq -\left(\sinh\left(\frac{\pi}{2}\right) - 1\right)^2 + 2\cosh\left(\frac{\pi}{2}\right) > 0$ for all $t \in [0,1]$.

Now, let us move continuously on M to obtain the different solutions of of the equation $T_4^0[M]u(t) = 0$, coupled with the boundary conditions (10.5). Let us see that it is not possible that these solutions begin to change sign on $(0,1)$. If this was the case, we would have that there exist $\widehat{M} \in (-\pi^4, 0)$ and $\hat{t} \in (0,1)$ such that $\hat{u}$ is a solution of $T_4^0[\widehat{M}]\hat{u}(t) = 0$ on $[\hat{t}, 1]$, verifying $\hat{u}(\hat{t}) = \hat{u}'(\hat{t}) = \hat{u}(1) = \hat{u}''(1) = 0$. Then, the function $\hat{y}(t) = \hat{u}((1-\hat{t})t+\hat{t})$ is an eigenfunction related to the eigenvalue $-(1-\hat{t})^4\,\widehat{M} \in (0, \pi^4)$ of the operator $T_4^0[0]$ on $X_{\{0,1\}}^{\{1,2\}}$ which is a contradiction.

Analogously, if there exists $\widehat{M} \in (-\pi^4, 0)$, for which there is a nontrivial solution of $T_4^0[\widehat{M}]u(t) = 0$ on $[0,1]$, satisfying $u(0) = 0$, coupled with the boundary conditions (10.5), then there is an eigenvalue $-\widehat{M} \in (0, \pi^4)$ of the operator $T_4^0[0]$ on $X_{\{0,1\}}^{\{1,2\}}$, which is again a contradiction.

Finally, since there is no positive eigenvalue of $T_4^0[0]$ on $X_{\{1\}}^{\{0,1,2\}}$, we can affirm that it is not possible that the sign change begins at $t = 1$. So, we have proved that every solution of $u^{(4)}(t) + Mu(t) = 0$ coupled with the boundary conditions (10.5) does not not have any zero on $(0,1)$ for $M \in (-\pi^4, 0)$. Thus, we also have arrived to a contradiction if $c^* > s$.

So, from Remark 10.6, Theorems 10.1 and 10.3, we can affirm that $T_4^0[M]$ is a strongly inverse negative operator on $X_{\{0,2\}}^{\{1,2\}}$ if and only if $M \in [-\pi^4, -m_1^4)$, where m_1 has been introduced in Example 6.6.

Example 10.8. Using a similar argument to Example 10.7, in [14] it is studied the strongly inverse negative character of the operator $T_4(p_1, p_2)[M]$ previously introduced in Section 9. There, a characterization of the parameter's set where $T_4(p_1, p_2)$ is strongly inverse negative on $X_{\{0,2\}}^{\{0,2\}}$ is obtained and several particular examples are given.

11. Characterization of strongly inverse positive (negative) character for non homogeneous boundary conditions

This section is devoted to the study of the operator $T_n[M]$, coupled with different non homogeneous boundary conditions. First, let us consider the set

$$\begin{aligned}\tilde{X}_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}} = \big\{u \in C^n(I) : {} & u^{(\sigma_1)}(a) = \cdots = u^{(\sigma_k-1)}(a) = 0\,,\\ & (-1)^{n-\sigma_k-1}u^{(\sigma_k)}(a) \geq 0\,,\ u^{(\varepsilon_1)}(b) = \cdots \\ & = u^{(\varepsilon_{n-k-1})}(b) = 0\,,\ u^{(\varepsilon_{n-k})}(b) \leq 0\big\}\,.\end{aligned} \tag{11.1}$$

That is, we consider a set where some of the boundary conditions do not have to be necessarily homogeneous. This information is very useful in order to apply the lower and upper solutions method and monotone iterative techniques for nonlinear boundary-value problems, see for instance [8].

So, we are interested in characterizing the parameter's set for which the operator $T_n[M]$ is strongly inverse positive or negative on $\tilde{X}_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$. We introduce the boundary conditions that a function $u \in \tilde{X}_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ must satisfy:

$$u^{(\sigma_1)}(a) = \cdots = u^{(\sigma_k-1)}(a) = 0\,, \quad u^{(\sigma_k)}(a) = c_1\,, \tag{11.2}$$

$$u^{(\varepsilon_1)}(b) = \cdots = u^{(\varepsilon_{n-k-1})}(b) = 0\,, \quad u^{(\varepsilon_{n-k})}(b) = c_2\,, \tag{11.3}$$

where $(-1)^{n-\sigma_k-1}c_1 \geq 0$ and $c_2 \leq 0$. We can relate problem (1.4), (11.2)-(11.3) with the homogeneous problem (1.4)–(1.6) by means of the following result.

Lemma 11.1. *If problem* (1.4)–(1.6) *has only the trivial solution. Then problem* $T_n[M]u(t) = h(t)$, $t \in I$, *coupled with boundary conditions* (11.2)-(11.3) *has a unique solution, which is given by*

$$u(t) = \int_a^b g_M(t,s)\, h(s)\, ds + c_1\, x_M(t) + c_2\, z_M(t)\,, \tag{11.4}$$

where $g_M(t,s)$ *is the related Green's function of* $T_n[M]$ *on* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ *and:*

- x_M *is defined as the unique solution of*

$$\begin{gathered} T_n[M]u(t) = 0\,, \quad t \in I\,, \\ u^{(\sigma_1)}(a) = \dots = u^{(\sigma_{k-1})}(a) = 0\,, \quad u^{(\sigma_k)}(a) = 1\,, \\ u^{(\varepsilon_1)}(b) = \dots = u^{(\varepsilon_{n-k})}(b) = 0\,. \end{gathered} \tag{11.5}$$

- z_M *is defined as the unique solution of*

$$\begin{gathered} T_n[M]u(t) = 0\,, \quad t \in I \\ u^{(\sigma_1)}(a) = \dots = u^{(\sigma_k)}(a) = 0\,, \\ u^{(\varepsilon_1)}(b) = \dots = u^{(\varepsilon_{n-k-1})}(b) = 0\,, \quad u^{(\varepsilon_{n-k})}(b) = 1\,. \end{gathered} \tag{11.6}$$

Using this Lemma we can obtain the following result which characterizes the strongly inverse positive (negative) character of $T_n[M]$ on $\tilde{X}_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.

Theorem 11.2. $T_n[M]$ *is strongly inverse positive (negative) on* $\tilde{X}_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ *if and only if it is strongly inverse positive (negative) on* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.

Proof. Since $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}} \subset \tilde{X}_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ the necessary condition is obvious.

Now, let us see the sufficiency part. From the strongly inverse positive (negative) character of $T_n[M]$ on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, using Theorem 2.14, we conclude that $g_M > 0$ (< 0) a.e. on $I \times I$. Then, from Lemma 11.1, we only need to study the sign of x_M and z_M. To do that, we establish a relationship between these functions and some derivatives of $g_M(t,s)$.

Taking into account the boundary conditions, it is clear that

$$x_M(t) = (-1)^{n-1-\sigma_k}\, w_M(t) \quad \text{and} \quad z_M(t) = (-1)^{n-\varepsilon_{n-k}}\, y_M(t)\,,$$

where w_M and y_M have been defined in the proof of Theorem 8.1 as follows:

$$w_M(t) = \frac{\partial^\eta}{\partial s^\eta} g_M^1(t,s)\big|_{s=a}\,, \quad y_M(t) = \frac{\partial^\gamma}{\partial s^\gamma} g_M^2(t,s)\big|_{s=b}\,.$$

If $T_n[M]$ is strongly inverse positive (negative) on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, then $w_M(t) \geq 0$ (≤ 0) and $(-1)^\gamma\, y_M(t) \geq 0$. Since $\gamma = n-1-\varepsilon_{n-k}$, it follows that $(-1)^{n-\varepsilon_{n-k}}\, y_M(t) \leq 0$ in both cases. Thus, the result is proved. ☐

12. Study of particular type of operators

In this section we consider a particular type of operators which satisfy property (T_d) on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, thus we can apply previous results to these operators. After that, we obtain some results which characterize either the strongly inverse positive character or the strongly inverse negative character of $T_n[M]$ if $n-k$ is even or odd, respectively, in different sets where more general non homogeneous boundary conditions are considered. First, we introduce the following notation.

Notation 12.1. First, let us denote $\alpha_2 \in \{-1,0,1,\dots,n-2\}$, such that $\alpha_2 \notin \{\sigma_1,\dots,\sigma_k\}$ and $\{\alpha_2+1,\alpha_2+2,\dots,\sigma_k\} \subset \{\sigma_1,\dots,\sigma_k\}$; and $\beta_2 \in \{-1,0,1,\dots,n-2\}$, such that $\beta_2 \notin \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ and $\{\beta_2+1,\beta_2+2,\dots,\varepsilon_{n-k}\} \subset \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$. We denote $\mu = \max\{\alpha_2,\beta_2\}$.

Remark 12.2. If $\sigma_k = k-1$ then $\alpha_2 = -1$. Otherwise, $\alpha_2 \geq \alpha \geq 0$. Moreover, if $\varepsilon_{n-k} = n-k-1$ then $\beta_2 = -1$. Otherwise, $\beta_2 \geq \beta \geq 0$.

Now, we introduce the following sufficient condition for an operator to satisfy property (T_d).

Proposition 12.3. *If the linear differential equation of $(n-\mu-1)^{\rm th}$-order:*

$$L_{n-\mu-1}u(t) \equiv u^{(n-\mu-1)}(t) + p_1(t)u^{(n-\mu-2)}(t) + \cdots + p_{n-\mu-1}(t)u(t) = 0\,, \quad (12.1)$$

with $p_j \in C^{n-j}(I)$, is disconjugate on I, then the operator:

$$\tilde{T}_n[0]u(t) = u^{(n)}(t) + p_1(t)u^{(n-1)}(t) + \cdots + p_{n-\mu-1}(t)u^{(\mu+1)}(t)\,,$$

satisfies property (T_d) on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.

Proof. From Theorems 2.3 and 2.4, since the linear differential equation (12.1) is disconjugate on I, there exist positive functions $v_1, \dots, v_{n-\mu-1}$ such that $v_k \in C^{n-\mu-k}(I)$ for $k = 1,\dots,n-\mu-1$, and

$$L_{n-\mu-1}u \equiv v_1 \cdots v_{n-\mu-1} \frac{d}{dt}\Big(\frac{1}{v_{n-\mu-1}} \frac{d}{dt}\Big(\cdots \frac{d}{dt}\big(\frac{u}{v_1}\big)\Big)\Big)\,.$$

Step 1. Let us see that, in fact, $v_k \in C^n(I)$ for $k = 1,\dots,n-\mu-1$. Since, $p_j \in C^{n-j}(I)$ for $j \in \{\mu+1,\dots,n-1\}$, every solution of (12.1) belongs to $C^n(I)$.

If we look at the proof of Theorem 2.4, given in [15, Chapter 3, Theorem 2], we observe that v_k is given by the recurrence formula

$$v_1 = y_1\,, \quad v_2 = \frac{W(y_1,y_2)}{y_1^2}\,,$$
$$v_k = \frac{W(y_1,\dots,y_k)W(y_1,\dots,y_{k-2})}{W(y_1,\dots,y_k-1)^2}\,, \quad \text{for } k \geq 2\,,$$

where $\{y_1,\dots,y_{n-\mu-1}\}$ is a Markov fundamental system of solutions of (12.1) and W the correspondent Wronskians. Thus, taking into account that $y_1,\dots,y_{n-\mu-1} \in C^n(I)$, we conclude that $v_1 \in C^n(I)$, $v_2 \in C^{n-1}(I)\,,\dots,v_{n-\mu-1} \in C^{\mu+2}(I)$.

Now, let us consider the expression (3.3), with $\ell = n-\mu-1$ and $p_{\ell j} = p_j \in C^{n-j}(I)$, $j \in \{\mu+1,\dots,n-1\}$ given by expressions (3.4)–(3.7). First, let us see that $v_1 \in C^n(I)$, $v_2 \in C^n(I)$, $v_3 \in C^{n-1}(I)\,,\dots,v_{n-\mu-1} \in C^{\mu+3}(I)$.

If $\mu = n-2$, then $n-\mu-1 = 1$ and the result is proved. Otherwise, $p_1 \in C^{n-1}(I) \subset C^{\mu+2}(I)$, since $v_1, \dots, v_{n-\mu-2} \in C^{\mu+3}(I)$ and $v_{n-\mu-1} \in C^{\mu+2}(I)$, from (3.4) we obtain that $v'_{n-\mu-1} \in C^{\mu+2}(I)$, then $v_{n-\mu-1} \in C^{\mu+3}(I)$.

Let us assume that $v_{k+1} \in C^{n-k}(I)$, $v_{k+2} \in C^{n-k-1}(I), \dots, v_{n-\mu-1} \in C^{\mu+3}(I)$, then since $v_k \in C^{n-k}(I)$ considering the expression of $p_{n-\mu-k}$, given in (3.7) for $\ell_\ell = n-\mu-k$, we obtain that $v_k^{(n-\mu-k)} \in C^{\mu+1}(I)$, hence $v_k \in C^{n-k+1}(I)$. Thus, we have proved by induction that $v_1 \in C^n(I)$, $v_2 \in C^n(I)$, $v_3 \in C^{n-1}(I), \dots, v_{n-\mu-1} \in C^{\mu+3}$. If $\mu = n-3$, then the result is proved, since v_1, $v_2 \in C^n(I)$.

