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EXISTENCE OF SOLUTIONS FOR NONCONVEX SECOND-ORDER DIFFERENTIAL INCLUSIONS IN THE INFINITE DIMENSIONAL SPACE

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Abstract. We prove the existence of solutions to the differential inclusion

$$\ddot{x}(t) \in F(x(t), \dot{x}(t)) + f(t, x(t), \dot{x}(t)), \quad x(0) = x_0, \quad \dot{x}(0) = y_0,$$

where f is a Carathéodory function and F with nonconvex values in a Hilbert space such that $F(x, y) \subset \gamma(\partial g(y))$, with g a regular locally Lipschitz function and γ a linear operator.

1. Introduction

In the present paper we consider the Cauchy problem for second-order differential inclusion

$$\begin{gathered} \ddot{x}(t) \in F(x(t), \dot{x}(t)) + f(t, x(t), \dot{x}(t)), \\ x(0) = x_0, \quad \dot{x}(0) = y_0 \end{gathered} \tag{1.1}$$

where $F(\cdot, \cdot)$ is a given set-valued map and f is a Carathéodory function. Second order differential inclusions have been studied by many authors, mainly in the case when the multifunction is convex valued. Several existence results may be found in [2, 8, 10, 13, 14].

Recently in [11] and [12], the situation when the multifunction is not convex valued is considered, the existence of solution for the problem (1.1) was obtained in the finite dimensional case by assuming $F(\cdot, \cdot)$ upper semicontinuous, compact valued multifunction such that $F(x, y) \subset \partial g(y)$ for some convex proper lower semicontinuous function g. In this paper we extend this result in two ways: we consider the infinite dimensional case and we relax the convexity assumption on the function g, namely we suppose that g is uniformly regular and so the usual subdifferentials will be replaced by the Clarke subdifferentials. The class of proper convex lower semicontinuous functions and the class of lower-C^2 functions (see examples 2.2, 2.3) are strictly contained within the class of uniformly regular functions. The paper is organized as follows: in Section 2 we recall some preliminary facts that we need in the sequel and in Section 3 we prove our main result.

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2. Preliminaries

Let $\mathbb{H}$ be a real separable Hilbert space with the norm $\|\cdot\|$ and scalar product $\langle\cdot,\cdot\rangle$. We denote by $\mathbb{B} := \mathbb{B}(0,1)$ the unit open ball of $\mathbb{H}$ and let $\overline{\mathbb{B}}$ be its closure. We denote by $\delta^*(.,A)$ the support function of A, by $d(x,A)$ the distance from $x \in \mathbb{H}$ to A. for any two subsets A, B of $\mathbb{H}$, $d_{\mathbb{H}}(A,B)$ stands to the Hausdorff distance between A and B.

Let σ the weak topology in $\mathbb{H}$. Let us $(e_n)_{n\geq 1}$ be a dense sequence in $\overline{\mathbb{B}}$ and we consider the linear application $\gamma : \mathbb{H} \to \mathbb{H}$ defined by

$$\forall x \in \mathbb{H}, \quad \gamma(x) = \sum_{n=1}^{\infty} 2^{-n}\langle x, e_n\rangle e_n.$$

Note that this series is absolutely convergent. According to the specialists of the theory of linear operators the application γ belongs to the class of the nuclear operators of $\mathbb{H}$. Further, γ satisfies the two following properties:

(a) The restriction of γ to $\overline{\mathbb{B}}$ is continuous from $(\overline{\mathbb{B}}, \sigma)$ into $\mathbb{H}$.

(b) For all $x \in \mathbb{H}\backslash\{0\}$, $\langle x, \gamma(x)\rangle > 0$.

Indeed b) is obvious. This condition is equivalent to

$$x \in \mathbb{H} \mapsto \langle x, \gamma(x)\rangle$$

is a strictly convex function (see [16]).

