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FAST HOMOCLINIC SOLUTIONS FOR DAMPED VIBRATION SYSTEMS WITH SUBQUADRATIC AND ASYMPTOTICALLY QUADRATIC POTENTIALS

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Abstract. In this article, we study the nonperiodic damped vibration problem

$$\ddot{u}(t) + q(t)\dot{u}(t) - L(t)u(t) + \nabla W(t, u(t)) = 0,$$

where $L(t)$ is uniformly positive definite for all $t \in \mathbb{R}$, and $W(t, x)$ is either subquadratic or asymptotically quadratic in x as $|x| \to \infty$. Based on the minimax method in critical point theory, we prove the existence and multiplicity of fast homoclinic solutions for the above problem.

1. Introduction and statement of main results

This article concerns the existence and multiplicity of homoclinic orbits for the damped vibration problem

$$\ddot{u}(t) + q(t)\dot{u}(t) - L(t)u(t) + \nabla W(t, u(t)) = 0, \tag{1.1}$$

where $t \in \mathbb{R}$, $u = (u_1, u_2, \dots, u_N) \in \mathbb{R}^N$, $q \in C(\mathbb{R}, \mathbb{R})$, $L \in C(\mathbb{R}, \mathbb{R}^{N^2})$ is a symmetric matrix for all $t \in \mathbb{R}$, $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ and $\nabla W(t, x)$ denotes the gradient of W with respect to x. As usual, we say that a solution u of problem (1.1) is homoclinic to 0 if $u(t) \to 0$ as $|t| \to \infty$. Furthermore, if $u \neq 0$, then u is called a nontrivial homoclinic solution.

Homoclinic orbits play an important role in the study of qualitative behavior of dynamical systems. They may be "organizing centers" for the dynamics in their neighborhood. Under certain conditions, their existence may imply the existence of chaos nearby or the bifurcation behavior of periodic orbits. Such orbits have been studied since the time of Poincaré, but mainly by perturbation techniques. During the last twenty more years, critical point theory and variational methods have been widely used in homoclinic motions.

If $q(t) \equiv 0$, problem (1.1) reduces to the second order Hamiltonian system

$$\ddot{u}(t) - L(t)u(t) + \nabla W(t, u(t)) = 0, \quad t \in \mathbb{R}. \tag{1.2}$$

There are many papers devoted to the study on the existence and multiplicity of homoclinic orbits of system (1.2) under various hypotheses on the nonlinearity, see,

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for example, [3, 6, 7, 8, 10, 12, 13, 15, 16, 17, 18, 19, 21, 22, 23] and the references therein. If $q(t) \neq 0$, only a few results are known for problem (1.1). Damped vibration problems with constant coefficient of the form

$$\ddot{u}(t) + A\dot{u}(t) - L(t)u(t) + \nabla W(t, u(t)) = 0, \quad \forall t \in \mathbb{R}, \tag{1.3}$$

have been studied by Zhang and Ruan [24], where the authors prove the existence of one nontrivial homoclinic orbit under the following conditions:

(A1) $L \in C(\mathbb{R}, \mathbb{R}^{N^2})$ is a symmetric and positive definite matrix for all $t \in \mathbb{R}$ and there exists a function $\alpha \in C(\mathbb{R}, (0, \infty))$ such that $\alpha(t) \to +\infty$ as $|t| \to \infty$ and

$$(L(t)x, x) \geq \alpha(t)|x|^2, \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N;$$

(A2) $W(t, x) = a(t)|x|^\gamma$, where $1 < \gamma < 2$ is a constant and $a \in C(\mathbb{R}, \mathbb{R}) \cap L^{\frac{2}{2-\gamma}}(\mathbb{R}, \mathbb{R})$ such that $a(t_0) > 0$ for some $t_0 \in \mathbb{R}$.

Zhu [26] assume that $L \in C(\mathbb{R}, \mathbb{R}^{N \times N})$ is T-periodic with $T > 0$, $W(t, x) \geq 0$ and there exist a symmetry T-periodic matrix valued function $M \in C(\mathbb{R}, \mathbb{R}^{N \times N})$ such that $|\nabla W(t, x) - M(t)x|/|x| \to 0$ as $|x| \to \infty$ and prove the existence of one homoclinic orbit of problem (1.3). The proof is based on a version of mountain pass theorem and the concentration-compactness principle. See also [20, 24] for the related results. Inspired by [1], Zhang and Yuan [25] introduce the concept of fast homoclinic orbits and investigate the existence of fast homoclinic orbits of a special case of problem (1.1),

$$\ddot{u}(t) + c\dot{u}(t) - L(t)u(t) + \nabla W(t, u(t)) = 0, \quad t \in \mathbb{R}, \tag{1.4}$$

where $c > 0$ is a constant. Precisely, they make the following assumptions:

(A1') $L \in C(\mathbb{R}, \mathbb{R}^{N^2})$ is a positive difinite symmetric matrix for all $t \in \mathbb{R}$ and there is $K_1 > -c^2/4$ such that

$$(L(t)x, x) \geq K_1|x|^2, \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N;$$

[(A2') $W(t, x) = a(t)|x|^\gamma$, where $1 < \gamma < 2$ is a constant and $a \in L^{2/2-\gamma}(e^{ct})$ such that $a(t_0) > 0$ for some $t_0 \in \mathbb{R}$.

Theorem 1.1 (see [25])**.** *Assume that $L(t)$ satisfies* (A1) *or* (A1') *and $W(t, x)$ satisfies* (A2')*. Then problem* (1.4) *has at least one nontrivial fast homoclinic solution.*

This result has been extended in [4, 5] to more general situations. Particularly, Chen and Tang [5] constructed the existence and multiplicity of fast homoclinic orbits of problem (1.1) under the following assumptions:

(A3) $L \in C(\mathbb{R}, \mathbb{R}^{N^2})$ is a positive definite symmetric matrix of all $t \in \mathbb{R}$, and there exists $\beta > 0$ such that

$$(L(t)x, x) \geq \beta|x|^2, \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N.$$

(A4) There exist two constants $1 < \gamma_1 < \gamma_2 < 2$ and two functions $a_1 \in L^{\frac{2}{2-\gamma_1}}(e^{Q(t)})$, $a_2 \in L^{\frac{2}{2-\gamma_2}}(e^{Q(t)})$ such that

$$|W(t, x)| \leq a_1(t)|x|^{\gamma_1}, \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N, \ |x| \leq 1,$$
$$|W(t, x)| \leq a_2(t)|x|^{\gamma_2}, \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N, \ |x| \geq 1.$$

(A5) There are two functions $b \in L^{\frac{2}{2-\gamma_1}}(e^{Q(t)})$ and $\varphi \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that

$$|\nabla W(t,x)| \le b(t)\varphi(|x|), \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N,$$

where $\varphi(s) = O(s^{\gamma_1 - 1})$ as $s \to 0^+$.

(A6) There is an open set $J \subset \mathbb{R}$ and two constants $\gamma_3 \in (1,2)$ and $\eta > 0$ such that

$$W(t,x) \ge \eta |x|^{\gamma_3}, \quad \forall (t,x) \in J \times \mathbb{R}^N, \ |x| \le 1.$$

Theorem 1.2 (see [5, Theorems 1.3 and 1.4]). *Assume that conditions* (A3)–(A6) *hold and $Q(t) := \int_0^t q(s)ds$ satisfies*

$$Q(t) \to +\infty \quad as\ |t| \to \infty. \tag{1.5}$$

Then (1.1) *has at least one nontrivial fast homoclinic solution. If moreover $W(t,x)$ is even in x, then* (1.1) *has infinitely many nontrivial fast homoclinic solutions.*

Motivated by [1, 4, 5, 25], in this article, we try to obtain new existence and multiplicity results of system (1.1) by imposing general subquadratic conditions on the potential W. Furthermore, we consider the situation where W is asymptotically quadratic as $|x| \to \infty$, and also establish the existence and multiplicity.

