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POSITIVE SOLUTIONS OF A NONLINEAR HIGHER ORDER BOUNDARY-VALUE PROBLEM

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Abstract. The authors consider the higher order boundary-value problem

$$u^{(n)}(t) = q(t)f(u(t)), \quad 0 \le t \le 1,$$
$$u^{(i-1)}(0) = u^{(n-2)}(p) = u^{(n-1)}(1) = 0, \quad 1 \le i \le n-2,$$

where $n \ge 4$ is an integer, and $p \in (1/2, 1)$ is a constant. Sufficient conditions for the existence and nonexistence of positive solutions of this problem are obtained. The main results are illustrated with an example.

1. Introduction

We consider the problem of the existence and nonexistence of positive solutions of the nonlinear n-th order ordinary differential equation

$$u^{(n)}(t) = g(t)f(u(t)), \quad 0 \le t \le 1, \tag{1.1}$$

subject to boundary conditions

$$\begin{gathered} u^{(i-1)}(0) = 0, \quad 1 \le i \le n-2, \\ u^{(n-2)}(p) = 0, \quad u^{(n-1)}(1) = 0, \end{gathered} \tag{1.2}$$

where

- (H1) $n \ge 4$ is a fixed integer, $p \in (1/2, 1)$ is constant, and
- (H2) $f : [0, \infty) \to [0, \infty)$ and $g : [0, 1] \to [0, \infty)$ are continuous, and $g(t) \not\equiv 0$ on $[0, 1]$.

Our interest here is in obtaining positive solutions to this boundary-value problem, that is, solutions $u(t)$ such that $u(t) > 0$ for $t \in (0, 1)$.

The importance of boundary-value problems in a wide variety of applications in the physical, biological and engineering sciences is now well documented in the literature, and in the last ten years this has become an extremely active area of research. The monographs of Agarwal [1] and Agarwal, O'Regan, and Wong [3] contain excellent surveys of known results. Recent contributions to the study of multipoint boundary-value problems can be found in the papers of Agarwal and Kiguradze [2], Anderson and Davis [4], Cao and Ma [5], Graef, Henderson and

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Yang [6], Graef, Qian, and Yang [7, 8], Graef and Yang [9, 10], Hu and Wang [12], Infante [13], Infante and Webb [14], Kong and Kong [15], Ma [17, 18, 19], Maroun [20], Raffoul [21], Wang [22], Webb [23, 24], and Zhou and Xu [25]. The three-point boundary conditions considered here, namely, conditions (1.2) above, have been used by many authors in the study of existence of positive solutions of second order problems. Here, we use these conditions but for problems involving higher order ($n \geq 4$) differential equations.

Let $G_3 : [0,1] \times [0,1] \to [0,\infty)$ be defined by

$$G_3(t,s) = \begin{cases} t(2s-t)/2, & t \leq s \leq p, \\ s^2/2, & s \leq t, \text{ and } s \leq p, \\ t(2p-t)/2, & t \leq s, \text{ and } s \geq p, \\ t(2p-t)/2 + (t-s)^2/2, & t \geq s \geq p. \end{cases}$$

For $n \geq 4$, we define

$$G_n(t,s) = \int_0^t G_{n-1}(v,s)dv, \quad (t,s) \in [0,1] \times [0,1].$$

Then, for $n \geq 4$, $G_n(t,s)$ is the Green's function for the equation

$$u^{(n)}(t) = 0$$

subject to the boundary conditions (1.2). Moreover, solving the problem (1.1)–(1.2) is equivalent to finding a solution to the integral equation

$$u(t) = \int_0^1 G_n(t,s)q(s)f(u(s))\,ds, \quad 0 \leq t \leq 1.$$

It is obvious that

$$G_n(t,s) > 0, \quad \text{for} \quad t,s \in (0,1) \text{ and } n \geq 3.$$

Throughout this paper, we let

$$F_0 = \limsup_{x \to 0^+}(f(x)/x), \quad f_0 = \liminf_{x \to 0^+}(f(x)/x),$$
$$F_\infty = \limsup_{x \to +\infty}(f(x)/x), \quad f_\infty = \liminf_{x \to +\infty}(f(x)/x).$$

To prove our results, we will use the following fixed point theorem known as the Guo-Krasnosel'skii fixed point theorem [11, 16].