In the sequel we note by $\Gamma(\mathbb{H})$ the set of linear applications $\gamma : \mathbb{H} \to \mathbb{H}$ verifying the conditions a) and b). $\Gamma(\mathbb{H}) \subset \mathbb{K}(\mathbb{H})$ the space of compact operators of $\mathbb{H}$. If $\mathbb{H} = \mathbb{R}^m$ then $\Gamma(\mathbb{H})$ coincides with the set of the automorphism of $\mathbb{R}^m$ associated to positive definite matrices.

Definition 2.1 ([5]). Let $f : \mathbb{H} \to R \cup \{+\infty\}$ be a lower semicontinuous function and let $\Omega \subset dom f$ be a nonempty open subset. We will say that f is uniformly regular over Ω if there exists a positive number $\beta \geq 0$ such that for all $x \in \Omega$ and for all $\xi \in \partial^P f(x)$ one has

$$\langle \xi, x' - x\rangle \leq f(x') - f(x) + \beta\|x' - x\|^2 \quad \text{for all } x' \in \Omega.$$

Here $\partial^P f(x)$ denotes the proximal subdifferential of f at x (for its definition the reader is refereed for instance to [7]). We will say that f is uniformly regular over closed set S if there exists an open set O containing S such that f is uniformly regular over O. The class of functions that are uniformly regular over sets is so large. We state here some examples.

Example 2.2. Any lower semicontinuous proper convex function f is uniformly regular over any nonempty subset of its domain with $\beta = 0$.

Example 2.3. Any lower-C^2 function f is uniformly regular over any nonempty convex compact subset of its domain. Indeed, let f be a lower-C^2 function over a nonempty convex compact set $S \subset dom f$. By Rockafellar's result (see for instance [14, Theorem 10.33]) there exists a positive real number β such that $g := f + \frac{\beta}{2}\|.\|^2$ is a convex function on S. Using the definition of the subdifferential of convex functions and the fact that the Clarke subdifferential of f is $\partial^C f(x) = \partial g(x) - \beta x$ for any $x \in S$, we get the inequality in definition 2.1 and so f is uniformly regular over S.

The following proposition summarizes some important properties for uniformly regular locally Lipschitz functions over sets needed in the sequel. For the proof of these results we refer the reader to [4, 6].

Proposition 2.4. *Let $g : \mathbb{H} \to \mathbb{R}$ be a locally Lipschitz function and Ω a nonempty open set. If f is uniformly regular over Ω, then the following hold:*

(i) *The proximal subdifferential of g is closed over Ω, that is, for every $x_n \to x \in \Omega$ with $x_n \in \Omega$ and every $\xi_n \to \xi$ with $\xi_n \in \partial^P g(x_n)$ one has $\xi \in \partial^P g(x)$*

(ii) *The proximal subdifferential of g coincides with $\partial^C g(x)$ the Clarke subdifferential for any point x (see for instance [7] for the definition of $\partial^C g$)*

(iii) *The proximal subdifferential of g is upper hemicontinuous over S, that is, the support function $x \mapsto \langle v, \partial^P g(x)\rangle$ is u.s.c. over S for every $v \in \mathbb{H}$*

(iv) *For any absolutely continuous map $x : [0, T] \to \Omega$ for which $\dot{x}(t)$ is absolutely continuous one has*

$$\frac{d}{dt}(f \circ \dot{x})(t) = \langle \partial^C f(\dot{x}(t)); \ddot{x}(t)\rangle.$$

For a multifunction $F : \Omega_1 \times \Omega_2 \subset \mathbb{H} \times \mathbb{H} \to 2^{\mathbb{H}}$ and for any $(x_0, y_0) \in \Omega_1 \times \Omega_2$ we consider the Cauchy problem

$$\ddot{x}(t) \in F(x(t), \dot{x}(t)) + f(t, x(t), \dot{x}(t)), x(0) = x_0, \dot{x}(0) = y_0$$

under the following assumptions:

(H1) Ω_1, Ω_2 are open subsets in $\mathbb{H}$ and $F : \Omega_1 \times \Omega_2 \to 2^{\mathbb{H}}$ is upper semicontinuous (i.e for all $\epsilon > 0$ there exists $\delta > 0$ such that $\|z - z'\| \le \delta$ implies $F(z') \subset F(z) + \epsilon\mathbb{B}$) with compact values.