Now, let us assume that $\mu < n-3$. Considering the expression of $p_{n-\mu-3} \in C^{\mu+3}(I)$, given in (3.7) for $\ell_\ell = n-\mu-k$. Since $v_2 \in C^n(I)$, we conclude that $v_3^{(n-\mu-3)} \in C^{\mu+3}(I)$; so, $v_3 \in C^n(I)$. If we suppose that $v_1, \dots, v_{k-1} \in C^n(I)$, then by considering the expression of $p_{n-\mu-k} \in C^{\mu+k}(I)$, we conclude that $v_k^{(n-\mu-k)} \in C^{\mu+k}(I)$, thus $v_k \in C^n(I)$. Then, we have proved that $v_1, \dots, v_{n-\mu-1} \in C^n(I)$.

Step 2. Construction of the decomposition satisfying property (T_d). Now, we consider the decomposition of $\tilde{T}[0]$ as follows:

$$\tilde{T}[0]u \equiv v_1 \dots v_{n-\mu-1} \frac{d}{dt}\Big(\frac{1}{v_{n-\mu-1}} \frac{d}{dt}\Big(\cdots \frac{d}{dt}\big(\frac{u^{(\mu+1)}}{v_1}\big)\Big)\Big).$$

Hence, if we denote $\tilde{v}_1 = \dots = \tilde{v}_{\mu+1} = 1$ and $\tilde{v}_{\mu+2} = v_1, \dots, \tilde{v}_n = v_{n-\mu-1}$, we can decompose $\tilde{T}_n[0]$ in the following sense:

$$\tilde{T}_0\, u = u\,, \quad \tilde{T}_k u = \frac{d}{dt}\Big(\frac{\tilde{T}_{k-1} u}{\tilde{v}_k}\Big), \quad k = 1, \dots, n\,.$$

Trivially $\tilde{T}_n[0]u = \tilde{v}_1 \dots \tilde{v}_n \tilde{T}_n u$.

Now, let us see that this decomposition satisfies the property (T_d). We have that $\tilde{T}_0 u = u$, $\tilde{T}_1 u = u', \dots, \tilde{T}_{\mu+1} u = u^{(\mu+1)}$. Hence, if $\sigma_i < \alpha_2 \le \mu$ then $\tilde{T}_{\sigma_i} u(a) = u^{(\sigma_i)}(a) = 0$.

Analogously, if $\varepsilon_i < \beta_2 \le \mu$, then $\tilde{T}_{\varepsilon_i} u(b) = u^{(\varepsilon_i)}(b) = 0$. If $h > \mu+1$, then

$$\tilde{T}_h u = \frac{u^{(h)}}{v_1 \dots v_h} + p_{h\,1} u^{(h-1)} + \dots + p_{h\,h-\mu-1} u^{(\mu+1)}\,,$$

where $p_{h\,i}$ is given by equations (3.4)–(3.7).

If $\sigma_i > \mu$, then by definition of μ, $u^{(\mu+1)}(a) = u^{(\mu+2)}(a) = \dots = u^{(\sigma_i)}(a) = 0$. Hence $\tilde{T}_{\sigma_i} u(a) = 0$. Analogously, if $\varepsilon_i > \mu$, then $u^{(\mu+1)}(b) = u^{(\mu+2)}(b) = \dots = u^{(\varepsilon_i)}(b) = 0$. Hence $\tilde{T}_{\varepsilon_i} u(b) = 0$. Thus, the result is proved. □

As consequence of this result, we can apply Theorems 8.1, 10.1, 10.3 and 11.2 to operator $\tilde{T}_n[M]$. Moreover, for this particular case, we will be able to obtain a characterization of strongly inverse positive (negative) character in different spaces with inhomogeneous boundary conditions.

Definition 12.4. Let us consider $\{\sigma_{\epsilon_1}, \dots, \sigma_{\epsilon_\ell}\} \subset \{\sigma_1, \dots, \sigma_k\}$ such that $\sigma_{\epsilon_1} < \sigma_{\epsilon_2} < \dots < \sigma_{\epsilon_\ell} = \sigma_k$, with $\sigma_{\epsilon_{\ell-1}} < \mu$. And $\{\varepsilon_{\kappa_1}, \dots, \varepsilon_{\kappa_h}\} \subset \{\varepsilon_1, \dots, \varepsilon_{n-k}\}$ such that $\varepsilon_{\kappa_1} < \varepsilon_{\kappa_2} < \dots < \varepsilon_{\kappa_h} = \varepsilon_{n-k}$, with $\varepsilon_{\kappa_{h-1}} < \mu$. Let us define the set of functions $X_{\{\sigma_1,\dots,\sigma_k\}_{\{\sigma_{\epsilon_1},\dots,\sigma_{\epsilon_\ell}\}}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}_{\{\varepsilon_{\kappa_1},\dots,\varepsilon_{\kappa_h}\}}}$ as the set of functions $u\in C^n(I)$ such that

$$\begin{gathered} u^{(\sigma_j)}(a)=\begin{cases}0 & j\notin\{\epsilon_1,\dots,\epsilon_\ell\}\\ (-1)^{n-\sigma_j-(k-j)+1}\varphi_j & j\in\{\epsilon_1,\dots,\epsilon_\ell\}\end{cases}\\ \text{for some } \varphi_j\ge 0,\ j=1,\dots,k\,; \text{and}\\ u^{(\varepsilon_i)}(b)=\begin{cases}0 & i\notin\{\kappa_1,\dots,\kappa_h\}\\ (-1)^{n-k+i-1}\psi_i & i\in\{\kappa_1,\dots,\kappa_h\}\end{cases}\\ \text{for some } \psi_i\ge 0\,,\ i=1,\dots,n-k\,. \end{gathered}\tag{12.2}$$

Now, we enunciate a similar result to Lemma 11.1 for this more general case

Lemma 12.5. *If problem* (1.4)–(1.6) *has only the trivial solution. Then problem* $T_n[M]u(t)=h(t)$, $t\in I$, *coupled with the boundary conditions*

$$u^{(\sigma_j)}(a)=\begin{cases}0\,, & j\notin\{\epsilon_1,\dots,\epsilon_\ell\}\,,\ j=1,\dots,k\,,\\ c_j\,, & j\in\{\epsilon_1,\dots,\epsilon_\ell\},\quad j=1,\dots,k\,;\end{cases}\tag{12.3}$$

$$u^{(\varepsilon_i)}(b)=\begin{cases}0\,, & i\notin\{\kappa_1,\dots,\kappa_h\},\ i=1,\dots,n-k\,,\\ d_i\,, & i\in\{\kappa_1,\dots,\kappa_h\},\ i=1,\dots,n-k\end{cases}\tag{12.4}$$

has a unique solution, which is given by

$$u(t)=\int_a^b g_M(t,s)\,h(s)\,ds+\sum_{j=1}^{\ell}c_{\epsilon_j}x_M^{\sigma_{\epsilon_j}}(t)+\sum_{i=1}^{h}d_{\kappa_i}z_M^{\varepsilon_{\kappa_i}}(t)\,,\tag{12.5}$$

where $g_M(t,s)$ *is the related Green's function of* $T_n[M]$ *on* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, *and*

- $x_M^{\sigma_{\epsilon_j}}$ *is the unique solution of*

$$\begin{gathered} T_n[M]u(t)=0\,,\quad t\in I\\ u^{(\sigma_{\epsilon_j})}(a)=1\,,\\ u^{(\sigma_1)}(a)=\dots=u^{(\sigma_{\epsilon_j-1})}(a)=u^{(\sigma_{\epsilon_j+1})}(a)=\dots=u^{(\sigma_k)}(a)=0\,,\\ u^{(\varepsilon_1)}(b)=\dots=u^{(\varepsilon_{n-k})}(b)=0\,. \end{gathered}\tag{12.6}$$

- $z_M^{\varepsilon_{\kappa_i}}$ *is the unique solution of*

$$\begin{gathered} T_n[M]u(t)=0\,,\quad t\in I\,,\\ u^{(\sigma_k)}(a)=\dots=u^{(\sigma_k)}=0\,,\\ u^{(\varepsilon_{\kappa_i})}(b)=1\,,\\ u^{(\varepsilon_1)}(b)=\dots=u^{(\varepsilon_{\kappa_i-1})}(b)=u^{(\varepsilon_{\kappa_i+1})}(b)=\dots=u^{(\varepsilon_{n-k})}(b)=0\,. \end{gathered}\tag{12.7}$$

We have the following results, which ensures the existence of the different eigenvalues.

Lemma 12.6.

- *If* $\sigma_{\epsilon_j}>\alpha$, *then* $\tilde T_n[0]$ *satisfies property* (T_d) *on* $X_{\{\sigma_1,\dots,\sigma_{\epsilon_j-1},\sigma_{\epsilon_j+1},\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.
- *Operator* $\tilde T_n[0]$ *satisfies property* (T_d) *on* $X_{\{\sigma_1,\dots,\sigma_{\epsilon_j-1},\sigma_{\epsilon_j+1},\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$.
- *If* $\varepsilon_{\kappa_i}>\beta$, *then* $\tilde T_n[0]$ *satisfies property* (T_d) *on* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{\kappa_i-1},\varepsilon_{\kappa_i+1},\dots,\varepsilon_{n-k}|\beta\}}$.

- *Operator* $\tilde{T}_n[0]$ *satisfies the property* (T_d) *on* $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{\kappa_i-1},\varepsilon_{\kappa_i+1},\dots,\varepsilon_{n-k}\}}$.

Proof. Let us see the different cases:

- If $\sigma_j < \mu$, then $T_{\sigma_j} u(a) = u^{(\sigma_j)}(a) = 0$.
- If $\sigma_j = \sigma_k$, then $T_{\sigma_k} u(a) = 0$, by the definition of μ.
- If $\varepsilon_i < \mu$, then $T_{\varepsilon_i} u(b) = u^{(\varepsilon_i)}(b) = 0$.
- If $\varepsilon_i = \varepsilon_{n-k}$, then $T_{\varepsilon_{n-k}} u(b) = 0$, by the definition of μ.
- If $u \in X_{\{\sigma_1,\dots,\sigma_{\epsilon_j-1},\sigma_{\epsilon_j+1},\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$ or $u \in X_{\{\sigma_1,\dots,\sigma_{\epsilon_j-1},\sigma_{\epsilon_j+1},\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ and $\sigma_{\epsilon_j} > \alpha$, then $T_\alpha u(a) = \frac{1}{v_1(a)\dots v_\alpha(a)} u^{(\alpha)}(a) = 0$.
- Analogously, if either $u \in X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{\kappa_i-1},\varepsilon_{\kappa_i+1},\dots,\varepsilon_{n-k}\}}$ or $\varepsilon_{\kappa_i} > \beta$ and $u \in X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{\kappa_i-1},\varepsilon_{\kappa_i+1},\dots,\varepsilon_{n-k}|\beta\}}$, then $T_\beta u(b) = \frac{1}{v_1(b)\dots v_\alpha(b)} u^{(\beta)}(b) = 0$.

□

Remark 12.7. If we can prove that either $T_{\sigma_j} u(a) = u^{(\sigma_j)}(a)$ or $T_{\varepsilon_i} u(b) = u^{(\varepsilon_i)}(b)$, we do not need the assumption that $\sigma_j < \mu$ or $\varepsilon_i < \mu$ given by the choice of $\{\epsilon_1,\dots,\epsilon_\ell\}$ and $\{\kappa_1,\dots,\kappa_h\}$ on Definition 12.4.

This is true, in particular, if we can choose on decomposition (3.1)-(3.2), $v_1 \equiv \dots \equiv v_{\sigma_j} \equiv 1$ or $v_1 \equiv \dots \equiv v_{\varepsilon_i} \equiv 1$. We note that such a choice is valid for the operator $T_n^0[M] = u^{(n)}(t) + Mu(t)$, where we can choose $v_1 \equiv \dots \equiv v_n \equiv 1$.

The following results are also true under the hypothesis of this remark.

Lemma 12.8.

- *Let $n-k$ be even, then the following assertions are satisfied:*
  - *If $\sigma_{\epsilon_j} > \alpha$, then there is $\lambda_{\sigma_{\epsilon_j}}^1 > 0$, the least positive eigenvalue of $\tilde{T}_n[0]$ on $X_{\{\sigma_1,\dots,\sigma_{\epsilon_j-1},\sigma_{\epsilon_j+1},\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*
  - *There is $\lambda_{\sigma_{\epsilon_j}}^2 < 0$ the largest negative eigenvalue of $\tilde{T}_n[0]$ on the set $X_{\{\sigma_1,\dots,\sigma_{\epsilon_j-1},\sigma_{\epsilon_j+1},\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$.*
  - *If $\varepsilon_{\kappa_i} > \beta$, then there is $\lambda_{\varepsilon_{\kappa_i}}^1 > 0$, the least positive eigenvalue of $\tilde{T}_n[0]$ on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{\kappa_i-1},\varepsilon_{\kappa_i+1},\dots,\varepsilon_{n-k}|\beta\}}$.*
  - *There exists $\lambda_{\varepsilon_{\kappa_i}}^2 < 0$ the largest negative eigenvalue of $\tilde{T}_n[0]$ on the set $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{\kappa_i-1},\varepsilon_{\kappa_i+1},\dots,\varepsilon_{n-k}\}}$.*
- *Let $n-k$ be odd, then the following assertions are satisfied:*
  - *If $\sigma_{\epsilon_j} > \alpha$, then there exists $\lambda_{\sigma_{\epsilon_j}}^1 < 0$, the largest negative eigenvalue of $\tilde{T}_n[0]$ on $X_{\{\sigma_1,\dots,\sigma_{\epsilon_j-1},\sigma_{\epsilon_j+1},\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*
  - *There exists $\lambda_{\sigma_{\epsilon_j}}^2 > 0$, the least positive eigenvalue of operator $\tilde{T}_n[0]$ on $X_{\{\sigma_1,\dots,\sigma_{\epsilon_j-1},\sigma_{\epsilon_j+1},\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$.*
  - *If $\varepsilon_{\kappa_i} > \beta$, then there exists $\lambda_{\varepsilon_{\kappa_i}}^1 < 0$, the largest negative eigenvalue of $\tilde{T}_n[0]$ on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{\kappa_i-1},\varepsilon_{\kappa_i+1},\dots,\varepsilon_{n-k}|\beta\}}$.*
  - *There exists $\lambda_{\varepsilon_{\kappa_i}}^2 > 0$, the least positive eigenvalue of operator $\tilde{T}[0]$ on $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{\kappa_i-1},\varepsilon_{\kappa_i+1},\dots,\varepsilon_{n-k}\}}$.*

Proof. Since $\{\sigma_1,\dots,\sigma_k\}-\{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ satisfy property (N_a), this property is satisfied in all the spaces involved in the result. Moreover, from Lemma 12.6, the property (T_d) is also satisfied. Then, by applying Theorems 5.1, 2.16 and 2.17, the result is proved. □

Now, let us see two results which allow us to ensure that functions $x_M^{\sigma_{\epsilon_j}}$ and $z_M^{\varepsilon_{\kappa_i}}$ are of constant sign for suitable values of M.