Theorem 1.1. *Let X be a Banach space over the reals, and let $P \subset X$ be a cone in X. Assume that Ω_1 and Ω_2 are bounded open subsets of X with $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$, and let*

$$L : P \cap (\overline{\Omega_2} - \Omega_1) \to P$$

be a completely continuous operator such that, either one of the following two conditions hold.

(K1) $\|Lu\| \leq \|u\|$ *for* $u \in P \cap \partial\Omega_1$ *and* $\|Lu\| \geq \|u\|$ *for* $u \in P \cap \partial\Omega_2$,
(K2) $\|Lu\| \geq \|u\|$ *for* $u \in P \cap \partial\Omega_1$ *and* $\|Lu\| \leq \|u\|$ *for* $u \in P \cap \partial\Omega_2$.

Then L has a fixed point in $P \cap (\overline{\Omega_2} - \Omega_1)$.

The next section contains some preliminary lemmas; our main results appear in Sections 3 and 4.

2. Preliminary Lemmas

The following lemmas will be used in the proofs of our main results.

Lemma 2.1. *If $u \in C^n[0,1]$ satisfies the boundary conditions* (1.2) *and*
$$u^{(n)}(t) \ge 0 \quad \text{for } 0 \le t \le 1, \tag{2.1}$$
then for each $i = 0, 1, 2, \dots, n-3$, we have
$$u^{(i)}(t) \ge 0 \quad \text{for } 0 \le t \le 1. \tag{2.2}$$

Proof. If we define $w(t) = u^{(n-3)}(t)$ for $0 \le t \le 1$, then we have
$$\begin{gathered} w'''(t) \ge 0 \quad \text{for } 0 \le t \le 1, \\ w(0) = w'(p) = w''(1) = 0. \end{gathered}$$
Therefore,
$$u^{(n-3)}(t) = w(t) = \int_0^1 G_3(t,s) w'''(t)\, dt \ge 0, \quad 0 \le t \le 1.$$
Since $u(0) = u'(0) = \dots = u^{(n-4)}(0) = 0$, we have
$$u^{(i)}(t) \ge 0 \quad \text{for} \quad 0 \le t \le 1 \text{ and } i = 0, 1, \dots, n-3,$$
which completes the proof of the lemma. □

The next two lemmas give estimates on the growth of $u(t)$.

Lemma 2.2. *If $u \in C^n[0,1]$ satisfies* (1.2) *and* (2.1)*, then*
$$u(t) \ge t^{n-2} u(1) \quad \text{for} \quad 0 \le t \le 1.$$

Proof. If we define
$$h(t) = u(t) - t^{n-2} u(1), \quad 0 \le t \le 1, \tag{2.3}$$
then
$$h^{(n)}(t) = u^{(n)}(t) \ge 0, \quad 0 \le t \le 1. \tag{2.4}$$
To prove the lemma, it suffices to show that $h(t) \ge 0$ for $0 \le t \le 1$. It is easy to see from (2.3) that
$$h(0) = h'(0) = \dots = h^{(n-3)}(0) = h(1) = 0.$$
Since $h(0) = h(1) = 0$, by the Mean Value Theorem, there exists $r_1 \in (0,1)$ such that $h'(r_1) = 0$. Similarly, $h'(0) = h'(r_1) = 0$ implies that there exists $r_2 \in (0, r_1)$ such that $h''(r_2) = 0$. Continuing this procedure, we can find a sequence of numbers
$$1 > r_1 > r_2 > \dots > r_{n-3} > 0$$
such that
$$h^{(i)}(r_i) = 0, \quad 0 \le i \le n-3.$$
It is also easy to see from (2.3) that $h^{(n-1)}(1) = 0$. Since
$$h^{(n)}(t) \ge 0 \quad \text{for } 0 \le t \le 1, \tag{2.5}$$
we have
$$h^{(n-1)}(t) \le 0 \quad \text{for } 0 \le t \le 1. \tag{2.6}$$
This implies that $h^{(n-3)}(t)$ is concave downward.