(H2) There exist $\gamma \in \Gamma(\mathbb{H})$ and a locally Lipschitz β-uniformly regular function $g : \mathbb{H} \to \mathbb{R}$ over Ω_2 such that

$$F(x, y) \subset \gamma(\partial^C g(y)) \quad \text{for all } (x, y) \in \Omega_1 \times \Omega_2. \tag{2.1}$$

(H3) $f : \mathbb{R}^+ \times \mathbb{H} \times \mathbb{H} \to \mathbb{H}$ is a Carathéodory function, (i.e. for every $x, y \in \mathbb{H}, t \longmapsto f(t, x, y)$ is measurable, for $t \in \mathbb{R}^+, (x, y) \longmapsto f(t, x, y)$ is continuous) and for any bounded subset B of $\mathbb{H} \times \mathbb{H}$, there is a compact set K such that $f(t, x, y) \in K$ for all $(t, x, y) \in \mathbb{R}^+ \times B$.

By a solution of problem (1.1) we mean an absolutely continuous function $x(.) : [0, T] \to \mathbb{H}$ with absolutely continuous derivative $\dot{x}(.)$ such that $x(0) = x_0, \dot{x}(0) = y_0$ and $\ddot{x}(t) \in F(x(t), \dot{x}(t)) + f(t, x(t), \dot{x}(t))$ a.e. on $[0, T]$. For more details on differential inclusions, we refer to [1].

3. Main result

Our main result is the following.

Theorem 3.1. *Consider $F : \Omega_1 \times \Omega_2 \to 2^{\mathbb{H}}$, $f : \mathbb{R} \times \mathbb{H} \times \mathbb{H} \to \mathbb{H}$, $g : \mathbb{H} \to \mathbb{R}$ and $\gamma \in \Gamma(\mathbb{H})$ satisfy Hypotheses (H1)-(H3). Then, for every $(x_0, y_0) \in \Omega_1 \times \Omega_2$ there exist $T > 0$ and $x(.) : [0, T] \to \mathbb{H}$ solution to problem* (1.1).

Proof. Let $r > 0$ be such that $\bar{\mathbb{B}}(y_0, r) \subset \Omega_2$ and g is L-Lipschitz on $\bar{\mathbb{B}}(y_0, r)$. Then we have that $\partial^C g(y) \subset L\bar{\mathbb{B}}$, whenever $y \in \bar{\mathbb{B}}(y_0, r)$. By our assumption (H3), there is a positive constant m such that $f(t, x, y) \in K \subset m\mathbb{B}$ for all $(t, x, y) \in$

$\mathbb{R}^+ \times \bar{\mathbb{B}}(x_0, r) \times \bar{\mathbb{B}}(y_0, r)$. Moreover, since $\gamma \in \Gamma(\mathbb{H})$, the set $K_1 := \gamma(L\bar{\mathbb{B}})$ is convex compact in $\mathbb{H}$ and so there exists $m_1 > 0$ such that $K_1 \subset m_1\mathbb{B}$. Choose T such that

$$0 < T < \min\{\frac{r}{m_1 + m}, \frac{r}{r + \|y_0\|}\}$$

Set $I := [0, T]$. For each integer $n \geq 1$ and for $1 \leq i \leq n-1$ we set $t_i^n := \frac{iT}{n}$, $I_i^n := [t_{i-1}^n, t_i^n[$ and $t_n^n = T$, $I_n^n = T$. Let define the following approximate sequences

$$y_n(t) = y_n(t_i^n) + \int_{t_i^n}^t [f(s, x_n(t_i^n), y_n(t_i^n)) + u_i^n] ds$$

$$x_n(t) = x_n(t_i^n) + \int_{t_i^n}^t y_n(s) ds$$

whenever $t \in I_{i+1}^n, 0 \leq i \leq n-1$, where $x_n(0) = x_0$, $y_n(0) = y_0$, and $u_i^n \in F(x_n(t_i^n), y_n(t_i^n))$.

