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ON COMMUTING DIFFERENTIAL OPERATORS

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Abstract. The theory of commuting linear differential expressions has received a lot of attention since Lax presented his description of the KdV hierarchy by Lax pairs (P, L). Gesztesy and the present author have established a relationship of this circle of ideas with the property that all solutions of the differential equations $Ly = zy$, $z \in \mathbb{C}$, are meromorphic. In this paper this relationship is explored further by establishing its existence for Gelfand-Dikii systems with rational and simply periodic coefficients.

1. Introduction

The theory of commuting linear differential expressions was begun by Floquet [6] in 1879 and advanced significantly when Wallenberg [20] and Schur [18] addressed it some 25 years later. An even bigger impact had Burchnall and Chaundy with a series of papers ([1], [2], [3]) in the 1920s when they discovered a relationship with algebraic geometry (see Section 2). The exploration of commuting differential expressions was again taken up in the 1970s and 1980s because of the connection with completely integrable partial differential equations. The ones in question here are the Gelfand-Dikii systems which may be represented by equations of the type $L_t = [P, L]$ where P and L are linear differential expressions. The most famous such equation is the Korteweg-de Vries (KdV) equation

$$q_t = \frac{1}{4} q_{xxx} + \frac{3}{2} q q_x$$

which is obtained by choosing $L = D^2 + q$ and $P = D^3 + \frac{3}{2} qD + \frac{3}{4} q_x$ when D denotes the differential expression d/dx. The letters P and L where chosen by Gelfand and Dikii in honor of Peter Lax who first represented the KdV equation using a Lax pair [14].

Only a select few expressions L will allow the existence of an expression P whose order is relatively prime to the order of L but which commutes with L and, due to the Burchnall-Chaundy theorem, such L are also called algebro-geometric (see Section 2 for precise statements and definitions). From the works of Its and Matveev [11] and Krichever [12], [13] it is clear that the coefficients of L should be given in terms of specific differential polynomials of a Riemann theta function (i.e., a

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polynomial in that function and its derivatives). However, to recognize whether a given differential expression is algebro-geometric, this knowledge is of little value.

The aim of the present paper is to give an easily verifiable sufficient condition to ensure that a given differential expression L (with rational or simply periodic coefficients) is algebro-geometric (see Theorem 1). A few years ago such a characterization was obtained by Gesztesy and myself for expressions of the form $L = D^2 + q$ with an elliptic potential q (see [7]). In fact, we found that L is algebro-geometric if and only if the equation $Ly = zy$ has only meromorphic solution regardless what z is. The corresponding relationship exists also for rational and simply periodic potentials of $D^2 + q$ (see [22]) and for the more general AKNS system (at least in the case of elliptic coefficients, see [8]). The clue in [7] was to consider the independent variable of the equation $y'' + qy = zy$ as a complex variable and use a classical theorem of Picard treating equations with elliptic coefficients. For a survey of this and related approaches to integrable systems see [9].

In retrospect it is clear from the work of Its and Matveev [11] and of Segal and Wilson [19] that the solutions of $Ly = zy$ are necessarily meromorphic if L is algebro-geometric. However, it seems that nobody thought that this was peculiar.

The following theorem, which establishes sufficient conditions for a differential expression to be algebro-geometric, will be proven in this paper:

Theorem 1. *Suppose that the coefficients of the differential expression*

$$L = D^n + q_{n-2}D^{n-2} + \dots + q_0$$

are either

- *rational functions, which are bounded at infinity, or else*
- *meromorphic, simply periodic functions with period p, which remain bounded as $|\Im(x/p)|$ tends to infinity.*

If, regardless of $z \in \mathbb{C}$, all solutions of the differential equation $Ly = zy$ are meromorphic then L is algebro-geometric.

Therefore, given a differential expression L in one of the classes indicated, it suffices to examine the behavior of the solutions of $Ly = zy$ near the finitely many singular points of the equation. This is a routine, if lengthy, task.

