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CONTINUABILITY OF SOLUTIONS TO FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. This article concerns the Caputo fractional differential equation

$$ {}^cD_a^\alpha x^{[n-1]}(t) = f(t, x(t)) + e(t), \quad n \geq 2 $$

where $x^{[n-1]}$ is the quasiderivative of x of order $(n-1)$ and ${}^cD_a^\alpha$ is the Caputo derivative of the order $\alpha \in (0,1)$. We study the continuability and noncontinuability of solutions.

1. Introduction

We consider the fractional differential equation

$$ {}^cD_a^\alpha x^{[n-1]}(t) = f\big(t, x(t)\big) + e(t) \tag{1.1} $$

where $a > 1$, $\alpha \in (0,1)$, $n \geq 2$ is an integer, ${}^cD_a^\alpha u(t)$ is the Caputo derivative of order α, defined as

$$ \begin{gathered} {}^cD_a^\alpha u(t) := \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} u'(s)\, ds\,, \\ \Gamma(x) = \int_0^\infty s^{x-1} e^{-s} ds\,, \quad x > 0 \end{gathered} \tag{1.2} $$

is the Gamma function and $u^{[i]}$, $i = 0, \dots, n-1$ are quasiderivatives of u defined as

$$ u^{[0]}(t) = u(t)\,, \quad u^{[i]}(t) = a_i(t)\big(u^{[i-1]}(t)\big)'\,, \quad i = 1, \dots, n-1\,. \tag{1.3} $$

Let $[a,b] \subset [a,\infty)$, and $AC[a,b]$ the set of all functions defined on $[a,b]$ that are absolutely continuous on $[a,b]$.

Let $[a,b) \subset [a,\infty)$. Then we denote by $AC_{\rm loc}[a,b)$ the set of all functions defined on $[a,b)$ that are absolutely continuous on every compact subinterval of $[a,b)$.

In the reminder of this article we assume the following:

- (H1) $a_i\colon [a,\infty) \to (0,\infty)$ are continuous functions for $i = 1, \dots, n-1$;
- (H2) $e\colon [a,\infty) \to \mathbb{R} = (\infty,\infty)$;
- (H3) $f\colon [a,\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous.

Note that

$$ x^{[n-1]}(t) = a_{n-1}(t)\big(a_{n-2}(t)(\dots(a_1 x'(t))' \dots\big)'\,. $$

In some places, the following assumptions will be used:

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(H4) There exist continuous functions $r\colon [a,\infty) \to \mathbb{R}_+ = [0,\infty)$ and $\omega\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\omega(x) > 0$ for $x > 0$, ω is nondecreasing and

$$\big|f(t,x)\big| \le r(t)\omega(|x|)\,, \quad \forall t \in [a,\infty)\,,\ x \in \mathbb{R}\,;$$

(H5) $e \in AC_{\rm loc}[a,\infty)$, $f(t,u) \in AC_{\rm loc}[a,\infty)$ for any fixed $u \in \mathbb{R}$, $f(t,u) \in AC_{\rm loc}(\mathbb{R})$ for any fixed $t \in [a,\infty)$.

The Caputo derivative given by (1.2) is the special case of Caputo derivative of order $\alpha > 0$, defined as

$${}^cD_a^\alpha u(t) := \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-\alpha-1} u^{(m)}(s)\, ds\,,$$

where m is the smallest integer greater than or equal to α, see e.g. [4, 5, 7]. Fractional differential equations have attract eda great attention in the last two decades because of their importance in applications in areas of physics, chemistry, aerodynamics, etc., see e.g. monographs [4, 5, 9] and the references therein.

There are a lot of papers devoted to the study of asymptotic behavior of solutions of fractal differential equations, see e.g. [6, 7, 8, 9, 10, 12]. But results of forced fractional differential equations are relatively scarece. Equation (1.1) is studied in [7] (when $n = 2$ or $n = 3$ and $a_2 \equiv 1$) where sufficient conditions for boundedness of all non-oscillatory solutions are given.

A function $x\colon [a,b) \to \mathbb{R}$, $b \le \infty$ is said to be the solution of (1.1) if $x^{[n-1]} \in AC_{\rm loc}[a,b)$ and (1.1) is valid on $[a,b)$. We will suppose that x is nonextendable to the right, i.e., if $b < \infty$, then x cannot be defined at $t = b$. Solution x is said to be continuable if $b = \infty$, otherwise it is said to be noncontinuable. A continuable solution x is said to be proper if it is nontrivial in any neighbourhood of ∞.

In this article we study problem (1.1) with

$$x^{[i]}(a) = d_i\,, \quad i = 0,\dots,n-1, \tag{1.4}$$

where $d_i \in \mathbb{R}$, $i = 0,\dots,n-1$.

Let (1.1), (1.4) have a solution x. We investigate whether or not, x is continuable.