For every $0 \leq i \leq n-1$, take $z_i^n \in \partial^C g(y_n(t_i^n))$ such that $u_i^n = \gamma(z_i^n)$. Now let us define the step functions from $[0, T]$ to $[0, T]$ by

$$\theta_n(t) = t_i^n,\ u_n(t) = u_i^n,\ z_n(t) = z_i^n \quad t \in I_{i+1}^n.$$

Then, for all $n \in \mathbb{N}^*$ and all $t \in [0, T]$, we have the following properties:

$$0 \leq t - \theta_n(t) \leq \frac{T}{n} \tag{3.1}$$

$$y_n(t) = y_0 + \int_0^t [f(s, x_n(\theta_n(s)), y_n(\theta_n(s))) + u_n(s)] ds \tag{3.2}$$

$$x_n(t) = x_0 + \int_0^t y_n(s) ds \tag{3.3}$$

$$u_n(t) \in F(x_n(\theta_n(t)), y_n(\theta_n(t))) \tag{3.4}$$

$$z_n(t) \in \partial^C g(y_n(\theta_n(t))) \tag{3.5}$$

$$u_n(t) = \gamma(z_n(t)). \tag{3.6}$$

Observe that $y_n(t) \in \bar{\mathbb{B}}(y_0, r)$ and $x_n(t) \in \bar{\mathbb{B}}(x_0, r)$ for all $n \in \mathbb{N}^*$ and all $t \in [0, T]$. Indeed it is obvious that

$$\|y_n(t) - y_0\| = \| \int_0^t [f(s, x_n(\theta_n(s)), y_n(\theta_n(s))) + u_n(s)] ds\| \leq (m_1 + m)T < r$$

and

$$\|x_n(t) - x_0\| = \| \int_0^t y_n(s) ds\| \leq (\|y_0\| + r)T < r$$

Hence

$$\|y_n(t) - y_n(t')\| \leq (m_1 + m)|t' - t|$$

whenever $0 \leq t \leq t' \leq T$ and $n \in \mathbb{N}^*$. On the other hand, we have

$$\|x_n(t) - x_n(t')\| \leq \int_t^{t'} \|y_n(s)\| ds \leq (r + \|y_0\|)\, |t - t'|$$

whenever $0 \leq t \leq t' \leq T$ and $n \in \mathbb{N}^*$. Hence $(x_n)_{n \in \mathbb{N}^*}$ and $(y_n)_{n \in \mathbb{N}^*}$ are equi-Lipschitz subsets of $C([0, T], \mathbb{H})$. The sets $\{x_n(t) : n \in \mathbb{N}^*\}$ and $\{y_n(t) : n \in \mathbb{N}^*\}$ are relatively compact in $\mathbb{H}$ for every $t \in [0, T]$. Indeed we have for all $n \in \mathbb{N}^*$ and all $t \in [0, T]$

$$y_n(t) \in y_0 + [0, T]\{K_1 + K\} := K_2$$

which is compact. We have also

$$x_n(t) \in x_0 + [0,T]K_2$$

for all $n \in \mathbb{N}^*$ and all $t \in [0,T]$. Then by Ascoli's theorem, $(x_n)_{n\in\mathbb{N}^*}$ and $(y_n)_{n\in\mathbb{N}^*}$ are relatively compact in the Banach space $C([0,T],\mathbb{H})$. Further, the sequences $(u_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ are relatively $\sigma(L^1([0,T],\mathbb{H});\ L^\infty([0,T],\mathbb{H}))$-compact and $\sigma(L^\infty([0,T],\mathbb{H}); L^1([0,T],\mathbb{H}))$-compact respectively since we have a.e.

$$\forall n \in \mathbb{N}^* \quad u_n(t) \in K_1 \quad \text{and} \quad z_n(t) \in L\bar{\mathbb{B}}\,.$$