Proof of Theorem 1. Theorem 3 gives a sufficient condition for L to be algebro-geometric provided the equation $Ly = zy$ has a solution of a certain form. That this is indeed so is guaranteed by Theorem 7 in the rational case (choose $t(x) = x$) and by Theorem 8 in the simply periodic case (choose $t(x) = \exp(2\pi ix/p)$). □

The proofs of Theorems 7 and 8 rely on results by Halphen and Floquet (concerned with the rational and simply periodic case, respectively). These, in turn, are modeled after the above mentioned theorem of Picard. The proof of Theorem 3 is suggested by the work of Burchnall and Chaundy [3].

While in the case of the KdV hierarchy the corresponding theorem was first proven for elliptic potentials the current methods are not easily adaptable to elliptic coefficients of L when $n > 2$. The reason is that the known proofs for the KdV hierarchy rely on the recursion relation through which the hierarchy may be defined. An analogous representation is unknown for general n (see however [4] for $n = 3$). The current proof, on the other hand, does not extend to the elliptic case because the relationship between λ and z, which is algebraic for rational and simply periodic coefficients, is transcendental in the case of elliptic coefficients.

In Section 2 we will review the theory of Burchnall and Chaundy and prove a characterization of algebro-geometric potentials. Section 3 presents the Halphen theorem and an analogous version of the Floquet theorem. Section 4 establishes that in the cases considered certain solutions are of the form required by Theorem 3. An important ingredient for this part is the asymptotic behavior of the solutions as the spectral parameter tends to infinity. This is, of course, a well researched subject and the reader is reminded of the basic facts, following Wasow [21], in the appendix.

2. Burchnall-Chaundy theory

Definition 1. A differential expression L of order $n \geq 2$ and leading coefficient one is called algebro-geometric if there exists a natural number m, relatively prime with respect to n, a polynomial $\mathcal{Q}$ of the form

$$\mathcal{Q}(p,\ell) = p^n - \ell^m + \sum_{\substack{a,b\geq 0\\ am+bn<nm}} c_{a,b}p^a\ell^b, \tag{1}$$

and a differential expression P of order m, such that
1. $Q(P,L)=0$ and
2. if $L \in \mathbb{C}[R]$ for some first order differential expression R then $P \notin \mathbb{C}[R]$.

The most trivial examples of algebro-geometric differential expressions are given by expressions with constant coefficients when one may choose $P = D$, the operator of taking a first derivative.

Theorem 2. *Suppose P and L are differential expressions of relatively prime orders m and n respectively. Then P and L commute if and only if there exists a polynomial of the form* (1) *such that $Q(P,L)=0$.*

This theorem, obtained in the early 1920s by Burchnall and Chaundy [1], may serve as a characterization for algebro-geometric differential expressions:

Corollary 1. *A differential expression L of order $n \geq 2$ and leading coefficient one is algebro-geometric if and only if there exists a natural number m, relatively prime with respect to n and a differential expression P of order m, such that*
1. $[P,L]=0$ and
2. if $L \in \mathbb{C}[R]$ for some first order differential expression R then $P \notin \mathbb{C}[R]$.

The restriction to $n \geq 2$ is due to the fact that $L = D+q(x)$ and $[P,L]=0$ imply that $P \in \mathbb{C}[L]$ regardless what q is. This is seen as follows: Suppose P commutes with L and is of order m. Without loss of generality we may assume that P has leading coefficient one. Then $P - L^m$ commutes with L, has order less than m and a constant leading coefficient. Induction proves the claim.

Now consider the differential expression $\hat{L} = D^n + \hat{q}_{n-1}D^{n-1} + ... + \hat{q}_0$ and let E be the operator of multiplication by $\exp(\int^x \hat{q}_{n-1}dt/n)$. Then

$$L = E\hat{L}E^{-1} = D^n + q_{n-2}D^{n-2} + ... + q_0 \tag{2}$$

for appropriate functions q_0, ..., q_{n-2}. We call L the normal form of $\hat{L}$. If $\hat{P}$ is some other differential expression and if $P = E\hat{P}E^{-1}$, then $[P,L]=0$ if and only if $[\hat{P},\hat{L}]=0$. Therefore, to characterize the algebro-geometric differential expressions we may restrict ourselves to those which are of the form (2).