When $\alpha = 1$, then (1.1) is the ordinary differential equation ($t \ge a$)

$$x^{[n]}(t) = f\big(t, x(t)\big) + e(t) \tag{1.5}$$

with $x^{[n]}(t) = \big(x^{[n-1]}(t)\big)'$. It is known that (1.5) can have noncontinuable solutions, see [2, 8]. A special case of (1.5) is the equation

$$x''(t) = r(t)h(x) \tag{1.6}$$

where $\lambda_1 > 1$, $\lambda_2 \in (0,1)$, $M > 0$, $r \in C^0[a,\infty)$, $h \in C^0(\mathbb{R})$, $r(t) \ge \frac{M}{t^2}$ for large t, $h(x)x > 0$ for $x \ne 0$,

$$\begin{aligned} \big|h(x)\big| &\ge |x|^{\lambda_1} \quad \text{for } |x| \ge 1\,,\\ \big|h(x)\big| &\le |x|^{\lambda_2} \quad \text{for } |x| < 1\,. \end{aligned}$$

Then, by [1, Lemma 4], equation (1.6) has no proper solution.

Some papers only study proper solutions of (1.1) because of their great importance. In this article, we study only the part corresponding to the continuability of solutions to (1.1). However, the methods used here can be applied for other types of Caputo differential equations.

Notation. We denote

$$\bar r(t)=\max_{a\le s\le t}|r(s)|\,,\quad \bar e(t)=\max_{a\le s\le t}|e(s)|\,,\quad t\ge a\,.$$

If x is a solution of (1.1) defined on $[a,b)$ with $b\le\infty$, we put

$$\bar x(t)=\max_{a\le s\le t}|x(s)|\,,\quad t\in[a,b)\,.$$

Let $1\le j\le i\le n-1$ be integers and $t\in[a,\infty)$. Then we put

$$J_{i,j}(t)=\int_a^t a_j^{-1}(s_{j+1})\int_a^{s_{j+1}} a_{j+1}^{-1}(s_{j+2})\int_a^{s_{j+2}}\cdots\int_a^{s_i} a_i^{-1}(\sigma)\,d\sigma\,ds_i\dots ds_{j+1}\,,$$
$$J_{j,i}(t)\equiv 1\quad\text{if } j>i\,.$$

If $i,j\in\{0,1,\dots\}$, $i<j$ and $c_k\in\mathbb{R}$ for $i\le k\le j$, then we put $\sum_{k=j}^{i}c_k=0$.

2. Preliminaries

The following lemmas state some properties of Caputo fractional differential equations . For this, we define the Riemann-Liouville fractional integral operator of order α on $L_1[a,b)$, $b\le\infty$ by

$$J_a^\alpha g(t):=\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}g(s)\,ds\,.$$

Let $Dg(t)=\frac{d}{dt}g(t)$.

Lemma 2.1. *Let $a<b\le\infty$. Then*

(i) *J_a^α maps $AC_{\rm loc}[a,b)$ to $AC_{\rm loc}[a,b)$.*

(ii) *If $g\in AC_{\rm loc}[a,b)$, then $J_a^{1-\alpha}J_a^\alpha g=J_a^1 g$, and*

$${}^cD_a^\alpha g(t)=DJ_a^{1-\alpha}\big[g(t)-g(a)\big]\,,\quad t\in[a,b)\,.$$

(iii) *If $g\in AC_{\rm loc}[a,b)$, then*

$$J_a^\alpha\,{}^cD_a^\alpha g(t)=g(t)-g(a)\,,\quad t\in[a,b)\,.$$

For the proof of (i), see [10, Lemma 2.3]. For (ii), see [5, Theorem 2.2, Definition 3.2 and Lemma 2.11]. For (iii), see [5, Theorem 3.8].

Lemma 2.2. (i) *Let x be a solution of* (1.1). *Then it is the solution of the nonlinear Volterra type integral equation* $(t\ge a)$

$$x^{[n-1]}(t)=x^{[n-1]}(a)+\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\big[f(s,x(s))+e(s)\big]\,ds\,. \tag{2.1}$$

Let (H5) *be valid. Then equation* (1.1) *is equivalent to* (2.1)*, i.e. every function x, defined on $[a,b)$, $b\le\infty$ such that $x^{[n-1]}\in AC^1_{\rm loc}[a,b)$ is the solution of* (1.1) *if, and only if it is the solution of* (2.1).

(ii) *Let a solution x of* (1.1) *be defined on $[a,b)$, $b<\infty$. If*

$$\limsup_{t\to b-}\sum_{i=0}^{n-1}|x^{[i]}(t)|=\infty \tag{2.2}$$

then it is noncontinuable. If (H5) *holds and x is noncontinuable then* (2.2) *holds and*

$$\lim_{t\to b-}\bar x(t)=\infty\,. \tag{2.3}$$

Proof. (i) Let x be a solution of (1.1) on $[a,b)$, $b \le \infty$. Then $x^{[n-1]} \in AC_{\rm loc}[a,b)$ and according to Lemma 2.1(iii) (with $g = x^{[n-1]}$)

$$x^{[n-1]}(t) - x^{[n-1]}(a) = J_a^{\alpha}\, {}^cD_a^{\alpha} x^{[n-1]}(t) = J_a^{\alpha}\big(f\big(t,x(t)\big) + e(t)\big)\,;$$

hence, (2.1) is valid.