Therefore, by extracting subsequences if necessary, we can assume that there exist x in $C([0,T],\mathbb{H})$, y in $C([0,T],\mathbb{H})$, u in $L^1([0,T],\mathbb{H})$ and z in $L^\infty([0,T],\mathbb{H})$ such that $x_n \to x$ in $C([0,T],\mathbb{H})$, $y_n \to y$ in $C([0,T],\mathbb{H})$, $u_n \to u$ for $\sigma(L^1([0,T],\mathbb{H});$ $L^\infty([0,T],\mathbb{H}))$-topology and $z_n \to z$ for $\sigma(L^\infty([0,T],\mathbb{H});\ L^1([0,T],\mathbb{H}))$-topology.
Also, we have $f(.,x_n(\theta_n(.)),y_n(\theta_n())) \to f(.,x(.),y(.))$ in the norm of the space $L^1([0,T],\mathbb{H})$. Consequently, for all $t \in [0,T]$,

$$x_0 + \int_0^t \dot{x}(s)ds = x(t) = \lim_{n\to\infty} x_n(t) = x_0 + \lim_{n\to\infty}\int_0^t y_n(s)ds = x_0 + \int_0^t y(s)ds$$

which gives the equality

$$\dot{x}(t) = y(t) \quad \text{for almost } t \in [0,T]. \tag{3.7}$$

Now we assert that $u = \gamma(z)$ a.e. Indeed, for any $w \in \mathbb{H}$ and any measurable set A in $[0,T]$, one has

$$\begin{aligned}
\langle w, \int_A u(\eta)d\eta\rangle &= \int_A \langle w, u(\eta)\rangle d\eta \\
&= \lim_{n\to\infty}\int_A \langle w, u_n(\eta)\rangle d\eta \\
&= \lim_{n\to\infty}\langle w, \int_A \gamma(z_n(\eta))d\eta\rangle \\
&= \lim_{n\to\infty}\langle w, \gamma(\int_A z_n(\eta)d\eta)\rangle \\
&= \langle w, \gamma(\int_A z(\eta)d\eta)\rangle \\
&= \langle w, \int_A \gamma(z(\eta)d\eta)\rangle.
\end{aligned}$$

Hence $u(t) = \gamma(z(t))$ for almost every $t \in [0,T]$. Note that $\lim_{n\to\infty} x_n(\theta_n(t)) = x(t)$ and $\lim_{n\to\infty} y_n(\theta_n(t)) = y(t)$, for all $t \in [0,T]$ where $y(t) = y_0 + \int_0^t \dot{y}(s)ds$, for all $t \in [0,T]$. Then it follows from (3.2) that $\dot{y}(t) = f(t,x(t),y(t)) + u(t)$ for almost $t \in [0,T]$ and by (3.7) we obtain that

$$\ddot{x}(t) = f(t,x(t),y(t)) + u(t)$$

for almost $t \in [0,T]$. By construction, we have for a.e $t \in [0,T]$,

$$\begin{aligned}
\ddot{x}_n(t) - f_n(t) = u_n(t) &\in F(x_n(\theta_n(t)), y_n(\theta_n(t))) \\
&\subset \gamma(\partial^C g(y_n(\theta_n(t)))) \\
&= \gamma(\partial^P g(y_n(\theta_n(t)))).
\end{aligned} \tag{3.8}$$

Since $y_n(\theta_n(t) \in \bar{\mathbb{B}}(y_0, r) \subset \Omega_2$, the last equality follows from the uniform regularity of g over Ω_2 and the part (ii) in proposition 2.4. The convergence of z_n to z for $\sigma(L^\infty([0,T],\mathbb{H}); L^1([0,T],\mathbb{H}))$-topology and Mazur's Lemma entails

$$z \in \bigcap_n \overline{co}^\sigma\{z_m : m \geq n\}, \quad \text{for a.e. } t \in [0,T]$$