If $L \in \mathbb{C}[R]$ for some first order differential expression $R = D + a(x)$ and if L is of the form (2) then a must be constant, that is L has constant coefficients and it commutes with $P = D$ and hence is algebro-geometric. On the other hand, if L is in the form (2) and does not have constant coefficients then it is not in $\mathbb{C}[R]$ for any first order expression R and we obtain the following characterization:

Corollary 2. *The differential expression L given in* (2) *is algebro-geometric if and only if there exists a natural number m, relatively prime with respect to n, and a differential expression P of order m such that $[P, L] = 0$.*

We will now give a sufficient condition for L to be algebro-geometric in terms of the solutions of the differential equations $Ly = zy$.

Theorem 3. *Let L be a differential expression of the form* (2) *and suppose that, for every $z \in \mathbb{C}$, the equation $Ly = zy$ has a solution of the form*

$$\psi(\lambda, x) = (\lambda^g + r_{g-1}(t(x))\lambda^{g-1} + \dots + r_0(t(x))) \exp(\lambda x)$$

where r_0, ..., r_{g-1} are rational functions, t is a meromorphic function, and

$$\lambda^n + \rho_{n-2}\lambda^{n-2} + \dots + \rho_0 = z$$

for certain complex numbers ρ_0, ..., ρ_{n-2}. Then there exists a differential expression P whose order m is relatively prime with respect to n such that $[P, L] = 0$. In particular, L is algebro-geometric.

Proof. Define

$$U = D^g + r_{g-1}(t(x))D^{g-1} + \dots + r_0(t(x))$$

and

$$L_0 = D^n + \rho_{n-2}D^{n-2} + \dots + \rho_0.$$

Then consider the differential expressions $V = LU - UL_0$. Since $L_0(\exp(\lambda x)) = z \exp(\lambda x)$ and $U(\exp(\lambda x)) = \psi(\lambda, x)$ we obtain

$$V(\exp(\lambda x)) = (L - z)U(\exp(\lambda x)) = (L - z)\psi(\lambda, x) = 0$$

for every $\lambda \in \mathbb{C}$. Since the functions $\exp(\lambda x)$ are linearly independent for distinct λ we obtain that V is the zero expression, that is,

$$LU = UL_0.$$

Let $\{y_1, ..., y_g\}$ be a basis of $\ker U$. To each element y_ℓ of this basis we may associate a differential expression H_ℓ with constant coefficients in the following way. Since $y_\ell \in \ker U$, so is $L_0 y_\ell$ and, in fact, $L_0^j y_\ell$ for every $j \in \mathbb{N}$. Since $\ker U$ is finite-dimensional there exists a $k \in \mathbb{N}$ and complex numbers $\beta_0, ..., \beta_k$ such that $\beta_0 = 1$ and

$$\sum_{j=0}^{k} \beta_{k-j} L_0^j y_\ell = 0.$$

Then define $H_\ell = \sum_{j=0}^{k} \beta_{k-j} L_0^j$. Since the expressions H_ℓ commute among themselves we obtain that $\ker U \subset \ker(D^j \prod_{\ell=1}^{g} H_\ell)$ for any nonnegative integer j. Let S_μ be the set of all differential expressions H of order μ with constant coefficients such that $\ker U \subset \ker H$. We have just shown that S_μ is not empty provided μ is sufficiently large. Hence there exists the number

$$m = \min\{\mu \in \mathbb{N} : \gcd(\mu, n) = 1, S_\mu \neq \emptyset\}.$$

Let P_0 be an element of S_m. Since $\ker U \subset \ker P_0$, we obtain that there exists an expression P such that $PU = UP_0$. Hence $[P, L]U = PLU - LPU = UP_0L_0 - UL_0P_0 = U[P_0, L_0] = 0$ and thus $[P, L] = 0$. In view of Corollary 2 this proves that L is algebro-geometric. □