Proposition 12.9. *Let $u\in C^n(I)$ be a solution of $\tilde T[M]u(t)=0$ for $t\in(a,b)$, which satisfies the boundary conditions*

$$\begin{gathered} u^{(\sigma_1)}(a)=\dots=u^{(\sigma_{\epsilon_j-1})}(a)=u^{(\sigma_{\epsilon_j+1})}(a)=\dots=u^{(\sigma_k)}(a)=0\,,\\ u^{(\varepsilon_1)}(b)=\dots=u^{(\varepsilon_{n-k})}(b)=0\,. \end{gathered}\tag{12.8}$$

Then, the function u does not have any zeros on (a,b) provided that one of the following assertions are fulfilled:

- *If $n-k$ is even, $k>1$, $\sigma_{\epsilon_j}>\alpha$ and $M\in[-\lambda^1_{\sigma_{\epsilon_j}},-\lambda^2_{\sigma_{\epsilon_j}}]$, where*
  - * $\lambda^1_{\sigma_{\epsilon_j}}>0$ *is the least positive eigenvalue of* $\tilde T_n[0]$ *in* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_{\epsilon_j-1},\sigma_{\epsilon_j+1},\dots,\sigma_k|\alpha\}}$,
  - * $\lambda^2_{\sigma_{\epsilon_j}}<0$ *is the largest negative eigenvalue of* $\tilde T_n[0]$ *on* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}_{\{\sigma_1,\dots,\sigma_{\epsilon_j-1},\sigma_{\epsilon_j+1},\dots,\sigma_k\}}$.
- *If $n-k$ is even, $k>1$, $\sigma_{\epsilon_j}<\alpha$ and $M\in[-\lambda_1,-\lambda^2_{\sigma_{\epsilon_j}}]$, where*
  - * $\lambda_1>0$ *is the least positive eigenvalue of* $\tilde T_n[0]$ *on* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$,
  - * $\lambda^2_{\sigma_{\epsilon_j}}<0$ *is the largest negative eigenvalue of* $\tilde T_n[0]$ *on* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}_{\{\sigma_1,\dots,\sigma_{\epsilon_j-1},\sigma_{\epsilon_j+1},\dots,\sigma_k\}}$.
- *If $k=1$, n odd, $\sigma_{\epsilon_j}>\alpha$ and $M\in[-\lambda^1_{\sigma_{\epsilon_j}},+\infty)$, where*
  - * $\lambda^1_{\sigma_{\epsilon_j}}>0$ *is the least positive eigenvalue of* $\tilde T_n[0]$ *on the set* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_{\epsilon_j-1},\sigma_{\epsilon_j+1},\dots,\sigma_k|\alpha\}}$.
- *If $k=1$, n odd, $\sigma_{\epsilon_j}<\alpha$ and $M\in[-\lambda_1,+\infty)$, where*
  - * $\lambda_1>0$ *is the least positive eigenvalue of* $\tilde T_n[0]$ *on* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$.
- *If $n-k$ is odd, $k>1$, $\sigma_{\epsilon_j}>\alpha$ and $M\in[-\lambda^2_{\sigma_{\epsilon_j}},-\lambda^1_{\sigma_{\epsilon_j}}]$, where*
  - * $\lambda^2_{\sigma_{\epsilon_j}}>0$ *is the least positive eigenvalue of* $\tilde T_n[0]$ *on the set* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}_{\{\sigma_1,\dots,\sigma_{\epsilon_j-1},\sigma_{\epsilon_j+1},\dots,\sigma_k\}}$,
  - * $\lambda^1_{\sigma_{\epsilon_j}}<0$ *is the largest negative eigenvalue of* $\tilde T_n[0]$ *on the set* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_{\epsilon_j-1},\sigma_{\epsilon_j+1},\dots,\sigma_k|\alpha\}}$.
- *If $n-k$ is odd, $k>1$, $\sigma_{\epsilon_j}<\alpha$ and $M\in[-\lambda^2_{\sigma_{\epsilon_j}},-\lambda_1]$, where*
  - * $\lambda^2_{\sigma_{\epsilon_j}}>0$ *is the least positive eigenvalue of* $\tilde T_n[0]$ *on the set* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}_{\{\sigma_1,\dots,\sigma_{\epsilon_j-1},\sigma_{\epsilon_j+1},\dots,\sigma_k\}}$,
  - * $\lambda_1<0$ *is the largest negative eigenvalue of* $\tilde T_n[0]$ *on* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$.
- *If $k=1$, n odd, $\sigma_{\epsilon_j}>\alpha$ and $M\in(-\infty,-\lambda^1_{\sigma_{\epsilon_j}}]$, where*

  * $\lambda^1_{\sigma_{\epsilon_j}} < 0$ *is the largest negative eigenvalue of* $\tilde{T}_n[0]$ *on the set* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_{\epsilon_j-1},\sigma_{\epsilon_j+1},\dots,\sigma_k|\alpha\}}$.
- *If* $k=1$, n *odd,* $\sigma_{\epsilon_j} < \alpha$ *and* $M \in (-\infty, -\lambda_1]$, *where*
  * $\lambda_1 < 0$ *is the largest negative eigenvalue of* $\tilde{T}_n[0]$ *on* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$.

Proof. Firstly, let us see what happens for $M=0$. As we have seen in the previous results, without taking into account the boundary conditions, if u is a solution of $\tilde{T}_n[0]u(t)=0$ on (a,b), then u has at most $n-1$ zeros. However, from the boundary conditions (12.8), we conclude that $\tilde{T}_\ell u(a)=0$ or $\tilde{T}_\ell u(b)=0$ at least $n-1$ times from $\ell=0$ to $n-1$. Thus, we lose the $n-1$ possible oscillations and u does not have any zero on (a,b).

Now, let us consider $u_M \in C^n(I)$ a solution of $\tilde{T}_n[0]u_M(t)=0$ on (a,b). Assume that $u_0>0$ on (a,b) (if $u_0<0$ on (a,b) the arguments are valid by multiplying by -1) and we move continuously on M to obtain u_M. We will see that while $u_M \geq 0$, it cannot have any double zero, which implies that it is positive on (a,b).

It is known that $\tilde{T}[0]u_M(t)=-Mu_M(t)$, on (a,b), hence $\tilde{T}_{n-1}u_M$ is a monotone function on I, with at most one zero. Then, arguing as before, we conclude, without taking into account the boundary conditions, that u_M can have at most n zeros. But, if we consider the boundary conditions (12.8), we lose $n-1$ possible oscillation and u_M is only allowed to have a simple zero on (a,b), which is not possible if it is of constant sign. Hence, we can affirm that $u_M>0$ on (a,b) up to one of the following assertions is satisfied:

- $\sigma_{\epsilon_j} > \alpha$ and $u_M^{(\alpha)}(a)=0$.
- $\sigma_{\epsilon_j} < \alpha$ and $u_M^{(\sigma_{\epsilon_j})}(a)=0$.
- $u_M^{(\beta)}(b)=0$.

Now, let us study separately the cases where $M>0$ or $M<0$ to see with which of the previous assertions the sign change begins in each case.

If $M \geq 0$, then $\tilde{T}[0]u_M(t)=-Mu_M(t)\leq 0$ for $t\in(a,b)$. Thus, $\tilde{T}_n u_M(a)\leq 0$ and $\tilde{T}_n u_M(b)\leq 0$. With maximal oscillation $\tilde{T}_{n-\ell}u_M(a)$ changes its sign each time that it is not null and $\tilde{T}_{n-\ell}u_M(b)$ changes its sign as many times as it vanishes.

- If $\sigma_{\epsilon_j} > \alpha$, from $\ell=0$ to $n-\alpha$, $\tilde{T}_{n-\ell}u_M(a)$ vanishes $k-1-\alpha$ times. If $\tilde{T}_{n-\ell}u_M(a)=0$ for $\ell<n-\alpha$ and $n-\ell \notin \{\sigma_1,\dots,\sigma_{\epsilon_j-1},\sigma_{\epsilon_j+1},\dots,\sigma_k|\alpha\}$, then $\tilde{T}_\alpha u_M(a)\neq 0$ and u_M remains positive on (a,b). So, we can assume that this situation cannot be fulfilled.

  Hence, with maximal oscillation, we have:
$$\tilde{T}_\alpha u_M(a)\begin{cases}\geq 0\,, & \text{if } n-\alpha-(k-1-\alpha)=n-k+1 \text{ is odd},\\ \leq 0\,, & \text{if } n-k+1 \text{ is even}.\end{cases}$$
  Since $\sigma_{\epsilon_j} > \alpha$, from (3.3), we have that
$$\tilde{T}_\alpha u_M(a)=\frac{u^{(\alpha)}(a)}{v_1(a)\dots v_\alpha(a)}\,,$$
  so, with maximal oscillation:
$$u_M^{(\alpha)}(a)\begin{cases}\geq 0\,, & \text{if } n-k \text{ is even},\\ \leq 0\,, & \text{if } n-k \text{ is odd}.\end{cases}$$

- If $\sigma_{\epsilon_j} < \alpha$, from $\ell = 0$ to $n - \sigma_{\epsilon_j}$, $\tilde{T}_{n-\ell} u_M(a)$ vanishes $k - 1 - \sigma_{\epsilon_j}$ times. Again, let us assume that $\tilde{T}_{n-\ell} u_M(a) \neq 0$ for $\ell < n - \sigma_{\epsilon_j}$ if $n - \ell \notin \{\sigma_1, \dots, \sigma_k\}$. Then, with maximal oscillation, we have
$$\tilde{T}_{\sigma_{\epsilon_j}} u_M(a) \begin{cases} \geq 0 & \text{if } n - \sigma_{\epsilon_j} - (k - 1 - \sigma_{\epsilon_j}) = n - k + 1 \text{ is odd,} \\ \leq 0 & \text{if } n - k + 1 \text{ is even.} \end{cases}$$
Since $\sigma_{\epsilon_j} < \alpha$, from (3.3), we have
$$\tilde{T}_{\sigma_{\epsilon_j}} u_M(a) = \frac{u^{(\sigma_{\epsilon_j})}(a)}{v_1(a) \dots v_{\sigma_{\epsilon_j}}(a)}\,.$$
In particular, if $\sigma_{\epsilon_j} < \mu$, then $v_1(t) \dots v_{\sigma_{\epsilon_j}}(t) = 1$. Thus, with maximal oscillation:
$$u_M^{(\sigma_{\epsilon_j})}(a) \begin{cases} \geq 0\,, & \text{if } n - k \text{ is even,} \\ \leq 0\,, & \text{if } n - k \text{ is odd.} \end{cases}$$

On the other hand, from $\ell = 0$ to $n - \beta$, $\tilde{T}_{n-\ell} u_M(b)$ vanishes $n - k - \beta$ times. We can also assume that $\tilde{T}_{n-\ell} u_M(b) \neq 0$ if $n - \ell \notin \{\varepsilon_1, \dots, \varepsilon_{n-k} | \beta\}$. Then, with maximal oscillation:
$$\tilde{T}_\beta u_M(b) \begin{cases} \geq 0\,, & \text{if } n - k - \beta \text{ is odd,} \\ \leq 0\,, & \text{if } n - k - \beta \text{ is even.} \end{cases}$$
From (3.3), we have that
$$\tilde{T}_\beta u_M(b) = \frac{u^{(\beta)}(b)}{v_1(b) \dots v_\beta(b)}\,.$$
Thus:

- if $n - k$ is even, to set maximal oscillation, we need
$$u_M^{(\beta)}(b) \begin{cases} \leq 0\,, & \text{if } \beta \text{ is even,} \\ \geq 0\,, & \text{if } \beta \text{ is odd.} \end{cases}$$
- if $n - k$ is odd, to ensure maximal oscillation is necessary:
$$u_M^{(\beta)}(b) \begin{cases} \geq 0\,, & \text{if } \beta \text{ is even,} \\ \leq 0\,, & \text{if } \beta \text{ is odd.} \end{cases}$$

Since, we are considering $u_M \geq 0$, it is known that
$$u_M^{(\alpha)}(a) \begin{cases} \geq 0\,, & \text{if } \sigma_{\epsilon_j} > \alpha, \\ \geq 0\,, & \text{if } \sigma_{\epsilon_j} < \alpha, \end{cases} \tag{12.9}$$
and
$$u_M^{(\beta)}(b) \begin{cases} \geq 0\,, & \text{if } \beta \text{ is even,} \\ \leq 0\,, & \text{if } \beta \text{ is odd,} \end{cases} \tag{12.10}$$
Taking into account that if $k = 1$, then $u_M^{(\beta)}(b) \neq 0$ for all $M \in \mathbb{R}$, we obtain the following conclusions for $M \geq 0$:

- If $n - k$ is odd and $\sigma_{\epsilon_j} > \alpha$, then $u_M \geq 0$ if $u_N^{(\alpha)}(a) \neq 0$ for all N between 0 and M; i.e., up to an eigenvalue of $\tilde{T}_n[0]$ on $X_{\{\sigma_1, \dots, \sigma_{\epsilon_j - 1}, \sigma_{\epsilon_j + 1}, \dots, \sigma_k | \alpha\}}^{\{\varepsilon_1, \dots, \varepsilon_{n-k}\}}$ is found.

