

## UNIQUENESS OF LIMITING SOLUTION TO A STRONGLY COMPETING SYSTEM

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ABSTRACT. We prove a uniqueness for the positive solution to a strongly competing system of Lotka-Volterra type problem in the limiting configuration, when the competition rate tends to infinity. We give an alternate proof of uniqueness based on properties of limiting solutions.

### 1. INTRODUCTION

The aim of this paper is to investigate the uniqueness of solution for a competition-diffusion system of Lotka-Volterra type, with Dirichlet boundary conditions as the competition rate tends to infinity. This model of strongly competing systems have been extensively studied from different point of views, see [3, 5, 7, 6, 8, 9] and references therein.

The model describes the steady state of  $m$  competing species coexisting in the same area  $\Omega$ . Let  $u_i(x)$  denote the population density of the  $i^{th}$  component. The following system shows the steady state of interaction between  $m$  components

$$\begin{aligned} \Delta u_i^\varepsilon &= \frac{1}{\varepsilon} u_i^\varepsilon \sum_{j \neq i} u_j^\varepsilon(x) \quad \text{in } \Omega, \\ u_i^\varepsilon &\geq 0, \quad i = 1, \dots, m \quad \text{in } \Omega, \\ u_i^\varepsilon(x) &= \phi_i(x), \quad i = 1, \dots, m \quad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

Here  $\Omega \subset \mathbb{R}^d$  is an open, bounded, and connected domain with smooth boundary;  $m$  is an integer;  $\phi_i$  are non-negative  $C^1$  functions with disjoint supports, that is,  $\phi_i \cdot \phi_j = 0$  almost everywhere on the boundary; and the term  $1/\varepsilon$  is the competition rate.

This model is also called adjacent segregation, modeling when particles annihilate each other on contact. The system (1.1) has been generalized for nonlinear diffusion or long segregation, where species interact at a distance from each other see [4]. Also in [10] the generalization of this problem has been considered for the extremal Pucci operator. The numerical treatment of the limiting case of system (1.1) is given in [2].

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The limiting configuration (solution) of (1.1) as  $\varepsilon$  tends to zero, is related to a free boundary problem and the densities  $u_i$  satisfy the system of differential inequalities. The uniqueness of limiting solution is proven for the cases  $m = 2$  in [5] and  $m = 3$  in planar domain, see [7]. Later in [11] these uniqueness results have been generalized to arbitrary dimension and arbitrary number of species.

In this work we give a new proof for uniqueness of the limiting configuration for arbitrary  $m$  competing densities. We use properties of limiting solution, which is rather different approach than the proof of uniqueness for limiting solution given in [11]. The outline of this article is as follows: In Section 2, we state the problem and provide mathematical background and known results, which will be used in our proof. In Section 3, we prove the uniqueness of the system (1.1) in the limiting case as  $\varepsilon$  tends to zero.

## 2. KNOWN RESULTS AND MATHEMATICAL BACKGROUND

In this section we recall some estimates and compactness properties that will play an important role in our study. As shown in [11], for each  $\varepsilon$ , system (1.1) has a unique solution. Their proof uses the sub- and sup-solution method for nonlinear elliptic systems to construct iterative monotone sequences which lead to the uniqueness for system (1.1).

Let  $U^\varepsilon = (u_1^\varepsilon, \dots, u_m^\varepsilon)$  be the unique solution of system (1.1) for a fixed  $\varepsilon$ . Then  $u_i^\varepsilon$ , for  $i = 1, \dots, m$ , satisfies the differential inequality

$$-\Delta u_i^\varepsilon \leq 0 \quad \text{in } \Omega. \quad (2.1)$$

We define

$$\widehat{u}_i^\varepsilon := u_i^\varepsilon - \sum_{j \neq i} u_j^\varepsilon,$$

then it is easy to verify the property

$$-\Delta \widehat{u}_i^\varepsilon = \sum_{j \neq i} \sum_{h \neq j} u_j^\varepsilon u_h^\varepsilon \geq 0. \quad (2.2)$$

By constructing of sub and super solution to the system (1.1), we can show that  $\frac{\partial u_i^\varepsilon}{\partial n}$  is bounded on  $\partial\Omega$  (independent of  $\varepsilon$ ). Then multiplying the inequality  $-\Delta u_i^\varepsilon \leq 0$  by  $u_i^\varepsilon$  and integrating by part yields that  $u_i^\varepsilon$  is bounded in  $H^1(\Omega)$  for each  $\varepsilon$ .