3. The theorems of Picard, Floquet, and Halphen

As mentioned in the introduction the proof of Theorem 1 relies on classical theorems by Floquet and Halphen concerning the linear differential equation

$$y^{(n)} + q_{n-1}y^{(n-1)} + ... + q_0y = 0 \tag{3}$$

with simply periodic and rational coefficients, respectively. These theorems, in turn, where inspired by a theorem of Picard which is concerned with the elliptic case.

While Picard's theorem [17] will not be used I state it for the sake of its historic significance.

Theorem 4. *Assume that the coefficients $q_0, ..., q_{n-1}$ in (3) are elliptic functions with common fundamental periods $2\omega_1$ and $2\omega_2$. If the differential equation (3) has only meromorphic solutions then it has a solution which is elliptic of the second kind*[1]*.*

Halphen's theorem is concerned with the rational case. A proof is given by Ince [10] and this proof can be used to state the following version which is different from Ince's version.

Theorem 5. *Let the coefficients $q_0, ..., q_{n-1}$ in (3) be rational functions which are bounded at infinity and define $\rho_j = \lim_{x\to\infty} q_j$. If the differential equation (3) has only meromorphic solutions then there is a solution $R(x)\exp(\lambda x)$ where R is a rational function and λ satisfies*

$$\lambda^n + \rho_1\lambda^{n-1} + ... + \rho_n = 0.$$

Floquet's famous theorem (see e.g. Eastham [5] or Magnus and Winkler [15]) on periodic differential equation, though inspired by Picard's results, has a broader scope but also gives less information on the structure of solutions when compared with the theorems by Picard and Halphen. We will therefore provide a below an analogue of Picard's or Halphen's theorem for the simply periodic case.

Let us first remember a few basic facts from the theory of meromorphic, simply periodic functions (for more information see, e.g., Markushevich [16], Chapter III.4). If f is a meromorphic periodic function with period 2π then

$$f^*(t) = f(-i\log(t))$$

is meromorphic on $\mathbb{C} - \{0\}$. If f is entire then f^* is analytic on $\mathbb{C} - \{0\}$.

A meromorphic simply periodic function q with period p which has only finitely many poles in the period strip $\{x \in \mathbb{C} : 0 \le \Re(x/p) < 1\}$ and which is bounded as $|\Im(x/p)|$ tends to infinity is of the form

$$q(x) = \frac{a_0 + a_1 e^{2\pi i x/p} + ... + a_m e^{2\pi i m x/p}}{b_0 + b_1 e^{2\pi i x/p} + ... + b_m e^{2\pi i m x/p}}.$$

[1]A function f is called elliptic of the second kind, if it is meromorphic and if there exist two numbers a_1 and a_2, independent over the real numbers, and two numbers ρ_1 and ρ_2 such that $f(x + a_1) = \rho_1 f(x)$ and $f(x + a_2) = \rho_2 f(x)$ for all x.

We will call such functions bounded at the ends of the period strip. Note that

$$\lim_{\Im(x/p)\to\infty} q(x) = \frac{a_0}{b_0} = q^*(0)$$

and

$$\lim_{\Im(x/p)\to-\infty} q(x) = \frac{a_m}{b_m} = q^*(\infty).$$

Theorem 6. *Let the coefficients $q_0, ..., q_{n-1}$ in (3) be meromorphic, simply periodic with period p, and bounded at the ends of the period strip. If the differential equation (3) has only meromorphic solutions then there is a solution $R(\mathrm{e}^{2\pi ix/p})\exp(i\lambda x)$ where R is a rational function and λ satisfies*

$$(i\lambda)^n + q_{n-1}^*(0)(i\lambda)^{n-1} + ... + q_0^*(0) = 0.$$

Since a proof of this theorem does not seem to be readily available I will give an outline below. But first I will present a few lemmas which will be needed.