Let (H5) hold and x be a solution of (2.1). Then $x \in C^1[a,b)$ and according to (H5), $f(t,x(t)) + e(t) \in AC_{\rm loc}[a,b)$. Using Lemma 2.1(i), $J_a^{\alpha}\big(f(t,x(t)) + e(t)\big) \in AC_{\rm loc}[a,b)$. From this and (2.1), we have $x^{[n-1]} \in AC_{\rm loc}[a,b)$. Applying Lemma 2.1(ii) and (2.1), we have

$$\begin{aligned}
{}^cD_a^{\alpha} x^{[n-1]} &= DJ_a^{1-\alpha}\big(x^{[n-1]}(t) - x^{[n-1]}(a)\big)\\
&= DJ_a^{1-\alpha}\Big(\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\big[f(s,x(s)) + e(s)\big]\,ds\Big)\\
&= DJ_a^{1-\alpha} J_a^{\alpha}\big(f(t,x(t)) + e(t)\big)\\
&= DJ_a^{1}\big(f(t,x(t)) + e(t)\big) = f(t,x(t)) + e(t)
\end{aligned}$$

for $t \in [a,b)$. Hence (1.1) holds.

(ii) If (2.2) holds then x is clearly noncontinuable. Let (H5) hold and let x be a noncontinuable solution of (1.1) defined on $[a,b)$, $b < \infty$. We prove (2.2). So, suppose, on the contrary, that $\sum_{i=0}^{n-1} |x^{[i]}(t)|$ is bounded on $[a,b)$. From this and from $b < \infty$, $\lim_{t\to b-} x^{[i]}(t)$ exist for $i = 0,1,\dots,n-2$. The existence of $\lim_{t\to b-} x^{[n-1]}(t)$ follows from (2.1). So, the solution x of (2.1) can be extended to $t = b$, $x^{[i]}(b) := \lim_{t\to b-} x^{[i]}(t)$, $i = 0,1,\dots,n-1$. Moreover, as $x \in C^1[a,b]$, $\big(f(t,x(t)+e(t)\big) \in AC[a,b]$, according to part (i), x is the solution of (1.1) on $[a,b]$. This contradicts the noncontinuability of x proves statement (2.2).

If (2.3) does not hold then (2.1) implies $x^{[n-1]}$ is bounded on $[a,b)$ and, hence, $x^{[i]}$, $i = 0,1,\dots,n-2$ are bounded on $[a,b)$ that contradicts (2.2). Thus, (2.3) is valid. □

Because of Lemma 2.2(i), we will investigate (2.1) instead of (1.1) without mention it. The proofs of the main results are based on the following lemmas.

Lemma 2.3. *Let $u\colon [a,b) \to \mathbb{R}$, $a < b \le \infty$ be a function such that $u^{[n-1]}$ exists on $[a,b)$ and let*

$$\big|u^{[n-1]}(t)\big| \le K(t)\,, \quad t \in [a,b) \tag{2.4}$$

where K is a nondecreasing, continuous function. Then

$$\big|u^{[1]}(t)\big| \le \sum_{i=1}^{n-2} J_{2,i}(t)\big|u^{[i]}(a)\big| + J_{2,n-1}(t)K(t) \quad \textit{for } t \in [a,b). \tag{2.5}$$

Proof. If $n = 2$, then (2.5) follows from (2.4). Hence, suppose $n \ge 3$. We prove that

$$\big|u^{[j]}(t)\big| \le \sum_{i=j}^{n-2} J_{j+1,i}(t)\big|u^{[i]}(a)\big| + K(t)J_{j+1,n-1}(t) \tag{2.6}$$

for $j = 1,2,\dots,n-2$. Using (1.3) we have

$$\big(u^{[n-2]}(t)\big)' = \frac{1}{a_{n-1}(t)} u^{[n-1]}(t)\,.$$

From this and from (2.4), the integration implies

$$\big|u^{[n-2]}(t) - u^{[n-2]}(a)\big| \le \int_a^t \frac{K(\sigma)}{a_{n-1}(\sigma)}\, d\sigma \le K(t)J_{n-1,n-1}(t)$$

and (2.6) holds for $j = n-2$. We apply mathematical induction. Suppose, that (2.6) holds for $j = n-2, n-3, \dots, k$. Then, by (1.3),

$$\big(u^{[k]}(t)\big)' = \frac{1}{a_{k+1}(t)}\, u^{[k+1]}(t)$$

and the integration on $[a,t]$ implies

$$\begin{aligned}
\big|u^{[k]}(t) - u^{[k]}(a)\big| &\le \int_a^t a_{k+1}^{-1}(\sigma)|u^{[k+1]}(\sigma)|\, d\sigma \\
&\le \int_a^t a_{k+1}^{-1}(\sigma)\Big[\sum_{i=k+1}^{n-2} J_{k+2,i}(\sigma)|u^{[i]}(a)| + K(\sigma)J_{k+2,n-1}(\sigma)\Big]\, d\sigma \\
&\le \sum_{i=k+1}^{n-2} J_{k+1,i}(t)|u^{[i]}(a)| + K(t)J_{k+1,n-1}(t)\,.
\end{aligned}$$