The above discussion shows that the solution of (1.1) belongs to the following class  $F$ , see [5, Lemma 2.1]:

$$F = \{(u_1, \dots, u_m) \in (H^1(\Omega))^m : u_i \geq 0, -\Delta u_i \leq 0, -\Delta \widehat{u}_i \geq 0, u_i = \phi_i \text{ on } \partial\Omega\},$$

where as in system (1.1) the boundary data  $\phi_i \in C^1(\partial\Omega)$ , nonnegative functions and  $\phi_i \cdot \phi_j = 0$ , almost everywhere on the boundary.

The following result in [3, 5] shows the asymptotic behavior of the system as  $\varepsilon \rightarrow 0$ . Let  $U^\varepsilon = (u_1^\varepsilon, \dots, u_m^\varepsilon)$  be the solution of system (1.1). If  $\varepsilon$  tends to zero, then there exists  $U = (u_1, \dots, u_m) \in (H^1(\Omega))^m$  such that for all  $i = 1, \dots, m$ :

- (1) up to a subsequences,  $u_i^\varepsilon \rightarrow u_i$  strongly in  $H^1(\Omega)$ ,
- (2)  $u_i \cdot u_j = 0$  if  $i \neq j$  a.e in  $\Omega$ ,
- (3)  $\Delta u_i = 0$  in the set  $\{u_i > 0\}$ ,

(4) Let  $x$  belong to the common interface of two components  $u_i$  and  $u_j$ , then

$$\lim_{y \rightarrow x} \nabla u_i(y) = - \lim_{y \rightarrow x} \nabla u_j(y).$$

From above, the limiting solution, as  $\varepsilon$  tends to zero, belongs to the class

$$S = \{(u_1, \dots, u_m) \in F : u_i \cdot u_j = 0 \text{ for } i \neq j\}.$$

Note that the inequalities in (2.1) and (2.2) hold as  $\varepsilon$  tends to zero. Also

$$-\Delta \widehat{u}_i = 0 \quad \text{on } \{x \in \Omega : u_i(x) > 0\}.$$

In this part we briefly review the known results about uniqueness of the limiting configuration of the system (1.1). In particular, for the case  $m = 2$ , the limiting solution and the rate of convergence are given (see [5, Theorem 2.1]). For the sake of clarity we recall the following result.

**Theorem 2.1.** *Let  $W$  be harmonic in  $\Omega$  with the boundary data  $\phi_1 - \phi_2$ . Let  $u_1 = W^+$ ,  $u_2 = -W^-$ , then the pair  $(u_1, u_2)$  is the limit configuration of any sequences  $(u_1^\varepsilon, u_2^\varepsilon)$  and*

$$\|u_i^\varepsilon - u_i\|_{H^1(\Omega)} \leq C \cdot \varepsilon^{1/6} \quad \text{as } \varepsilon \rightarrow 0, \quad i = 1, 2.$$

For the case  $m = 3$ , the uniqueness of the limiting configuration, as  $\varepsilon$  tends to zero, is shown in [7] on a planar domain, with appropriate boundary conditions. More precisely, the authors prove that the limiting configuration of the system

$$\begin{aligned} \Delta u_i^\varepsilon &= \frac{u_i^\varepsilon(x)}{\varepsilon} \sum_{j \neq i}^3 u_j^\varepsilon(x) \quad \text{in } \Omega, \\ u_i^\varepsilon(x) &= \phi_i(x) \quad \text{on } \partial\Omega, \\ & i = 1, 2, 3, \end{aligned}$$

minimizes the energy

$$E(u_1, u_2, u_3) = \int_{\Omega} \sum_{i=1}^3 \frac{1}{2} |\nabla u_i|^2 dx,$$

among all segregated states  $u_i \cdot u_j = 0$ , a.e. with the same boundary conditions.

**Remark 2.2.** System (1.1) is not in a variational form. Existence and uniqueness for a class of segregation states governed by a variational principle are proved in [6].