Lemma 1. *Let v be a polynomial with $v(0) \neq 0$, abbreviate e^{ix} by t, and suppose that*

$$y(x) = \frac{u(t)}{v(t)} t^\lambda$$

is meromorphic with respect to x. Then

$$y^{(k)}(x) = \frac{t^\lambda}{v(t)^{k+1}} \sum_{j=0}^{k} t^j f_{j,k}(\lambda, t) u^{(j)}(t)$$

where the $f_{j,k}$ are polynomials in both of their variables and $u^{(j)}$ denotes the j-th derivative of u with respect to t. In particular, $f_{j,j}(\lambda, t) = (iv(t))^j$. Moreover, $\deg f_{j,k}(\lambda, \cdot) \leq k \deg v$.

Proof. The first statement follows immediately from an induction over k. In fact

$$f_{j,k+1} = iv((\lambda + j)f_{j,k} + f_{j-1,k}) + it(vf'_{j,k} - (k + 1)v' f_{j,k})$$

where primes denote derivatives with respect to t and $f_{j,k} = 0$ unless $j \in \{0, ..., k\}$.

This implies that $f_{j,j}(\lambda, t) = (iv(t))^j$. The statement about the degree of $f_{j,k}(\lambda, \cdot)$ follows now, for fixed j, by another induction over k. □

Lemma 2. *The polynomials $f_{j,k}$ in Lemma 1 have the following property:*

$$f_{j,k}(\lambda, 0) = (iv(0))^k g_{j,k}(\lambda)$$

where $g_{j,k}$ is a polynomial of degree $k - j$ with leading coefficient $\binom{k}{j}$. Moreover, $g_{0,k}(\lambda) = \lambda^k$.

Proof. These statements are also proven by induction. □

Proof of Theorem 6. Without loss of generality we assume that $p = 2\pi$. For convenience we also introduce $q_n = 1$.

Each of the coefficients q_j has at most finitely many poles in the period strip $\{x \in \mathbb{C} : 0 \leq \Re(x) < 2\pi\}$. These poles will be denoted by x_1, ..., x_m. From Floquet's theorem we know that there is a solutions of $Ly = zy$ of the form

$$\psi(x) = \phi(x)\mathrm{e}^{i\lambda x}$$

where ϕ is a periodic function with period 2π and λ is a suitable complex number which is determined up to addition of an arbitrary integer. By hypothesis ϕ is a meromorphic function and its poles may occur only at the points x_1, ..., x_m and their translates. Therefore there exist positive integers s_j and a polynomial

$$v(t) = \prod_{j=1}^{m} (t - \mathrm{e}^{ix_j})^{s_j}$$

such that $v(\mathrm{e}^{ix})\phi(x)$ is an entire meromorphic function which is periodic with period 2π. This implies that there is a function u_0 which is analytic on $\mathbb{C} - \{0\}$ such that $u_0(\mathrm{e}^{ix}) = v(\mathrm{e}^{ix})\phi(x)$. We want to show that u_0 is a rational function.

Now multiply (3) by $v(\mathrm{e}^{ix})^{n+1}$ and perform the substitution $y(x) = t^\lambda u(t)/v(t)$ with $t = \mathrm{e}^{ix}$.

With the aid of Lemma 1 equation (3) turns into

$$\sum_{j=0}^{n} t^j p_j(t) u^{(j)}(t) = 0 \tag{4}$$

where

$$p_j(t) = \sum_{k=j}^{n} v(t)^{n-k} q_k^*(t) f_{j,k}(\lambda, t)$$

and where, of course, $q_k^*(t) = q_k(-i\log(t))$.