- If $n-k$ is odd and $\sigma_{\epsilon_j} < \alpha$, then $u_M \geq 0$ if $u_N^{(\beta)}(b) \neq 0$ for all N between 0 and M; i.e., up to an eigenvalue of $\tilde{T}_n[0]$ on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ is found.
- If $n-k$ is even and $k > 1$, then $u_M \geq 0$ up to $u_M^{(\beta)}(b) = 0$; i.e., up to an eigenvalue of $\tilde{T}_n[0]$ on $X_{\{\sigma_1,\dots,\sigma_{\epsilon_j-1},\sigma_{\epsilon_j+1},\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$ is found.
- If $k=1$ and n is odd, then $u_M \geq 0$ for all $M \geq 0$.

Now, let us see what happens for $M \leq 0$. In this case, we have that $\tilde{T}_n[0]u_M(t) = -Mu_M(t) \geq 0$ for $t \in (a,b)$. Then, $\tilde{T}_n u_M(a) \geq 0$ and $\tilde{T}_n u_M(b) \geq 0$. Hence, we conclude that with maximal oscillation, the inequalities are reversed from the case $M \geq 0$. So, we obtain that:

- If $\sigma_{\epsilon_j} > \alpha$, with maximal oscillation
$$u_M^{(\alpha)}(a) \begin{cases} \leq 0\,, & \text{if } n-k \text{ is even,} \\ \geq 0\,, & \text{if } n-k \text{ is odd.} \end{cases}$$
- If $\sigma_{\epsilon_j} < \alpha$, with maximal oscillation
$$u_M^{(\sigma_{\epsilon_j})}(a) \begin{cases} \leq 0\,, & \text{if } n-k \text{ is even,} \\ \geq 0\,, & \text{if } n-k \text{ is odd,} \end{cases}$$

and

- if $n-k$ is even, with maximal oscillation:
$$u_M^{(\beta)}(b) \begin{cases} \geq 0\,, & \text{if } \beta \text{ is even,} \\ \leq 0\,, & \text{if } \beta \text{ is odd.} \end{cases}$$
- If $n-k$ is odd, with maximal oscillation:
$$u_M^{(\beta)}(b) \begin{cases} \leq 0\,, & \text{if } \beta \text{ is even,} \\ \geq 0\,, & \text{if } \beta \text{ is odd.} \end{cases}$$

Then, taking into account that $u_M \geq 0$, (12.9) and (12.10) are also satisfied.

Hence, using that if $k=1$, then $u_M^{(\beta)}(b) \neq 0$ for all $M \in \mathbb{R}$, we obtain the following conclusions for $M \leq 0$:

- If $n-k$ is even and $\sigma_{\epsilon_j} > \alpha$, then $u_M \geq 0$ if $u_N^{(\alpha)}(a) \neq 0$ for all N between 0 and M; i.e., up to an eigenvalue of $\tilde{T}_n[0]$ on $X_{\{\sigma_1,\dots,\sigma_{\epsilon_j-1},\sigma_{\epsilon_j+1},\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ is found.
- If $n-k$ is even and $\sigma_{\epsilon_j} < \alpha$, then $u_M \geq 0$ if $u_N^{(\sigma_{\epsilon_j})}(a) \neq 0$ for all N between 0 and M; i.e., up to an eigenvalue of $\tilde{T}_n[0]$ on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ is found.
- If $n-k$ is odd and $k > 1$, then $u_M \geq 0$ if $u_N^{(\beta)}(b) \neq 0$ for all N between 0 and M; i.e., up to an eigenvalue of $\tilde{T}_n[0]$ on $X_{\{\sigma_1,\dots,\sigma_{\epsilon_j-1},\sigma_{\epsilon_j+1},\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$ is found.
- If $k=1$ and n is even, then $u_M \geq 0$ for all $M \leq 0$.

The result is proved. □

Proposition 12.10. *Let $u \in C^n(I)$ be a solution of $\tilde{T}[M]u(t) = 0$ for $t \in (a,b)$, which satisfies the boundary conditions*

$$\begin{gathered} u^{(\sigma_1)}(a) = \cdots = u^{(\sigma_k)}(a) = 0\,, \\ u^{(\varepsilon_1)}(b) = \cdots = u^{(\varepsilon_{\kappa_i}-1)}(b) = u^{(\varepsilon_{\kappa_i}+1)}(b) = \cdots = u^{(\varepsilon_{n-k})}(b) = 0\,. \end{gathered} \tag{12.11}$$

Then, u does not have any zeros on (a,b) provided that one of the following assertions is fulfilled:

- *If $n-k$ is even, $\varepsilon_{\kappa_i} > \beta$ and $M \in [-\lambda^1_{\varepsilon_{\kappa_i}}, -\lambda^2_{\varepsilon_{\kappa_i}}]$, where*
  - * $\lambda^1_{\varepsilon_{\kappa_i}} > 0$ *is the least positive eigenvalue of $\tilde{T}_n[0]$ on the set* $X^{\{\varepsilon_1,\dots,\varepsilon_{\kappa_i}-1,\varepsilon_{\kappa_i}+1,\dots,\varepsilon_{n-k}|\beta\}}_{\{\sigma_1,\dots,\sigma_k\}}$,
  - * $\lambda^2_{\varepsilon_{\kappa_i}} < 0$ *is the largest negative eigenvalue of $\tilde{T}_n[0]$ on the set* $X^{\{\varepsilon_1,\dots,\varepsilon_{\kappa_i}-1,\varepsilon_{\kappa_i}+1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k|\alpha\}}$.
- *If $n-k$ is even, $\varepsilon_{\kappa_i} < \alpha$ and $M \in [-\lambda_1, -\lambda^2_{\varepsilon_{\kappa_i}}]$, where*
  - * $\lambda_1 > 0$ *is the least positive eigenvalue of $\tilde{T}_n[0]$ on* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$,
  - * $\lambda^2_{\varepsilon_{\kappa_i}} < 0$ *is the largest negative eigenvalue of $\tilde{T}_n[0]$ on the set* $X^{\{\varepsilon_1,\dots,\varepsilon_{\kappa_i}-1,\varepsilon_{\kappa_i}+1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k|\alpha\}}$.
- *If $n-k$ is odd, $k < n-1$, $\varepsilon_{\kappa_i} > \beta$ and $M \in [-\lambda^2_{\varepsilon_{\kappa_i}}, -\lambda^1_{\varepsilon_{\kappa_i}}]$, where*
  - * $\lambda^2_{\varepsilon_{\kappa_i}} > 0$ *is the least positive eigenvalue of $\tilde{T}_n[0]$ on the set* $X^{\{\varepsilon_1,\dots,\varepsilon_{\kappa_i}-1,\varepsilon_{\kappa_i}+1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k|\alpha\}}$,
  - * $\lambda^1_{\varepsilon_{\kappa_i}} < 0$ *is the largest negative eigenvalue of $\tilde{T}_n[0]$ on the set* $X^{\{\varepsilon_1,\dots,\varepsilon_{\kappa_i}-1,\varepsilon_{\kappa_i}+1,\dots,\varepsilon_{n-k}|\beta\}}_{\{\sigma_1,\dots,\sigma_k\}}$.
- *If $n-k$ is odd, $k < n-1$, $\varepsilon_{\kappa_i} < \alpha$ and $M \in [-\lambda^2_{\varepsilon_{\kappa_i}}, -\lambda_1]$, where*
  - * $\lambda^2_{\varepsilon_{\kappa_i}} > 0$ *is the least positive eigenvalue of $\tilde{T}_n[0]$ on the set* $X^{\{\varepsilon_1,\dots,\varepsilon_{\kappa_i}-1,\varepsilon_{\kappa_i}+1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k|\alpha\}}$,
  - * $\lambda_1 < 0$ *is the largest negative eigenvalue of $\tilde{T}_n[0]$ on* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$.
- *If $k = n-1$, $\varepsilon_{\kappa_i} > \alpha$ and $M \in (-\infty, -\lambda^1_{\sigma_{\epsilon_j}}]$, where*
  - * $\lambda^1_{\varepsilon_{\kappa_i}} < 0$ *is the largest negative eigenvalue of $\tilde{T}_n[0]$ on the set* $X^{\{\varepsilon_1,\dots,\varepsilon_{\kappa_i}-1,\varepsilon_{\kappa_i}+1,\dots,\varepsilon_{n-k}|\beta\}}_{\{\sigma_1,\dots,\sigma_k\}}$.
- *If $k = n-1$, $\varepsilon_{\kappa_i} < \alpha$ and $M \in (-\infty, -\lambda_1]$, where*
  - * $\lambda_1 < 0$ *is the largest negative eigenvalue of $\tilde{T}_n[0]$ on* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$.

The proof of the above proposition is analogous to the proof of Proposition 12.9, and is omitted here. Now, we are in a position to prove a result which gives a relationship on the eigenvalues of the different spaces $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_{\epsilon_j-1},\sigma_{\epsilon_j+1},\dots,\sigma_k|\alpha\}}$ with the closest to zero eigenvalue of $\tilde{T}_n[0]$ on $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$. The result is the following.

Proposition 12.11. *Let $j_1 \in \{\epsilon_1,\dots,\epsilon_\ell\}$ be such that $\alpha < \sigma_{j_1}$, then the following assertions are true:*

- *If $n-k$ is even, then $0 < \lambda_1 < \lambda^1_{\sigma_{j_1}}$, where*

* $\lambda^1_{\sigma_{j_1}} > 0$ *is the least positive eigenvalue of* $\tilde{T}_n[0]$ *on the set* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_{j_1-1},\sigma_{j_1+1},\dots,\sigma_k|\alpha\}}$.
* $\lambda_1 > 0$ *is the least positive eigenvalue of* $\tilde{T}_n[0]$ *on* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$.

- *If* $n-k$ *is odd, then* $\lambda^1_{\sigma_{j_1}} < \lambda_1 < 0$, *where*
  * $\lambda^1_{\sigma_{j_1}} < 0$ *is the largest negative eigenvalue of* $\tilde{T}_n[0]$ *on the set* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_{j_1-1},\sigma_{j_1+1},\dots,\sigma_k|\alpha\}}$.
  * $\lambda_1 < 0$ *is the largest negative eigenvalue of* $\tilde{T}_n[0]$ *on* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$.

Proof. To prove this result, let us denote $v_M \in C^n(I)$ as a solution of $\tilde{T}[M]v_M(t) = 0$ on (a,b), coupled with the following boundary conditions

$$\begin{gathered} v_M^{(\sigma_j)}(a) = 0\,, \quad j = 0,\dots,k,\ j \neq j_1\,, \\ v_M^{(\varepsilon_i)}(b) = 0, \quad i = 0,\dots,n-k\,. \end{gathered} \tag{12.12}$$

Let us study v_0, with the arguments used before. We know that, without taking into account the boundary conditions, v_0 has at most $n-1$ zeros. However, from the boundary conditions (12.12), we conclude that $n-1$ possible oscillations are lost. Hence, since v_0 is a nontrivial function, the boundary conditions for the maximal oscillation are verified.

Let us choose $v_0 \geq 0$ (if $v_0 \leq 0$, the arguments are valid by multiplying by -1), then $v_0^{(\alpha)}(a) \geq 0$. From (3.3) $T_\alpha v_0(a)$ also satisfies this inequality.

Let us study the sign of $v_M^{(\sigma_{j_1})}(a)$. Realize, that, to achieve the maximal oscillation, $T_\ell v_M(a)$ must change its sign each time that it is non null.

From $\ell = \alpha$ to σ_{j_1}, $T_\ell v_0(a)$ vanishes $j_1 - 1 - \alpha$ times, then, with maximal oscillation:

$$T_{\sigma_{j_1}} v_0(a) \begin{cases} > 0\,, & \text{if } \sigma_{j_1} - \alpha - (j_1 - 1 - \alpha) = \sigma_{j_1} - j_1 + 1 \text{ is even,} \\ < 0\,, & \text{if } \sigma_{j_1} - j_1 + 1 \text{ is odd.} \end{cases}$$

From the choice of $j_1 \in \{\epsilon_1,\dots,\epsilon_\ell\}$, we can affirm that

$$v_0^{(\sigma_{j_1})}(a) \begin{cases} < 0\,, & \text{if } \sigma_{j_1} - j_1 \text{ is even,} \\ > 0\,, & \text{if } \sigma_{j_1} - j_1 \text{ is odd.} \end{cases} \tag{12.13}$$

Now, let us move with continuity on M up to $-\lambda_{\sigma_{j_1}}$ and study the sign of $v^{(\sigma_{j_1})}_{-\lambda^1_{\sigma_{j_1}}}(a)$.