Because $h^{(n-3)}(0) = h^{(n-3)}(r_{n-3}) = 0$, we have

$$h^{(n-3)}(t) \geq 0 \quad \text{on } [0, r_{n-3}] \quad \text{and} \quad h^{(n-3)}(t) \leq 0 \quad \text{on } [r_{n-3}, 1]. \tag{2.7}$$

In view of (2.7) and the fact that $h^{(n-4)}(0) = h^{(n-4)}(r_{n-4}) = 0$, we have

$$h^{(n-4)}(t) \geq 0 \quad \text{on } [0, r_{n-4}] \quad \text{and} \quad h^{(n-4)}(t) \leq 0 \quad \text{on } [r_{n-4}, 1].$$

If we continue this procedure, we finally obtain

$$h'(t) \geq 0 \quad \text{on } [0, r_1] \quad \text{and} \quad h'(t) \leq 0 \quad \text{on } [r_1, 1]. \tag{2.8}$$

Combining (2.8) with the fact that $h(0) = h(1) = 0$ yields

$$h(t) \geq 0 \quad \text{for } 0 \leq t \leq 1,$$

which completes the proof of the lemma. □

Lemma 2.3. *If $u \in C^n[0, 1]$ satisfies* (1.2) *and* (2.1)*, then*

$$u(t) \leq t^{n-4} u(1) \quad \textit{for } t \in [0, 1].$$

Proof. If we define

$$h(t) = t^{n-4} u(1) - u(t), \quad t \in [0, 1]. \tag{2.9}$$

then

$$h^{(n)}(t) = -u^{(n)}(t) \leq 0, \quad 0 \leq t \leq 1. \tag{2.10}$$

To prove the lemma, it suffices to show that $h(t) \geq 0$ for $0 \leq t \leq 1$. It is easy to see from (2.9) that

$$h(0) = h'(0) = \cdots = h^{(n-5)}(0) = h(1) = 0.$$

By the Mean Value Theorem, in view of the fact that $h(0) = h(1) = 0$, there exists $r_1 \in (0, 1)$ such that $h'(r_1) = 0$. Because $h'(0) = h'(r_1) = 0$, there exists $r_2 \in (0, r_1)$ such that $h''(r_2) = 0$. If we continue this procedure, then we can find a sequence of numbers

$$1 > r_1 > r_2 > \cdots > r_{n-4} > 0$$

such that

$$h^{(i)}(r_i) = 0, \quad 0 \leq i \leq n - 4.$$

We can also see from (2.9) that

$$h^{(n-3)}(0) = h^{(n-2)}(p) = h^{(n-1)}(1) = 0.$$

Therefore, we have

$$h^{(n-3)}(t) = \int_0^1 G_3(t, s) h^{(n)}(s)\, ds \leq 0, \quad 0 \leq t \leq 1.$$

This means that $h^{(n-4)}(t)$ is nonincreasing. Since $h^{(n-4)}(r_{n-4}) = 0$, we have

$$h^{(n-4)}(t) \geq 0 \quad \text{on } [0, r_{n-4}] \quad \text{and} \quad h^{(n-4)}(t) \leq 0 \quad \text{on } [r_{n-4}, 1].$$

Since $h^{(n-5)}(0) = h^{(n-5)}(r_{n-5}) = 0$, we have

$$h^{(n-5)}(t) \geq 0 \quad \text{on } [0, r_{n-5}] \quad \text{and} \quad h^{(n-5)}(t) \leq 0 \quad \text{on } [r_{n-5}, 1].$$

If we continue this procedure, we finally obtain

$$h'(t) \geq 0 \quad \text{on } [0, r_1] \quad \text{and} \quad h'(t) \leq 0 \quad \text{on } [r_1, 1]. \tag{2.11}$$

Combining (2.11) with the fact that $h(0) = h(1) = 0$ yields

$$h(t) \geq 0 \quad \text{for } 0 \leq t \leq 1,$$

which completes the proof of the lemma. □

The next theorem is a direct consequence of Lemmas 2.1, 2.2, and 2.3.

