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DISTRIBUTIONAL SOLUTIONS FOR DAMPED WAVE EQUATIONS

MARC NUALART

Abstract. This work presents results on solutions to the one-dimensional damped wave equation, also called telegrapher's equation, when the initial conditions are general distributions. We make a complete deduction of its fundamental solutions, both for positive and negative times. To obtain them we only use self-similarity arguments and distributional calculus, making no use of Fourier or Laplace transforms. We next use these fundamental solutions to prove both the existence and the uniqueness of solutions to the distributional initial value problem. As applications we recover the semi-group property for initial data in classical function spaces, and we find the probability distribution function for a recent financial model of evolution of prices.

1. Introduction

The one dimensional damped wave equation $u_{tt} + ku_t = c^2 u_{xx}$ for $k, c > 0$ has been vastly studied in [1, 2, 8] and related to several important phenomena. These are, for example, the mechanical oscillations of a string with friction [10], the four examples considered in [4, Section 2], and, of more interest in the last decade, the persistent motion in movement ecology [6, 9] and the probability density function for a price evolution model of a financial asset [3].

The equation has been considered either in the whole real line or in an interval with boundary conditions. In both cases, many properties of the solution, particularly of their decay on time, are well-known for initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ belonging to different classes of functions, including distributions. In particular, in [4], the attention is focussed to the case of the whole real line and $f(x) = \delta(x)$, the Dirac's delta function and $g(x) = 0$, obtaining

$$\begin{aligned} u(x,t) = {}& \frac{1}{2} e^{-kt/2} \{\delta(x-ct) + \delta(x+ct)\} \\ & + k\frac{e^{-kt/2}}{8c}\big[I_0(\xi) + k\frac{I_1(\xi)}{2\xi}\big] H(ct-|x|), \end{aligned}$$

where I_0, I_1 are Bessel functions, $H(x)$ is the Heaviside function and

$$\xi = \frac{\sqrt{c^2t^2 - x^2}}{2ct}.$$

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This is a solution to be compared, in the context of random walks, to the Gaussian solution of the Diffusion Equation $u_t = Du_{xx}$ with $u(x,0) = \delta(x)$, namely

$$u(x,t) = \frac{1}{\sqrt{4\pi Dt}} \exp\Big(- \frac{x^2}{4Dt}\Big).$$

In [3], the attention is focused on the initial conditions $f(x) = \delta(x)$ and $g(x) = -c\delta'(x)$, a case that will be considered in detail in Section 6 below.

In the whole real line setting, the usual methods often include the Fourier and Laplace transforms. This is the approach of Masoliver in [5], where the solution to the problem with initial conditions $f(x) = \delta(x)$ and $g(x) = 0$ is obtained by means of the Laplace-Fourier transform. In the same reference it is stressed that

> although the solution has been known since a very long time ago, its derivation has remained quite obscure in the literature.

In this article we solve the equation in the whole real line both for positive and negative times, a setting that has not always been studied in detail, when the initial conditions are general distributions. Furthermore, we prove that this solution is unique, a result that we have not been able to find explicitly in the literature for general initial conditions. This clearly differs from the non-uniqueness phenomena that takes place for the one-dimensional heat equation in the real line, when one accepts solutions that can grow very fast near infinity.

Our results are based on deducing first the fundamental solutions of the equation. We emphasize that we obtain them without any use of the Fourier or Laplace transforms, but only with self-similar arguments and distributional calculus. This allows us not to be forced to restrict ourselves to only tempered distributions. We believe that this procedure is more complete and clear than in the previous literature.

The initial-value problem is

$$\begin{gathered} u_{tt} + ku_t = c^2 u_{xx} \\ u(0) = f, \quad u_t(0) = g \end{gathered} \tag{1.1}$$

which is equivalent to

$$\begin{gathered} v_{tt} = c^2 v_{xx} + \frac{k^2}{4} v \\ v(0) = f, \quad v_t(0) = g + \frac{k}{2} f \end{gathered} \tag{1.2}$$

for $v = e^{kt/2}u$.

In the following results, the meaning of solution is always in $\mathcal{D}'(\mathbb{R}\times\mathbb{R})$, but we will prove that the solution belongs to the space $\mathcal{C}^1(\mathbb{R}, \mathcal{D}'(\mathbb{R}))$ of $\mathcal{C}^1$ functions on time with values on the space of distributions in $x \in \mathbb{R}$. This gives us a stronger regularity for the solution than just the $\mathcal{D}'(\mathbb{R}\times\mathbb{R})$ meaning.

Theorem 1.1. *Let us consider the function*

$$\psi(x,t) = \operatorname{sgn}(t) \frac{1}{2c} I_0 \left(2\alpha\sqrt{c^2t^2 - x^2}\right) \mathcal{X}_{[-c|t|, c|t|]}(x),$$

with $\alpha = k/(4c)$*,* I_0 *the modified Bessel function of first kind and parameter 0,* $\operatorname{sgn}(t)$ *the sign function, and* $\mathcal{X}_\Omega$ *the characteristic function of* Ω*. Then* $\psi(x,t)$

belongs to $\mathcal{C}^1(\mathbb{R}, \mathcal{D}'(\mathbb{R}))$ *and solves* (1.2) *in the sense of distributions for* $f = 0$ *and* $g = \delta$ *the Dirac delta centered at* $x = 0$*. Furthermore, its time-derivative is*

$$\psi_t(\cdot, t) = \frac{1}{2}[\delta(\cdot - ct) + \delta(\cdot + ct)] + \alpha c|t| \frac{I_0'(2\alpha\sqrt{c^2t^2 - \cdot^2})}{\sqrt{c^2t^2 - \cdot^2}} \mathcal{X}_{[-c|t|, c|t|]}(\cdot),$$

in the distributional sense.

This solution ψ will be called the fundamental solution of problem (1.2). From ψ and ψ_t we are able to obtain solutions of (1.2) when the initial conditions f and g are general distributions, as the next result shows.

Theorem 1.2. *Let* $f, g \in \mathcal{D}'(\mathbb{R})$*, the solution* $v \in \mathcal{C}^1(\mathbb{R}, \mathcal{D}'(\mathbb{R}))$ *to the initial value problem* (1.2) *in the sense of distributions is given by*

$$\langle v, \varphi \rangle := \langle f, \psi_t * \varphi \rangle + \langle g + \frac{k}{2} f, \psi * \varphi \rangle,$$

for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$.

Theorem 1.3. *Let* $f, g \in \mathcal{D}'(\mathbb{R})$*. Then the distribution* v *given in Theorem 1.2 is the unique solution in* $\mathcal{C}^1(\mathbb{R}, \mathcal{D}'(\mathbb{R}))$ *of problem* (1.2)*.*

These results allow us to solve problem (1.1) uniquely as stated next.