Before stating our main results, we describe some properties of the weighted Sobolev space E on which the variational functional associated to problem (1.1) will be defined. Let

$$E := \Big\{ u \in H^1(\mathbb{R}, \mathbb{R}^N) : \int_{\mathbb{R}} e^{Q(t)} [|\dot{u}(t)|^2 + (L(t)u(t), u(t))] dt < +\infty \Big\},$$

where $Q(t)$ is given in (1.5). Then E is a Hilbert space with the inner product and norm

$$(u,v) = \int_{\mathbb{R}} e^{Q(t)} [(\dot{u}(t), \dot{v}(t)) + (L(t)u(t), v(t))] dt, \quad \|u\| = (u,u)^{1/2}.$$

Define

$$L^p(e^{Q(t)}) := \{ u : \mathbb{R} \to \mathbb{R}^N \text{is Lebesgue measurable}, \quad \|u\|_p < +\infty \},$$

where

$$\|u\|_p := \Big(\int_{\mathbb{R}} e^{Q(t)} |u(t)|^p dt \Big)^{1/p}, \quad 2 \le p < +\infty.$$

Clearly, under (A3), the embedding of $E \hookrightarrow L^2(e^{Q(t)})$ is continuous, and hence there exists $\tau > 0$ such that

$$\|u\|_2 \le \tau \|u\|, \quad \forall u \in E. \tag{1.6}$$

Definition 1.3. If (1.5) holds, then a solution u ($\in E$) of problem (1.1) is called a fast homoclinic solution.

We use the following hypotheses:

(A7) There exist constants σ, $\gamma \in (1,2)$ and functions $m \in L^{\frac{2}{2-\sigma}}(e^{Q(t)})$, $h \in L^{\frac{2}{2-\gamma}}(e^{Q(t)})$ such that

$$|\nabla W(t,x)| \le m(t)|x|^{\sigma-1} + h(t)|x|^{\gamma-1}, \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N.$$

(A8) There exist $t_0 \in \mathbb{R}$, two sequences $\{\delta_n\}$, $\{M_n\}$ and constants a, δ, $d > 0$ such that $\delta_n > 0$, $M_n > 0$ and
$$\lim_{n\to\infty} \delta_n = 0, \quad \lim_{n\to\infty} M_n = +\infty,$$
$$|x|^{-2}W(t,x) \geq M_n \quad \text{for } |t-t_0| \leq d \text{ and } |x| = \delta_n,$$
$$|x|^{-2}W(t,x) \geq -a \quad \text{for } |t-t_0| \leq d \text{ and } |x| \leq \delta.$$

Theorem 1.4 (Subquadratic case)**.** *Assume that* (A3), (A7), (A8) *hold and* $q(t)$ *satisfies* (1.5)*. Then* (1.1) *possesses at least one nontrivial fast homoclinic solution. If moreover* $W(t,x)$ *is even in* x*, then* (1.1) *possesses infinitely many nontrivial fast homoclinic solutions.*

Remark 1.5. Comparing Theorem 1.4 with Theorem 1.2, our condition (A8) is much weaker than (A6), since (A6) implies $\lim_{|x|\to 0} \frac{W(t,x)}{|x|^2} = +\infty$ uniformly for $t \in J$.

Next we study the asymptotically quadratic problem. Let $A := -d^2/dt^2 - q(t)d/dt + L(t)$,
$$\widetilde{W}(t,x) = \frac{1}{2}(\nabla W(t,x), x) - W(t,x), \quad \forall (t,x) \in \mathbb{R}\times\mathbb{R}^N,$$
and denote the spectrum of A in $L^2(\mathbb{R},\mathbb{R}^N)$ by $\sigma(A)$. We assume the following:

(A9) $W(t,x) \geq 0$ for all (t,x) and $\nabla W(t,x) = o(|x|)$ as $x \to 0$ uniformly in t.

(A10) $\nabla W(t,x) = L_\infty(t)x + \nabla R(t,x)$, where $L_\infty(t)$ is a bounded continuous $N\times N$ matrix-valued function and $\nabla R(t,x) = o(|x|)$ uniformly in t as $|x| \to \infty$.

(A11) $l_0 := \inf_{t\in\mathbb{R},|x|=1}(L_\infty(t)x, x) > \inf \sigma(A)$.

(A12) $\gamma < \beta$, where $\gamma := \sup_{|t|\geq t_0, x\neq 0} |\nabla W(t,x)|/|x|$ for some $t_0 > 0$.

(A13) Either (i) $0 \notin \sigma(A - L_\infty)$, or (ii) $\widetilde{W}(t,x) \geq 0$ for all (t,x) and $\widetilde{W}(t,x) \geq \delta_0$ for some $\delta_0 > 0$ and all (t,x) with $|x|$ sufficiently large.

From conditions (A9)–(A13) we infer that for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that
$$|\nabla W(t,x)| \leq \varepsilon|x| + C_\varepsilon |x|^{p-1}, \tag{1.7}$$
$$|W(t,x)| \leq \varepsilon|x|^2 + C_\varepsilon |x|^p \tag{1.8}$$
for all $(t,x) \in \mathbb{R}\times\mathbb{R}^N$, where $p \geq 2$. Let m denote the number of eigenfunctions with corresponding eigenvalues of A lying in $(0, l_0)$. We have the following theorem.

Theorem 1.6 (Asymptotically quadratic case)**.** *Assume (A3), (A9)–(A13) hold and* $q(t)$ *satisfies* (1.5)*. Then* (1.1) *possesses at least one nontrivial fast homoclinic solution. Moreover, if* $W(t,x)$ *is even in* x*, then* (1.1) *possesses at least* m *pairs of nontrivial fast homoclinic solutions.*

The rest of this article is organized as follows. In Section 2 we introduce some preliminary results and prove Theorem 1.4. Section 3 is concerned with the asymptotically quadratic case and the proof of Theorem 1.6 is complete. Finally, we give two typical examples to illustrate our results.

Throughout this article, we denote by c and c_i ($i = 1, 2, \dots$) various positive constants, which may vary from line to line. "$\to$" (resp. "$\rightharpoonup$") denotes the strong (resp. weak) convergence.

2. Proof of Theorem 1.4

Lemma 2.1. *If $u \in E$, then*

$$\|u\|_\infty \le \frac{1}{\sqrt{2e_0\sqrt{\beta}}}\|u\| = \frac{1}{\sqrt{2e_0\sqrt{\beta}}}\Big(\int_{\mathbb{R}} e^{Q(t)}[|\dot{u}(t)|^2 + (L(t)u(t), u(t))]dt\Big)^{1/2},$$

where $\|u\|_\infty = \mathrm{ess}, \sup_{t\in\mathbb{R}} |u(t)|$ *and* $e_0 = e^{\min\{Q(t):t\in\mathbb{R}\}}$.