Hence, (2.6) is valid for $j = k$. Now, (2.5) is given by (2.6) for $j = 1$. □

Lemma 2.4. *Let* (H4) *hold and let x be a solution of* (1.1) *defined on $[a,b)$, $b \le \infty$. Then*

$$\bar{x}(t) \le M_1(t) + \int_a^t M_2(s)\omega\big(\bar{x}(s)\big)\, ds \tag{2.7}$$

for $t \in [a,b)$, where

$$\begin{aligned}
M_1(t) &= \big|x^{[0]}(a)\big| + \int_a^t a_1^{-1}(s)\Big[\sum_{i=1}^{n-2} J_{2,i}(s)\big|x^{[i]}(a)\big| \\
&\quad + \Big(|x^{[n-1]}(a)| + \frac{\bar{e}(s)}{\alpha\Gamma(\alpha)}(s-a)^{\alpha}\Big)J_{2,n-1}(s)\Big]\, ds\,, \\
M_2(t) &= \frac{\bar{r}(t)}{\alpha\Gamma(\alpha)} a_1^{-1}(t)(t-a)^{\alpha} J_{2,n-1}(t)\,.
\end{aligned} \tag{2.8}$$

Proof. By (2.1) and (H4), we hve

$$\begin{aligned}
\big|x^{[n-1]}(t)\big| &\le \big|x^{[n-1]}(a)\big| + \frac{\bar{e}(t)}{\alpha\Gamma(\alpha)}(t-a)^{\alpha} \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\, r(s)\omega\big(|x(s)|\big)\, ds \\
&\le \big|x^{[n-1]}(a)\big| + \frac{\bar{e}(t)}{\alpha\Gamma(\alpha)}(t-a)^{\alpha} + \frac{\bar{r}(t)}{\alpha\Gamma(\alpha)}(t-a)^{\alpha}\omega\big(\bar{x}(t)\big)\,.
\end{aligned} \tag{2.9}$$

Applying Lemma 2.3 for $u = x$, $b = t$ and

$$K(t) = \big|x^{[n-1]}(a)\big| + \frac{\bar{e}(t)}{\alpha\Gamma(\alpha)}(t-a)^{\alpha} + \frac{\bar{r}(t)}{\alpha\Gamma(\alpha)}(t-a)^{\alpha}\omega\big(\bar{x}(t)\big),$$

from (2.9) we obtain

$$\big|(x^{[0]}(t))'\big| = \frac{|x^{[1]}(t)|}{a_1(t)} \le \bar{M}_1(t) + M_2(t)\omega\big(\bar{x}(t)\big) \tag{2.10}$$

with

$$\bar{M}_1(t) = a_1^{-1}(t)\Big\{\sum_{i=1}^{n-2} J_{2,i}(t)\big|x^{[i]}(a)\big| + \Big(\big|x^{[n-1]}(a)\big| + \frac{\bar{e}(t)}{\alpha\Gamma(\alpha)}(t-a)^\alpha\Big)J_{2,n-1}(t)\Big\}\,.$$

Hence, using the first equality in (1.3), the integration of (2.10) on $[a,\tau)$, $a<\tau\le t$ implies

$$|x(\tau)| \le M_1(t) + \int_a^t M_2(s)\omega\big(\bar{x}(s)\big)\,ds\,,$$

or

$$\bar{x}(t) \le M_1(t) + \int_a^t M_2(s)\omega\big(\bar{x}(s)\big)\,ds\,.$$

Hence, (2.7) is valid. □

The following two lemmas are well known.

Lemma 2.5 ([11, Lemma 2.1])**.** *Let $k>0$, $\lambda>1$, $t_0\ge 0$ be constants, F be a continuous, nonnegative function on $\mathbb{R}_+$ and v be a continuous, nonnegative function on $\mathbb{R}_+$ satisfying the inequality*

$$v(t) \le k + \int_{t_0}^t F(s)v^\lambda(s)\,ds\,,\quad t\ge t_0\,. \tag{2.11}$$

If

$$(\lambda-1)k^{\lambda-1}\int_{t_0}^\infty F(s)\,ds < 1 \tag{2.12}$$

then

$$v(t) \le k\Big(1-(\lambda-1)k^{\lambda-1}f_{t_0}^t F(s)\,ds\Big)^{-\frac{1}{\lambda-1}}$$

for $t\ge t_0$.

Lemma 2.6 ([8, Lemma 9.2])**.** *Let $k>0$, $g>0$ be a continuous function on $[t_0,b)$, $b\le\infty$ and $\omega(t)>0$ for $t\ge k$ be a continuous function such that $\int_k^\infty \frac{ds}{\omega(s)}=\infty$. Then for any continuous function $x\colon[t_0,b)\to\mathbb{R}_+$ fulfilling*

$$x(t) \le k + \int_{t_0}^t g(s)\omega\big(x(s)\big)\,ds\,,\quad t\in[t_0,b)$$

the estimation

$$x(t) \le \Omega^{-1}\Big(\int_{t_0}^t g(s)\,ds\Big)\,,\quad t\in[t_0,b)$$

holds where Ω^{-1} is the inverse function to $\Omega(s)=\int_k^s \frac{d\tau}{\omega(\tau)}$.