Remark 1.4. Let $f, g \in \mathcal{D}'(\mathbb{R})$. The distribution u given by

$$\begin{aligned} \langle u, \varphi \rangle :=& e^{-kt/2} \big\{ \langle f, \big(\psi_t + \frac{k}{2}\psi\big) * \varphi \rangle + \langle g, \psi * \varphi \rangle \big\} \\ =& e^{-kt/2} \big\{ \langle f, \psi_t * \varphi \rangle + \langle \frac{k}{2} f + g, \psi * \varphi \rangle \big\}, \end{aligned}$$

for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ is the unique solution in $\mathcal{C}^1(\mathbb{R}, \mathcal{D}'(\mathbb{R}))$ of problem (1.1).

The structure of this article is as follows. In Section 2 we deduce heuristically a possible solution to problem 1.2 for $f = 0$ and $g = \delta$. In Section 3, inspired by the heuristics, we prove rigorously Theorem 1.1. Then, in Section 4 we prove Theorems 1.2 and 1.3. Finally, in Section 5 we give some properties of the semi-group that generates the solution when the initial conditions $(f, g) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ and in Section 6 we apply Remark 1.4 to the price evolution model of a financial asset proposed in [3].

While many results are thoroughly proved here, for the sake of simplicity the complete calculations of some proofs have been omitted but can be found in the author's Bachelor Degree Thesis [7] which was advised by Joan Solà-Morales.

2. Heuristics

Inspired by [2], we introduce the characteristic coordinates for the 1-D wave equation, which are $\zeta = ct - x$ and $\eta = ct + x$. If $\psi(x, t)$ solves $\psi_{tt} - c^2\psi_{xx} = k^2\psi/4$ then $v(\zeta, \eta) := \psi(x, t)$ solves $v_{\zeta\eta} = \frac{k^2}{16c^2} v$. At this point [8] uses the Fourier transform, although it also mentions the possibility of proceeding in other ways.

We note that for all $a \neq 0$, the function $w(\zeta, \eta) := v(a\zeta, \eta/a)$ also solves $w_{\zeta\eta} = \frac{k^2}{16c^2} w$. Therefore, we have a family of solutions that depends on the parameter a, the equation is invariant under the transformation $(\zeta, \eta) \to (a\zeta, \eta/a)$. For this reason, we look for solutions invariant under this kind of transformations, for example we look for solutions of the form $\psi(x, t) = v(\zeta, \eta) = f(\zeta\eta) = f(c^2t^2 - x^2) =$

$f(\lambda) = h(2\alpha\sqrt{\lambda}) = h(\xi)$ for some f and h to be found, where we write $\lambda = c^2t^2 - x^2$ and $\xi = 2\alpha\sqrt{\lambda}$. This way,

$$\begin{aligned} 0 = \psi_{tt} - c^2\psi_{xx} - \frac{k^2}{4}\psi &= 4c^2\left[\lambda f'' + f' - \alpha^2 f\right] \\ &= \frac{c^2}{\lambda}\left[\xi^2 h''(\xi) + \xi h'(\xi) - \xi^2 h(\xi)\right]. \end{aligned}$$

The solution for h is a linear combination of the modified Bessel functions of order $n = 0$,

$$h(\xi) = AI_0(\xi) + BK_0(\xi),$$

when $\lambda \geq 0$. For $\lambda < 0$, we extend h by 0 and we write ψ as

$$\psi(x,t) = [AI_0(2\alpha\sqrt{\lambda}) + BK_0(2\alpha\sqrt{\lambda})]\mathcal{X}_{[-c|t|,c|t|]}(x).$$

Let $g(x)$ be continuous and take

$$v(x,t) := (g*\psi)(x)(t) = \int_{x-ct}^{x+ct} [AI_0(2\alpha\sqrt{\lambda}) + BK_0(2\alpha\sqrt{\lambda})]g(y)dy,$$

for $\lambda = c^2t^2 - (x-y)^2$ and with A, B still to be determined. We compute

$$\begin{aligned} v_t(x,t) &= c\,[AI_0(0) + BK_0(0)]\,[g(x+ct) + g(x-ct)] \\ &\quad + \int_{x-ct}^{x+ct}\left[AI_0'(2\alpha\sqrt{\lambda}) + BK_0'(2\alpha\sqrt{\lambda})\right]\frac{2\alpha c^2 t}{\sqrt{\lambda}}g(y)dy \end{aligned} \tag{2.1}$$

since $\lambda = 0$ when $y = x \pm ct$. We have that $I_0(0) = 1$ and $K_0(z) \to \infty$ when $z \to 0$, so for v_t to exist and be bounded we impose $B = 0$. Therefore, (2.1) reduces to

$$v_t(x,t) = cA[g(x+ct) + g(x-ct)] + \int_{x-ct}^{x+ct} AI_0'(2\alpha\sqrt{\lambda})\frac{2\alpha c^2 t}{\sqrt{\lambda}}g(y)dy.$$

We have $\lim_{z\to 0} I_0'(2\alpha z)/z = \alpha$, which implies

$$\lim_{t\to 0}\int_{x-ct}^{x+ct} AI_0'(2\alpha\sqrt{\lambda})\frac{2\alpha c^2 t}{\sqrt{\lambda}}g(y)dy = 0$$

because the interval of integration reduces only to the point x and the integrand is not only bounded but also tends to 0 as t does so. Hence, we have $\lim_{t\to 0} v_t(x,t) = 2cAg(x) = g(x)$ for $A = \frac{1}{2c}$, we deduce that $v(x,t)$ satisfies $v_t(x,0) = g(x)$. From

$$v(x,t) = \int_{x-ct}^{x+ct}\frac{1}{2c}I_0(2\alpha\sqrt{\lambda})g(y)dy$$

and using the same argument, when $t \to 0$ the integral reduces to the single point x and the integrand is bounded becuase $I_0(z) \to 1$ as $z \to 0$, we can easily show that $v(x,0) = 0$. All we have done is quite heuristic, not very rigorous. However, it has provided us a useful insight into the equation and its properties, as well as a good candidate for the actual solution of the problem.

3. Proof of Theorem 1.1

We must prove that $\psi(x,t)$ belongs to $\mathcal{C}^1(\mathbb{R},\mathcal{D}'(\mathbb{R}))$ and as a two-variables distribution, $\psi(x,t) \in \mathcal{D}'(\mathbb{R}\times\mathbb{R})$ solves (1.2) in the sense of distributions when $f = 0$ and $g = \delta$ the Dirac delta centered at $x = 0$. We first check that ψ solves the differential equation and afterwards we prove that it belongs to $\mathcal{C}^1(\mathbb{R},\mathcal{D}'(\mathbb{R}))$, satisfies the required initial conditions, and we compute its time-derivative.

Proof that ψ solves (1.2). First of all, when $x = \pm ct$ we have $\psi(x,t) \equiv \operatorname{sgn}(t)\frac{1}{2c}$, notice there is a discontinuity in the straight lines $\{x = ct\}$ and $\{x = -ct\}$, the characteristics of our equation, because of the discontinuity of $\mathcal{X}_{[-c|t|,c|t|]}(x)$. Hence, we cannot expect our candidate to be a classical solution of the problem. However, let us see it is a solution in the sense of distributions.