Proof. Fix $t \in \mathbb{R}$. For each $k \in \mathbb{N}$, we have

$$|u(t)|^2 = \int_k^t 2(\dot{u}(s), u(s))ds + |u(k)|^2, \quad |u(t)|^2 = \int_{-k}^t 2(\dot{u}(s), u(s))ds + |u(-k)|^2.$$

Thus

$$\begin{aligned} 2|u(t)|^2 &\le \int_k^t 2(\dot{u}(s), u(s))ds + \int_{-k}^t 2(\dot{u}(s), u(s))ds + |u(k)|^2 + |u(-k)|^2 \\ &\le \int_{-k}^k 2|\dot{u}(s)||u(s)|ds + |u(k)|^2 + |u(-k)|^2. \end{aligned}$$

Letting $k \to \infty$, we obtain

$$\begin{aligned} |u(t)|^2 &\le \int_{\mathbb{R}} |\dot{u}(s)||u(s)|ds \\ &\le \frac{1}{2\sqrt{\beta}}\int_{\mathbb{R}} \left(|\dot{u}(s)|^2 + \beta|u(s)|^2\right) ds \\ &\le \frac{1}{2\sqrt{\beta}}\int_{\mathbb{R}} \left[|\dot{u}(s)|^2 + (L(s)u(s), u(s))\right] ds \\ &\le \frac{1}{2e_0\sqrt{\beta}}\int_{\mathbb{R}} e^{Q(t)} \left[|\dot{u}(s)|^2 + (L(s)u(s), u(s))\right] ds \end{aligned}$$

for every $t \in \mathbb{R}$, where $e_0 = e^{\min\{Q(t):t\in\mathbb{R}\}}$. This completes the proof. □

We remark that the above lemma was stated in [5] without proof. We include its proof here for the readers' convenience.

Lemma 2.2. *Suppose that* (A7) *is satisfied and* $u_n \rightharpoonup u$ *in* E. *Then*

$$\nabla W(t, u_n) \to \nabla W(t, u) \quad \text{in } L^2(e^{Q(t)}). \tag{2.1}$$

Proof. By the properties of the functions m and h, we have that for every $\varepsilon > 0$, there exists $T_\varepsilon > 0$ such that

$$\Big(\int_{|t|\ge T_\varepsilon} e^{Q(t)}|m(t)|^{\frac{2}{2-\sigma}}dt\Big)^{\frac{2-\sigma}{2}} < \sqrt{\varepsilon}, \quad \Big(\int_{|t|\ge T_\varepsilon} e^{Q(t)}|h(t)|^{\frac{2}{2-\gamma}}dt\Big)^{\frac{2-\gamma}{2}} < \sqrt{\varepsilon}.$$

Hence, using (A7), the boundedness of (u_n) and Hölder's inequality, we obtain

$$
\begin{aligned}
&\int_{|t|\ge T_\varepsilon} e^{Q(t)}|\nabla W(t,u_n)-\nabla W(t,u)|^2dt\\
&\le \int_{|t|\ge T_\varepsilon} e^{Q(t)}[m(t)(|u_n|^{\sigma-1}+|u|^{\sigma-1})+h(t)(|u_n|^{\gamma-1}+|u|^{\gamma-1})]^2dt\\
&\le 4\int_{|t|\ge T_\varepsilon} e^{Q(t)}m^2(t)(|u_n|^{2\sigma-2}+|u|^{2\sigma-2})dt\\
&\quad +4\int_{|t|\ge T_\varepsilon} e^{Q(t)}h^2(t)(|u_n|^{2\gamma-2}+|u|^{2\gamma-2})dt \qquad (2.2)\\
&\le 4\Big(\int_{|t|\ge T_\varepsilon} e^{Q(t)}|m(t)|^{\frac{2}{2-\sigma}}dt\Big)^{2-\sigma}(\|u_n\|_2^{2\sigma-2}+\|u\|_2^{2\sigma-2})\\
&\quad +4\Big(\int_{|t|\ge T_\varepsilon} e^{Q(t)}|h(t)|^{\frac{2}{2-\gamma}}dt\Big)^{2-\gamma}(\|u_n\|_2^{2\gamma-2}+\|u\|_2^{2\gamma-2})\\
&\le c\varepsilon.
\end{aligned}
$$

It follows from the boundedness of (u_n), Lemma 2.1 and the dominated convergence theorem that

$$\int_{|t|\le T_\varepsilon} e^{Q(t)}|\nabla W(t,u_n)-\nabla W(t,u)|^2dt\to 0 \quad \text{as } n\to\infty,$$

which, together with (2.2), shows that (2.1) holds. □

Consider the functional φ defined on $(E,\|\cdot\|)$ by

$$\varphi(u)=\frac{1}{2}\|u\|^2-\int_{\mathbb{R}} e^{Q(t)}W(t,u(t))dt.$$

It follows from (A7), (1.6) and the Hölder inequality that

$$
\begin{aligned}
\int_{\mathbb{R}} e^{Q(t)}W(t,u)dt &\le \int_{\mathbb{R}} e^{Q(t)}(m(t)|u|^\sigma+h(t)|u|^\gamma)dt\\
&\le \Big(\int_{\mathbb{R}} e^{Q(t)}|m(t)|^{\frac{2}{2-\sigma}}dt\Big)^{\frac{2-\sigma}{2}}\Big(\int_{\mathbb{R}} e^{Q(t)}|u(t)|^2dt\Big)^{\sigma/2}\\
&\quad +\Big(\int_{\mathbb{R}} e^{Q(t)}|h(t)|^{\frac{2}{2-\gamma}}dt\Big)^{\frac{2-\gamma}{2}}\Big(\int_{\mathbb{R}} e^{Q(t)}|u(t)|^2dt\Big)^{\gamma/2} \qquad (2.3)\\
&\le \tau^\sigma\|m\|_{\frac{2}{2-\sigma}}\|u\|^\sigma+\tau^\gamma\|h\|_{\frac{2}{2-\gamma}}\|u\|^\gamma.
\end{aligned}
$$

Hence φ is well defined. In addition, we have the following lemma.

Lemma 2.3. *Let* (A3) *and* (A7) *be satisfied. Then* $\varphi\in C^1(E,\mathbb{R})$ *and*

$$\langle\varphi'(u),v\rangle=\int_{\mathbb{R}} e^{Q(t)}[(\dot u(t),\dot v(t))+(L(t)u(t),v(t))-(\nabla W(t,u(t)),v(t))]dt$$

for all u, $v\in E$. *The critical point* u *of* φ *is a classical solution of problem* (1.1) *with* $u(\pm\infty)=0$.

Proof. In view of Lemma 2.2 and (1.6), the proof is standard and we refer to [14]. □

We shall use the following two propositions to prove Theorem 1.4.

Proposition 2.4 (see [13]). *Let E be a real Banach space and $\Phi \in C^1(E, \mathbb{R})$ satisfy the (PS) condition, i.e., $(u_n) \subset E$ has a convergent subsequence whenever $\{\Phi(u_n)\}$ is bounded and $\Phi'(u_n) \to 0$ as $n \to \infty$. If Φ is bounded from below, then $c^* = \inf_E \Phi$ is a critical value of Φ.*

To prove the existence of infinitely many homoclinic orbits, we require the new version of symmetric mountain pass lemma by Kajikiya (see [11]). Let E be a Banach space and

$$\Gamma := \{A \subset E \backslash \{0\} : A \text{ is closed and symmetric with respect to the origin}\}.$$

We define $\Gamma_k := \{A \in \Gamma : \gamma(A) \geq k\}$, where

$$\gamma(A) := \inf \big\{ m \in \mathbb{N} : \exists h \in C(A, \mathbb{R}^m \backslash \{0\}), -h(x) = h(-x) \big\}.$$

If there is no such mapping h for any $m \in \mathbb{N}$, we set $\gamma(A) = +\infty$.