Consider the auxilliary system of differential equations

$$y_i' = b_i(t)y_{i+1}\,,\quad i=1,\dots,n-1\,,\quad y_n' = F(t,y_1)\,, \tag{2.13}$$

where $b_i\in C^0[a,\infty)$, $b_i>0$ on $[a,\infty)$, $i=1,\dots,n-1$ and $F\in C^0([a,\infty),\mathbb{R})$.

Furthermore, suppose $y_0>0$, $b_n\in C^0[a,\infty)$, $b_n>0$, $\lambda>1$, $\beta\in\{-1,1\}$ exist such that

$$\beta F(t,u) \ge b_n(t)|u|^\lambda \quad \text{for } t\ge a,\ \beta u>y_0\,. \tag{2.14}$$

A solution $\{y_i\}_1^u$ of (2.13), defined on $[a,b)$ with $b<\infty$, is called noncontinuable if it can not be extended to $t=b$. In this case

$$\limsup_{t\to b-}\sum_{i=1}^{u}|y_i(t)|=\infty\,.$$

The following lemma states sufficient conditions for the existence of noncontinuable solutions of (2.13) with (2.14).

Lemma 2.7. *Suppose* (2.14) *holds.*

(i) *If* $t_1\in(a,\infty)$*, then* (2.13) *possesses a noncontinuable solution* $\{y_i\}_{i=1}^u$ *that is defined on a subinterval* $[a,b)\subset[a,t_1)$ *and*

$$\beta y_i(t)\ge y_0 \quad \text{for} \quad t\in[a,b)\,,\quad i=1,\dots,n\,.$$

(ii) *Let* $\delta>0$*,* $\mu_i\in\mathbb{R}$ *for* $i=1,\dots,n$*,*

$$b_i(t)\ge\delta t^{\mu_i}\,,\quad i=1,\dots,n$$

and let

$$\mu_n+\lambda\sum_{i=1}^{n-1}(1+\mu_i)+1>0\,. \tag{2.15}$$

Then any solution $\{y_i\}_1^n$ *of* (2.13)*, satisfying the initial conditions*

$$\beta y_i(a)>y_0\,,\quad i=1,\dots,n\,,$$

is noncontinuable.

(iii) *Let* $\int_a^\infty b_i(t)\,dt=\infty$ *for* $i=1,\dots,n$*. Then the statement in* (ii) *is valid.*

The above lemma follows [2, Theorems 3, 4 (for $l=n$)] or [3, Theorems 1, 2, 3].

3. Continuable solutions

The first theorem gives a sufficient condition for all solutions of (1.1) be continuable. It is a generalization of well known theorem by Winter and Osgood [8] for differential equations.

Theorem 3.1. *Suppose* (H4) *and*

$$\int_1^\infty\frac{dx}{\omega(x)}=\infty\,. \tag{3.1}$$

Then every solution of (1.1) *is continuable.*

Proof. Suppose, on the contrary, that x is a noncontinuable solution of (1.1) defined on $[a,b)$. Then according to Lemma 2.2(ii), $b<\infty$ and

$$\lim_{t\to b-}\bar{x}(t)=\infty\,. \tag{3.2}$$

Lemma 2.4 implies

$$\bar{x}(t)\le M_1(t)+\int_a^t M_2(s)\omega\big(\bar{x}(s)\big)\,ds\le M_1(b)+M\int_a^t\omega\big(\bar{x}(s)\big)\,ds$$

on $[a,b)$ where M_1 and M_2 are given by (2.8) and $M=\max_{a\le s\le b}M_2(s)$. From this, (3.2) and Lemma 2.6 (with $t_0=a$, $k=M_1(b)$, $g(t)\equiv M$, $x(t)=\bar{x}(t)$) we obtain

$$\int_a^\infty\frac{d\tau}{\omega(\tau)}=\lim_{t\to b-}\int_a^{\bar{x}(t)}\frac{d\tau}{\omega(\tau)}\le\lim_{t\to b-}\int_a^t M\,ds=M(b-a)<\infty\,.$$

This contradicts (3.1) and proves that x is continuable. □

If (3.1) does not hold, then noncontinuable solutions may exist (see Theorem 3.3 below). The following theorem gives us a set of initial conditions under which solutions are continuable.

Theorem 3.2. *Let $\lambda > 1$, (H4) and (H5) hold with $\omega(x) = x^\lambda$ for $x \in \mathbb{R}_+$ and let x be a solution of (1.1) satisfying the initial conditions $d_j \in \mathbb{R}$,*

$$x^{[j]}(a) = d_j\,, \quad j = 0, \dots, n-1\,. \tag{3.3}$$

If

$$\begin{aligned} k := &|d_0| + \int_a^\infty a_1^{-1}(s)\Big\{\sum_{i=1}^{n-2} J_{2,i}(s)|d_i| \\ &+ \Big(|d_{n-1}| + \frac{\bar{e}(s)}{\alpha\Gamma(\alpha)}(s-a)^\alpha\Big)J_{2,n-1}(s)\Big\}\,ds < \infty\,, \end{aligned} \tag{3.4}$$

and

$$\frac{(\lambda-1)k^{\lambda-1}}{\alpha\Gamma(\alpha)}\int_a^\infty \frac{\bar{r}(s)(t-a)^\alpha}{a_1(t)}\,J_{2,n-1}(t)\,dt < 1\,, \tag{3.5}$$

then x is continuable.