Let L be the differential operator defined by

$$L(u) := u_{tt} - c^2 u_{xx} - \frac{k^2}{4} u,$$

the aim of the proof is to show that $\psi(x,t)$ is such that $\langle L(\psi), \varphi \rangle = 0$, for all $\varphi \in \mathcal{C}_0^\infty\left(\mathbb{R}^2\right)$, that is,

$$\begin{aligned} \langle L(\psi), \varphi \rangle := \langle \psi, L^*(\varphi) \rangle &= \int_{\mathbb{R}^2} \psi L^*(\varphi)\, dx\, dt \\ &= \int_{\mathbb{R}^2} \psi \Big(\varphi_{tt} - c^2 \varphi_{xx} - \frac{k^2}{4}\varphi \Big)\, dx\, dt = 0, \end{aligned} \tag{3.1}$$

by definition of the adjoint of the operator, which in this case it is itself. Note that this last integral is well defined.

Let then $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ be a smooth function with compact support K, say included in the rectangle $R = [-a, a] \times [-\frac{a}{c}, \frac{a}{c}] \subset \mathbb{R} \times \mathbb{R}$ for some $a > 0$. Below there is a picture of the situation. The rectangle represents R and the shaded area the region where ψ is not 0, see Figure 1.

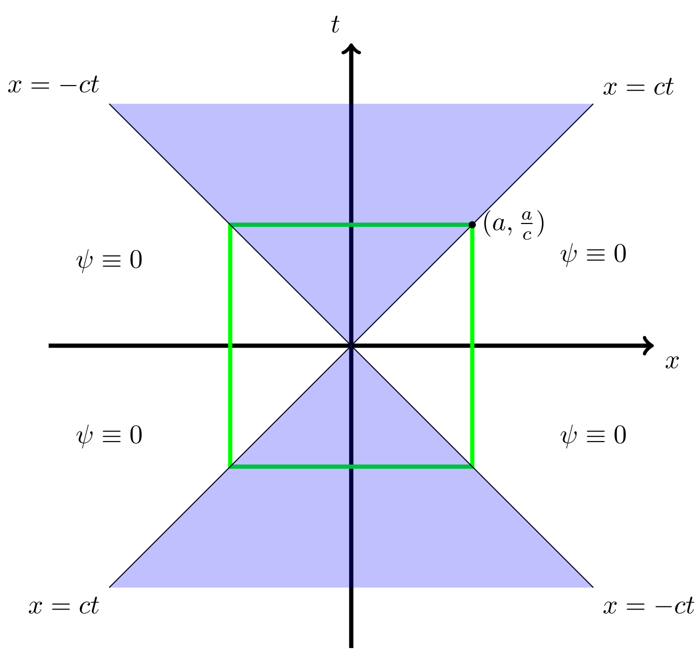

Figure 1. Scheme of the situation when integrating

Because of the compact support $K \subset R$ of φ and the fact that ψ is identically zero whenever $|x| > c|t|$, we have that the domain of integration of the last integral in (3.1) can be reduced to two distinct open triangles inside the rectangle R, which are

$$\nabla := \big\{(x,t) \in \mathbb{R}^2 : 0 < t < \frac{a}{c},\ -ct < x < ct\big\},$$
$$\Delta := \big\{(x,t) \in \mathbb{R}^2 : -\frac{a}{c} < t < 0,\ ct < x < -ct\big\}.$$

Then, we can write

$$\int_{\mathbb{R}^2} \psi\Big(\varphi_{tt} - c^2\varphi_{xx} - \frac{k^2}{4}\varphi\Big) = \int_\nabla \psi \operatorname{div} X - \int_\nabla \frac{k^2}{4}\psi\varphi + \int_\Delta \psi \operatorname{div} X - \int_\Delta \frac{k^2}{4}\psi\varphi, \tag{3.2}$$

where $X = \begin{pmatrix} -c^2\varphi_x \\ \varphi_t \end{pmatrix}$.

Let us reason for the integral on ∇, the argument is the same for the other. The vector field ψX is continuously differentiable up to the boundary because both ψ and X are so inside the triangle. Hence, we can use the divergence theorem,

$$\int_\nabla \psi \operatorname{div} X = \int_\nabla \operatorname{div}(\psi X) - \int_\nabla \nabla\psi \cdot X = \int_\vee \psi X \cdot n \, d\ell - \int_\nabla \nabla\psi \cdot X. \tag{3.3}$$

Notice also that

$$\begin{aligned} \int_\nabla \nabla\psi \cdot X &= \int_\nabla \begin{pmatrix} \psi_x & \psi_t \end{pmatrix} \cdot \begin{pmatrix} -c^2\varphi_x \\ \varphi_t \end{pmatrix} \\ &= \int_\nabla Y \cdot \nabla\varphi = \int_\nabla \operatorname{div}(\varphi Y) - \int_\nabla \varphi \operatorname{div} Y, \end{aligned} \tag{3.4}$$

where $Y = \begin{pmatrix} -c^2\psi_x \\ \psi_t \end{pmatrix}$.

The vector field φY is continuously differentiable up to the boundary because both φ and Y are so *inside* the triangle. Hence, we can use the divergence theorem again in (3.4) to obtain

$$\int_\nabla \nabla\psi \cdot X = \int_\vee \varphi Y \cdot n \, d\ell - \int_\nabla \varphi \operatorname{div} Y.$$

This way we can write (3.3) as

$$\int_\nabla \psi \operatorname{div} X = \int_\vee \psi X \cdot n \, d\ell - \int_\vee \varphi Y \cdot n \, d\ell + \int_\nabla \varphi \operatorname{div} Y. \tag{3.5}$$

The same reasoning holds for the integral on Δ, namely

$$\int_\Delta \psi \operatorname{div} X = \int_\wedge \psi X \cdot n \, d\ell - \int_\wedge \varphi Y \cdot n \, d\ell + \int_\Delta \varphi \operatorname{div} Y, \tag{3.6}$$

The boundary terms are integrated on

$$\vee := \{(x,t) \in \mathbb{R}^2 : 0 < t < \frac{a}{c},\ x = \pm ct\},$$
$$\wedge := \{(x,t) \in \mathbb{R}^2 : -\frac{a}{c} < t < 0,\ x = \pm ct\}.$$

The boundaries of the triangles also contain the segments $[-a,a]\times\{\frac{a}{c}\}$ and $[-a,a]\times\{-\frac{a}{c}\}$. However, the integrals there are 0 because the segments fall outside the compact support of ψX (due to φ) and therefore we just integrate on $\vee$ and $\wedge$ taking n the exterior normal unit vector on such sets. See Figure 1 to clarify the situation.