Proposition 2.5 (Symmetric mountain pass lemma). *Let E be an infinite dimensional Banach space and $\Phi \in C^1(E, \mathbb{R})$ be even, $\Phi(0) = 0$ and satisfies the following conditions:*

(i) *Φ is bounded from below and satisfies the Palais-Smale condition (PS).*
(ii) *For each $k \in \mathbb{N}$, there exists an $A_k \in \Gamma_k$ such that $\sup_{u \in A_k} \Phi(u) < 0$.*

Then either one of the following two conditions holds:

(1) *There exists a sequence $\{u_k\}$ such that $\Phi'(u_k) = 0$, $\Phi(u_k) < 0$ and $\{u_k\}$ converges to zero.*
(2) *There exist two sequence $\{u_k\}$ and $\{v_k\}$ such that $\Phi'(u_k) = 0$, $\Phi(u_k) = 0$, $u_k \neq 0$, $\lim_{k\to\infty} u_k = 0$, $\Phi'(v_k) = 0$, $\Phi(v_k) < 0$, $\lim_{k\to\infty} \Phi(v_k) = 0$ and $\{v_k\}$ converges to a non-zero limit.*

Remark 2.6. From Proposition 2.5, we deduce a sequence $\{u_k\}$ of critical points such that $I(u_k) \leq 0$, $u_k \neq 0$ and $\lim_{k\to\infty} u_k = 0$.

Lemma 2.7. *Let* (A3) *and* (A7) *hold. Then φ is bounded from below and satisfies the (PS) condition.*

Proof. By (2.3), we obtain

$$\varphi(u) \geq \frac{1}{2}\|u\|^2 - (\tau^\sigma \|m\|_{\frac{2}{2-\sigma}} \|u\|^\sigma + \tau^\gamma \|h\|_{\frac{2}{2-\gamma}} \|u\|^\gamma)$$

for all $u \in E$. Since σ, $\gamma \in (1,2)$, it follows that

$$\varphi(u) \to +\infty \quad \text{as } \|u\| \to \infty. \tag{2.4}$$

Hence φ is bounded from below.

Let $(u_n) \subset E$ be a (PS)-sequence of φ. From (2.4), we know that (u_n) is bounded, and then, passing to a subsequence, $u_n \rightharpoonup u$ in E for some $u \in E$. By Lemma 2.2, we have

$$\begin{aligned}
&\|u_n - u\|^2 \\
&= \langle \varphi'(u_n) - \varphi'(u), u_n - u\rangle + \int_{\mathbb{R}} e^{Q(t)} (\nabla W(t, u_n) - \nabla W(t, u), u_n - u) dt \\
&\leq \|\varphi'(u_n)\|_{E^*} \|u_n - u\| - \langle \varphi'(u), u_n - u\rangle \\
&\quad + \Big(\int_{\mathbb{R}} e^{Q(t)} |\nabla W(t, u_n) - \nabla W(t, u)|^2 dt \Big)^{1/2} \Big(\int_{\mathbb{R}} e^{Q(t)} |u_n - u|^2 dt \Big)^{1/2} \to 0
\end{aligned}$$

as $n \to \infty$. Therefore the (PS) condition holds. $\square$

Proof of Theorem 1.4. (Existence) Lemmas 2.3 and 2.7 enable us to apply Proposition 2.4 to φ to get a critical point u^* such that $\varphi(u^*) = \inf_E \varphi$. Now we show $u^* \neq 0$. Without loss of generality, we set $t_0 = 0$. Choose $u_0 \in (W_0^{1,2}(J) \cap E)\backslash\{0\}$ with $\|u_0\|_\infty \leq 1$ and $|u_0(t)| = 1$ for $t \in [-d/2, d/2]$, where $J = [-d, d]$. Hence, for $\zeta \in (0, \delta)$ (δ is given by (A8)), we obtain

$$\begin{aligned}
\varphi(\zeta u_0) &= \frac{\zeta^2}{2}\|u_0\|^2 - \int_{\mathbb{R}} e^{Q(t)} W(t, \zeta u_0) dt \\
&= \frac{\zeta^2}{2}\|u_0\|^2 - \int_J e^{Q(t)} W(t, \zeta u_0) dt \\
&\leq \frac{\zeta^2}{2}\|u_0\|^2 - \int_{-d/2}^{d/2} e^{Q(t)} W(t, \zeta u_0) dt + a\zeta^2 \int_{J\backslash[-d/2,d/2]} e^{Q(t)} dt
\end{aligned}$$

by (A8). Substituting $\zeta = \delta_n$ and noting that $|\delta_n u_0(t)| = \delta_n$ for $t \in [-d/2, d/2]$, we obtain

$$\varphi(\delta_n u_0) \leq \delta_n^2 \Big(\frac{\|u_0\|^2}{2} + a \int_{J\backslash[-d/2,d/2]} e^{Q(t)} dt - M_n \int_{-d/2}^{d/2} e^{Q(t)} dt \Big).$$

Since $\delta_n \to 0$ and $M_n \to +\infty$, we can choose $n_0 > 0$ large enough such that the right side of the above inequality is negative. So

$$\varphi(u^*) \leq \varphi(\delta_{n_0} u_0) < 0,$$

which implies that $u^* \neq 0$. Hence u^* is a nontrivial homoclinic solution of problem (1.1).

(Multiplicity) From Lemmas 2.3, 2.7 and the evenness of W, we know that $\varphi \in C^1(E, \mathbb{R})$, satisfies the condition (i) of Proposition 2.5 and $\varphi(-u) = \varphi(u)$. It remains to verify that condition (ii) of Proposition 2.5 is satisfied. We adapt an argument in [11].

For simplicity, we assume that $t_0 = 0$ in (A8). For arbitrary $k \in \mathbb{N}$, we shall construct an $A_k \in \Gamma_k$ satisfying $\sup_{u \in A_k} \varphi(u) < 0$. Divide $[-d, d]$ equally into k closed subintervals and denote them by I_i with $1 \leq i \leq k$. Setting $a = 2d/k$, then the length of each I_i is a. For $1 \leq i \leq k$, let t_i be the center of I_i and J_i be the closed interval centered at t_i with length $a/2$. Choose a function $\xi \in C_0^\infty(\mathbb{R}, \mathbb{R}^N)$ such that $|\xi(t)| \equiv 1$ for $t \in [-a/4, a/4]$, $\xi(t) \equiv 0$ for $t \in \mathbb{R}\backslash[-a/2, a/2]$ and $|\xi(t)| \leq 1$ for $t \in \mathbb{R}$. Now for each $1 \leq i \leq k$, define $\xi_i \in C_0^\infty(\mathbb{R}, \mathbb{R}^N)$ by

$$\xi_i(t) = \xi(t - t_i), \quad t \in \mathbb{R}.$$

We see that

$$\operatorname{supp} \xi_i \subset I_i, \quad \operatorname{supp} \xi_i \cap \operatorname{supp} \xi_j = \emptyset \quad (i \neq j), \tag{2.5}$$

$|\xi_i(t)| = 1$ ($t \in J_i$), and $|\xi_i(t)| \leq 1$ ($t \in \mathbb{R}$). Let

$$V_k = \big\{ (s_1, s_2, \dots, s_k) \in \mathbb{R}^k : \max_{1 \leq i \leq k} |s_i| = 1 \big\}, \tag{2.6}$$

$$W_k = \big\{ \sum_{i=1}^k s_i \xi_i(t) : (s_1, s_2, \dots, s_k) \in V_k \big\}. \tag{2.7}$$

Since V_k is homeomorphic to the unit sphere in $\mathbb{R}^k$ by an odd mapping, we obtain $\gamma(V_k) = k$. Besides, $\gamma(W_k) = \gamma(V_k) = k$ because the mapping $(s_1, s_2, \dots, s_k) \mapsto$