Theorem 2.4. *If $u \in C^n[0,1]$ satisfies* (1.2) *and* (2.1)*, then $0 \le u(t) \le u(1)$ for $0 \le t \le 1$, and*

$$t^{n-4}u(1) \ge u(t) \ge t^{n-2}u(1) \quad \text{for } 0 \le t \le 1. \tag{2.12}$$

In particular, if $u(t)$ is a nonnegative solution to the problem (1.1)–(1.2)*, then $u(t)$ satisfies* (2.12).

Note that Theorem 2.4 provides both an upper and a lower estimate to each positive solution to the problem (1.1)–(1.2).

3. Existence of Positive Solutions

We begin by introducing some notation. Define

$$A = \int_0^1 G_n(1,s)g(s)s^{n-2}\,ds \quad \text{and} \quad B = \int_0^1 G_n(1,s)g(s)s^{n-4}\,ds.$$

Let $X = C[0,1]$ with the supremum norm

$$\|v\| = \max_{t\in[0,1]} |v(t)|, \quad v \in X,$$

and let

$$P = \big\{v \in X : v(1) \ge 0,\ t^{n-2}v(1) \le v(t) \le v(1)t^{n-4} \text{ on } [0,1]\big\}.$$

Obviously X is a Banach space and P is a positive cone of X. Define the operator $T : P \to X$ by

$$Tu(t) = \int_0^1 G_n(t,s)g(s)f(u(s))ds, \quad 0 \le t \le 1,\ u \in P.$$

By a standard argument we can show that $T : P \to X$ is a completely continuous operator. It is obvious that if $u \in P$, then $u(1) = \|u\|$. We see from Theorem 2.4 that if $u(t)$ is a nonnegative solution to the problem (1.1)–(1.2), then $u \in P$. In a similar fashion to the proof of Theorem 2.4, we can show that $T(P) \subset P$. To find a positive solution to the problem (1.1)–(1.2), we only need to find a fixed point u of T such that $u \in P$ and $u(1) = \|u\| > 0$.

We now give our first existence result.

Theorem 3.1. *If $BF_0 < 1 < Af_\infty$, then the problem* (1.1)–(1.2) *has at least one positive solution.*

Proof. Choose $\varepsilon > 0$ such that $(F_0 + \varepsilon)B \le 1$. There exists $H_1 > 0$ such that

$$f(x) \le (F_0 + \varepsilon)x \quad \text{for } 0 < x \le H_1.$$

For each $u \in P$ with $\|u\| = H_1$, we have

$$\begin{aligned}
(Tu)(1) &= \int_0^1 G_n(1,s)g(s)f(u(s))\,ds \\
&\le (F_0+\varepsilon)\int_0^1 G_n(1,s)g(s)u(s)\,ds \\
&\le (F_0+\varepsilon)\|u\|\int_0^1 G_n(1,s)g(s)s^{n-4}ds \\
&\le (F_0+\varepsilon)\|u\|B \le \|u\|,
\end{aligned}$$

which means $\|Tu\| \le \|u\|$. If we let $\Omega_1 = \{u \in X : \|u\| < H_1\}$, then

$$\|Tu\| \le \|u\| \quad \text{for } u \in P \cap \partial\Omega_1.$$

Next we construct Ω_2. Since $1 < Af_\infty$, we can choose $c \in (0, 1/4)$ and $\delta > 0$ such that

$$(f_\infty - \delta)\int_c^1 G_n(1,s)g(s)s^{n-2}\,ds > 1.$$

There exists $H_3 > 0$ such that

$$f(x) \ge (f_\infty - \delta)x \quad \text{for } x \ge H_3.$$

Let $H_2 = \max\{H_3 c^{2-n}, 2H_1\}$. Now if $u \in P$ with $\|u\| = H_2$, then for $c \le t \le 1$, we have