We compute the integrals on $\vee$, the computation on $\wedge$ is similar and can be found in [7]. To compute $\int_\vee \psi X\cdot n\,d\ell$ we split the domain of integration $\vee$ according to the sign of x and we apply the corresponding values of n in each case. First, we parametrize the segment $\{x=-ct: 0<t<\frac{a}{c}\}$ by $\sigma(s)=\big(s,\frac{-s}{c}\big)$ with $s\in[-a,0]$, giving

$$\begin{aligned}
&\int_{-a}^{0}\Big[\psi\begin{pmatrix}-c^2\varphi_x\\ \varphi_t\end{pmatrix}\cdot\begin{pmatrix}-\frac{1}{c} & -1\end{pmatrix}\Big](\sigma(s))ds\\
&= c\int_{-a}^{0}\Big[\psi\Big(\varphi_x-\frac{1}{c}\varphi_t\Big)\Big](\sigma(s))ds\\
&= c\int_{-a}^{0}[\psi\frac{d\varphi}{ds}](\sigma(s))ds\\
&= c\Big((\psi\varphi)(0,0^+)-\int_{-a}^{0}[\varphi\frac{d\psi}{ds}](s,\frac{-s}{c})ds\Big).
\end{aligned}$$

We parametrize the other segment by $\sigma(s)=(s,\frac{s}{c})$, with $s\in[0,a]$. Using the same parts integration and directional derivative strategy as before, we find

$$\int_0^a\Big[\psi\begin{pmatrix}-c^2\varphi_x\\ \varphi_t\end{pmatrix}\cdot\begin{pmatrix}\frac{1}{c}-1\end{pmatrix}\Big]\big(\sigma(s)\big)ds = c\Big((\psi\varphi)(0,0^+)+\int_0^a[\varphi\frac{d\psi}{ds}]\big(s,\frac{s}{c}\big)ds\Big).$$

Consequently,

$$\int_\vee \psi X\cdot n\,d\ell = 2c(\psi\varphi)(0,0^+)+c\int_0^a[\varphi\frac{d\psi}{ds}]\big(s,\frac{s}{c}\big)ds - c\int_{-a}^0[\varphi\frac{d\psi}{ds}]\big(s,\frac{-s}{c}\big)ds \quad (3.7)$$

For $\wedge$ a similar calculation yields

$$\int_\wedge \psi X\cdot n\,d\ell = 2c(\psi\varphi)(0,0^-)+c\int_0^a[\varphi\frac{d\psi}{ds}]\big(s,-\frac{s}{c}\big)ds - c\int_{-a}^0[\varphi\frac{d\psi}{ds}]\big(s,\frac{s}{c}\big)ds. \quad (3.8)$$

On the other hand, to compute $\int_\vee \varphi Y\cdot n\,d\ell$ we parametrize again one segment by $\sigma(s)=\big(s,\frac{-s}{c}\big)$ with $s\in[-a,0]$. Then

$$\begin{aligned}
&\int_{-a}^{0}\Big[\varphi\begin{pmatrix}-c^2\psi_x\\ \psi_t\end{pmatrix}\cdot\begin{pmatrix}-\frac{1}{c}-1\end{pmatrix}\Big]\big(\sigma(s)\big)ds\\
&= \int_{-a}^{0}[\varphi(c\psi_x-\psi_t)]\big(\sigma(s)\big)ds\\
&= c\int_{-a}^{0}\Big[\varphi\Big(\psi_x-\frac{1}{c}\psi_t\Big)\Big]\big(\sigma(s)\big)ds\\
&= c\int_{-a}^{0}[\varphi\frac{d\psi}{ds}]\big(s,-\frac{s}{c}\big)ds.
\end{aligned}$$

The other segment is parametrized by $\sigma(s)=\big(s,\frac{s}{c}\big)$ with $s\in[0,a]$. Using the same strategy as above, we have

$$\int_0^a\Big[\varphi\begin{pmatrix}-c^2\psi_x\\ \psi_t\end{pmatrix}\cdot\begin{pmatrix}\frac{1}{c} & -1\end{pmatrix}\Big]\big(\sigma(s)\big)ds = -c\int_0^a[\varphi\frac{d\psi}{ds}]\big(s,\frac{s}{c}\big)ds.$$

Therefore,

$$\int_\vee \varphi Y\cdot n\,d\ell = c\int_{-a}^0[\varphi\frac{d\psi}{ds}]\big(s,-\frac{s}{c}\big)ds - c\int_0^a[\varphi\frac{d\psi}{ds}]\big(s,\frac{s}{c}\big)ds \quad (3.9)$$

and for $\wedge$ a similar calculation yields

$$\int_{\wedge} \varphi Y \cdot n \, d\ell = c \int_{-a}^{0} [\varphi \frac{d\psi}{ds}]\big(s, \frac{s}{c}\big) ds - c \int_{0}^{a} [\varphi \frac{d\psi}{ds}]\big(s, -\frac{s}{c}\big) ds. \tag{3.10}$$

Finally, combining (3.7) and (3.8) in (3.5) and combining (3.9) and (3.10) in (3.6) we have that (3.2) reads

$$\begin{aligned}
&\int_{\mathbb{R}^2} \psi\Big(\varphi_{tt} - c^2\varphi_{xx} - \frac{k^2}{4}\varphi\Big)\, dx\, dt \\
&= 2c(\psi\varphi)(0, 0^+) + 2c(\psi\varphi)(0, 0^-) \\
&\quad + 2c \int_{0}^{a} [\varphi \frac{d\psi}{ds}]\big(s, \frac{s}{c}\big) ds - 2c \int_{-a}^{0} [\varphi \frac{d\psi}{ds}]\big(s, \frac{-s}{c}\big) ds \\
&\quad - 2c \int_{-a}^{0} [\varphi \frac{d\psi}{ds}]\big(s, \frac{s}{c}\big) ds + 2c \int_{0}^{a} [\varphi \frac{d\psi}{ds}]\big(s, -\frac{s}{c}\big) ds \\
&\quad + \int_{\nabla \cup \Delta} \varphi\Big(\psi_{tt} - c^2\psi_{xx} - \frac{k^2}{4}\psi\Big)\, dx\, dt.
\end{aligned}$$

The notation 0^+ represents the limit going to zero from above and 0^- going from below. This distinction is crucial since we have ψ defined differently whereas $t > 0$ or not. Indeed, we have $\psi(0, 0^+) = \frac{1}{2c}$ and $\psi(0, 0^-) = -\frac{1}{2c}$. Since $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$, we have $2c(\psi\varphi)(0, 0^+) + 2c(\psi\varphi)(0, 0^-) = 0$.