From Proposition 12.9, $v_{-\lambda_{\sigma_{j_1}}} > 0$ on (a,b). Moreover, $v^{(\alpha)}_{-\lambda_{\sigma_{j_1}}}(a) = 0$. Thus, with the calculations done before, we conclude that the maximal oscillation is satisfied too. So, we can study in this case the sign of $v^{(\sigma_{j_1})}_{-\lambda^1_{\sigma_{j_1}}}(a)$. Let us consider $\alpha_1 \in \{0,\dots,n-1\}$, previously introduced in the proof of Proposition 6.5. Since $v_{-\lambda_{\sigma_{j_1}}} \geq 0$ on I, we can affirm that $v^{(\alpha_1)}_{-\lambda_{\sigma_{j_1}}}(a) > 0$.

From $\ell = \alpha_1$ to σ_{j_1}, there are $j_1 - \alpha_1$ zeros for $T_\ell v_{-\lambda^1_{\sigma_{j_1}}}(a)$, then, with maximal oscillation:

$$T_{\sigma_{j_1}} v_{-\lambda^1_{\sigma_{j_1}}}(a) \begin{cases} > 0\,, & \text{if } \sigma_{j_1} - \alpha_1 - (j_1 - \alpha_1) = \sigma_{j_1} - j_1 \text{ is even,} \\ < 0\,, & \text{if } \sigma_{j_1} - j_1 \text{ is odd.} \end{cases}$$

From the choice of $j_1 \in \{\epsilon_1, \dots, \epsilon_\ell\}$, we can affirm that

$$v_{-\lambda^1_{\sigma_{j_1}}}^{(\sigma_{j_1})}(a) \begin{cases} > 0\,, & \text{if } \sigma_{j_1} - j_1 \text{ is even,} \\ < 0\,, & \text{if } \sigma_{j_1} - j_1 \text{ is odd.} \end{cases} \tag{12.14}$$

Hence, in this case, since we have been moving continuously on M, we can affirm that there exist $-\tilde{\lambda}_1$ between 0 and $-\lambda^1_{\sigma_{j_1}}$ such that $v_{-\tilde{\lambda}_1}^{(\sigma_{j_1})}(a) = 0$, i.e. we have proved the existence on an eigenvalue of $\tilde{T}_n[0]$ on $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$ between 0 and $-\lambda^1_{\sigma_{j_1}}$, and the result is proved. □

In an analogous way, we can prove the following result for the eigenvalues of $\tilde{T}[0]$ on $X^{\{\varepsilon_1,\dots,\varepsilon_{\kappa_i-1},\varepsilon_{\kappa_i+1},\dots,\varepsilon_{n-k}|\beta\}}_{\{\sigma_1,\dots,\sigma_k\}}$, comparing them with the closest to zero eigenvalue on $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$.

Proposition 12.12. *Let $i_1 \in \{\kappa_1, \dots, \kappa_h\}$ be such that $\varepsilon_{i_1} > \beta$, then the following assertions are true:*

- *If $n-k$ is even, then $0 < \lambda_1 < \lambda^1_{\varepsilon_{i_1}}$, where*
  - *$\lambda^1_{\varepsilon_{i_1}} > 0$ is the least positive eigenvalue of $\tilde{T}_n[0]$ on the set $X^{\{\varepsilon_1,\dots,\varepsilon_{i_1-1},\varepsilon_{i_1+1},\dots,\varepsilon_{n-k}|\beta\}}_{\{\sigma_1,\dots,\sigma_k\}}$.*
  - *$\lambda_1 > 0$ is the least positive eigenvalue of $\tilde{T}_n[0]$ on $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$.*
- *If $n-k$ is odd, then $0 > \lambda_1 > \lambda^1_{\varepsilon_{i_1}}$, where*
  - *$\lambda^1_{\varepsilon_{i_1}} < 0$ is the largest negative eigenvalue of $\tilde{T}_n[0]$ on the set $X^{\{\varepsilon_1,\dots,\varepsilon_{i_1-1},\varepsilon_{i_1+1},\dots,\varepsilon_{n-k}|\beta\}}_{\{\sigma_1,\dots,\sigma_k\}}$.*
  - *$\lambda_1 < 0$ is the largest negative eigenvalue of $\tilde{T}_n[0]$ on $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$.*

The proof of the above proposition is analogous to the one of Proposition 12.11, and is omitted here. Now, let us establish a comparison between the eigenvalues in the different spaces $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}_{\{\sigma_1,\dots,\sigma_{\epsilon_j-1},\sigma_{\epsilon_j+1},\dots,\sigma_k\}}$.

Proposition 12.13. *Let $\sigma_{j_1}, \sigma_{j_2} \in \{\sigma_{\epsilon_1}, \dots, \sigma_{\epsilon_\ell}\}$ be such that $j_1 < j_2$. Then the following assertions are fulfilled:*

- *If $n-k$ is even and $k > 1$, then $0 > \lambda^2_{\sigma_{j_1}} > \lambda^2_{\sigma_{j_2}}$, where*
  - *$\lambda^2_{\sigma_{j_1}} < 0$ is the largest negative eigenvalue of $\tilde{T}_n[0]$ on the set $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}_{\{\sigma_1,\dots,\sigma_{j_1-1},\sigma_{j_1+1},\dots,\sigma_k\}}$.*
  - *$\lambda^2_{\sigma_{j_2}} < 0$ is the largest negative eigenvalue of $\tilde{T}_n[0]$ on the set $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}_{\{\sigma_1,\dots,\sigma_{j_2-1},\sigma_{j_2+1},\dots,\sigma_k\}}$.*
- *If $n-k$ is odd and $k > 1$, then $0 < \lambda^2_{\sigma_{j_1}} < \lambda^2_{\sigma_{j_2}}$, where*
  - *$\lambda^2_{\sigma_{j_1}} > 0$ is the least positive eigenvalue of $\tilde{T}_n[0]$ on the set $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}_{\{\sigma_1,\dots,\sigma_{j_1-1},\sigma_{j_1+1},\dots,\sigma_k\}}$.*
  - *$\lambda^2_{\sigma_{j_2}} > 0$ is the least positive eigenvalue of $\tilde{T}_n[0]$ on the set $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}_{\{\sigma_1,\dots,\sigma_{j_2-1},\sigma_{j_2+1},\dots,\sigma_k\}}$.*

Proof. To prove this result, we denote by $v_{1M} \in C^n(I)$ a solution of $\tilde{T}[M]\, v_{1M}(t) = 0$ on (a,b), coupled with the following boundary conditions:

$$\begin{gathered} v_{1M}^{(\sigma_j)}(a) = 0\,,\ j = 0,\dots,k, \quad \text{if } j \neq j_1\,,\ j_2\,, \\ v_{1M}^{(\epsilon_i)}(b) = 0\,, \quad \text{if } i = 0,\dots,n-k\,, \\ v_{1M}^{(\beta)}(b) = 0\,. \end{gathered} \tag{12.15}$$

Again, from the boundary conditions (12.12), to ensure that is is a nontrivial solution, v_{10} satisfies the conditions of maximal oscillation at $t = a$ and $t = b$.

First, let us see what happens if $\sigma_{j_1} > \alpha$. Let us choose $v_{10} \ge 0$ (if $v_{10} \le 0$, then the arguments are valid by multiplying by -1), then $v_{10}^{(\alpha)}(a) \ge 0$. From (3.3) we have $T_\alpha v_{10}(a) \ge 0$.

To study the sign of $v_0^{(\sigma_{j_2})}(a)$, realize that, to achieve the maximal oscillation, $T_\ell v_M(a)$ changes its sign each time that it is non null.

From $\ell = \alpha$ to σ_{j_2}, there are $j_2 - 2 - \alpha$ zeros for $T_\ell v_{10}(a)$, then, with maximal oscillation:

$$T_{\sigma_{j_2}} v_{10}(a) \begin{cases} > 0\,, & \text{if } \sigma_{j_2} - \alpha - (j_2 - 2 - \alpha) = \sigma_{j_2} - j_2 + 2 \text{ is even}, \\ < 0\,, & \text{if } \sigma_{j_2} - j_2 + 2 \text{ is odd}. \end{cases}$$

From the choice of $j_2 \in \{\epsilon_1, \dots, \epsilon_\ell\}$, we can affirm that

$$v_{10}^{(\sigma_{j_2})}(a) \begin{cases} > 0\,, & \text{if } \sigma_{j_2} - j_2 \text{ is even}, \\ < 0\,, & \text{if } \sigma_{j_2} - j_2 \text{ is odd}. \end{cases} \tag{12.16}$$

Now, let us move with continuity on M up to $-\lambda_{\sigma_{j_2}}^2$ and analyze the sign of $v_{1\,-\lambda_{\sigma_{j_2}}^2}^{(\sigma_{j_2})}(a)$. Let us denote $\bar{\lambda_2} = -\lambda_{\sigma_{j_2}}^2$, from Proposition 12.9, it is known that $v_{1\bar{\lambda_2}} > 0$ on (a,b). Moreover, $v_{1\bar{\lambda_2}}^{(\sigma_{j_1})}(a) = 0$. Thus, since another possible zero on the boundary will imply that $v_{1\bar{\lambda_2}} \equiv 0$, we conclude that the maximal oscillation is satisfied too.

So, we can study, in this case, the sign of $v_{1\bar{\lambda_2}}(a)$. Since $v_{1\bar{\lambda_2}} \ge 0$ on I, we can affirm that, as for $M = 0$, $v_{1\bar{\lambda_2}}^{(\alpha)}(a) > 0$.

From $\ell = \alpha$ to σ_{j_2}, there are $j_2 - 1 - \alpha$ zeros for $T_\ell v_{1\bar{\lambda_2}}(a)$, then, with maximal oscillation:

$$T_{\sigma_{j_2}} v_{1\bar{\lambda_2}}(a) \begin{cases} > 0\,, & \text{if } \sigma_{j_2} - \alpha - (j_2 - 1 - \alpha) = \sigma_{j_2} - j_2 + 1 \text{ is even}, \\ < 0\,, & \text{if } \sigma_{j_2} - j_2 + 1 \text{ is odd}. \end{cases}$$

From the choice of $j_2 \in \{\epsilon_1, \dots, \epsilon_\ell\}$, we can affirm that

$$v_{1\,-\lambda_{\sigma_{j_2}}^2}^{(\sigma_{j_2})}(a) \begin{cases} < 0\,, & \text{if } \sigma_{j_2} - j_2 \text{ is even}, \\ > 0\,, & \text{if } \sigma_{j_2} - j_2 \text{ is odd}. \end{cases} \tag{12.17}$$

Now, let us see what happens if $\sigma_{j_1} < \alpha < \sigma_{j_2}$. In this case, $\sigma_{j_1} = j_1 - 1$. For $M = 0$, since $v_{10} \ge 0$, we have that $v_{10}^{(\sigma_{j_1})}(a) \ge 0$. From (3.3) we have that $T_{\sigma_{j_1}} v_{10}(a) \ge 0$. Let us study the sign of $v_{10}^{(\sigma_{j_2})}(a)$ in this case.

From $\ell=\sigma_{j_1}$ to σ_{j_2}, there are $j_2-2-(j_1-1)=j_2-j_1-1$ zeros of $T_\ell v_{10}(a)$. Then, with maximal oscillation:

$$T_{\sigma_{j_2}}v_{10}(a)\begin{cases}>0\,, & \text{if } \sigma_{j_2}-j_1-1-(j_2-j_1-1)=\sigma_{j_2}-j_2 \text{ is even},\\ <0\,, & \text{if } \sigma_{j_2}-j_2 \text{ is odd}.\end{cases}$$

From the choice of $j_2\in\{\epsilon_1,\dots,\epsilon_\ell\}$, we can affirm that (12.16) holds.

Now, we study the sign of $v_{1\bar{\lambda}_2}(a)$ if the conditions to allow the maximal oscillation hold. Since $v_{1-\lambda^2_{\sigma_{j_2}}}\geq 0$ on I, we can affirm that $v_{1-\lambda_{\sigma_{j_1}}}^{(\alpha)}(a)>0$. From $\ell=\alpha$ to σ_{j_2}, there are $j_2-1-\alpha$ zeros for $T_\ell v_{1-\lambda^2_{\sigma_{j_2}}}(a)$, then, with maximal oscillation and repeating the previous arguments, we obtain that (12.17) is satisfied.

Finally, let us study the case where $\sigma_{j_2}<\alpha$. In this situation, $\sigma_{j_1}=j_1-1$ and $\sigma_{j_2}=j_2-1$. For $M=0$, since $v_{10}\geq 0$, we have that $v_{10}^{(\sigma_{j_1})}(a)\geq 0$ and from (3.3) $T_\alpha v_{10}(a)\geq 0$.

Let us study the sign of $v_{10}^{(\sigma_{j_2})}(a)$ in this situation. Since $\alpha>\sigma_{j_2}$, for all $\ell=\sigma_{j_1},\dots,,\sigma_{j_2}$, we have that $\tilde{T}_\ell v_{10}(a)=0$. So, to allow the maximal oscillation, it must be satisfied that $T_{\sigma_{j_2}}v_{10}(a)<0$. And this inequality also holds for $v_{10}^{(\sigma_{j_2})}(a)$. In this case, for $M=-\lambda^2_{\sigma_{j_2}}$, since $v_{1-\lambda_{\sigma^2_{j_2}}}>0$ on (a,b) and $v_{1-\lambda_{\sigma^2_{j_2}}}^{(\sigma_{j_1})}(a)=0$, we have that $v_{1-\lambda^2_{\sigma_{j_2}}}^{(\sigma_{j_2})}(a)>0$.