Proof. Let x be a solution of (1.1) with (3.3), (3.4) and (3.5). Suppose, on the contrary, that x is noncontinuable and it is defined on $[a, b)$, $b < \infty$. Then according to Lemma 2.2(ii)

$$\lim_{t\to b-} \bar{x}(t) = \infty\,. \tag{3.6}$$

Lemma 2.4 implies

$$\bar{x}(t) \le M_1(t) + \int_a^t M_2(s)\,\bar{x}^\lambda(s)\,ds \tag{3.7}$$

for $t \in [a, b)$ where M_1 and M_2 are given by (2.8). As M_1 is nondecreasing, (3.4) implies $k = M_1(\infty)$ is finite.

Let $T \in [a, b)$ be fixed. We define

$$v(t) = \begin{cases} \bar{x}(t) & \text{if } t \in [a, T) \\ \bar{x}(T) & \text{if } t > T\,. \end{cases} \tag{3.8}$$

Then with respect to (3.7),

$$v(t) \le k + \int_a^t M_2(s)v^\lambda(s)\,ds\,, \quad t \in [a, \infty)\,.$$

Now, according to Lemma 2.5 (with $t_0 = a$, $F = M_2$, condition (2.12) follows from (3.5)) we have

$$v(t) \le k\Big(1 - (\lambda-1)k^{\lambda-1}\int_a^\infty M_2(s)\,ds\Big)^{-\frac{1}{\lambda-1}} =: k_1 < \infty$$

for $t \ge a$. Hence, by (3.8),

$$\bar{x}(t) \le k_1\,, \quad t \in [a, T]\,.$$

As $T \in [a, b)$ is arbitrary, $\bar{x}(t) \le k_1$ for $t \in [a, b)$. The contradiction with (3.6) proves that x is continuable. □

The following two theorems give us sets of initial conditions for which the solutions are noncontinuable.

Theorem 3.3. *Let $\lambda > 1$, $x_0 > 0$, $\beta \in \{-1, 1\}$, $t_1 > a$ and a continuous function $r\colon [a, t_1] \to (0, \infty)$ exist such that*

$$\begin{gathered} \beta f(t,x) \geq r(t)|x|^\lambda \quad \text{for } t \in [a, t_1]\,, \beta x \geq x_0\,, \\ \beta e(t) \geq -\frac{x_0^\lambda}{2}\, r(t) \quad \text{for } t \in [a, t_1]\,. \end{gathered} \tag{3.9}$$

Then there exists $D > 0$ such that any solution of (1.1) satisfying $\beta x^{[i]}(a) \geq D$, $i = 0, \dots, n-1$ is noncontinuable.

Proof. Let $\beta = 1$. Consider the auxiliary differential equations

$$y^{[n]} = \frac{t_1^{\alpha-1}}{2\Gamma(\alpha)}\, r_0 |y(t)|^\lambda \operatorname{sgn} y(t) \tag{3.10}$$

for $t \in [a, t_1)$, $y^{[n]}(t) = \big(y^{[n-1]}(t)\big)'$, $r_0 = \min_{a \leq t \leq t_1} r(t) > 0$. This equation can be transformed into

$$\begin{gathered} y_i' = \frac{1}{a_i(t)} y_{i+1}\,, \quad i = 1, 2, \dots, n-1\,, \\ y_n' = \frac{t_1^{\alpha-1}}{2\Gamma(\alpha)} r_0 |y_1(t)|^\lambda \operatorname{sgn} y(t) \end{gathered} \tag{3.11}$$

with $y_i = y^{[i-1]}$, $i = 1, 2, \dots, n$. Then, according to Lemma 2.7(i) (with $t_1 = t_1$, $y_0 = x_0$, $b_i(t) = \big(a_i(t)\big)^{-1}$ for $i = 1, \dots, n-1$, $b_n = \frac{t_1^{\alpha-1}}{2\Gamma(\alpha)} r_0$), (3.11) has a noncontinuable solution y defined on $[a, b) \subset [a, t_1)$ such that $y_i(t) \geq x_0$ for $t \in [a, b)$. Denote by $d_i = y_{i+1}(a)$, $i = 0, \dots, n-1$. Hence, (3.10) has the solution y with the initial conditions

$$y^{[i]}(a) = d_i\,, \quad i = 0, \dots, n-1 \tag{3.12}$$

and (3.11) implies all quasiderivatives are increasing. At the same time

$$\limsup_{t \to b} \sum_{i=0}^{n-1} y^{[i]}(t) = \infty\,. \tag{3.13}$$

Let x be a solution of (1.1) with the initial conditions

$$x^{[i]}(a) > d_i\,, \quad i = 0, \dots, n-1\,. \tag{3.14}$$

We denote by I the intervals where both functions y and x are defined. We prove that

$$x^{[i]}(t) > y^{[i]}(t)\,, \quad t \in I\,,\ i = 0, \dots, n-1\,. \tag{3.15}$$

Because of the initial conditions (3.12) and (3.13), equation (3.15) is valid in a right neigbourhood of a. Suppose, that it is not valid on the whole interval I. Then there is a $t_2 \in I$ and an index $j \in \{0, \dots, n-1\}$ exist such that

$$x^{[j]}(t_2) = y^{[j]}(t_2)\,, \quad x^{[i]}(t) > y^{[i]}(t) \quad \text{for } t \in [a, t_2)\,, \tag{3.16}$$