We recall ψ is either $\frac{1}{2c}$ or $-\frac{1}{2c}$ on the lines $\{x = \pm ct\}$. This means ψ is constant on such lines and, as a result,

$$\frac{d\psi}{ds}\big(s, \frac{s}{c}\big) = \frac{d\psi}{ds}\big(s, -\frac{s}{c}\big) = 0,$$

for all $s \in \mathbb{R}$. Hence, the line integrals are all 0 and what remains to be seen is that

$$\int_{\nabla \cup \Delta} \varphi\Big(\psi_{tt} - c^2\psi_{xx} - \frac{k^2}{4}\psi\Big)\, dx\, dt = 0,$$

which is true provided that $\psi_{tt} - c^2\psi_{xx} - \frac{k^2}{4}\psi = 0$ in the interior of the two triangles. We just check the result for positive times. For negatives times, the minus sign does not affect at all the result. Let us recall our candidate of solution inside the integrating region for positive times is

$$\psi(x, t) = \frac{1}{2c} I_0\Big(2\alpha\sqrt{c^2t^2 - x^2}\Big).$$

This function is infinitely differentiable in both variables in this region. To verify it satisfies the PDE, we use the same reasoning as in the Heuristics Section 2. That is, for $\lambda = c^2t^2 - x^2$ we write $\psi(x, t) = \frac{1}{2c} I_0(2\alpha\sqrt{\lambda}) = f(\lambda) = g(2\alpha\sqrt{\lambda}) = g(\xi)$ for some f and g to determine. Hence, recalling $\alpha = \frac{k}{4c}$, and using the relations between f and g we deduce that

$$\begin{aligned}
\psi_{tt} - c^2\psi_{xx} - \frac{k^2}{4}\psi &= 4c^2[\lambda f'' + f' - \alpha^2 f] \\
&= \frac{c^2}{\lambda}[\xi^2 g''(\xi) + \xi g'(\xi) - \xi^2 g(\xi)] = 0,
\end{aligned}$$

because $g(\xi) = \frac{1}{2c} I_0(\xi)$ is a multiple of the modified Bessel equation of order 0 and parameter 1, which precisely satisfies this last ODE and this finishes the proof. □

We now prove the other part of Theorem 1.1, which refers to ψ belonging to $\mathcal{C}^1(\mathbb{R}, \mathcal{D}'(\mathbb{R}))$ and the initial conditions that $\psi(x,t)$ satisfy. This result was inspired by [11].

Proof of $\psi(x,t) \in \mathcal{C}^1(\mathbb{R}, \mathcal{D}'(\mathbb{R}))$. We have to see that there exists

$$\lim_{h\to 0} \frac{\psi(\cdot, t+h) - \psi(\cdot, t)}{h} = \psi_t(\cdot, t) \in \mathcal{C}^0\left(\mathbb{R}, \mathcal{D}'(\mathbb{R})\right)$$

with the limits computed using the topology of the arrival space $\mathcal{D}'(\mathbb{R})$, that is

$$\langle \frac{\psi(\cdot, t+h) - \psi(\cdot, t)}{h}, \varphi\rangle \xrightarrow{h\to 0} \langle \psi_t(\cdot, t), \varphi\rangle,$$

for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ and $t \in \mathbb{R}$, and that this limit is continuous, that is

$$\lim_{h\to 0} \langle \psi_t(\cdot, t+h), \varphi\rangle = \langle \psi_t(\cdot, t), \varphi\rangle,$$

for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ and $t \in \mathbb{R}$, where the duality product notation $\langle \psi(\cdot, t), \varphi\rangle$ refers here to $\int_{\mathbb{R}} \psi(x,t)\varphi(x)dx$. We just prove the result for $t = 0$. For $t > 0$ and $t < 0$ it is done similarly and it can be found in [7]. Notice $\psi(x, 0) = 0$ is expected from the odd symmetry of the function. Indeed, we see that with this choice, we have a differentiable application, in particular a continuous one.

Let us first take $1 \gg h > 0$ and $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$,

$$\begin{aligned}
\langle \frac{\psi(\cdot, h)}{h}, \varphi\rangle &= \int_{\mathbb{R}} \frac{1}{h} \psi(x,h)\varphi(x)ds \\
&\overset{h\geq 0}{=} \int_{\mathbb{R}} \frac{1}{2c} \frac{I_0\left(2\alpha\sqrt{c^2h^2 - x^2}\right)}{h} \mathcal{X}_{[-ch,ch]}(x)\varphi(x)dx \\
&= \frac{1}{h} \int_{-ch}^{ch} \frac{1}{2c} I_0\left(2\alpha\sqrt{c^2h^2 - x^2}\right)\varphi(x)dx.
\end{aligned}$$

When $h \to 0$, the interval of the integral reduces to 0 while the integrand tends to $\frac{1}{2c}$, it is bounded. So, the integral goes to 0 and then the quotient tends to $\frac{0}{0}$. Using Hopital's Rule,

$$\begin{aligned}
&\frac{d}{dh}\Big(\int_{-ch}^{ch} \frac{1}{2c} I_0\big(2\alpha\sqrt{c^2h^2 - x^2}\big)\varphi(x)dx\Big) \\
&= \frac{1}{2c}\{I_0(0)\varphi(ch)c - I_0(0)\varphi(-ch)(-c)\} \\
&\quad + \int_{-ch}^{ch} \frac{1}{2c} \frac{I_0'(2\alpha\sqrt{c^2h^2 - x^2})}{2\sqrt{c^2h^2 - x^2}} 2\alpha 2c^2 h\varphi(x)dx \\
&= \frac{1}{2}\{\varphi(ch) + \varphi(-ch)\} + \int_{-ch}^{ch} \alpha ch \frac{I_0'(2\alpha\sqrt{c^2h^2 - x^2})}{2\sqrt{c^2h^2 - x^2}}\varphi(x)dx,
\end{aligned}$$

after applying $I_0(0) = 1$ and simplifying terms. Letting $h \to 0$ the integral vanishes since the integrand goes to 0 and the interval collapses and so

$$\lim_{h\to 0^+} \langle \frac{\psi(\cdot, h)}{h}, \varphi\rangle = \frac{1}{2}\{\varphi(0) + \varphi(0)\} = \varphi(0) = \langle \delta, \varphi\rangle.$$

We obtain the same result when $h \to 0^-$, concluding that $\psi_t(\cdot, 0) = \delta(\cdot) \in \mathcal{D}'(\mathbb{R})$.

To show the continuity, we need

$$\lim_{h\to 0} \langle \psi_t(\cdot, h), \varphi\rangle = \langle \psi_t(\cdot, 0), \varphi\rangle = \langle \delta, \varphi\rangle.$$

Here we just present the case where $h \to 0^+$, the other one is similar. We have

$$\langle \psi_t(\cdot, h), \varphi \rangle = \frac{1}{2}\varphi(ch) + \frac{1}{2}\varphi(-ch) + \int_{-ch}^{ch} \alpha ch \frac{I_0'(2\alpha\sqrt{c^2h^2 - x^2})}{\sqrt{c^2h^2 - x^2}} \varphi(x) dx.$$

which is precisely what we obtained a few lines above, so we conclude that indeed $\lim_{h\to 0} \langle \psi_t(\cdot, h), \varphi \rangle = \langle \psi_t(\cdot, 0), \varphi \rangle = \langle \delta, \varphi \rangle$. □

4. Initial value problem

Here we prove Theorems 1.2 and 1.3. To do so, we use the next intermediate results.