Hence, in all the cases, since we have been moving continuously on M, we can affirm that there exists $-\tilde{\lambda}_1$ lying between 0 and $-\lambda^2_{\sigma_{j_2}}$, such that $v_{1-\tilde{\lambda}_1}^{(\sigma_{j_2})}(a)=0$. As consequence, we have proved the existence on an eigenvalue of $\tilde{T}_n[0]$ on $X_{\{\sigma_1,\dots,\sigma_{j_1-1},\sigma_{j_1+1},\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ between 0 and $-\lambda^2_{\sigma_{j_2}}$, and the result is proved. □

Before introducing the final result which characterizes the strongly inverse positive (negative) character in the different spaces $X_{\{\sigma_1,\dots,\sigma_k\}_{\{\sigma_{\epsilon_1},\dots,\sigma_{\epsilon_\ell}\}}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}_{\{\varepsilon_{\kappa_1},\dots,\varepsilon_{\kappa_h}\}}}$, we show a result which gives an order on the eigenvalues associated to different spaces $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{\kappa_i-1},\varepsilon_{\kappa_i+1},\dots,\varepsilon_{n-k}\}}$.

Proposition 12.14. *Let $i_1,i_2\in\{\kappa_1,\dots,\kappa_h\}$ be such that if $i_1<i_2$, then the following assertions hold:*

- *If $n-k$ is even, then $0>\lambda^2_{\varepsilon_{i_1}}>\lambda^2_{\varepsilon_{i_2}}$, where*
    - * $\lambda^2_{\varepsilon_{i_1}}<0$ *is the largest negative eigenvalue of* $\tilde{T}_n[0]$ *on the set* $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{i_1-1},\varepsilon_{i_1+1},\dots,\varepsilon_{n-k}\}}$.
    - * $\lambda^2_{\varepsilon_{i_2}}<0$ *is the largest negative eigenvalue of* $\tilde{T}_n[0]$ *on the set* $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{i_2-1},\varepsilon_{i_2+1},\dots,\varepsilon_{n-k}\}}$.
- *If $n-k$ is odd and $k<n-1$, then $0<\lambda^2_{\varepsilon_{i_1}}<\lambda^2_{\varepsilon_{i_2}}$, where*
    - * $\lambda^2_{\varepsilon_{i_1}}>0$ *is the least positive eigenvalue of* $\tilde{T}_n[0]$ *on the set* $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{i_1-1},\varepsilon_{i_1+1},\dots,\varepsilon_{n-k}\}}$.
    - * $\lambda^2_{\varepsilon_{i_2}}>0$ *is the least positive eigenvalue of* $\tilde{T}_n[0]$ *on the set* $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{i_2-1},\varepsilon_{i_2+1},\dots,\varepsilon_{n-k}\}}$.

The proof of the above proposition follows the same structure and arguments as Proposition 12.13, and is omitted here. Once we have obtained the previous results, which allow us to characterize the constant sign of the functions $x_M^{\sigma_{\epsilon_j}}$ and $z_M^{\varepsilon_{\kappa_i}}$ for $j = 1, \dots, \ell$ and $i = 0, \dots, h$, respectively, we can obtain a characterization of the strongly inverse positive (negative) character of operator $\tilde{T}_n[M]$ in the spaces $X_{\{\sigma_1,\dots,\sigma_k\}_{\{\sigma_{\epsilon_1},\dots,\sigma_{\epsilon_\ell}\}}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}_{\{\varepsilon_{\kappa_1},\dots,\varepsilon_{\kappa_h}\}}}$ as follows.

Theorem 12.15. *If $n-k$ is even, then the operator $\tilde{T}[M]$ is strongly inverse positive on $X_{\{\sigma_1,\dots,\sigma_k\}_{\{\sigma_{\epsilon_1},\dots,\sigma_{\epsilon_\ell}\}}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}_{\{\varepsilon_{\kappa_1},\dots,\varepsilon_{\kappa_h}\}}}$ if and only if one of the following assertions is satisfied:*

- *If $k > 1$ and $M \in (-\lambda_1, -\lambda_2]$, where:*
  - *$\lambda_1 > 0$ is the least positive eigenvalue of $\tilde{T}_n[0]$ on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*
  - *$\lambda_2 < 0$ is the maximum between,*
    $\lambda^2_{\sigma_{\epsilon_1}} < 0$, the largest negative eigenvalue of $\tilde{T}_n[0]$ on the set $X_{\{\sigma_1,\dots,\sigma_{\epsilon_1-1},\sigma_{\epsilon_1+1},\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$.
    $\lambda^2_{\varepsilon_{\kappa_1}} < 0$, the largest negative eigenvalue of $\tilde{T}_n[0]$ on the set $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{\kappa_1-1},\varepsilon_{\kappa_1+1},\dots,\varepsilon_{n-k}\}}$.
- *If $k = 1$ and $M \in (-\lambda_1, -\lambda_2]$, where:*
  - *$\lambda_1 > 0$ is the least positive eigenvalue of $\tilde{T}_n[0]$ on $X_{\{\sigma_1\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-1}\}}$.*
  - *$\lambda_2 = \lambda^2_{\varepsilon_{\kappa_1}} < 0$, the largest negative eigenvalue of $\tilde{T}_n[0]$ on the set $X_{\{\sigma_1|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{\kappa_1-1},\varepsilon_{\kappa_1+1},\dots,\varepsilon_{n-1}\}}$.*

If $n-k$ is odd, then the operator $\tilde{T}[M]$ is strongly inverse negative on the set $X_{\{\sigma_1,\dots,\sigma_k\}_{\{\sigma_{\epsilon_1},\dots,\sigma_{\epsilon_\ell}\}}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}_{\{\varepsilon_{\kappa_1},\dots,\varepsilon_{\kappa_h}\}}}$ if and only if one of the following assertions is satisfied:

- *If $1 < k < n-1$ and $M \in [-\lambda_2, -\lambda_1)$, where:*
  - *$\lambda_1 < 0$ is the largest negative eigenvalue of $\tilde{T}_n[0]$ on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*
  - *$\lambda_2 > 0$ is the minimum between,*
    $\lambda^2_{\sigma_{\epsilon_1}} > 0$, the least positive eigenvalue of $\tilde{T}_n[0]$ on the set $X_{\{\sigma_1,\dots,\sigma_{\epsilon_1-1},\sigma_{\epsilon_1+1},\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$.
    $\lambda^2_{\varepsilon_{\kappa_1}} > 0$, the least positive eigenvalue of $\tilde{T}_n[0]$ on the set $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{\kappa_1-1},\varepsilon_{\kappa_1+1},\dots,\varepsilon_{n-k}\}}$.
- *If $k = 1 < n-1$ and $M \in [-\lambda_2, -\lambda_1)$, where:*
  - *$\lambda_1 < 0$ is the largest negative eigenvalue of $\tilde{T}_n[0]$ on $X_{\{\sigma_1\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-1}\}}$.*
  - *$\lambda_2 = \lambda^2_{\varepsilon_{\kappa_1}} > 0$ is the least positive eigenvalue of $\tilde{T}_n[0]$ on the set $X_{\{\sigma_1|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{\kappa_1-1},\varepsilon_{\kappa_1+1},\dots,\varepsilon_{n-1}\}}$.*
- *If $1 < k = n-1$ and $M \in [-\lambda_2, -\lambda_1)$, where:*
  - *$\lambda_1 < 0$ is the largest negative eigenvalue of $\tilde{T}_n[0]$ on $X_{\{\sigma_1,\dots,\sigma_{n-1}\}}^{\{\varepsilon_1\}}$.*
  - *$\lambda_2 = \lambda^2_{\sigma_{\epsilon_1}} > 0$, the least positive eigenvalue of $\tilde{T}_n[0]$ on the set $X_{\{\sigma_1,\dots,\sigma_{\epsilon_1-1},\sigma_{\epsilon_1+1},\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$.*
- *If $n = 2$ and $M \in (-\infty, -\lambda_1)$, where:*

* $\lambda_1 < 0$ *is the largest negative eigenvalue of* $\tilde{T}_n[0]$ *on* $X_{\{\sigma_1\}}^{\{\varepsilon_1\}}$.

Proof. From Lemma 12.5, we only have to study the sign of $g_M(t,s)$, $x_M^{\sigma_{\epsilon_j}}$ for $j = 0,\dots,\ell$ and $z_M^{\varepsilon_{\kappa_i}}$ for $i = 0,\dots,h$.

First, let us see that if M belongs to the given intervals, then the operator is strongly inverse positive or negative in each case. And, finally, we will see that this interval cannot be increased. Taking into account Theorem 8.1, $(-1)^{n-k} g_M(t,s) > 0$ on the given intervals. Moreover, if either $n-k$ is even and $M < 0$ or $n-k$ is odd and $M > 0$, the intervals cannot be increased.

Now, let us study the sign of $x_0^{\sigma_{\epsilon_j}}$ and $z_0^{\sigma_{\kappa_i}}$.

It is known that $x_M^{\sigma_{\epsilon_j}}$ satisfies the boundary conditions (12.12) introduced in the proof of Proposition 12.11. Then, for $M = 0$, the maximal oscillation is satisfied. So, we can study the sign of $x_0^{\sigma_{\epsilon_j}(\sigma_{\epsilon_j})}$ taking into account that $x_0^{\sigma_{\epsilon_j}(\sigma_{\epsilon_j})}(a) = 1$.

If $\sigma_{\epsilon_j} < \alpha$, then $x_0^{\sigma_{\epsilon_j}} > 0$.

If $\sigma_{\epsilon_j} > \alpha$, from $\ell = \alpha$ to σ_{ϵ_j}, there are $\epsilon_j - 1 - \alpha$ zeros for $T_\ell x_0^{\sigma_{\epsilon_j}}(a)$.

From the choice of ϵ_j, we have that $T_{\sigma_{\epsilon_j}} x_0^{\sigma_{\epsilon_j}}(a) > 0$. So, to have maximal oscillation, we need

$$T_\alpha x_0^{\sigma_{\epsilon_j}}(a) \begin{cases} > 0\,, & \text{if } \sigma_{\epsilon_j} - \alpha - (\epsilon_j - 1 - \alpha) = \sigma_{\epsilon_j} - \epsilon_j + 1 \text{ is even,} \\ < 0\,, & \text{if } \sigma_{\epsilon_j} - \epsilon_j + 1 \text{ is odd.} \end{cases}$$

These inequalities are also satisfied by $x_0^{\sigma_{\epsilon_j}(\alpha)}(a)$, thus

$$x_0^{\sigma_{\epsilon_j}} \begin{cases} > 0 \text{ on } I, & \text{if } \sigma_{\epsilon_j} - \epsilon_j + 1 \text{ is even,} \\ < 0 \text{ on } I, & \text{if } \sigma_{\epsilon_j} - \epsilon_j + 1 \text{ is odd.} \end{cases} \tag{12.18}$$

Note that if $\sigma_{\epsilon_j} < \alpha$, then $\sigma_{\epsilon_j} = \epsilon_j - 1$. Hence, $\sigma_{\epsilon_j} - \epsilon_j + 1 = 0$ is an even number. Thus, equation (12.18) is satisfied for all σ_{ϵ_j}, with $j = 1,\dots,\ell$.

Moreover, from Propositions 12.9, 12.11 and 12.13, inequalities (12.18) are satisfied on the whole intervals given in the result. Thus, for those M, we have

$$(-1)^{n-\sigma_{\epsilon_j}-(k-j)+1} x_M^{\sigma_{\epsilon_j}} \begin{cases} > 0 \text{ on } I, & \text{if } n-k \text{ is even,} \\ < 0 \text{ on } I, & \text{if } n-k \text{ is odd.} \end{cases} \tag{12.19}$$

In an analogous way, we can study $z_M^{\varepsilon_{\kappa_i}}$ to conclude that for all M on the intervals given on the result, it is satisfied:

$$(-1)^{n-k-\kappa_i+1} z_0^{\varepsilon_{\kappa_i}} \begin{cases} > 0\,, & \text{if } n-k \text{ is even,} \\ < 0\,, & \text{if } n-k \text{ is odd.} \end{cases} \tag{12.20}$$

So, we have proved that if M belongs to those intervals, operator $\tilde{T}_n[M]$ is strongly inverse negative (positive). Moreover, we have also seen that if either $n-k$ is even and $M < 0$ or $n-k$ is odd and $M > 0$ the intervals cannot be increased, since g_M is not of constant sign. So, we only need to prove that if $n-k$ is even and $M > 0$ or $n-k$ is odd and $M < 0$ the intervals cannot be increased too. To this end, we study the functions $x_M^{\sigma_{\epsilon_1}}$ and $z_M^{\varepsilon_{\kappa_1}}$. In particular, we will verify that if either $k \neq 1$ or $k \neq n-1$, one of them must necessarily change its sign for $M > -\lambda_2$ if $n-k$ is even or for $M < -\lambda_2$ if $n-k$ is odd.

If $\sigma_{\epsilon_1} = \sigma_k$ and $\varepsilon_{\kappa_1} = \varepsilon_{n-k}$ the result follows from Theorem 11.2. Otherwise, either $\lambda_2 = \lambda_{\sigma_{\epsilon_1}}$ or $\lambda_2 = \lambda_{\varepsilon_{\kappa_1}}$.

First, let us assume that $n-k$ is even. Suppose that there exists $M^* > -\lambda_2$ such that $\tilde{T}_n[M]$ is inverse positive on $X_{\{\sigma_1,\dots,\sigma_k\}_{\{\sigma_{\epsilon_1},\dots,\sigma_{\epsilon_\ell}\}}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}_{\{\varepsilon_{\kappa_1},\dots,\varepsilon_{\kappa_h}\}}}$. We will arrive to a contradiction.