$i = 0, \dots, n-1$. First, we prove that $j \neq n-1$. Using (3.9) and (3.16), for $t \in [a, t_2)$ we have

$$\begin{aligned} x^{[n-1]}(t) &> d_{n-1} + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \big[e(t) + f\big(s, x(s)\big)\big]\, ds \\ &\geq d_{n-1} + \frac{t_1^{\alpha-1}}{\Gamma(\alpha)} \int_a^t \Big[-\frac{r(s)}{2} x_0^\lambda + r(s) x^\lambda(s)\Big]\, ds \end{aligned}$$

$$\begin{aligned}&\ge d_{n-1} + \frac{t_1^{\alpha-1}}{\Gamma(\alpha)}\int_a^t \frac{r(s)}{2}x^\lambda(s)\,ds\\ &\ge d_{n-1} + \frac{t_1^{\alpha-1}r_0}{2\Gamma(\alpha)}\int_a^t y^\lambda(s)\,ds = y^{[n-1]}(t)\,.\end{aligned}$$

Hence, $j \in \{0,\dots,n-2\}$. If $w(t) = x^{[j]}(t) - y^{[j]}(t)$, then $w(a) > 0$, $w(t_2) = 0$ and there exists $t_3 \in (a, t_2)$ such that $w'(t_3) < 0$, i.e.,

$$\big(x^{[j]}(t_3) - y^{[j]}(t_3)\big)' = \frac{1}{a_{j+1}(t_3)}\big[x^{[j+1]}(t_3) - y^{[j+1]}(t_3)\big] < 0\,.$$

This contradicts (3.16) and implies (3.15) is valid. Now, according to (3.13), (3.15) and Lemma 2.2(ii), x is noncontinuable. So the statement of the theorem holds with $D = \max(d_0,\dots,d_{n-1}) + 1$.

When $\beta = -1$, the proof is similar. □

Theorem 3.4. *Let $\lambda > 1$, $\beta \in \{-1, 1\}$, $x_0 > 0$ and let a continuous function $r\colon [a,\infty) \to (0,\infty)$ be such that*

$$\beta f(t,x) \ge r(t)|x|^\lambda \quad \text{for } t \in [a,\infty),\ \beta x \ge x_0\,,$$

$$\beta e(t) \ge -\frac{x_0^\lambda}{2} r(t) \quad \text{for } t \in [a,\infty)\,.$$

Let one of the following two assumptions hold:

(i) *Let $C_j \in \mathbb{R}_+$, $\lambda_j \in \mathbb{R}$, $j = 1,\dots,n$ be such that*

$$a_i(t) \le C_i t^{\lambda_i}\,, \quad i = 1,\dots,n-1, r(t) \ge C_n t^{\lambda_u} \tag{3.17}$$

for $t \ge a$ and

$$\lambda_n > -1 + \lambda\Big[1 - \alpha - \sum_{i=1}^{n-1}(1-\lambda_i)\Big]\,. \tag{3.18}$$

(ii) *Let $\int_a^\infty a_i^{-1}(t)\,dt = \infty$ for $i = 1,\dots,n-2$, $\int_a^\infty t^{\alpha-1}a_{n-1}^{-1}(t)\,dt = \infty$ and $\int_a^\infty r(t)\,dt = \infty$.*

Then any solution x of (1.1) *satisfying the initial conditions*

$$\beta x^{[i]}(a) > x_0 a^{1-\alpha}\,, \quad i = 0,\dots,n-2, \beta x^{[n-1]}(a) > x_0$$

is noncontinuable.

Proof. (i) Let $\beta = 1$. Consider the auxilliary integro-differential equation

$$y^{[n-1]}(t) = y^{[n-1]}(a) + \frac{t^{\alpha-1}}{2\Gamma(\alpha)}\int_a^t r(s)|y(s)|^\lambda \operatorname{sgn} y(s)\,ds \tag{3.19}$$

and its solution with the initial conditions

$$y^{[j]}(a) = d_j > 0\,, \quad j = 0,\dots,n-1\,. \tag{3.20}$$

This equation is equivalent to the system

$$\begin{aligned} y_j' &= \frac{1}{a_j(t)}\,y_{j+1}\,, \quad j = 1,\dots,n-2\,,\\ y_{n-1}' &= \frac{1}{a_{n-1}(t)}t^{\alpha-1}y_n\,, \end{aligned} \tag{3.21}$$

$$y_n' = \frac{(1-\alpha)d_{n-1}}{t^\alpha} + \frac{1}{2\Gamma(\alpha)}r(t)|y_1|^\lambda \operatorname{sgn} y_1 > \frac{1}{2\Gamma(\alpha)}r(t)|y_1|^\lambda \operatorname{sgn} y_1$$

with

$$y_i = y^{[i-1]}\,,\ i=1,\dots,n-1\,,\quad y_n = t^{1-\alpha}y^{[n-1]}\,. \tag{3.22}$$