Proposition 4.1. *Let* $\psi(x,t) = \operatorname{sgn}(t)\frac{1}{2c} I_0\left(2\alpha\sqrt{c^2t^2 - x^2}\right) \mathcal{X}_{[-c|t|, c|t|]}(x)$ *and* $g \in \mathcal{D}'(\mathbb{R})$ *a distribution. Then the distribution* v *defined by*

$$\langle v(t), \varphi \rangle := \langle g, \psi(t) * \varphi \rangle,$$

for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ *is such that* $v \in \mathcal{C}^1(\mathbb{R}, \mathcal{D}'(\mathbb{R}))$, *as a two variable distribution it is a solution in the distributional sense of*

$$\begin{gathered} v_{tt} = c^2 v_{xx} + \frac{k^2}{4} v \\ v(0) = 0, \quad v_t(0) = g \end{gathered}$$

and its time-derivative is the distribution v_t *defined by*

$$\langle v_t(t), \varphi \rangle := \langle g, \psi_t(t) * \varphi \rangle$$

for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$.

Before the proof, let us state these rather useful remarks.

Remark 4.2. This candidate to solution v may be understood as a resulting distribution on the x variable for each $t \in \mathbb{R}$ or also as a two dimensional distribution. We will use this double meaning in different parts of the proof.

Remark 4.3. From Theorem 1.1, we know that

$$\langle L(\psi), \phi \rangle := \int_{\mathbb{R}^2} \psi L^*(\phi)\, dx\, dt = 0,$$

for all $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$, a property that will be essential in the proof.

Remark 4.4. In certain cases, we may write

$$\begin{aligned} v(x,t) &= \int_{\mathbb{R}} \psi(x - y, t) g(y) dy \\ &= \int_{x-ct}^{x+ct} \frac{1}{2c} I_0(2\alpha\sqrt{c^2t^2 - (x-y)^2}) g(y) dy, \end{aligned}$$

when g is such that this expression makes sense, for example when $g \in L^2(\mathbb{R})$.

Proof of Proposition 4.1. Observe that $\psi(t)$ is a function with compact support and continuous inside $[-c|t|, c|t|]$. Therefore, $\psi(t) * \varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ and it makes sense to compute the action of the distribution g on $\psi(t) * \varphi$, rendering v a well-defined distribution.

Let us now think our candidate to solution as a two dimensional distribution. For the same differential operator L as before, we must give some sense to $\langle L(v), \phi\rangle = 0$, a distribution acting on a two-variable test function ϕ. A reasonable definition is

$$\langle L(v), \phi\rangle := \int_{\mathbb{R}} \langle v(t), L^*(\phi)\rangle dt$$

because it is fully well-defined and coincides with the action of any $h \in L^1_{\rm loc}(\mathbb{R}^2)$ against a two-variable test function. We proceed

$$\begin{aligned}\int_{\mathbb{R}} \langle v(t), L^*(\phi)\rangle dt &= \int_{\mathbb{R}} \langle g, \psi(t) * L^*(\phi(t))\rangle dt \\ &= \Big\langle g, \int_{\mathbb{R}} \psi(t) * L^*(\phi(t)) dt\Big\rangle.\end{aligned}$$

Notice we can enter the integral inside the duality product $\langle \cdot, \cdot\rangle$ because it is indeed a linear continuous form and the integral is a limiting process based on sums that only concerns $\psi(t) * L^*(\phi(t))$. Now,

$$\int_{\mathbb{R}} \psi(t) * L^*(\phi(t)) dt = \int_{\mathbb{R}^2} \psi(y,t) L^*(\phi(x-y,t))\, dy\, dt.$$

For a fixed $x \in \mathbb{R}$, writing $\phi(x-y,t) =: \varphi(y,t)$, we have that $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ and $L^*(\phi(x-y,t)) = L^*(\varphi(y,t)) \in \mathcal{C}_0^\infty(\mathbb{R}^2)$. Thus,

$$\Big\langle g, \underbrace{\int_{\mathbb{R}^2} \psi(y,t) L^*(\varphi(y,t))\, dy\, dt}_{0} \Big\rangle = 0,$$

thanks to Remark 4.3. Consequently, the distribution v is a solution in the sense of distributions of the differential equation.

For the first initial condition $\langle v(0), \varphi\rangle := \langle g, \psi(0) * \varphi\rangle = \langle g, 0 * \varphi\rangle = 0$, for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, so that $v(0) = 0$ as a distribution. For the second initial condition, $v_t(0) = g$ as a consequence of $v(t) \in \mathcal{C}^1(\mathbb{R}, \mathcal{D}'(\mathbb{R}))$. Thanks to Theorem 1.1, we know that $\psi(t) \in \mathcal{C}^1(\mathbb{R}, \mathcal{D}'(\mathbb{R}))$ and hence we can write

$$\langle v_t(t), \varphi\rangle := \langle g, \psi_t(t) * \varphi\rangle.$$

This expression is well defined since ψ_t is compactly supported both in the distributional and functional sense, from which we deduce that $\psi_t * \varphi \in \mathcal{C}_0^\infty(\mathbb{R})$. Moreover, $\psi(t) \in \mathcal{C}^1(\mathbb{R}, \mathcal{D}'(\mathbb{R}))$ implies $v(t) \in \mathcal{C}^1(\mathbb{R}, \mathcal{D}'(\mathbb{R}))$ and, as a result, we have $\langle v_t(0), \varphi\rangle := \langle g, \psi_t(0) * \varphi\rangle = \langle g, \delta * \varphi\rangle = \langle g, \varphi\rangle$ so we conclude that $v_t(0) = g$ as a distribution. □

Remark 4.5. Following the proof of Proposition 4.1, any distribution v defined by $\langle v(t), \varphi\rangle := \langle g, \psi(t) * \varphi\rangle$ satisfies any PDE in the sense of distributions if its kernel or fundamental solution $\psi(t)$ also does so.

Lemma 4.6. *Let $g \in \mathcal{D}'(\mathbb{R})$. Then the distribution w defined by*

$$\langle w, \varphi\rangle := \langle g, \psi_t * \varphi\rangle$$

for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ solves the problem

$$\begin{gathered} w_{tt} = c^2 w_{xx} + \frac{k^2}{4} w, \\ w(0) = g, \quad w_t(0) = 0 \end{gathered}$$

in the sense of distributions.

Proof. It is reasonable to define w this way. Indeed, notice ψ_t is compactly supported as a distribution and, as a result, $\psi_t * \varphi \in \mathcal{C}_0^\infty(\mathbb{R})$. We know $\psi(t)$ solves the PDE of (1.2) and differentiating the equation with respect to time we obtain that ψ_t also solves it. Consequently, Remark 4.5 implies that w satisfies the PDE in the sense of distributions. As for the initial conditions, they are easily deduced (remember that $\psi_t(0) = \delta$). □

With these partial results we are now able to prove Theorem 1.2.

Proof of Theorem 1.2. We divide the problem into two smaller ones, one with homogeneous first initial condition and the other with homogeneous second initial condition. We use Lemma 4.6 for the problem that has a distribution as its first initial condition and we use Proposition 4.1 for the problem that has a distribution as its second initial condition. Then the solution for the general problem is the sum of these two partial solutions acting on the same test function. □

To prove uniqueness of solutions we present the following result.