Because all solutions of (3) are meromorphic any pole of any of the coefficients must be a regular singular point of the differential equation. Therefore the poles of q_j have order $n-j$ at worst and the functions $v(t)q_{n-1}^*(t)$, ..., $v(t)^n q_0^*(t)$ are polynomials. This implies that the coefficients p_j in (4) are polynomials. Zero and infinity are singular points of the equation (4) and, since $p_n(0) = (iv(0))^n \neq 0$, we obtain that zero is in fact a regular singular point. This, in turn, implies that the isolated singularity $t = 0$ of u_0 can not be an essential singularity, i.e., u_0 is analytic in $\mathbb{C}$ with the exception of a possible pole at zero.

Moreover, at least one of the indices of the singular point $t = 0$ of equation (4) must be an integer because zero is an isolated singularity for the solution u_0. Remember that λ is only determined up to the addition of an integer. Therefore and because $u_0(t)t^\lambda = (t^{-m}u_0(t))t^{\lambda+m}$ we can and will choose the smallest integer index to be zero. Having made this convention u_0 is now analytic at zero, i.e., u_0 is an entire function. The product of all the indices, which equals the constant term of the indicial equation, must now be zero, too. Hence

$$0 = \sum_{k=0}^{n} i^k q_k^*(0) g_{0,k}(\lambda) = \sum_{k=0}^{n} q_k^*(0)(i\lambda)^k$$

which is the desired relationship for λ. The theorem will now be proved once we show that u_0 is a polynomial. To see this we have to study its behavior at infinity.

Let $s = 1/t$ and define integers $a_{j,k}$ by the equality

$$u^{(j)}(t) = \sum_{k=1}^{j} a_{j,k} s^{j+k} w^{(k)}$$

where $u(t) = w(s)$. This yields, in particular, $a_{j,j} = (-1)^j$. Also define $a_{0,0} = 1$ and $a_{j,0} = 0$ for any $j \in \mathbb{N}$.

Introducing s as the independent variable we find

$$0 = v(t)^{-n} \sum_{j=0}^{n} t^j p_j(t) u^{(j)} = \sum_{k=0}^{n} s^k \tilde{p}_k(s) w^{(k)} \tag{5}$$

where

$$\tilde{p}_k(s) = v(1/s)^{-n} \sum_{j=k}^{n} a_{j,k} p_j(1/s) = \sum_{j=k}^{n} \sum_{m=j}^{n} a_{j,k} q_m^*(1/s) \frac{f_{j,m}(\lambda, 1/s)}{v(1/s)^m}.$$

Recall from Lemma 1 that the degree of $f_{j,m}(\lambda, \cdot)$ is not larger than the degree of v^m. Hence the functions $\tilde{p}_k$ are bounded at zero. In particular, $\tilde{p}_n(s) = (-i)^n$ is bounded but also different from zero. It now follows that $s = 0$ is a regular singular point of equation (5). Therefore, and since $s = 0$ must be an isolated singularity of $w_0(s) = u_0(1/s)$, the function w_0 behaves like an integer power near zero. This, in turn implies that the entire function u_0 behaves like an integer power at infinity, i.e., u_0 is a polynomial. □

4. The structure of solutions

4.1. **The rational case.**

Theorem 7. *Consider the differential expression*

$$L = D^n + q_{n-2} D^{n-2} + \dots + q_0$$

where q_{n-2}, ..., q_0 are rational functions which have respectively the limits ρ_{n-2}, ..., ρ_0 at infinity. Assume that, for all $z \in \mathbb{C}$ all solutions of the equation $Ly = zy$ are meromorphic. Then $Ly = zy$ has a solution of the form

$$\psi(\lambda, x) = (\lambda^g + r_{g-1}(x)\lambda^{g-1} + \dots + r_0(x)) \exp(\lambda x)$$

where $r_0, \dots, r_{g-1}$ are rational functions and

$$z = \lambda^n + \rho_{n-2}\lambda^{n-2} + \dots + \rho_0.$$

Proof. If L has constant coefficients the theorem is trivially true with $g = 0$. Hence assume that L does not have constant coefficients.