If $\lambda_2 = \lambda_{\sigma_{\epsilon_1}}$, let us consider the function $x_M^1(t) = (-1)^{n-\sigma_{\epsilon_j}-(k-j)+1} x_M^{\sigma_{\epsilon_1}}(t)$. Trivially, $x_M^1 \in X_{\{\sigma_1,\dots,\sigma_k\}_{\{\sigma_{\epsilon_1},\dots,\sigma_{\epsilon_\ell}\}}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}_{\{\varepsilon_{\kappa_1},\dots,\varepsilon_{\kappa_h}\}}}$ and $\tilde{T}[M^*]x_{M^*}^1(t) = 0$. Then, we have that $x_{M^*}^1 \geq 0$ on I.

Let us see that necessarily $x_0^1 \geq x_{-\lambda_2}^1 \geq x_{M^*}^1$ on I. Indeed, let us construct the sequence

$$\alpha_0 = x_0^1 , \quad \tilde{T}_n[M^*]\alpha_{n+1} = (M^* + \lambda_2)\alpha_n , \quad n \geq 0 ,$$

where $\alpha_n^{(\sigma_j)}(a) = 0$, if $j \neq \epsilon_1$ for $j = 1, \dots, k$, $\alpha_n^{(\sigma_{\epsilon_1})}(a) = (-1)^{n-\sigma_{\epsilon_1}-(k-\epsilon_1)+1}$ and $\alpha_n^{(\varepsilon_i)}(b) = 0$ for $i = 1, \dots, n-k$. In particular, $\alpha_n \in X_{\{\sigma_1,\dots,\sigma_k\}_{\{\sigma_{\epsilon_1},\dots,\sigma_{\epsilon_\ell}\}}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}_{\{\varepsilon_{\kappa_1},\dots,\varepsilon_{\kappa_h}\}}}$ for $n = 0, 1, \dots$

Let us see that this sequence is non-increasing and bounded from below by zero clearly.

$$\tilde{T}[M^*]\, \alpha_1 = (M^* + \lambda_2)\, x_0^1 \geq 0 .$$

Since $\alpha_1 \in X_{\{\sigma_1,\dots,\sigma_k\}_{\{\sigma_{\epsilon_1},\dots,\sigma_{\epsilon_\ell}\}}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}_{\{\varepsilon_{\kappa_1},\dots,\varepsilon_{\kappa_h}\}}}$ and we are working under the assumption that $\tilde{T}[M^*]$ is inverse positive in such set, we have that $\alpha_1 \geq 0$. Now, $\tilde{T}_n[M^*](\alpha_0 - \alpha_1) = -\lambda_2\, x_0^1 \geq 0$. In this case $\frac{d^{\sigma_j}}{dt^{\sigma_j}}(\alpha_0 - \alpha_1)\big|_{t=a} = 0$ for $j = 1, \dots, k$ and $\frac{d^{\varepsilon_i}}{dt^{\varepsilon_i}}(\alpha_0 - \alpha_1)\big|_{t=b} = 0$ for $i = 1, \dots, n-k$, then $\alpha_0 - \alpha_1 \in X_{\{\sigma_1,\dots,\sigma_k\}_{\{\sigma_{\epsilon_1},\dots,\sigma_{\epsilon_\ell}\}}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}_{\{\varepsilon_{\kappa_1},\dots,\varepsilon_{\kappa_h}\}}}$. So, $\alpha_0 \geq \alpha_1$.

Proceeding analogously for $n \geq 1$, we obtain that $\{\alpha_n\}$ is a non-increasing and nonnegative sequence.

Now, let us consider the sequence

$$\beta_0 = x_{M^*}^1 , \quad \tilde{T}_n[M^*]\, \beta_{n+1} = (M^* + \lambda_2)\beta_n , \ n \geq 0 ,$$

where $\beta_n^{(\sigma_j)}(a) = 0$, if $j \neq \epsilon_1$ for $j = 1, \dots, k$, $\beta_n^{(\sigma_{\epsilon_1})}(a) = (-1)^{n-\sigma_{\epsilon_1}-(k-\epsilon_1)+1}$ and $\beta_n^{(\varepsilon_i)}(b) = 0$ for $i = 1, \dots, n-k$. As consequence, $\beta_n \in X_{\{\sigma_1,\dots,\sigma_k\}_{\{\sigma_{\epsilon_1},\dots,\sigma_{\epsilon_\ell}\}}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}_{\{\varepsilon_{\kappa_1},\dots,\varepsilon_{\kappa_h}\}}}$ for $n = 0, 1, \dots$.

Let us check that this sequence is nondecreasing. By definition, $\tilde{T}_n[M^*](\beta_1 - \beta_0) = (M^* + \lambda_2)x_{M^*}^1 \geq 0$. In this case, $\frac{d^{\sigma_j}}{dt^{\sigma_j}}(\beta_1 - \beta_0)\big|_{t=a} = 0$ for $j = 1, \dots, k$ and $\frac{d^{\varepsilon_j}}{dt^{\varepsilon_j}}(\beta_1 - \beta_0)\big|_{t=b} = 0$ for $i = 1, \dots, n-k$, then $\beta_1 - \beta_0 \in X_{\{\sigma_1,\dots,\sigma_k\}_{\{\sigma_{\epsilon_1},\dots,\sigma_{\epsilon_\ell}\}}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}_{\{\varepsilon_{\kappa_1},\dots,\varepsilon_{\kappa_h}\}}}$. So, $\beta_1 \geq \beta_0$.

Analogously, for $n \geq 1$, we conclude that $\{\beta_n\}$ is a nondecreasing sequence. Moreover, by properties of the related Green's function, which is continuous on $I \times I$, it is bounded from above.

Since $\tilde{T}_n[-\lambda_2]$ is strongly inverse positive on $X_{\{\sigma_1,\dots,\sigma_k\}_{\{\sigma_{\epsilon_1},\dots,\sigma_{\epsilon_\ell}\}}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}_{\{\varepsilon_{\kappa_1},\dots,\varepsilon_{\kappa_h}\}}}$, $x_{-\lambda_2}^1$ is the unique solution of $\tilde{T}_n[-\lambda_2]u(t) = 0$, coupled with the boundary conditions imposed to α_n and β_n. Thus, we can affirm that

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = x_{-\lambda_2}^1 ,$$

and $\alpha_0 = x_0^1 \geq x_{-\lambda_2}^1 \geq x_{M^*}^1 = \beta_0 \geq 0$ on I.

Repeating the previous arguments, we can conclude that for all $M \in [-\lambda_2, M^*]$, we have:

$$x^1_{-\lambda_2} \geq x^1_M \geq x^1_{M^*} \geq 0 \quad \text{on } I\,. \tag{12.21}$$

On the other hand, it is known that ${x^1_{-\lambda_2}}^{(\beta)}(b) = 0$. From inequality (12.21), we have ${x^1_M}^{(\beta)}(b) = 0$ for all $M \in [-\lambda_2, M^*]$, which contradicts the discrete character of the spectrum $\tilde{T}_n[0]$ on $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}_{\{\sigma_1,\dots,\sigma_{\epsilon_1-1},\sigma_{\epsilon_1+1},\dots,\sigma_k\}}$. Thus, we arrive to a contradiction by supposing that there exists $M^* > -\lambda_2$ such that $\tilde{T}_n[M^*]$ is inverse positive on $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}_{\{\varepsilon_{\kappa_1},\dots,\varepsilon_{\kappa_h}\}}}_{\{\sigma_1,\dots,\sigma_k\}_{\{\sigma_{\epsilon_1},\dots,\sigma_{\epsilon_\ell}\}}}$.

Analogously, if $\lambda_2 = \lambda_{\varepsilon_{\kappa_1}}$, it can be proved that there does not exist any $M^* > -\lambda_2$ such that $\tilde{T}_n[M^*]$ is inverse positive on $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}_{\{\varepsilon_{\kappa_1},\dots,\varepsilon_{\kappa_h}\}}}_{\{\sigma_1,\dots,\sigma_k\}_{\{\sigma_{\epsilon_1},\dots,\sigma_{\epsilon_\ell}\}}}$.

Finally, we can proceed analogously when $n-k$ is odd to conclude that there is no $M^* < -\lambda_2$ such that $\tilde{T}_n[M^*]$ is inverse negative on $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}_{\{\varepsilon_{\kappa_1},\dots,\varepsilon_{\kappa_h}\}}}_{\{\sigma_1,\dots,\sigma_k\}_{\{\sigma_{\epsilon_1},\dots,\sigma_{\epsilon_\ell}\}}}$. □

12.1. **Particular cases.** This section is devoted to show the applicability of the previous results to some examples. Note that most of the examples given in Section 9 follow the structure given on this section. So, we will be able to obtain the characterization of the strongly inverse positive (negative) character for those operators in different spaces with non homogeneous boundary conditions.

- n^{th}-order operators with $(k, n-k)$ boundary conditions. In this case $\mu = \max\{\alpha_2, \beta_2\} = -1$. So, since the largest set where we can apply Theorem 12.15 is $X^{\{0,\dots,n-k-1\}_{\{n-k-1\}}}_{\{0,\dots,k-1\}_{\{k-1\}}}$, Theorem 12.15 is equivalent to Theorem 11.2. However, in many cases, we can be under the conditions of Remark 12.7 which allows us to apply Theorem 12.15 in bigger sets with more non homogeneous boundary conditions.
- Operator $T_4(p_1,p_2)[M]u(t) = u^{(4)}(t) + p_1(t)u^{(3)}(t) + p_2(t)u^{(2)}(t) + Mu(t)$ on $X^{\{0,2\}}_{\{0,2\}}$. The study of this type of operators on $X^{\{0,2\}^{\{2\}}}_{\{0,2\}_{\{2\}}}$ can be deduced from Theorem 11.2. But, in such a case, since $\mu = \max\{\alpha_2, \beta_2\} = 1$, by studying the different eigenvalues, we can characterize the strongly inverse positive character of $T_4(p_1,p_2)[M]$ in the different subsets of $X^{\{0,2\}_{\{0,2\}}}_{\{0,2\}_{\{0,2\}}}$. Let us consider, for instance, the operator

$$T_4[p,M]u(t) \equiv u^{(4)}(t) - pu''(t) + Mu(t)\,, \quad t \in I \equiv [a,b]\,,$$

where $p \geq 0$. In [14], there are obtained some of the related eigenvalues:

- The least positive eigenvalue of $T_4[p,0]$ on $X^{\{0,2\}}_{\{0,2\}}$ is $\lambda_1^p = \left(\frac{\pi}{b-a}\right)^4 + p\left(\frac{\pi}{b-a}\right)^2$.
- The largest negative eigenvalue of $T_4[p,0]$ on $X^{\{0,1,2\}}_{\{0\}}$ and on $X^{\{0\}}_{\{0,1,2\}}$ coincide and are $-\lambda_2^p$, where λ_2^p is the least positive solution of

$$\frac{\tan\left(\frac{b-a}{2}\sqrt{2\sqrt{\lambda}-p}\right)}{\sqrt{2\sqrt{\lambda}-p}} = \frac{\tanh\left(\frac{b-a}{2}\sqrt{2\sqrt{\lambda}+p}\right)}{\sqrt{2\sqrt{\lambda}+p}}\,.$$

Now, let us obtain the missing eigenvalues:

- The largest negative eigenvalues of $T_4[p,0]$ on $X_{\{2\}}^{\{0,1,2\}}$ and on $X_{\{0,1,2\}}^{\{2\}}$ coincide and are given by $-\lambda_{2_0}^p$, where $\lambda_{2_0}^p$ is the least positive solution of

$$\frac{\tan\Big(\frac{b-a}{2}\sqrt{2\sqrt{\lambda}-p}\Big)}{\sqrt{2\sqrt{\lambda}-p}}+\frac{\tanh\Big(\frac{b-a}{2}\sqrt{2\sqrt{\lambda}+p}\Big)}{\sqrt{2\sqrt{\lambda}+p}}=0\,.$$

Thus, we obtain the following conclusions:

- $T_4[p,M]$ is strongly inverse positive on ${X_{\{0,2\}}^{\{0,2\}}}_{\{2\}}^{\{2\}}$ if and only if $M\in(-\lambda_1^p,\lambda_2^p]$.
- $T_4[p,M]$ is strongly inverse positive on ${X_{\{0,2\}}^{\{0,2\}}}_{\{0,2\}}^{\{0,2\}}$ if and only if $M\in(-\lambda_1^p,\lambda_{2_0}^p]$.

• Operator $T_n^0[M]u(t)=u^{(n)}(t)+Mu(t)$. Now we treat some of this types of problems which have been introduced in Section 9.

Second order. The only possibility in this case is to consider $k=1$. Then, the characterization is obtained by applying Theorem 11.2 and the parameters set for the strongly inverse positive character is the same as in the homogeneous case which has been obtained in Section 9.

Third order. Let us consider, for instance, $\{\sigma_1,\sigma_2\}=\{1,2\}$ and $\{\varepsilon_1\}=\{0\}$. In such a case, $\mu=\max\{\alpha_2,\beta_2\}=\max\{-1,0\}=0$. Then, we obtain the characterization on ${X_{\{1,2\}}^{\{0\}}}_{\{2\}}^{\{0\}}$ from Theorem 11.2 or Theorem 12.15 equivalently.

From Remark 12.7, we are able to obtain the characterization on ${X_{\{1,2\}}^{\{0\}}}_{\{1,2\}}^{\{0\}}$ given as follows: $T_3^0[M]$ is strongly inverse negative on ${X_{\{1,2\}}^{\{0\}}}_{\{1,2\}}^{\{0\}}$ if and only if $M\in[-\lambda_2,-\lambda_1)$ where $\lambda_1=-m_4^3$, with $m_4\approxeq 1.85$ the least positive solution of (9.1), is the largest negative eigenvalue of $T_3^0[0]$ on $X_{\{1,2\}}^{\{0\}}$ and $\lambda_2=m_7$, with $m_7\approxeq 1.84981$ the least positive solution of

$$2e^{3m/2}\cos\big(\frac{\sqrt{3}m}{2}\big)+1=0\,,$$

is the least positive eigenvalue of $T_3^0[0]$ on $X_{\{2\}}^{\{0,1\}}$.