$$u(t) \ge t^{n-2}\|u\| \ge c^{n-2}H_2 \ge H_3,$$

and

$$\begin{aligned}
(Tu)(1) &\ge \int_c^1 G_n(1,s)g(s)f(u(s))ds \\
&\ge (f_\infty-\delta)\int_c^1 G_n(1,s)g(s)u(s)ds \\
&\ge (f_\infty-\delta)\|u\|\int_c^1 G_n(1,s)g(s)s^{n-2}\,ds \ge \|u\|,
\end{aligned}$$

which means $\|Tu\| \ge \|u\|$. So, if we let $\Omega_2 = \{u \in X \mid \|u\| < H_2\}$, then $\overline{\Omega_1} \subset \Omega_2$ and

$$\|Tu\| \ge \|u\| \quad \text{for } u \in P \cap \partial\Omega_2.$$

Since the condition (K1) of Theorem 1.1 is satisfied, there exists a fixed point of T in P, and this completes the proof of the theorem. $\square$

Theorem 3.2. *If $BF_\infty < 1 < Af_0$, then the problem* (1.1)–(1.2) *has at least one positive solution.*

Proof. Choose $\varepsilon > 0$ such that $(f_0 - \varepsilon)A \ge 1$. There exists $H_1 > 0$ such that

$$f(x) \ge (f_0 - \varepsilon)x \quad \text{for } 0 < x \le H_1.$$

So, for each $u \in P$ with $\|u\| = H_1$, we have

$$
\begin{aligned}
(Tu)(1) &= \int_0^1 G_n(1,s)g(s)f(u(s))\,ds \\
&\geq (f_0 - \varepsilon)\int_0^1 G_n(1,s)g(s)u(s)\,ds \\
&\geq (f_0 - \varepsilon)\|u\|\int_0^1 G_n(1,s)g(s)s^{n-2}\,ds \\
&\geq A(f_0 - \varepsilon)\|u\| \geq \|u\|,
\end{aligned}
$$

which means $\|Tu\| \geq \|u\|$. If we let $\Omega_1 = \{u \in X : \|u\| < H_1\}$, then

$$
\|Tu\| \geq \|u\|, \quad \text{for } u \in P \cap \partial\Omega_1.
$$

To construct Ω_2, we choose $\delta \in (0,1)$ such that $((F_\infty + \delta)B + \delta) \leq 1$. There exists $H_3 > 0$ such that

$$
f(x) \leq (F_\infty + \delta)x \quad \text{for } x \geq H_3.
$$

If we let $M = \max_{0 \leq x \leq H_3} f(x)$, then $f(x) \leq M + (F_\infty + \delta)x$ for $x \geq 0$. Let

$$
K = M\int_0^1 G(1,s)g(s)ds
$$

and let $H_2 = \max\{2H_1, K(1-(F_\infty+\delta)B)^{-1}\}$. Now for each $u \in P$ with $\|u\| = H_2$, we have

$$
\begin{aligned}
(Tu)(1) &= \int_0^1 G_n(1,s)g(s)f(u(s))\,ds \\
&\leq \int_0^1 G_n(1,s)g(s)(M + (F_\infty + \delta)u(s))\,ds \\
&\leq K + (F_\infty + \delta)\int_0^1 G_n(1,s)g(s)u(s)\,ds \\
&\leq K + (F_\infty + \delta)H_2\int_0^1 G_n(1,s)g(s)s^{n-4}\,ds \\
&\leq K + (F_\infty + \delta)BH_2 \\
&\leq (1 - (F_\infty + \delta)B)H_2 + (F_\infty + \delta)BH_2 = H_2,
\end{aligned}
$$

which means $\|Tu\| \leq \|u\|$. So, if we let $\Omega_2 = \{u \in X \mid \|u\| < H_2\}$, then $\overline{\Omega_1} \subset \Omega_2$ and

$$
\|Tu\| \leq \|u\|, \quad \text{for } u \in P \cap \partial\Omega_2.
$$

By Theorem 1.1, T has a fixed point in $P \cap (\overline{\Omega_2} - \Omega_1)$. Therefore, problem (1.1)–(1.2) has at least one positive solution. This completes the proof of the theorem. □

4. Nonexistence Results and Example

In this section, we establish some nonexistence results for the positive solutions of the problem (1.1)–(1.2).