$\sum_{i=1}^k s_i\xi_i(t)$ is odd and homeomorphic. Noting W_k is compact, there exists $C_k > 0$ such that

$$\|u\| \le C_k, \quad \forall u \in W_k. \tag{2.8}$$

For $0 < \zeta < \delta$ (δ is the constant given in (A8)) and $u = \sum_{i=1}^k s_i\xi_i(t) \in W_k$, we obtain

$$\begin{aligned}\varphi(\zeta u) &= \frac{1}{2}\|\zeta u\|^2 - \int_{\mathbb{R}} e^{Q(t)} W\Big(t, \zeta\sum_{i=1}^k s_i\xi_i(t)\Big)dt \\ &\le \frac{\zeta^2}{2}C_k^2 - \sum_{i=1}^k \int_{I_i} e^{Q(t)} W(t, \zeta s_i\xi_i(t))dt\end{aligned}$$

by (2.8) and (2.5). Noting (2.6), there exists an integer $i_0 \in [1, k]$ such that $|s_{i_0}| = 1$. Then it follows that

$$\begin{aligned}&\sum_{i=1}^k \int_{I_i} e^{Q(t)} W(t, \zeta s_i\xi_i(t))dt \\ &= \int_{J_{i_0}} e^{Q(t)} W(t, \zeta s_{i_0}\xi_{i_0}(t))dt + \int_{I_{i_0}\setminus J_{i_0}} e^{Q(t)} W(t, \zeta s_{i_0}\xi_{i_0}(t))dt \\ &\quad + \sum_{i\ne i_0} \int_{I_i} e^{Q(t)} W(t, \zeta s_i\xi_i(t))dt.\end{aligned} \tag{2.9}$$

By (A8), one has

$$\int_{I_{i_0}\setminus J_{i_0}} e^{Q(t)} W(t, \zeta s_{i_0}\xi_{i_0}(t))dt + \sum_{i\ne i_0} \int_{I_i} e^{Q(t)} W(t, \zeta s_i\xi_i(t))dt \ge -a\zeta^2 \int_{-d}^{d} e^{Q(t)}dt.$$

Combining this with (2.9), (A8) and the fact $|\zeta s_{i_0}\xi_{i_0}(t)| = \zeta$ for $t \in J_{i_0}$, we have

$$\begin{aligned}\varphi(\delta_n u) &\le \frac{\delta_n^2}{2}C_k^2 + a\delta_n^2 \int_{-d}^{d} e^{Q(t)}dt - \int_{J_{i_0}} e^{Q(t)} W(t, \delta_n s_{i_0}\xi_{i_0}(t))dt \\ &\le \delta_n^2\Big(\frac{C_k^2}{2} + a\int_{-d}^{d} e^{Q(t)}dt - M_n \int_{J_{i_0}} e^{Q(t)}dt\Big).\end{aligned}$$

Since $\delta_n \to 0$ and $M_n \to +\infty$ as $n \to \infty$, we can choose $n_1 > 0$ large enough such that the right side of the last inequality is negative. Take

$$A_k = \delta_{n_1} W_k.$$

Then we have $\gamma(A_k) = \gamma(W_k) = k$ and $\sup_{u\in A_k} \varphi(u) < 0$. Consequently, by Proposition 2.5, problem (1.1) has infinitely many nontrivial homoclinic solutions. This completes the proof. □

3. Proof of Theorem 1.6

It follows from (A3) and (1.7) that the functional $\varphi : E \to \mathbb{R}$ given by

$$\varphi(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} e^{Q(t)} W(t, u)dt$$

is of class C^1, and

$$\langle \varphi'(u), v\rangle = \int_{\mathbb{R}} e^{Q(t)}[(\dot u(t), \dot v(t)) + (L(t)u(t), v(t)) - (\nabla W(t, u(t)), v(t))]dt \tag{3.1}$$

for all u, $v \in E$. The critical point u of φ is a classical solution of (1.1) with $u(\pm\infty) = 0$.

We shall apply the following two propositions to prove Theorem 1.6. The first one is Rabinowitz's mountain pass theorem which can be found in [14], and the second is a result by Bartolo et al. [2, Theorem 2.4]. In the linking theorem it is usually required that the functional Φ satisfies the stronger Palais-Smale condition. Nevertheless, the Cerami condition is sufficient for the deformation lemma, and hence for the linking theorem to hold (see [2]).

Proposition 3.1 (see [14]). *Let E be a real Banach space and $\Phi \in C^1(E, \mathbb{R})$ with $\Phi(0) = 0$. Suppose that*

(i) *there are constants ρ, $\alpha > 0$ such that $\Phi|_{\partial B_\rho} \geq \alpha$;*
(ii) *there is an $e \in E \backslash B_\rho$ such that $\Phi(e) < 0$;*
(iii) *Φ satisfies the (C) condition, i.e., $(u_n) \subset E$ has a convergent subsequence whenever $\{\Phi(u_n)\}$ is bounded and $(1 + \|u_n\|)\|\Phi'(u_n)\| \to 0$ as $n \to \infty$.*

Then Φ has a critical value $c \geq \alpha$ given by

$$c = \inf_{g \in \Gamma} \max_{s \in [0,1]} \Phi(g(s)),$$

where $\Gamma = \{g \in C([0, 1], E) : g(0) = 0, \quad g(1) = e\}$.

Proposition 3.2 (see [2]). *Suppose that $\Phi \in C^1(E, \mathbb{R})$ is even, $\Phi(0) = 0$ and there exist closed subspaces E_1, E_2 such that $\operatorname{codim} E_1 < \infty$, $\inf \Phi(E_1 \cap S_\rho) \geq \alpha$ for some α, $\rho > 0$ and $\sup \Phi(E_2) < +\infty$. If Φ satisfies the $(C)_c$ condition for all $c \in [\alpha, \sup \Phi(E_2)]$, then Φ has at least $dimE_2 - codimE_1$ pairs of critical points with corresponding critical values in $[\alpha, \sup \Phi(E_2)]$.*

Next we give Lemmas 3.3 and 3.4 which ensure that the functional φ satisfies the (C) condition.

Lemma 3.3. *Assume that conditions* (A9)-(A12) *are satisfied. Then any bounded (C) sequences of φ has a strongly convergent subsequence in E.*

Proof. Let $(u_n) \subset E$ be a bounded sequence such that $\{\varphi(u_n)\}$ is bounded and $(1 + \|u_n\|)\|\varphi'(u_n)\| \to 0$ as $n \to \infty$. We claim that for any $\varepsilon > 0$, there exists $R_\varepsilon > 2t_0$ (the constant t_0 appears in (A12)) and $n_\varepsilon > 0$ such that

$$\int_{|t| \geq R} e^{Q(t)}[|\dot{u}_n|^2 + (L(t)u_n, u_n)]dt < \varepsilon \quad \text{for all } R \geq R_\varepsilon \text{ and } n \geq n_\varepsilon. \tag{3.2}$$

Indeed, choose $\xi_R \in C^\infty(\mathbb{R}, [0, 1])$ such that

$$\xi_R(t) = \begin{cases} 0, & |t| \leq R/2, \\ 1, & |t| \geq R, \end{cases} \tag{3.3}$$

and there exists c_0 independent of R such that

$$|\frac{d}{dt}\xi_R(t)| \leq \frac{c_0}{R}, \quad \forall t \in \mathbb{R}. \tag{3.4}$$