Proposition 4.7. *Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, let $f, g \in \mathcal{D}'(\mathbb{R})$ and $v \in \mathcal{C}^1(\mathbb{R}, \mathcal{D}'(\mathbb{R}))$ a solution in the sense of distributions of problem* (1.2)*. Then the function $s(x) = v * \varphi(x) := \langle v, \varphi(x - \cdot)\rangle \in \mathcal{C}^\infty(\mathbb{R})$ is well defined and it is a classical solution in the sense of distributions of*

$$\begin{gathered} s_{tt} = c^2 s_{xx} + \frac{k^2}{4} s, \\ s(x,0) = (f * \varphi)(x), \quad s_t(x,0) = ((g + \frac{k}{2} f) * \varphi)(x) \end{gathered} \tag{4.1}$$

Proof. Both initial conditions are satisfied easily thanks to the initial conditions that v satisfies. As for the differential equation, we need

$$\int_{\mathbb{R}^2} L(s)\phi \, dx \, dt := \int_{\mathbb{R}^2} sL^*(\phi) \, dx \, dt = 0,$$

for all $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$. The integration is already well defined, since $s \in \mathcal{C}^\infty(\mathbb{R})$. Let $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$, we can write

$$\begin{aligned} \int_{\mathbb{R}^2} sL^*(\phi) \, dx \, dt &= \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} sL^*(\phi) dx \Big) dt \\ &= \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \langle v, \varphi(x - \cdot)\rangle L^*(\phi) dx \Big) dt \\ &= \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \langle v, L^*(\phi)\varphi(x - \cdot)\rangle dt \Big) dx \\ &= \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \langle v, L^*(\phi(x,t))\varphi(x - \cdot)\rangle dt \Big) dx \\ &\overset{x - \cdot = z}{\underset{dx = dz}{=}} \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \langle v, L^*(\phi(z + \cdot, t))\varphi(z)\rangle dt \Big) dz \\ &= \int_{\mathbb{R}} \varphi(z) \Big(\int_{\mathbb{R}} \langle v, L^*(\phi(z + \cdot, t))\rangle dt \Big) dz. \end{aligned}$$

Fixing $z \in \mathbb{R}$ and writing $\phi(z+y,t) =: \gamma(y,t)$, we have that $\gamma \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ and $L^*(\phi(z+y,t)) = L^*(\gamma(y,t)) \in \mathcal{C}_0^\infty(\mathbb{R}^2)$. Then

$$\int_{\mathbb{R}^2} sL^*(\phi)\,dx\,dt = \int_{\mathbb{R}} \varphi(z) \underbrace{\Big(\int_{\mathbb{R}} \langle v(t), L^*(\gamma)\rangle dt\Big)}_{0} dz = 0$$

because v is a distributional solution of problem (1.2). Consequently, s solves the problem in the classical sense. □

Now we prove uniqueness of solutions using convolutions and classical solutions.

Proof of Theorem 1.3. Assume $v,\, w \in \mathcal{C}^1(\mathbb{R}, \mathcal{D}'(\mathbb{R}))$ are both solutions of (1.2) and let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$. We define $s(x) := v * \varphi(x)$ and $r(x) := w * \varphi(x)$, both belong to $\mathcal{C}^\infty(\mathbb{R})$ and, by Proposition 4.7, they both are classical solutions of problem (4.1). If we see that $u(x) = s(x) - r(x)$ is the zero function then we are done.

First of all, $u(x,t)$ is continuous in $\mathbb{R}^2$. Continuity on the space variable x is granted by the convolution properties. Continuity on the time variable t is deduced thanks to Theorem 1.2 and the definition of continuity in the space of distributions. In particular, u is bounded in any compact set.

Let us take $T = \frac{2}{k}$ and $x \in \mathbb{R}$, we consider the characteristic triangle

$$\mathcal{T} = \{(y,s) \in \mathbb{R}^2 :\ s \in [0,T], y \in [x - c(T-s), x + c(T-s)]\}.$$

This triangle is compact, so there exists $M > 0$ such that $|u(y,s)| \le M$ for all (y,s) in $\mathcal{T}$. Since u solves $u_{tt} = c^2 u_{xx} + \frac{k^2}{4} u$ with homogeneous initial conditions, D'Alembert formula gives

$$u(x,t) = \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} \frac{k^2}{4} u(y,s)\,dy\,ds.$$

The pair (y,s) being in $\mathcal{T}$ implies $|u(y,s)| \le M$ and thus

$$\begin{aligned}
|u(x,t)| &\le \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} \frac{k^2}{4} M\,dy\,ds \\
&\le \frac{1}{2c} \frac{k^2}{4} M \int_0^T \int_{x-c(T-s)}^{x+c(T-s)} dy\,ds \\
&= \frac{1}{2c} \frac{k^2}{4} c T^2 M = \frac{M}{2}.
\end{aligned}$$

This way, if $|u(x,t)| \le M$ then $|u(x,t)| \le M/2$ from which we conclude that $u(x,t) = 0$ for all $0 \le t \le T$. Notice this argument can be used at any $x \in \mathbb{R}$ because despite M may depend on $x \in \mathbb{R}$, the time T for which we obtain this contraction does not. Hence, $u(x,t) = 0$ for all $(x,t) \in \mathbb{R} \times [0,T]$.

Now consider $\widetilde{u}(x,t) = u(x, T+t)$, whose first and second initial conditions are $\widetilde{u}(x,0) = u(x,T) = 0$ and $\widetilde{u}_t(x,0) = u_t(x,T) = 0$. We can repeat the same argument and get that $\widetilde{u}(x,t) = 0$, for all $(x,t) \in \mathbb{R} \times [0,T]$, that is, $u(x,t) = 0$ for all $(x,t) \in \mathbb{R} \times [0, 2T]$. By induction, it follows easily that $u(x,t) = 0$ for all $(x,t) \in \mathbb{R} \times \mathbb{R}^+$. The same can be done with negative times, $u(x,t) = 0$ in $\mathbb{R}^2$ and we are done. □

Remark 4.8. The formula given by (1.4) is the unique solution in $\mathcal{C}^1(\mathbb{R},\mathcal{D}'(\mathbb{R}))$ of problem (1.1). It easily follows from taking $v = e^{\frac{k}{2}t}u$ and applying the previous results.

5. Application 1: semi-groups

We now study problem (1.1) when (f,g) belong to the Hilbert space $H^1(\mathbb{R}) \times L^2(\mathbb{R})$. We apply the results found above for general distributions to recover properties of the solution in this special case.

Remark 5.1. Given $(f,g) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$, the solution to the initial value problem (1.1) is

$$\begin{aligned} u(x,t) = e^{\frac{-k}{2}t}\Big\{\frac{1}{2}[f(x+ct)+f(x-ct)] + \alpha ct\int_{x-ct}^{x+ct}\frac{I_0'(2\alpha\sqrt{\lambda})}{\sqrt{\lambda}}f(y)\,dy \\ + \int_{x-ct}^{x+ct}\frac{1}{2c}I_0(2\alpha\sqrt{\lambda})[\frac{k}{2}f(y)+g(y)]dy\Big\}, \end{aligned}$$

with $\lambda = c^2t^2-(x-y)^2$.