In [11], the uniqueness of the limiting configuration and least energy property are generalized to arbitrary dimension and for arbitrary number of components. Following the notation in [11], we have the matrix space

$$\Sigma = \{(u_1, u_2, \dots, u_m) \in \mathbb{R}^m : u_i \geq 0, u_i \cdot u_j = 0 \text{ for } i \neq j\}.$$

The authors in [11] show that the solution of the limiting problem  $(u_1, \dots, u_m) \in S$  is a harmonic map into the space  $\Sigma$ . The harmonic map is the critical point (in weak sense) of the energy functional

$$\int_{\Omega} \sum_{i=1}^m \frac{1}{2} |\nabla u_i|^2 dx,$$

among all nonnegative segregated states  $u_i \cdot u_j = 0$ , a.e. with the same boundary conditions, see [11, Theorem 1.6]. Their proof is based on computing the derivative of the energy functional with respect to the geodesic homotopy between  $u$  and a comparison to an energy minimizing map  $v$  with same boundary values. This demands some procedures to avoid singularity of free boundary. Unlike their approach, our proof is more direct and based on properties of limiting solutions and doesn't require results from regularity theory or harmonic maps.

### 3. UNIQUENESS

In this section we prove the uniqueness for the limiting case as  $\varepsilon$  tends to zero. Our approach is motivated from the recent work related to the numerical analysis of a certain class of the spatial segregation of reaction-diffusion systems (see [1]). We use the notation

$$\widehat{w}_i(x) := w_i(x) - \sum_{p \neq i} w_p(x),$$

for every  $1 \leq i \leq m$ .

**Lemma 3.1.** *Let two elements  $(u_1, \dots, u_m)$  and  $(v_1, \dots, v_m)$  belong to  $S$ . Then the following equation for each  $1 \leq i \leq m$  holds:*

$$\max_{\overline{\Omega}} (\widehat{u}_i(x) - \widehat{v}_i(x)) = \max_{\{u_i(x) \leq v_i(x)\}} (\widehat{u}_i(x) - \widehat{v}_i(x)).$$

*Proof.* We argue by contradiction. Assume there exists an  $i_0$  such that

$$\max_{\overline{\Omega}} (\widehat{u}_{i_0} - \widehat{v}_{i_0}) = \max_{\{u_{i_0} > v_{i_0}\}} (\widehat{u}_{i_0} - \widehat{v}_{i_0}) > \max_{\{u_{i_0} \leq v_{i_0}\}} (\widehat{u}_{i_0} - \widehat{v}_{i_0}). \quad (3.1)$$

Assume  $D = \{x \in \Omega : u_{i_0}(x) > v_{i_0}(x)\}$ , then in  $D$  we have

$$\begin{aligned} -\Delta \widehat{u}_{i_0}(x) &= 0, \\ -\Delta \widehat{v}_{i_0}(x) &\geq 0, \end{aligned} \quad (3.2)$$

which implies that

$$\Delta(\widehat{u}_{i_0}(x) - \widehat{v}_{i_0}(x)) \geq 0.$$

The weak maximum principle yields

$$\max_D (\widehat{u}_{i_0} - \widehat{v}_{i_0}) \leq \max_{\partial D} (\widehat{u}_{i_0} - \widehat{v}_{i_0}) \leq \max_{\{u_{i_0} = v_{i_0}\}} (\widehat{u}_{i_0} - \widehat{v}_{i_0}),$$

which is inconsistent with our assumption (3.1). It is clear that we can interchange the role of  $\widehat{u}_i$  and  $\widehat{v}_i$ . Thus, we also have

$$\max_{\overline{\Omega}} (\widehat{v}_i(x) - \widehat{u}_i(x)) = \max_{\{v_i(x) \leq u_i(x)\}} (\widehat{v}_i(x) - \widehat{u}_i(x)),$$

for  $1 \leq i \leq m$ . □

In view of Lemma 3.1 we define the following quantities

$$\begin{aligned} P &:= \max_{1 \leq i \leq m} \left( \max_{\overline{\Omega}} (\widehat{u}_i(x) - \widehat{v}_i(x)) \right) = \max_{1 \leq i \leq m} \left( \max_{\{u_i \leq v_i\}} (\widehat{u}_i(x) - \widehat{v}_i(x)) \right), \\ Q &:= \max_{1 \leq i \leq m} \left( \max_{\overline{\Omega}} (\widehat{v}_i(x) - \widehat{u}_i(x)) \right) = \max_{1 \leq i \leq m} \left( \max_{\{v_i \leq u_i\}} (\widehat{v}_i(x) - \widehat{u}_i(x)) \right). \end{aligned}$$