(here $\sigma = \sigma(L^\infty([0,T],\mathbb{H}); L^1([0,T],\mathbb{H}))$. Fix any such t and consider any $\xi \in \mathbb{H}$. Then, the last relation above yields

$$\langle \xi, z(t)\rangle \leq \inf_n \sup_{m\geq n} \langle \xi, z_n(t)\rangle$$

and by proposition 2.4 part (iii) and (3.5) yield

$$\begin{aligned}\langle \xi, z(t)\rangle &\leq \limsup_n \delta^*(\xi, \partial^P g(y_n(\theta_n(t)))) \\ &\leq \delta^*(\xi, \partial^p g(y(t))) \quad \text{for any } \xi \in H,\end{aligned}$$

So, by [9, Theorem VI.4], the convexity and the closeness of the set $\partial^p g(y(t))$ ensures

$$z(t) \in \partial^p g(y(t))$$

Now, since g is uniformly regular over Ω_2 and $\dot{x}(t) = y(t) \in \bar{\mathbb{B}}(y_0, r) \subset \Omega_2$ for all $t \in [0,T]$ we have by proposition 2.4 part (iv)

$$\begin{aligned}\frac{d}{dt}(g \circ \dot{x})(t) &= \langle \partial^p g(\dot{x}(t)), \ddot{x}(t)\rangle = \langle z(t), \ddot{x}(t)\rangle \\ &= \langle z(t), f(t, x(t), y(t)) + u(t)\rangle.\end{aligned}$$

Consequently,

$$g(\dot{x}(T)) - g(y_0) = \int_0^T \langle z(t), f(t, x(t), y(t))\rangle dt + \int_0^T \langle z(t), u(t)\rangle dt \tag{3.9}$$

On the other hand, since $y_n(\theta_n(t) \in \bar{\mathbb{B}}(y_0, r) \subset \Omega_2$ and by (3.8) and definition 2.1, we have for all $i \in \{0, \dots, n-1\}$

$$\begin{aligned}
&g(y_{i+1}^n) - g(y_i^n) \\
&\geq \langle z_i^n, y_{i+1}^n - y_i^n\rangle - \beta\|y_{i+1}^n - y_i^n\|^2 \\
&= \langle z_i^n, \int_{t_i^n}^{t_{i+1}^n} \ddot{x}_n(s) ds\rangle - \beta\|y_{i+1}^n - y_i^n\|^2 \\
&= \langle z_n(t), \int_{t_i^n}^{t_{i+1}^n} [f(s, x_n(\theta_n(s)), y_n(\theta_n(s))) + u_n(s)] ds\rangle - \beta\|y_{i+1}^n - y_i^n\|^2 \\
&\geq \int_{t_i^n}^{t_{i+1}^n} \langle z_n(s), f(s, x_n(\theta_n(s)), y_n(\theta_n(s)))\rangle ds + \int_{t_i^n}^{t_{i+1}^n} \langle z_n(s), u_n(s)\rangle ds \\
&\quad - \beta(m_1 + m)^2 (t_{i+1}^n - t_i^n)^2
\end{aligned}$$

By adding, we obtain

$$\begin{aligned}
&g(\dot{x}_n(T)) - g(y_0) \\
&\geq \int_0^T \langle z_n(s), f(s, x_n(\theta_n(s)), y_n(\theta_n(s)))\rangle ds + \int_0^T \langle z_n(s), u_n(s)\rangle ds - \varepsilon_n
\end{aligned} \tag{3.10}$$

with

$$\varepsilon_n = \frac{\beta(m_1+m)^2T^2}{n} \to 0$$

as $n \to \infty$. We have also have

$$\lim_{n\to\infty}\int_0^T \langle z_n(s), f(s, x_n(\theta_n(s)), y_n(\theta_n(s)))\rangle ds = \int_0^T \langle z(s), f(s, x(s), y(s))\rangle ds.$$

Indeed, for all $t \in [0,T]$ and all $n \in \mathbb{N}^*$,

$$\langle z_n(t), f(t, x_n(\theta_n(t)), y_n(\theta_n(t)))\rangle - \langle z(t), f(t, x(t), y(t))\rangle = \alpha_n(t) + \beta_n(t)$$

where

$$\alpha_n(t) = \langle z_n(t), f(t, x_n(\theta_n(t)), y_n(\theta_n(t))) - f(t, x(t), y(t))\rangle,$$
$$\beta_n(t) = \langle z_n(t) - z(t), f(t, x(t), y(t))\rangle$$