Fourth order. Let us consider again fourth-order problems introduced in Section 9, $X_{\{0\}}^{\{1,2,3\}}$ and $X_{\{0,2\}}^{\{1,3\}}$. In the first case we cannot apply directly Theorem 12.15, since $\mu=0$. However, with the same argument as in Remark 12.7, Theorem 12.15 is still true for $\sigma_{\epsilon_{\ell-1}}\geq\mu$ or $\varepsilon_{\kappa_{h-1}}\geq\mu$.

- The largest negative eigenvalue of $T_4^0[0]$ on $X_{\{0\}}^{\{1,2,3\}}$ is $\lambda_1=-\frac{\pi^4}{4}$.
- The least positive eigenvalue of $T_4^0[0]$ on $X_{\{0,1\}}^{\{0,3\}}$ is $\lambda_1^2=\pi^4$
- The least positive eigenvalue of $T_4^0[0]$ on $X_{\{0,1\}}^{\{1,3\}}$ is $\lambda_0^2=m_1^4$, where $m_1\approxeq 2.36502$ is the least positive solution of (6.2).

Thus, we conclude that $T_4^0[M]$ is strongly inverse negative on ${X_{\{0\}}^{\{1,2,3\}}}_{\{0\}}^{\{2,3\}}$ if and only if $M\in[-\pi^4,\frac{\pi^4}{4})$. Moreover, $T_4^0[M]$ is strongly inverse negative on ${X_{\{0\}}^{\{1,2,3\}}}_{\{0\}}^{\{1,2,3\}}$ if and only if $M\in[-m_1^4,\frac{\pi^4}{4})$.

For $X_{\{0,2\}}^{\{1,3\}}$, we have $\mu = \max\{1,2\} = 2$. Let us study the strongly inverse positive character on $X_{\{0,2\}_{\{0,2\}}}^{\{1,3\}_{\{1,3\}}}$.

- The least positive eigenvalue of $T_4^0[0]$ on $X_{\{0,2\}}^{\{1,3\}}$ is $\lambda_1 = \frac{\pi^4}{16}$.
- The largest negative eigenvalue of $T_4^0[0]$ on $X_{\{2\}}^{\{0,1,3\}}$ is $\lambda_0^2 = -\frac{\pi^4}{4}$
- The largest negative eigenvalue of $T_4^0[0]$ on $X_{\{0,1,2\}}^{\{1\}}$ is $\lambda_1^2 = -4\pi^4$.

Thus, $\lambda_2 = -\frac{\pi^4}{4}$ and we can conclude that $T_4^0[M]$ is strongly inverse positive on $X_{\{0,2\}_{\{0,2\}}}^{\{1,3\}_{\{1,3\}}}$ if and only if $M \in (-\frac{\pi^4}{16}, \frac{\pi^4}{4}]$.

Higher order. Now, let us analyze the sixth order operator given in Subsection 9. That is, the operator $T_6^0[M]$ defined on $X_{\{0,2,4\}}^{\{0,2,4\}}$. In this case, $\mu = \max\{3,3\} = 3$, so we can apply Theorem 12.15 in different spaces. Let us obtain the different eigenvalues:

- The largest negative eigenvalue of $T_6^0[0]$ on $X_{\{0,2,4\}}^{\{0,2,4\}}$ is $\lambda_1 = -\pi^6$.
- The least positive eigenvalue of $T_6^0[0]$ on $X_{\{0,4\}}^{\{0,1,2,4\}}$ is $\lambda_2^2 = m_8^6$, where $m_8 \approxeq 4.14577$ is the least positive solution of
$$\sqrt{3}e^{m/2}\left(e^{2m}+1\right) - 3\left(e^m+1\right)^2\left(e^m-1\right)\sin\left(\frac{\sqrt{3}m}{2}\right)$$
$$+\sqrt{3}\left(e^m+1\right)\left(e^m-1\right)^2\cos\left(\frac{\sqrt{3}m}{2}\right) - 2\sqrt{3}e^{3m/2}\cos\left(\sqrt{3}m\right) = 0\,.$$
- The least positive eigenvalue of $T_6^0[0]$ on $X_{\{0,1,2,4\}}^{\{0,4\}}$ is $\lambda_2^2 = m_8^6$.
- The least positive eigenvalue of $T_6^0[0]$ on $X_{\{2,4\}}^{\{0,1,2,4\}}$ is $\lambda_0^2 = m_9^6$, where $m_9 \approxeq 3.17334$ is the least positive solution of
$$-\sqrt{3}e^{m/2}\left(e^{2m}+1\right) - 3\left(e^m+1\right)^2\left(e^m-1\right)\sin\left(\frac{\sqrt{3}m}{2}\right)$$
$$-\sqrt{3}\left(e^m+1\right)\left(e^m-1\right)^2\cos\left(\frac{\sqrt{3}m}{2}\right) + 2\sqrt{3}e^{3m/2}\cos\left(\sqrt{3}m\right) = 0\,.$$
- The least positive eigenvalue of $T_6^0[0]$ on $X_{\{0,1,2,4\}}^{\{2,4\}}$ is $\lambda_0^2 = m_9^6$.

Thus, we conclude that $T_6^0[M]$ is strongly inverse negative on $X_{\{0,2,4\}_{\{2,4\}}}^{\{0,2,4\}_{\{2,4\}}}$ if and only if $M \in [-m_8^6, \pi^6)$. Moreover, $T_6^0[M]$ is strongly inverse negative on $X_{\{0,2,4\}_{\{0,2,4\}}}^{\{0,2,4\}_{\{0,2,4\}}}$ if and only if $M \in [-m_9^6, \pi^6)$.

Operators with non constant coefficients To complete this work we show an example where a fourth order operator with non constant coefficients is considered. Let us define the operator
$$T_4^{nc}[M] = u^{(4)} + e^{2\,t}\sin(2\,t)u'''(t) + Mu(t)\,,\quad t \in [0,1]$$
defined on $X_{\{0,2\}}^{\{1,2\}}$. In such a space, we have $\mu = \max\{1,0\} = 1$, and the linear differential equation
$$u''(t) + e^{2\,t}\sin(2\,t)u'(t) = 0\,,$$
is disconjugate on $[0,1]$, since it is a composition of two first order linear differential equations. Thus, we can apply all previous results to characterize the strongly inverse positive character of $T_4^{nc}[M]$ on $X_{\{0,2\}}^{\{1,2\}}$.

First, we obtain numerically, by means of Mathematica software, the different eigenvalues of $T_4^{nc}[0]$.

- The least positive eigenvalue of $T_4^{nc}[0]$ on $X_{\{0,2\}}^{\{1,2\}}$ is $\lambda_1 \approxeq 2.62355^4$.
- The largest negative eigenvalue of $T_4^{nc}[0]$ on $X_{\{0,1,2\}}^{\{1\}}$ is $\lambda_2'' \approxeq -4.69621^4$.
- The largest negative eigenvalue of $T_4^{nc}[0]$ on $X_{\{0\}}^{\{0,1,2\}}$ is $\lambda_2' \approxeq -6.18170^4$.
- The largest negative eigenvalue of $T_4^{nc}[0]$ on $X_{\{0,1,2\}}^{\{2\}}$ is $\lambda_1^2 \approxeq -3.45041^4$.
- The largest negative eigenvalue of $T_4^{nc}[0]$ on $X_{\{2\}}^{\{0,1,2\}}$ is $\lambda_0^2 \approxeq -4.20409^4$.

Thus, by Theorems 8.1 and 12.15, we conclude:

- $T_4^{nc}[M]$ is strongly inverse positive on $X_{\{0,2\}_{\{2\}}}^{\{1,2\}_{\{2\}}}$ if and only if $M \in \big(-2.62355^4, 4.69621^4\big]$.
- $T_4^{nc}[M]$ is strongly inverse positive on $X_{\{0,2\}_{\{0,2\}}}^{\{1,2\}_{\{2\}}}$ if and only if $M \in \big(-2.62355^4, 4.20409^4\big]$.
- $T_4^{nc}[M]$ is strongly inverse positive either on $X_{\{0,2\}_{\{2\}}}^{\{1,2\}_{\{1,2\}}}$ or $X_{\{0,2\}_{\{0,2\}}}^{\{1,2\}_{\{1,2\}}}$ if and only if $M \in \big(-2.62355^4, 3.45041^4\big]$.

To use Theorem 10.1, we can obtain the needed eigenvalues of $T_4^{nc}[0]$:

- The least positive eigenvalue of $T_4^{nc}[0]$ on $X_{\{0,1\}}^{\{1,2\}}$ is $\lambda_3' \approxeq 3.22872^4$.
- The least positive eigenvalue of $T_4^{nc}[0]$ on $X_{\{0,2\}}^{\{0,1\}}$ is $\lambda_3'' \approxeq 4.33768^4$.

Thus, from Theorem 10.1, if $T_4^{nc}[M]$ is a strongly inverse negative operator on $X_{\{0,2\}}^{\{1,2\}}$, then $M \in [-3.22872^4, -2.62355^4)$.

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References

[1] D. R. Anderson, J. Hoffacker; *Existence of solutions for a cantilever beam problem*, J. Math. Anal. Appl., 323 (2006), 958-973

[2] A. Cabada; *Green's Functions in the Theory of Ordinary Differential Equations*, Springer Briefs in Mathematics, 2014.

[3] A. Cabada; *The method of lower and upper solutions for second, third, fourth, and higher order boundary-value problems*, J. Math. Anal. Appl., 185 (1994), 302-320.

[4] A. Cabada, J. A. Cid; *Existence and multiplicity of solutions for a periodic Hill's equation with parametric dependence and singularities*, Abstr. Appl. Anal., 2011, Art. ID 545264, 19 pp.

[5] A. Cabada, J. A. Cid; *Existence of a non-zero fixed point for non-decreasing operators via Krasnoselskii's fixed point theorem,* Nonlinear Anal., 71 (2009), 2114–2118.

[6] A. Cabada, J. A. Cid, G. Infante; *New criteria for the existence of non-trivial fixed points in cones*, Fixed Point Theory and Appl., 2013:125, (2013), 12 pp.

[7] A. Cabada, J. A. Cid, B. Máquez-Villamarín; *Computation of Green's functions for boundary value problems with Mathematica*, Applied Mathematics and Computation, 219 (2012), 1919-1936.

[8] A. Cabada, J.A. Cid, L. Sanchez; *Positivity and lower and upper solutions for fourth order boundary value problems*, Nonlinear Anal., 67 (2007), 1599-1612.

[9] A. Cabada, R. R. Enguiça; *Positive solutions of fourth order problems with clamped beam boundary conditions*, Nonlinear Anal., 74 (2011), 3112-3122.

[10] A. Cabada, C. Fernández-Gómez; *Constant Sign Solutions of Two-Point Fourth Order Problems*, Appl. Math. Comput., 263 (2015), 122-133.

[11] A. Cabada, R. Precup, L. Saavedra, S. A. Tersian; *Multiple solutions to a fourth-order boundary-value problem*, Electron. J. Differential Equations, Vol. 2016 (2016), No. 254, pp. 1-18.

[12] A. Cabada, L. Saavedra; *The eigenvalue characterization for the constant sign Green's functions of $(k, n-k)$ problems*, Boundary Value Problems, 2016:44 (2016) 35pp.

[13] A. Cabada, L. Saavedra; *Disconjugacy characterization by means of spectral $(k, n-k)$ problems*, Appl. Math. Lett., 52 (2016), 21-29.

[14] A. Cabada, L. Saavedra; *Constant sign Green's function for simply supported beam equation*, Advances in differential equations, 2017, 22, 403-432.

[15] W. A. Coppel; *Disconjugacy*, Lecture Notes in Mathematics, Vol. 220. Springer-Verlag, Berlin-New York, 1971.

[16] J. A. Cid, D. Franco, F. Minhós; *Positive fixed points and fourth-order equations*, Bull. Lond. Math. Soc., 41 (2009), 72–78.

[17] C. De Coster, P. Habets; *Two-Point Boundary Value Problems: Lower and Upper Solutions*, Mathematics in Science and Engineering 205, Elsevier B. V., Amsterdam, 2006.

[18] D. Franco, G. Infante, J. Perán; *A new criterion for the existence of multiple solutions in cones*, Proc. Roy. Soc. Edinburgh Sect. A, 142 (2012), 1043–1050.

[19] J. R. Graef, L. Kong, H. Wang; *A periodic boundary value problem with vanishing Green's function*, Applied Mathematics Letters, 21 (2008), 176-180.

[20] M. A. Krasnosel'skiĭ; *Positive Solutions of Operator Equations*, Noordhoff, Groningen, 1964.

[21] G. S. Ladde, V. Lakshmikantham, A. S. Vatsala; *Monotone Iterative Techniques for Nonlinear Differential Equations*; Pitman, Boston M. A. 1985.

[22] R. Ma, Y. Lu; *Disconjugacy and monotone iteration method for third-order equations*, Commun. Pure Appl. Anal., 13, 3, (2014), 1223-1236.

[23] H. Persson; *A fixed point theorem for monotone functions*, Appl. Math. Lett., 19 (2006), 1207–1209.

[24] P. J. Torres; *Existence of One-Signed Periodic Solutions of Some Second-Order Differential Equations via a Krasnoselskii Fixed Point Theorem*, J. Differential Equations 190 (2003), 2, 643–662.

[25] E. Zeidler; *Nonlinear Functional Analysis and its Applications. I. Fixed-Point Theorems.* Translated from the German by Peter R. Wadsack, Springer, New York (1986).

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