**Lemma 3.2.** *Let two elements  $(u_1, \dots, u_m)$  and  $(v_1, \dots, v_m)$  belong to  $S$ , and let  $P$  and  $Q$  be as defined above. If  $P > 0$  is attained for some index  $1 \leq i_0 \leq m$ , then we have  $P = Q > 0$ . Moreover, there exist another index  $j_0 \neq i_0$  and a point  $x_0 \in \Omega$ , such that*

$$P = Q = \max_{\{u_{i_0} \leq v_{i_0}\}} (\widehat{u}_{i_0} - \widehat{v}_{i_0}) = \max_{\{u_{i_0} = v_{i_0} = 0\}} (\widehat{u}_{i_0} - \widehat{v}_{i_0}) = v_{j_0}(x_0) - u_{j_0}(x_0).$$

*Proof.* Let the maximum  $P > 0$  be attained for the  $i_0^{\text{th}}$  component. According to the previous lemma, we know that  $(\widehat{u}_{i_0}(x) - \widehat{v}_{i_0}(x))$  attains its maximum on the set  $\{u_{i_0}(x) \leq v_{i_0}(x)\}$ . Let that maximum point be  $x^* \in \{u_{i_0}(x) \leq v_{i_0}(x)\}$ . It is easy to see that  $\widehat{u}_{i_0}(x^*) - \widehat{v}_{i_0}(x^*) = P > 0$ , implies  $u_{i_0}(x^*) = v_{i_0}(x^*) = 0$ . Indeed, if  $u_{i_0}(x^*) = v_{i_0}(x^*) > 0$ , then in light of disjointness property of the components of  $u_i$  and  $v_i$  we get  $P = \widehat{u}_{i_0}(x^*) - \widehat{v}_{i_0}(x^*) = u_{i_0}(x^*) - v_{i_0}(x^*) = 0$  which is a contradiction. If  $u_{i_0}(x^*) < v_{i_0}(x^*)$ , then again because of the disjointness of the densities  $u_i, v_i$ , we have

$$0 < P = \widehat{u}_{i_0}(x^*) - \widehat{v}_{i_0}(x^*) = \widehat{u}_{i_0}(x^*) - v_{i_0}(x^*) \leq u_{i_0}(x^*) - v_{i_0}(x^*) < 0.$$

this again leads to a contradiction. Therefore  $u_{i_0}(x^*) = v_{i_0}(x^*) = 0$ .

Now assume by contradiction that  $Q \leq 0$ . Then by definition of  $Q$  we should have

$$\widehat{v}_j(x) \leq \widehat{u}_j(x), \quad \forall x \in \Omega, \quad j = 1, \dots, m.$$

This apparently yields

$$v_j(x) \leq u_j(x), \quad \forall x \in \Omega, \quad j = 1, \dots, m.$$

Let  $D_{i_0} = \{u_{i_0}(x) = v_{i_0}(x) = 0\}$ , then we have

$$0 < P = \max_{D_{i_0}} (\widehat{u}_{i_0}(x) - \widehat{v}_{i_0}(x)) = \max_{D_{i_0}} \left( \sum_{j \neq i_0} (v_j(x) - u_j(x)) \right) \leq 0.$$

This contradiction implies that  $Q > 0$ . By analogous proof, one can see that if  $P$  be non-positive then  $Q$  will be non-positive as well. Next, assume the maximum  $P$  is attained at a point  $x_0 \in D_{i_0}$ . Then, we get

$$\begin{aligned} 0 < P &= \widehat{u}_{i_0}(x_0) - \widehat{v}_{i_0}(x_0) \\ &= (u_{i_0}(x_0) - v_{i_0}(x_0)) + \sum_{j \neq i_0} (v_j(x_0) - u_j(x_0)) \\ &= \sum_{j \neq i_0} (v_j(x_0) - u_j(x_0)). \end{aligned}$$