Since $\varphi'(u_n) \to 0$ and $(u_n) \subset E$ is bounded, we obtain, for any $\varepsilon > 0$, there exists $n_\varepsilon > 0$ such that

$$\begin{aligned}\frac{\varepsilon}{2} &\geq \langle \varphi'(u_n), \xi_R u_n \rangle \\ &= \int_{\mathbb{R}} e^{Q(t)}[|\dot{u}_n|^2 + (L(t)u_n, u_n)]\xi_R dt + \int_{\mathbb{R}} e^{Q(t)} \dot{\xi}_R(\dot{u}_n, u_n) dt \\ &\quad - \int_{\mathbb{R}} e^{Q(t)}(\nabla W(t, u_n), u_n)\xi_R \, dt\end{aligned} \tag{3.5}$$

for $n \geq n_\varepsilon$. It follows from (3.3), (3.4) and (A12) that there exists $R_\varepsilon \geq 2t_0$ such that

$$\begin{aligned}\int_{\mathbb{R}} e^{Q(t)}|\dot{\xi}_R(\dot{u}_n, u_n)| dt &\leq \frac{c_0}{R}\int_{\mathbb{R}} e^{Q(t)}(|\dot{u}_n|^2 + |u_n|^2) dt \\ &\leq \frac{c_0}{R}(\|u_n\|^2 + \|u_n\|_2^2) \leq \frac{c}{R} \leq \frac{\varepsilon}{2}\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\Big|\int_{\mathbb{R}} e^{Q(t)}(\nabla W(t, u_n), u_n)\xi_R dt\Big| &\leq \gamma \int_{|t|\geq R/2} e^{Q(t)}|u_n|^2 \xi_R dt \\ &\leq \frac{\gamma}{\beta}\int_{|t|\geq R/2} e^{Q(t)}(L(t)u_n, u_n)\xi_R dt \\ &\leq \frac{\gamma}{\beta}\int_{\mathbb{R}} e^{Q(t)}(L(t)u_n, u_n)\xi_R dt\end{aligned} \tag{3.7}$$

for $R \geq R_\varepsilon$. Thus, combining (3.5)-(3.7) implies

$$(1 - \frac{\gamma}{\beta})\int_{\mathbb{R}} e^{Q(t)}[|\dot{u}_n|^2 + (L(t)u_n, u_n)]\xi_R dt \leq \varepsilon$$

for all $R \geq R_\varepsilon$ and $n \geq n_\varepsilon$. Hence (3.2) holds.

Since (u_n) is bounded, we may assume that, up to a subsequence, $u_n \rightharpoonup u$ in E for some $u \in E$. To prove our lemma, it suffices to show that $\|u_n\| \to \|u\|$ as $n \to \infty$. From (3.1), we obtain

$$o(1) = \langle \varphi'(u_n), u_n \rangle = (u_n, u_n) - \int_{\mathbb{R}} e^{Q(t)}(\nabla W(t, u_n), u_n) dt,$$
$$o(1) = \langle \varphi'(u_n), u \rangle = (u_n, u) - \int_{\mathbb{R}} e^{Q(t)}(\nabla W(t, u_n), u) dt.$$

So showing $\|u_n\| \to \|u\|$ is equivalent to proving that

$$\int_{\mathbb{R}} e^{Q(t)}(\nabla W(t, u_n), u_n - u) dt = o(1). \tag{3.8}$$

By (3.2), we have

$$\begin{aligned}&\Big|\int_{|t|\geq R} e^{Q(t)}(\nabla W(t, u_n), u_n - u) dt\Big| \\ &\leq \gamma \int_{|t|\geq R} e^{Q(t)}|u_n||u_n - u| dt \\ &\leq c\int_{|t|\geq R} e^{Q(t)}|u_n|^2 dt + c\int_{|t|\geq R} e^{Q(t)}|u|^2 dt \\ &\leq \frac{c}{\beta}\int_{|t|\geq R} e^{Q(t)}(L(t)u_n, u_n) dt + c\int_{|t|\geq R} e^{Q(t)}|u|^2 dt \leq c\varepsilon\end{aligned}$$

for all $R \geq R_\varepsilon$ and $n \geq n_\varepsilon$ large enough. This, together with the compactness of the embedding $E \hookrightarrow L^2_{loc}(e^{Q(t)})$, implies (3.8). This completes the proof. $\square$

Lemma 3.4. *Suppose that* (A3), (A9)-(A13) *are satisfied. Then* φ *satisfies the (C) condition.*

Proof. Let (u_n) be a Cerami sequence of φ. In view of Lemma 3.3, it suffices to show that (u_n) is bounded. Arguing indirectly, assume that $\|u_n\| \to \infty$. Take $w_n = u_n/\|u_n\|$. Then $\|w_n\| = 1$ and there is $w \in E$ such that

$$w_n \rightharpoonup w \text{ in } E, \quad w_n \to w \text{ in } L^2_{loc}(e^{Q(t)}), \quad w_n(t) \to w(t) \text{ a.e. } t \in \mathbb{R}, \tag{3.9}$$

after passing to a subsequence. We claim that

$$\ddot{w}(t) + q(t)\dot{w}(t) - (L(t) - L_\infty(t))w(t) = 0. \tag{3.10}$$

In fact, for each $\psi \in C_0^\infty(\mathbb{R}, \mathbb{R}^N)$, there holds

$$o(1) = \frac{\langle \varphi'(u_n), \psi\rangle}{\|u_n\|} = (w_n, \psi) - \int_{\mathbb{R}} e^{Q(t)} \frac{(\nabla W(t, u_n), \psi)}{\|u_n\|} dt \tag{3.11}$$

Noticing that $|\nabla R(t,x)| \leq c|x|$ for all (t,x), $|\nabla R(t, u_n(t))|/|u_n(t)| \to 0$ if $w(t) \neq 0$ and $\|w_n\|_\infty \leq (\sqrt{2e_0\sqrt{\beta}})^{-1}$ for all n, we have

$$\begin{aligned}
&\Big|\int_{\mathbb{R}} e^{Q(t)} \frac{(\nabla R(t, u_n), \psi)}{\|u_n\|} dt\Big| \\
&\leq \int_{\operatorname{supp}\psi} e^{Q(t)} \frac{|\nabla R(t, u_n)|}{|u_n|} |w_n||\psi| dt \\
&\leq \Big(\int_{supp\psi \cap \{w=0\}} + \int_{supp\psi\cap\{w\neq 0\}}\Big) e^{Q(t)} \frac{|\nabla R(t, u_n)|}{|u_n|} |w_n||\psi| dt \\
&\to 0 \quad \text{as } n \to \infty,
\end{aligned} \tag{3.12}$$

by the dominated convergence theorem. It follows from the second limit of (3.9) and Hölder's inequality that

$$\begin{aligned}
&\Big|\int_{\mathbb{R}} e^{Q(t)} (L_\infty(t)(w_n - w), \psi) dt\Big| \\
&\leq c \int_{\operatorname{supp}\psi} e^{Q(t)} |w_n - w||\psi| dt \\
&\leq c\Big(\int_{\operatorname{supp}\psi} e^{Q(t)} |w_n - w|^2 dt\Big)^{1/2} \Big(\int_{\operatorname{supp}\psi} e^{Q(t)} |\psi|^2 dt\Big)^{1/2} \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Combining this with (3.12) and (3.11), we obtain

$$\begin{aligned}
(w, \psi) &= \lim_{n\to\infty} \int_{\mathbb{R}} e^{Q(t)} \frac{(\nabla W(t, u_n), \psi)}{\|u_n\|} dt \\
&= \lim_{n\to\infty} \int_{\mathbb{R}} e^{Q(t)} \big[(L_\infty(t) w_n, \psi) + \frac{(\nabla R(t, u_n), \psi)}{\|u_n\|}\big] dt \\
&= \int_{\mathbb{R}} e^{Q(t)} (L_\infty(t) w, \psi) dt.
\end{aligned}$$

Therefore (3.10) holds.