Theorem 4.1. *Suppose that (H1) and (H2) hold. If $Bf(x) < x$ for all $x > 0$, then problem* (1.1)–(1.2) *has no positive solutions.*

Proof. Assume to the contrary that $u(t)$ is a positive solution of the problem (1.1)–(1.2). Then $u \in P$, $u(t) > 0$ for $0 < t \le 1$, and

$$\begin{aligned} u(1) &= \int_0^1 G_n(1,s)g(s)f(u(s))\,ds \\ &< B^{-1}\int_0^1 G_n(1,s)g(s)u(s)\,ds \\ &\le B^{-1}u(1)\int_0^1 G_n(1,s)g(s)s^{n-4}\,ds = u(1), \end{aligned}$$

which is a contradiction. □

Similarly, we have the following result.

Theorem 4.2. *Suppose that (H1) and (H2) hold. If $Af(x) > x$ for all $x > 0$, then problem (1.1)–(1.2) has no positive solutions.*

The proof of Theorem 4.2 is quite similar to that of Theorem 4.1 and is therefore omitted.

In [6], the present authors considered this same boundary-value problem and obtained sufficient conditions for the existence of at least one positive solution and sufficient conditions for there to be no positive solutions. The approach used in [6] was an adaptation of the technique used in [10]. The following example not only illustrates the main results in this paper but in fact shows that the results here are better than those obtained in [6].

Example 4.3. Consider the boundary-value problem

$$u^{(6)}(t) = (2t+t^2)\frac{\lambda u(t)(1+3u(t))}{1+u(t)}, \quad 0 < t < 1, \tag{4.1}$$

$$u(0) = u'(0) = u''(0) = u'''(0) = u^{(4)}(3/4) = u^{(5)}(1) = 0. \tag{4.2}$$

This problem is a special case of the problem (1.1)–(1.2), in which $n = 6$, $p = 3/4$, $g(t) = 2t + t^2$, and

$$f(u) = \lambda u(1+3u)/(1+u) \quad \text{for } u \ge 0.$$

Here $\lambda > 0$ is a parameter. It is easy to see that $F_0 = f_0 = \lambda$, $F_\infty = f_\infty = 3\lambda$, and $\lambda u \le f(u) \le 3\lambda u$ for $u \ge 0$. For the problem (4.1)–(4.2), calculations show that

$$A = 1926477939/70000000000 \quad \text{and} \quad B = 1284866333/35000000000.$$

By Theorem 3.1, we have that if

$$12.1120 \approx \frac{1}{3A} < \lambda < \frac{1}{B} \approx 27.2401,$$

then problem (1.1)–(1.2) has at least one positive solution. By Theorems 4.1 and 4.2, we see that if either

$$\lambda < \frac{1}{3B} \approx 9.0800 \quad \text{or} \quad \lambda > \frac{1}{A} \approx 36.3358$$

then (4.1)–(4.2) has no positive solutions.

If we use the definitions for A and B in [6], then

$$A = \frac{3445801}{314572800} \quad \text{and} \quad B = \frac{717}{2560}.$$

It is easy to see that Theorems 3.1 and 3.2 of [6] do not apply to problem (4.1)–(4.2). If we apply [6, Theorem 4.1] to (4.1)–(4.2), we have that if either

$$\lambda < \frac{1}{3B} \approx 1.1901 \quad \text{or} \quad \lambda > \frac{1}{A} \approx 91.2917,$$

then (4.1)–(4.2) has no positive solutions. Clearly, the results obtained in this paper improve those obtained in [6].

We wish to point out that Maroun [20] also considered the problem of existence of positive solutions of the problem (1.1)–(1.2) in the cases where $g(t)$ is singular at $t = 0$ and $t = 1$ and where $f(u)$ is singular at $u = 0$.

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