Since $z_n(t) - z(t) \to 0$ for $\sigma(L^\infty([0,T],\mathbb{H}); L^1([0,T],\mathbb{H}))$,

$$\int_0^T \beta_n(s)ds \to 0$$

and $f_n \to f$ strongly in $L^1([0,T],\mathbb{H})$ which implies

$$\int_0^T \alpha_n(s)ds \to 0.$$

Taking the limit superior in (3.10) when $n \to \infty$ and using the continuity of g, we obtain

$$g(\dot{x}(T)) - g(y_0) \geq \int_0^T \langle z(s), f(s, x(s), y(s))\rangle ds + \limsup_n \int_0^T \langle z_n(s), u_n(s)\rangle ds$$

This inequality compared with (3.9) yields

$$\limsup_n \int_0^T \langle z_n(s), u_n(s)\rangle ds \leq \int_0^T \langle z(s), u(s)\rangle ds \tag{3.11}$$

The values of the function z_n are in the convex weakly compact $C := L\bar{\mathbb{B}}$, further the application $\Lambda : (\mathbb{H}, \sigma) \to [0. + \infty]$ defined by

$$\Lambda(\alpha) = \begin{cases} \langle \alpha, \gamma(\alpha)\rangle & \text{if } \alpha \in C \\ +\infty & \text{otherwise} \end{cases}$$

is lower semicontinuous and strictly convex on C (According to a) and b)). The condition (3.11) is equivalent to

$$\limsup_n \int_0^T \Lambda(z_n(s))ds \leq \int_0^T \Lambda(z(s))ds\,.$$

Then [3, Proposition 3.2] yields

$$z(t) \in \bigcap_n \overline{co}^\sigma\{z_m(t) : m \geq n\}, \quad \text{for a.e } t \in [0,T].$$

Hence there is a negligible N such that for $t \notin N$, we have

$$u(t) = \gamma(z(t))$$
$$z(t) \in \bigcap_n \overline{co}^\sigma\{z_m(t) : m \geq n\}.$$

Now let $t \notin N$ be fixed. Then we can extract from $(z_n(t))_{n\in\mathbb{N}}$ a subsequence $(z_{n_k}(t))_{k\in\mathbb{N}}$, such that $z_{n_k}(t) \rightharpoonup z(t)$ weakly in $\mathbb{H}$ so that $\gamma(z_{n_k}(t)) \to \gamma(z(t))$ for the norm topology since $\gamma \in \Gamma(\mathbb{H})$. By (3.4) and (3.6), recalling that

$$u_n(t) = \gamma(z_n(t)) \in F(x_n(\theta_n(t)), y_n(\theta_n(t)))$$

for every $t \in [0,T]$ and every $n \in \mathbb{N}^*$, that $\lim_{n\to\infty} x_n(\theta_n(t)) = x(t)$, $\lim_{n\to\infty} y_n(\theta_n(t)) = y(t) = \dot{x}(t)$, for all $t \in [0,T]$ and that the graph of F is closed, we obtain

$$u(t) = \gamma(z(t)) \in F(x(t)), \dot{x}(t)) \quad \text{a.e.} \tag{3.12}$$

Since $\ddot{x}(t) = f(t, x(t), y(t)) + u(t)$ for almost $t \in [0,T]$ it follows from (3.12) that

$$\ddot{x}(t) \in F(x(t), \dot{x}(t)) + f(t, x(t), \dot{x}(t)) \quad \text{a.e. on } [0,T].$$

Therefore, differential inclusion (1.1) admits a solution. □

Remark 3.2. An inspection of the proof of Theorem 3.1 shows that the uniformity of the constant β was needed only over the ball $B(y_0, \rho)$ and so it was not necessary over all the set Ω_2.

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