The solution y of (3.19) and (3.20), and the solution $\{y_i\}_{i=1}^n$ of (3.21) with the initial conditions

$$y_i(a) = d_{i-1}\,,\ i=1,\dots,n-1\,,\quad y_n(a) = a^{1-\alpha}d_{n-1} \tag{3.23}$$

satisfy (3.22). We apply Lemma 2.7(ii) to (3.21) and (3.23) with

$$y_0 = x_0 a^{1-\alpha}\,,\quad b_i(t) = a_i^{-1}(t)\,,\quad i=1,\dots,n-2\,,$$

$$b_{n-1}(t) = t^{\alpha-1}a_{n-1}^{-1}\,,\quad b_n(t) = \frac{1}{2\Gamma(\alpha)}r(t)\,,\quad \mu_i = -\lambda_i, i=1,\dots,n-2\,,$$

$$\mu_{n-1} = -\lambda_{n-1} - 1 + \alpha\,,\quad \mu_n = \lambda_n\,,\quad \delta = \min\Big(C_1^{-1},\dots,C_{n-1}^{-1},\frac{C_n}{2\Gamma(\alpha)}\Big)\,.$$

Note, by (3.17) and (3.18), condition (2.15) is valid. Now, Lemma 2.7(ii) implies the solutions of (3.21) and (3.23) and of (3.19) and (3.20) are noncontinuable. The rest of the proof is similar as the one of Theorem 3.3; only (3.17) has to be replaced by

$$x^{[n-1]}(t) \ge \dots \ge d_{n-1} + \frac{t^{\alpha-1}}{2\Gamma(\alpha)}\int_a^t r(s)y^{\lambda}(s)\,ds = y^{[n-1]}(t)\,.$$

If $\beta = -1$, the proof is similar.

(ii) The proof is similar, we use Lemma 2.7(iii) instead of Lemma 2.7(ii). □

4. Special case

Consider the special case of (1.1), (1.4) (for $n=2$)

$$\begin{gathered} {}^cD_a^{\alpha}\big(a_1(t)x'\big) = r(t)|x|^{\lambda}\operatorname{sgn} x\,,\\ x(a) = d_0\,,\quad x^{[1]}(a) = d_1\,, \end{gathered} \tag{4.1}$$

where $\lambda > 0$, $d_0 \in \mathbb{R}$, $d_1 \in \mathbb{R}$, $r \in C[a,\infty)$, $a_1 \in C[a,\infty)$ and $a_1(t) > 0$ for $t \ge a$.

Corollary 4.1.

(i) *If $\lambda \le 1$, then any solution of (4.1) is continuable.*

(ii) *Let $\lambda > 1$ and $r > 0$ on $[a,\infty)$. Then there exists $D > 0$ such that any solution of (4.1) satisfying $|d_0| \ge D$, $|d_1| \ge D$ and $d_0d_1 > 0$ is noncontinuable.*

(iii) *Let $\lambda > 1$, $C_1 > 0$, $C_2 > 0$, $\lambda_1 \in \mathbb{R}$, $\lambda_2 \in \mathbb{R}$, either $\lambda_2 > -1 + \lambda(\lambda_1 - \alpha)$ or $\lambda_1 \le \alpha$, $\lambda_2 \ge -1$, and let*

$$a_1(t) \le C_1 t^{\lambda_1}\,,\quad r(t) \ge C_2 t^{\lambda_2}\quad \text{for } t \ge a\,. \tag{4.2}$$

If $d_0d_1 > 0$, then any solution of (4.1) is noncontinuable.

(iv) *Let $\lambda > 1$, $r \in AC_{\rm loc}[a,\infty)$, and d_0, d_1 be such that*

$$k = |d_0| + |d_1|\int_a^{\infty} a_1^{-1}(s)\,ds < \infty$$

and

$$\frac{(\lambda-1)k^{\lambda-1}}{\alpha\Gamma(\alpha)}\int_a^{\infty}\frac{\bar r(s)}{a_1(s)}\,(s-a)^{\alpha}\,ds < 1\,. \tag{4.3}$$

Then any solution x of (4.1) is continuable.

Proof. In cases (i), (ii), (iii) and (iv), the proofs follow from Theorems 3.1, 3.3, 3.4 and 3.2, respectively. In Theorem 3.4 we put $x_0 = \frac{1}{2}\min(|d_0|a^{\alpha-1}, |d_1|)$. □

Note that cases (iii) and (iv) of Corollary 4.1 are not in a contradiction. Let (4.2) be valid. If (iii) holds, then $\lambda_2 > -1 + \lambda(\lambda_1 - \alpha)$ is supposed. If (iv) is valid, then according to (4.3) we have $\lambda_2 < -1 + \lambda_1 - \alpha$. So, the relationships between λ_1 and λ_2 are different in these two cases.

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