By Halphen's theorem the equation $Ly = zy$ has a solution of the form $\psi(\lambda, x) = R_\lambda(x) \exp(\lambda x)$ where R_λ is a rational function and $z = \lambda^n + \rho_{n-2}\lambda^{n-2} + \dots + \rho_0$. The only thing left to investigate is the behavior of R_λ in terms of λ.

All finite singular points of the equation $Ly = zy$ must be regular singular points. Therefore q_j has no poles of order larger than $n - j$. Also the poles of R_λ must be located at the poles of the coefficients q_j. Hence there exist positive integers s_j and a polynomial

$$v(x) = \prod_{j=1}^{m} (x - x_j)^{s_j}.$$

such that the function $v\psi(\lambda, \cdot)$ is entire and the functions $v^2 q_{n-2}$, ..., $v^n q_0$ are polynomials. Therefore

$$\psi(\lambda, x) = \frac{p(\lambda, x)}{v(x)} \exp(\lambda x)$$

where

$$p(\lambda, x) = \sum_{j=0}^{N} c_j(\lambda) x^j.$$

Hence

$$0 = \psi^{(n)} + q_{n-2}\psi^{(n-2)} + ... + (q_0 - \lambda^n - ... - \rho_0)\psi = \frac{\exp(\lambda x)}{v(x)^{n+1}} F(\lambda, x).$$

Since $v^j q_{n-j}$ are polynomials the function $F(\lambda, \cdot)$ is a polynomial whose coefficients are polynomials in $c_0, ..., c_n$, and λ. In fact, as polynomials in $c_0, ..., c_n$ these coefficients are homogeneous of degree one. Each of these coefficients must be zero and therefore the coefficients c_j satisfy a system of linear homogeneous algebraic equations with coefficients in $\mathbb{C}[\lambda]$. A nontrivial solution exists and its components (the coefficients c_j) are rational functions of λ. Hence

$$p(\lambda, x) = \frac{1}{h(\lambda)} \sum_{j=0}^{N} \tilde{c}_j(\lambda) x^j$$

where $\tilde{c}_0, ..., \tilde{c}_n$, and h are polynomials. Without loss of generality we may assume that h is a constant. Hence, for some integer g,

$$p(\lambda, x) = v_g(x)\lambda^g + v_{g-1}(x)\lambda^{g-1} + ... + v_0(x)$$

and

$$\psi(\lambda, x) = (r_g(x)\lambda^g + r_{g-1}(x)\lambda^{g-1} + ... + r_0(x)) \exp(\lambda x)$$

where $r_j = v_j/v$ for $j = 0, ..., g$. Since $z^{1/n} = \lambda + O(\lambda^{-1})$ as λ tends to infinity, asymptotic considerations along the lines of Wasow [21] prove that $r_g(x) = 1$. For the sake of completeness Wasow's technique is outlined in the appendix. □

4.2. **The simply periodic case.**

Theorem 8. *Consider the differential expression*

$$L = D^n + q_{n-2}D^{n-2} + ... + q_0$$

where q_{n-2}, ..., q_0 are simply periodic meromorphic functions which are bounded at the ends of the period strip. Let the period be p and define $\rho_k = \lim_{\Im(x/p)\to\infty} q_k(x)$ for $k = 0, ..., n-2$. Assume that, for all $z \in \mathbb{C}$, all solutions of the equation $Ly = zy$ are meromorphic. Then $Ly = zy$ has a solution of the form

$$\psi(\lambda, x) = (\lambda^g + r_{g-1}(t(x))\lambda^{g-1} + ... + r_0(t(x))) \exp(\lambda x),$$

where $t(x) = \exp(2\pi i x/p)$, $r_0, ..., r_{g-1}$ are rational functions, and

$$z = \lambda^n + \rho_{n-2}\lambda^{n-2} + ... + \rho_0.$$

Proof. If L has constant coefficients the theorem is trivially true with $g = 0$. Hence assume that L does not have constant coefficients.