This shows that

$$\sum_{j \neq i_0} v_j(x_0) = \sum_{j \neq i_0} u_j(x_0) + P > 0.$$

Since  $(v_1, \dots, v_m) \in S$ , then there exists  $j_0 \neq i_0$  such that  $v_{j_0}(x_0) > 0$ . This implies

$$0 < P = \widehat{u}_{i_0}(x_0) - \widehat{v}_{i_0}(x_0) = v_{j_0}(x_0) - \sum_{j \neq i_0} u_j(x_0) \leq \widehat{v}_{j_0}(x_0) - \widehat{u}_{j_0}(x_0) \leq Q.$$

The same argument shows that  $Q \leq P$  which yields  $P = Q$ . Hence, we can write

$$P = v_{j_0}(x_0) - \sum_{j \neq i_0} u_j(x_0) = \widehat{v}_{j_0}(x_0) - \widehat{u}_{j_0}(x_0) = Q.$$

This gives us  $2 \sum_{j \neq j_0} u_j(x_0) = 0$ , and therefore

$$u_j(x_0) = 0, \quad \forall j \neq j_0,$$

which completes the last statement of the proof.  $\square$

We are ready to prove the uniqueness of a limiting configuration.

**Theorem 3.3.** *There exists a unique vector  $(u_1, \dots, u_m) \in S$ , which satisfies the limiting solution of (1.1).*

*Proof.* To show the uniqueness of the limiting configuration, we assume that two  $m$ -tuples  $(u_1, \dots, u_m)$  and  $(v_1, \dots, v_m)$  are the solutions of system (1.1) as  $\varepsilon$  tends to zero. These two solutions belong to the class  $S$ . For them we set  $P$  and  $Q$  as above. Then, we consider two cases  $P \leq 0$  and  $P > 0$ . If we assume that  $P \leq 0$  then Lemma 3.2 implies that  $Q \leq 0$ . This leads to

$$0 \leq -Q \leq \widehat{u}_i(x) - \widehat{v}_i(x) \leq P \leq 0,$$

for every  $1 \leq i \leq m$ , and  $x \in \Omega$ . This provides that

$$\widehat{u}_i(x) = \widehat{v}_i(x) \quad i = 1, \dots, m,$$

which in turn implies

$$u_i(x) = v_i(x).$$

Now, suppose  $P > 0$ . We show that this case leads to a contradiction. Let the value  $P$  is attained for some  $i_0$ , then due to Lemma 3.2 there exist  $x_0 \in \Omega$  and  $j_0 \neq i_0$  such that:

$$0 < P = Q = \widehat{u}_{i_0}(x_0) - \widehat{v}_{i_0}(x_0) = \max_{\{u_{i_0}=v_{i_0}=0\}} (\widehat{u}_{i_0}(x) - \widehat{v}_{i_0}(x)) = v_{j_0}(x_0) - u_{j_0}(x_0).$$

Let  $\Gamma$  be a fixed curve starting at  $x_0$  and ending on the boundary of  $\Omega$ . Since  $\Omega$  is connected, then one can always choose such a curve belonging to  $\overline{\Omega}$ . By the disjointness and smoothness of  $v_{j_0}$  and  $u_{j_0}$  there exists a ball centered at  $x_0$ , and with radius  $r_0$  ( $r_0$  depends on  $x_0$ ) which we denote  $B_{r_0}(x_0)$ , such that

$$v_{j_0}(x) - u_{j_0}(x) > 0 \quad \text{in } B_{r_0}(x_0).$$

This yields

$$\Delta(\widehat{v}_{j_0}(x) - \widehat{u}_{j_0}(x)) \geq 0 \quad \text{in } B_{r_0}(x_0).$$

The maximum principle implies that

$$\frac{\max_{B_{r_0}(x_0)} (\widehat{v}_{j_0}(x) - \widehat{u}_{j_0}(x))}{\max_{\partial B_{r_0}(x_0)} (\widehat{v}_{j_0}(x) - \widehat{u}_{j_0}(x))} \leq P.$$