For the function ξ_R given by (3.3), we have

$$\begin{aligned} o(1) &= \langle \varphi'(u_n), \xi_R u_n\rangle \\ &= \int_{\mathbb{R}} e^{Q(t)}[|\dot{u}_n|^2 + (L(t)u_n, u_n)]\xi_R dt + \int_{\mathbb{R}} e^{Q(t)}\dot{\xi}_R(\dot{u}_n, u_n)dt \\ &\quad - \int_{\mathbb{R}} e^{Q(t)}(\nabla W(t, u_n), u_n)\xi_R dt \end{aligned}$$

which implies that

$$\begin{aligned} o(1) = \int_{\mathbb{R}} e^{Q(t)}[|\dot{w}_n|^2 + (L(t)w_n, w_n)]\xi_R dt + \int_{\mathbb{R}} e^{Q(t)}\dot{\xi}_R(\dot{w}_n, w_n)dt \\ - \int_{\mathbb{R}} e^{Q(t)}\frac{(\nabla W(t, u_n), w_n)}{\|u_n\|}\xi_R dt. \end{aligned} \tag{3.13}$$

As in (3.2), by (A9) and (A12), for every $\varepsilon > 0$, there exists $\bar{R}_\varepsilon \geq 2t_0$ and $\bar{n}_\varepsilon > 0$ such that

$$\int_{|t|\geq R} e^{Q(t)}[|\dot{w}_n|^2 + (L(t)w_n, w_n)]dt < \varepsilon \quad \text{for all } R \geq \bar{R}_\varepsilon \text{ and } n \geq \bar{n}_\varepsilon.$$

Combining this with $(A3)$, we obtain

$$\int_{|t|\geq R} e^{Q(t)}|w_n|^2 dt \leq \frac{1}{\beta}\int_{|t|\geq R} e^{Q(t)}(L(t)w_n, w_n)dt \leq c\varepsilon$$

for all $R \geq \bar{R}_\varepsilon$ and $n \geq \bar{n}_\varepsilon$. This, jointly with the second limit of (3.9), shows that

$$w_n \to w \quad \text{in } L^2(e^{Q(t)}). \tag{3.14}$$

Moreover, it follows from (A10) and the dominated convergence theorem that

$$\begin{aligned} &\Big|\int_{\mathbb{R}} e^{Q(t)}\frac{(\nabla R(t, u_n), w_n)}{\|u_n\|}dt\Big| \\ &\leq \Big(\int_{w=0} + \int_{w\neq 0}\Big)e^{Q(t)}\frac{|\nabla R(t, u_n)|}{|u_n|}|w_n|^2 dt \to 0 \end{aligned}$$

as $n \to \infty$. Combining this with (3.10), (3.13) (with ξ_R replaced by 1), (3.14) and the Hölder inequality, we obtain

$$\begin{aligned} &\|w_n - w\|^2 \\ &= (w_n, w_n) - (w, w) + o(1) \\ &= \int_{\mathbb{R}} e^{Q(t)}\frac{(\nabla W(t, u_n), w_n)}{\|u_n\|}dt - \int_{\mathbb{R}} e^{Q(t)}(L_\infty(t)w, w)dt + o(1) \\ &\leq \int_{\mathbb{R}} e^{Q(t)}(L_\infty(t)w_n, w_n)dt - \int_{\mathbb{R}} e^{Q(t)}(L_\infty(t)w, w)dt + o(1) \\ &= \int_{\mathbb{R}} e^{Q(t)}(L_\infty(t)(w_n - w), w_n)dt + \int_{\mathbb{R}} e^{Q(t)}(L_\infty(t)w, w_n - w)dt + o(1) \\ &\leq c\,(\|w_n - w\|_2\|w_n\|_2 + \|w\|_2\|w_n - w\|_2) + o(1) \\ &= o(1), \end{aligned}$$

i.e., $w_n \to w$ in E and hence $w \neq 0$. This is a contradiction if (i) of (A13) holds. Now we assume that (ii) is satisfied. Then $\widetilde{W}(t, x) \geq 0$ and there exists $\eta > 0$ such that $\widetilde{W}(t,x) \geq \delta_0$ whenever $|x| \geq \eta$. Thus there exists $C_1 > 0$ such that

$$\begin{aligned} C_1 &\geq \varphi(u_n) - \frac{1}{2}\langle \varphi'(u_n), u_n\rangle \\ &= \int_{\mathbb{R}} e^{Q(t)}\Big(\frac{1}{2}(\nabla W(t,u_n),u_n) - W(t,u_n)\Big)dt \\ &\geq e_0 \int_{|u_n|\geq\eta} \widetilde{W}(t,u_n)dt \\ &\geq e_0\delta_0 \operatorname{meas}\{t\in\mathbb{R} : |u_n(t)| \geq \eta\}, \end{aligned}$$

and then

$$\operatorname{meas}\{t\in\mathbb{R} : |u_n(t)|\geq\eta\} \leq \frac{C_1}{e_0\delta_0},$$

where e_0 is the constant appears in Lemma 2.1. It follows form (3.10) and the unique continuation arguments similar to Heinz [9] that $w(t) \neq 0$ a.e. $t\in\mathbb{R}$. Hence there exist $\varepsilon > 0$ and $\Omega \subset \mathbb{R}$ such that $|w(t)| \geq 2\varepsilon$ in Ω and $C_1/(e_0\delta_0) < |\Omega| < +\infty$. Since $\|w_n - w\|_\infty \leq \big(\sqrt{2e_0\sqrt{\beta}}\big)^{-1}\|w_n - w\| \overset{n}{\to} 0$, we obtain, for almost all n, $|w_n(t)| \geq \varepsilon$ and hence $|u_n(t)| \geq \eta$ in Ω. Thus

$$\frac{C_1}{e_0\delta_0} < \operatorname{meas}\Omega \leq \operatorname{meas}\{t\in\mathbb{R} : |u_n(t)|\geq\eta\} \leq \frac{C_1}{e_0\delta_0},$$

a contradiction again. Consequently, (u_n) is bounded in E. □

Now we study the linking structure of φ. We arrange all the eigenvalues (counted with multiplicity) of A in $(0, l_0)$ by $0 < \mu_1 \leq \mu_2 \leq \dots \leq \mu_m < l_0$, and e_j denotes the corresponding eigenfunctions, that is, $Ae_j = \mu_j e_j$, $1 \leq j \leq m$. Set $\widetilde{E} = \operatorname{span}\{e_1, e_2, \dots e_m\}$. Obviously,

$$\mu_1\|u\|_2^2 \leq \|u\|^2 \leq \mu_m\|u\|_2^2, \quad \forall u \in \widetilde{E}.$$

Lemma 3.5. *If conditions* (A3) *and* (A9) *hold, then there exist* ρ, $\alpha > 0$ *such that* $\varphi(u)|_{\|u\|=\rho} \geq \alpha$.