Theorem 6 applies and gives us a solution

$$\frac{u_\lambda(t(x))}{v(t(x))} \exp(i\lambda x)$$

where u_λ and v are polynomials. Again, we only have to study the behavior of u_λ with respect to λ. A similar proof as above, another call on Wasow's theorem, and a replacing $i\lambda$ by λ shows the validity of the present claim. □

Appendix A. Asymptotic behavior

Suppose the functions q_{n-2}, ..., q_0 are analytic in some open set Ω containing x_0. We want to study the behavior of solutions of the differential equation

$$Ly = y^{(n)} + q_{n-2}y^{(n-2)} + ... + q_0 y = \mu^n y$$

as μ tends to infinity.

Let

$$T = \begin{pmatrix} 1 & ... & 1 \\ \mu\sigma_1 & ... & \mu\sigma_n \\ \vdots & & \vdots \\ (\mu\sigma_1)^{n-1} & ... & (\mu\sigma_n)^{n-1} \end{pmatrix}$$

where σ_1, ..., σ_n denote the different n-th roots of one. The substitution

$$(y(x), ..., y^{(n-1)}(x))^t = Tu(x - x_0)$$

and letting $\varepsilon = 1/\mu$ transforms the equation $Ly = \mu^n y$ into a system $\varepsilon u' = A(\varepsilon, \cdot)u$ where

$$A(\varepsilon, t) = \sum_{j=0}^{n} A_j(t)\varepsilon^j$$

with $A_0 = \operatorname{diag}(\sigma_1, ..., \sigma_n)$, $A_1 = 0$, and A_j analytic in a vicinity of zero.

When r is a positive number and I a real open interval we denote by $S(r, I)$ the set $\{z \in \mathbb{C} : |z| < r, \arg(z) \in I\}$ and by $K(r)$ the set $\{z \in \mathbb{C} : |z| < r\}$.

Then, by a repeated application of Theorem 26.2 of Wasow [21] and its proof (in particular, the formulas 25.19 – 25.22) and because of the absence of a term $\varepsilon A_1(t)$, there exist numbers ρ and δ, an interval I, and matrix-valued functions $P : K(\rho) \times S(\delta, I) \to \mathbb{C}$ and $B : K(\rho) \times S(\delta, I) \to \mathbb{C}$ such that

1. P is holomorphic in both variables.
2. Asymptotically, as ε tends to zero in $S(\delta, I)$,
$$P(t, \varepsilon) \sim I + \sum_{j=2}^{\infty} P_j(t)\varepsilon^j.$$
3. The transformation $u = Pw$ takes the equation $\varepsilon u' = A(\varepsilon, \cdot)u$ into the completely decoupled system
$$\varepsilon w' = B(\varepsilon, \cdot)w$$
where B is diagonal and has the asymptotic expansion
$$B(\varepsilon, t) \sim A_0 + \sum_{j=2}^{\infty} B_j(t)\varepsilon^j$$
as ε tends to zero in $S(\delta, I)$.

A fundamental matrix of $\varepsilon w' = B(\varepsilon, \cdot)w$ is

$$w(\varepsilon, t) = \exp(\varepsilon^{-1} \int_0^t B(\varepsilon, s)ds) = \exp(A_0\mu t)\exp(\varepsilon C(t))$$

for a suitable diagonal matrix $C(t)$. Since $P_0(t) = I$ and $\exp(\varepsilon C(t)) = I + O(\varepsilon)$ we obtain for the asymptotic behavior of u

$$u(\varepsilon, t) = (I + O(\varepsilon))\exp(A_0\mu x).$$

Linear independent solutions of $Ly = \mu^n y$ are given by

$$\begin{aligned} y_j(\mu, x) &= \sum_{k=1}^{n} u_{k,j}(\varepsilon, x - x_0) = \sum_{k=1}^{n} (\delta_{j,k} + O(\varepsilon)) \exp(\mu\sigma_j(x - x_0)) \\ &= (1 + O(\mu^{-1})) \exp(\mu\sigma_j(x - x_0)). \end{aligned}$$

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