Definition 5.2. A family $\{\Gamma_t\}_{t\geq 0}$ of bounded linear operators on a Banach space X into itself is said to have the semi-group property if

$$\Gamma_0 = I_d \quad \text{and} \quad \Gamma_t(\Gamma_s) = \Gamma_{t+s} \quad \forall t,s \geq 0.$$

Let us consider $X = H^1(\mathbb{R}) \times L^2(\mathbb{R})$ and the Γ_t operator such that, for each $t \geq 0$ and for each $(f,g) \in X$, $\Gamma_t(f,g)$ gives the solution and its time derivative of problem (1.1) at time t, with f,g as the initial conditions. To ease the notations, we write $u(t)$ and $u_t(t)$ to refer to such solutions.

Theorem 5.3. *Let $(f,g) \in X$ and $\|\cdot\|_2$ denote the $L^2(\mathbb{R})$ norm. Then*

1. $\Gamma_t(f,g) = (u(t),u_t(t)) \in X$.
2. Γ_t *has the semi-group property.*
3. $\forall t \in \mathbb{R} \quad \exists M_1,M_2,M_3,N_1,N_2,N_3,P_1,P_2 > 0$ *such that*
   (a) $\|u\|_2 \leq e^{\frac{-k}{2}t}\{M_1\|f\|_2 + N_1\|g\|_2\}$,
   (b) $\|u_t\|_2 \leq e^{\frac{-k}{2}t}\{M_2\|f\|_2 + N_2\|g\|_2 + P_1\|f'\|_2\}$,
   (c) $\|u_x\|_2 \leq e^{\frac{-k}{2}t}\{M_3\|f\|_2 + N_3\|g\|_2 + P_2\|f'\|_2\}$,
   the semi-group acts continuously.

Remark 5.4. The proof of Theorem 5.3 can be found in [7], it relies on the fact that $u(x,t)$ is a combination of convolutions, the L^2 estimates of u and its derivatives are obtained applying Young's Inequality properly on each convolution.

6. Application 2: Financial models

Let us assume a particle moves on the discrete set $\{k\Delta x \mid k \in \mathbb{Z}\} \subset \mathbb{R}$ at time intervals of length Δt. Let us suppose that with probability p it repeats the same move as in the previous jump and with probability $1-p$ does the contrary move. In the limiting process, when Δx and Δt are small, we will assume that p is near 1, representing this way some kind of inertia in the movement.

To deduce which laws do the movement follow, let us denote the pair (k,n) =the particle is in the position $x = k\Delta x$ at time $t = n\Delta t$. Let us also define

$$\alpha(k,n) = \mathrm{P}\left(\text{the particle is in } (k,n) \text{ and comes from } (k-1,n-1)\right),$$

$$\beta(k,n) = \mathrm{P}\left(\text{the particle is in } (k,n) \text{ and comes from } (k+1,n-1)\right).$$

We are interested in deducing a law for $\gamma(k,n) = \alpha(k,n) + \beta(k,n)$, which is the probability of the particle being in (k,n). It is well known that γ satisfies

$$\frac{1}{c^2}\gamma_{tt} + \frac{k}{c^2}\gamma_t = \gamma_{xx},$$

where $c = \frac{\Delta x}{\Delta t}$ and $k = \lim_{\Delta t \to 0} \frac{2-2p}{\Delta t} \geq 0$. Reorganizing and denoting $\gamma = u$ we are left with

$$u_{tt} + ku_t = c^2 u_{xx}.$$

Let us now consider a financial asset, a stock share, for example, whose price resembles the motion of the described particle, i.e., it shows certain tendency to repeat the movement previously done. Then, given suitable initial conditions this equation models the probability density function of the price of the asset, a random variable. We study

$$\begin{gathered} u_{tt} + ku_t = c^2 u_{xx} \\ u(0) = \delta, \quad u_t(0) = -c\delta' \end{gathered} \tag{6.1}$$

The first initial condition $u(x,0) = \delta(x)$ means that the particle, in this case the price of the asset is 0 at time $t = 0$. The second initial condition $u_t(x,0) = -c\delta'(x)$ indicates that the price of the asset has an initial tendency to go upwards, to increase its value.

Let us explain $u_t(x,0)$ more carefully. We assume an initial tendency to go upwards, that is, at time Δt we know the particle is in Δx, so we also have $u(x,\Delta t) = \delta(x - \Delta x)$. By definition,

$$u_t(x,0) = \lim_{\Delta t \to 0} \frac{u(x,\Delta x) - u(x,0)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\delta(x - \Delta x) - \delta(x)}{\Delta t},$$

and since we are dealing with distributions, we compute its distributional derivative,

$$\begin{aligned} \Big\langle \frac{\delta(x-\Delta x) - \delta(x)}{\Delta t}, \varphi(x) \Big\rangle &= \int_{\mathbb{R}} \frac{\delta(x-\Delta x) - \delta(x)}{\Delta t} \varphi(x) dx \\ &= \frac{\varphi(\Delta x) - \varphi(0)}{\Delta t} \xrightarrow[\Delta t \to 0]{\Delta x = c\Delta t} c\varphi'(0) \\ &=: \langle -c\delta', \varphi \rangle, \end{aligned}$$

for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, from which we deduce that $u_t(x,0) = -c\delta'(x)$.

Remark 6.1. This is a well-known price evolution model thoroughly studied in [3]. Here we present an alternative construction of the solution using the results we have found for the damped wave equation with general distributions, in this case $(f,g) = (\delta, -c\delta')$, as its initial values.

To solve problem (6.1), we can use the formula we found out for the complete problem applying the considered initial conditions. Let us remember the general solution of the problem when the initial conditions are distributions is

$$\langle u, \varphi \rangle := e^{-kt/2}\Big(\big\langle f, \big(\psi_t + \frac{k}{2}\psi\big) * \varphi \big\rangle + \langle g, \psi * \varphi \rangle \Big).$$

In our case, $f = \delta$ and $g = -c\delta'$ and so we have the following result.

Theorem 6.2. *The solution to the asset price problem* (6.1) *is a probability density function given by*

$$\begin{aligned} u(x,t) &= e^{\frac{-k}{2}t}\delta(x-ct) + e^{\frac{-k}{2}t}\Big(\alpha(x+ct)\frac{I_0'(2\alpha\sqrt{\lambda_0})}{\sqrt{\lambda_0}} \\ &\quad + \frac{k}{4c}I_0(2\alpha\sqrt{\lambda_0})\Big)\mathcal{X}_{[-ct,ct]}(x), \end{aligned}$$

for all $t \geq 0$*, with* $\lambda = c^2t^2 - x^2$.

Remark 6.3. This probability density function is of a mixed type: it has a discrete part governed by the Dirac delta $\delta(x-ct)$ and a continuous part supported in the interval $[-ct, ct]$.

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Marc Nualart
Department of Mathematics, Imperial College London, London, SW7 2AZ, UK
Email address: `m.nualart-batalla20@imperial.ac.uk`