One the other hand, in view of Lemma 3.2 we have

$$\widehat{v}_{j_0}(x_0) - \widehat{u}_{j_0}(x_0) = v_{j_0}(x_0) - u_{j_0}(x_0) = P,$$

which implies that  $P$  is attained at the interior point  $x_0 \in B_{r_0}(x_0)$ . Thus,

$$\widehat{v}_{j_0}(x) - \widehat{u}_{j_0}(x) \equiv P > 0 \quad \text{in } \overline{B_{r_0}(x_0)}.$$

Next let  $x_1 \in \Gamma \cap \partial B_{r_0}(x_0)$ . We get  $\widehat{v}_{j_0}(x_1) - \widehat{u}_{j_0}(x_1) = P > 0$ , which leads to  $v_{j_0}(x_1) \geq u_{j_0}(x_1)$ . We proceed as follows: If  $v_{j_0}(x_1) > u_{j_0}(x_1)$ , then as above  $v_{j_0}(x) > u_{j_0}(x)$  in  $B_{r_1}(x_1)$ . This in turn implies

$$\Delta(\widehat{v}_{j_0}(x) - \widehat{u}_{j_0}(x)) \geq 0 \quad \text{in } B_{r_1}(x_1).$$

Again following the maximum principle and recalling that  $\widehat{v}_{j_0}(x_1) - \widehat{u}_{j_0}(x_1) = P$  we conclude that

$$\widehat{v}_{j_0}(x) - \widehat{u}_{j_0}(x) = P > 0 \quad \text{in } \overline{B_{r_1}(x_1)}.$$

If  $v_{j_0}(x_1) = u_{j_0}(x_1)$ , then clearly the only possibility is  $v_{j_0}(x_1) = u_{j_0}(x_1) = 0$ . Thus,

$$0 < P = \widehat{v}_{j_0}(x_1) - \widehat{u}_{j_0}(x_1) = \sum_{j \neq j_0} (u_j(x_1) - v_j(x_1)).$$

Following the lines of the proof of Lemma 3.2, we find some  $k_0 \neq j_0$ , such that

$$P = u_{k_0}(x_1) - v_{k_0}(x_1) = \widehat{u}_{k_0}(x_1) - \widehat{v}_{k_0}(x_1).$$

It is easy to see that there exists a ball  $B_{r_1}(x_1)$  (without loss of generality one keeps the same notation)

$$\Delta(\widehat{u}_{k_0}(x) - \widehat{v}_{k_0}(x)) \geq 0 \quad \text{in } B_{r_1}(x_1).$$

In view of the maximum principle and above steps we obtain:

$$\widehat{u}_{k_0}(x) - \widehat{v}_{k_0}(x) = P > 0 \quad \text{in } \overline{B_{r_1}(x_1)}.$$

Then we take  $x_2 \in \Gamma \cap \partial B_{r_1}(x_1)$  such that  $x_1$  stands between the points  $x_0$  and  $x_2$  along the given curve  $\Gamma$ . According to the previous arguments for the point  $x_2$  we will find an index  $l_0 \in \{1, \dots, m\}$  and corresponding ball  $B_{r_2}(x_2)$ , such that

$$|\widehat{u}_{l_0}(x) - \widehat{v}_{l_0}(x)| = P \quad \text{in } \overline{B_{r_2}(x_2)}.$$

We continue this way and obtain a sequence of points  $x_n$  along the curve  $\Gamma$ , which are getting closer to the boundary of  $\Omega$ . Since for all  $j = 1, \dots, m$  and  $x \in \partial\Omega$  we have

$$\widehat{u}_j(x) - \widehat{v}_j(x) = \widehat{v}_j(x) - \widehat{u}_j(x) = 0,$$

then obviously after finite steps  $N$  we find the point  $x_N$ , which will be very close to the  $\partial\Omega$  and for all  $j = 1, \dots, m$

$$|\widehat{u}_j(x_N) - \widehat{v}_j(x_N)| < P/2.$$

On the other hand, according to our construction for the point  $x_N$ , there exists an index  $1 \leq j_N \leq m$  such that

$$|\widehat{u}_{j_N}(x_N) - \widehat{v}_{j_N}(x_N)| = P,$$

which is a contradiction. This completes the proof.  $\square$

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