Proof. By (A9), for any $\varepsilon > 0$ $(< \beta/4)$, there exists $\delta > 0$ such that

$$|\nabla W(t,x)| \leq \varepsilon|x|, \quad \forall t\in\mathbb{R},\ |x|\leq\delta,$$

and then

$$|W(t,x)| \leq \varepsilon|x|^2, \quad \forall t\in\mathbb{R},\ |x|\leq\delta.$$

Hence, using (A3) and Lemma 2.1, we obtain

$$\varphi(u) \geq \frac{1}{2}\|u\|^2 - \frac{\varepsilon}{\beta}\int_{\mathbb{R}} e^{Q(t)}(L(t)u,u)dt \geq \frac{1}{4}\|u\|^2$$

for $u \in E$ with $\|u\| \leq \delta\sqrt{2e_0\sqrt{\beta}}$. Taking $\rho := \delta\sqrt{2e_0\sqrt{\beta}}$ and $\alpha := \rho^2/4$, we have $\varphi(u)|_{\|u\|=\rho} \geq \alpha$. □

Lemma 3.6. *Suppose that* (A3), (A9)–(A12) *are satisfied. Then* $\varphi(u) \to -\infty$ *as* $\|u\| \to \infty$ *in* $\widetilde{E}$.

Proof. We suppose by contradiction that there exists a sequence $(u_n) \subset \widetilde{E}$ with $\|u_n\| \to \infty$ such that $\varphi(u_n) \geq -a$ for some $a > 0$. Take $v_n = u_n/\|u_n\|$. Then $\|v_n\| = 1$ and there is $v_0 \in \widetilde{E}\backslash\{0\}$ such that

$$v_n \to v_0 \text{ in } \widetilde{E}, \quad v_n \to v_0 \text{ in } L^2(e^{Q(t)}), \quad v_n(t) \to v_0(t) \text{ a.e. } t\in\mathbb{R}.$$

Since

$$\|v_0\|^2 - \int_{\mathbb{R}} e^{Q(t)}(L_\infty(t)v_0, v_0)dt \le \|v_0\|^2 - l_0\|v_0\|_2^2 \le \mu_m\|v_0\|_2^2 - l_0\|v_0\|_2^2 < 0,$$

there is $L > 0$ such that

$$1 - \int_{-L}^{L} e^{Q(t)}(L_\infty(t)v_0, v_0)dt = \|v_0\|^2 - \int_{-L}^{L} e^{Q(t)}(L_\infty(t)v_0, v_0)dt < 0. \tag{3.15}$$

From (A9) and (A12) it folows that $|R(t,x)| \le c|x|^2$ for all (t,x), and that

$$R(t, u_n(t))/|u_n(t)|^2 \to 0$$

if $v_0(t) \neq 0$. Hence, using the dominated convergence theorem and the fact $\|v_n\|_\infty \le (\sqrt{2e_0\sqrt{\beta}})^{-1}$,

$$\lim_{n\to\infty} \int_{-L}^{L} e^{Q(t)} \frac{R(t,u_n)}{\|u_n\|^2} dt = \lim_{n\to\infty} \int_{-L}^{L} e^{Q(t)} \frac{R(t,u_n)}{|u_n|^2} |v_n|^2 dt = 0. \tag{3.16}$$

Consequently, (3.16), (3.15) and (A9) imply that

$$\begin{aligned}
0 &\le \lim_{n\to\infty} \frac{\varphi(u_n)}{\|u_n\|^2} \\
&= \lim_{n\to\infty} \Big[\frac{1}{2} - \int_{\mathbb{R}} e^{Q(t)} \frac{W(t,u_n)}{\|u_n\|^2} dt\Big] \\
&\le \lim_{n\to\infty} \Big(\frac{1}{2} - \int_{-L}^{L} e^{Q(t)} \frac{W(t,u_n)}{|u_n|^2} |v_n|^2 dt\Big) \\
&\le \frac{1}{2} - \frac{1}{2}\int_{-L}^{L} e^{Q(t)}(L_\infty(t)v_0, v_0)dt < 0,
\end{aligned}$$

a contradiction. □

As a special case, we have the following lemma.

Lemma 3.7. *Suppose that* (A3), (A9)–(A12) *are satisfied. Then there is $e \in E$ with $\|e\| > \rho$ such that $\varphi(e) < 0$.*

Proof of Theorem 1.6. (Existence) Lemmas 3.5 and 3.7 yield that φ possesses the linking structure, and the (C) condition is satisfied by Lemma 3.4. Hence, using Proposition 3.1, we know that φ has at least one nontrivial critical point.

(Multiplicity) Take $E_1 = E$ and $E_2 = \widetilde{E}$. Assume that W is even in x, then φ is even. Lemma 3.6 says that $\varphi|_{\widetilde{E}} < +\infty$. Therefore, φ has at least m pairs of nontrivial critical points by Lemmas 3.4, 3.5 and Proposition 3.2. □

4. Examples

Example 4.1. Consider the second-order system

$$\ddot{u}(t) + 2t\dot{u}(t) - L(t)u(t) + \nabla W(t, u(t)) = 0, \qquad t \in \mathbb{R}, \quad u \in \mathbb{R}^N, \tag{4.1}$$

where $L(t) = (1+t^2)I_N$, I_N denotes the unit matrix of order N and

$$W(t,x) = \begin{cases} |t|e^{-t^2}|x|^\alpha \sin^2\big(\frac{1}{|x|^\varepsilon}\big), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

$\varepsilon > 0$ small enough and $\alpha \in (1+\varepsilon, 2)$. A direct calculation shows that

$$\nabla W(t,x) = |t|e^{-t^2}\big[\alpha|x|^{\alpha-2}x\sin^2\big(\frac{1}{|x|^{\varepsilon}}\big) - \varepsilon|x|^{\alpha-\varepsilon-2}x\sin\big(\frac{2}{|x|^{\varepsilon}}\big)\big],$$

for all $(t,x) \in \mathbb{R}\times\mathbb{R}^N$. So it is easy to check that W satisfies (A7) and (A8) with $0 \notin [t_0-d, t_0+d]$, $m(t) = h(t) = |t|e^{-t^2}$,

$$\delta_n = \Big(\frac{2}{(2n+1)\pi}\Big)^{1/\varepsilon}, \quad M_n = \min_{|t-t_0|\le d}|t|e^{-t^2}\Big(\frac{(2n+1)\pi}{2}\Big)^{\frac{2-\alpha}{\varepsilon}}.$$

Hence Theorem 1.4 applies. However, it does not satisfy Theorem 1.2, because (A6) fails.

Example 4.2. Consider the second-order system

$$\ddot{u}(t) + q(t)\dot{u}(t) - L(t)u(t) + \nabla W(t,u(t)) = 0, \quad \forall t\in\mathbb{R},\ u\in\mathbb{R}^N,$$

where $L(t) = \ln(e^2+t^2)I_N$, $q \in C(\mathbb{R},\mathbb{R})$, $Q(t) := \int_0^t q(s)ds$ with $\lim_{|t|\to\infty} Q(t) = +\infty$ and

$$W(t,x) = a(t)|x|^2\Big(1 - \frac{1}{\ln(e+|x|)}\Big), \quad \forall (t,x)\in\mathbb{R}\times\mathbb{R}^N.$$

Here $a \in C(\mathbb{R},(0,1])$ with $\inf_{t\in\mathbb{R}} a(t) > 0$. A simple computation yields

$$\nabla W(t,x) = 2a(t)x\Big(1 - \frac{1}{\ln(e+|x|)}\Big) + \frac{a(t)|x|x}{(e+|x|)\ln^2(e+|x|)}$$

and

$$\widetilde{W}(t,x) = \frac{a(t)|x|^3}{2(e+|x|)\ln^2(e+|x|)}$$

for all $(t,x)\in\mathbb{R}\times\mathbb{R}^N$. Therefore, it is easy to see that conditions (A9)–(A13) are satisfied, and Theorem 1.6 applies.

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