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ELLIPTIC PERTURBATIONS FOR HAMMERSTEIN EQUATIONS WITH SINGULAR NONLINEAR TERM

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In memory of Professor Aldo Cossu

Abstract. We consider a singular elliptic perturbation of a Hammerstein integral equation with singular nonlinear term at the origin. The compactness of the solutions to the perturbed problem and, hence, the existence of a positive solution for the integral equation are proved. Moreover, these results are applied to nonlinear singular homogeneous Dirichlet problems.

1. Introduction

In this paper, using elliptic perturbations, we show the existence of a positive solution to the Hammerstein equation

$$u(x) = \int_\Omega K(x,y)g\big(y,u(y)\big)dy, \quad x \in \Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a bounded open set with smooth boundary and $g(y,s)$, $y \in \Omega$, $s > 0$, is a positive function that is bounded in a neighborhood of $+\infty$ and possibly nonsmooth as $s \to 0^+$, in particular we do not exclude that

$$\liminf_{s\to 0^+} g(y,s) = 0; \quad \limsup_{s\to 0^+} g(y,s) = +\infty.$$

Moreover, we do not assume anything about the existence of super or sub solutions to (1.1).

The literature on Hammerstein equations with integrand depending on the reciprocal of the solution is rather limited, nevertheless they arise, more or less directly, in a variety of settings: semilinear boundary value problems with a nonlinear term depending on the reciprocal of the solution, see [4, 5, 6, 10, 12, 14] and Theorem 2.4 in the following section; mathematical models of signal theory, see [18]; ecological models, see [19, pg. 103-104]; Boussinesq's equation in filtration theory, see [16].

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In literature some existence results for (1.1) are already present (see [3, 7, 8, 9, 17]. In [9, 17], the solutions are obtained via the perturbed problem

$$u_\varepsilon(x) = \int_\Omega K(x,y) g\big(y, \varepsilon + u_\varepsilon(y)\big) dy, \quad u_\varepsilon \in L^1(\Omega). \tag{1.2}$$

The argument of this paper consists in the approximation of (1.1) with the following elliptic integro-differential problem

$$\begin{gathered} -\varepsilon^\alpha \Delta u_\varepsilon(x) + u_\varepsilon(x) = \int_\Omega K(x,y) g\big(y, \varepsilon + u_\varepsilon(y)\big) dy \quad x \in \Omega, \\ u_\varepsilon(x) \geq 0 \quad x \in \Omega, \\ u_\varepsilon(x) = 0 \quad x \in \partial\Omega, \end{gathered} \tag{1.3}$$

where $\alpha > 0$. The elliptic perturbations (1.3) are interesting from both the mathematical and the physical point of view. Indeed, the solutions to (1.3) belong to $W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ on the other hand the ones of (1.2) are merely in $L^1(\Omega)$. The convergence of the solutions of the approximated problems (1.3) to one of the integral problem (1.1) makes easier the implementation of robust numerical schemes. In the fluidodynamic interpretation of (1.1) in filtration theory (see [16]) the perturbation $-\varepsilon^\alpha \Delta u_\varepsilon$ represents a small viscosity. This approach to the existence of solutions for (1.1) has been used extensively in the last years in various frameworks, in particular it gives physically meaningful solutions to Conservation Laws (see e.g. [2]).

Let us be more precise regarding our results. We prove that there exist an infinitesimal sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ and a nontrivial solution u_0 to (1.1) such that

$$\lim_k \int_\Omega \eta(x) \big| u_0(x) - u_{\varepsilon_k}(x) \big| dx = 0,$$

where $u_{\varepsilon_k} \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ solves (1.3) with ε_k instead of ε, $\eta(x)$ is a positive function depending on $K(x,y)$ (see the assumption (K2) in the following section), and the exponent q depends on the regularity of $K(x,y)$ (see (K1). Moreover, we prove that

$$\int_\Omega \eta(x) u_0(x) dx < +\infty$$

and we give an estimate on the first and second derivatives of the solutions to (1.3) as $\varepsilon \to 0$.

Finally, we consider the particular case in which $K(x,y)$ is the Green's function of $-\Delta$ on Ω. We prove an existence result for homogeneous semilinear Dirichlet problems with integrand depending on the reciprocal of the solution, our result is a bit more general than the ones present in the literature, see for example [4, 5, 6, 10, 12, 14].

The starting points of our analysis are the estimates for the solutions of singular linear elliptic perturbations proved by the Huet [15] and Friedman [11].

The paper is organized as follows. Section 2 is dedicated to the assumptions and results. In Section 3 we prove the existence principle for the integral equation (1.1). In Section 4 we apply that result to semilinear homogeneous Dirichlet problem for $-\Delta$ with singular nonlinear term in the origin. Finally, in the appendix some convergence results are present.

2. Assumptions and results

Let us list the notation used in this paper.

$$\mathbb{R}_+ := [0,+\infty[;\quad \mathbb{R}_+^* :=]0,+\infty[;\quad \mathbb{N}^* := \mathbb{N}\setminus\{0\}.$$

Let $E \subset \mathbb{R}^k, k \geq 1$, be a measurable set (we will consider only measurable sets). $|E|$ is the measure of E, χ_E is the characteristic map of E and $|\cdot|_{\rho,E}, 1 \leq \rho \leq \infty$, is the $L^\rho(E)$ norm. $L^\rho_+(E)$ is the cone of all $\phi \in L^\rho(E), \phi \geq 0$ almost everywhere in E and $L^\rho_+(\theta,E), \theta$ measurable, is the cone of all measurable $\phi, \phi \geq 0$ almost everywhere in E, such that $\theta\phi \in L^\rho(E)$. $W^{\frac{1}{\rho},\rho}(E)$ is the space of the maps $\phi \in L^\rho(E)$ such that

$$\int_{E\times E} \frac{|\phi(x)-\phi(y)|^\rho}{|x-y|^{k+1}}dx\,dy < +\infty.$$

Let u, v be two maps, $u \leq v$ is the set of all points $x \in \Omega$ such that $u(x) \leq v(x)$. Analogously, we define $u < v$, $u \geq v$, $u > v$.

We continue with the assumptions on the nonlinear term $g(y,s)$ and the kernel $K(x,y)$. Let $g : \Omega \times \mathbb{R}_+^* \to \mathbb{R}$ be a positive Carathèodory function (namely $g(\cdot,s)$ is measurable in Ω for each $s > 0$; $g(y,\cdot)$ is continuous in $\mathbb{R}_+^*$ for almost every $y \in \Omega$).

(G1) There exist $\phi_0 \in L^r(\Omega)$, $1 \leq r \leq +\infty$, and $p > 0$ such that

$$0 \leq g^*(y,s) \leq \frac{\phi_0(y)}{s^p},\quad y \in \Omega,\quad 0 < s \leq 1,$$

where $g^*(y,s) := \sup_{s\leq t} g(y,t) \in \mathbb{R}$, $(y,s) \in \Omega \times \mathbb{R}_+^*$.

(G2) There exist $\mu_0 > 0$ and $\Omega_0 \subset \Omega$, $|\Omega_0| > 0$, such that

$$\liminf_{s\to 0^+} \frac{g(y,s)}{s} \geq \mu_0,$$

uniformly with respect to $y \in \Omega_0$.

Let $K(x,y)$, $(x,y) \in \Omega\times\Omega$, be a nonnegative kernel and introduce the notation

$$K(\phi) := \int_\Omega K(\cdot,y)\phi(y)dy.$$

(K1) $K \in W^{\frac{1}{q},q}(\Omega\times\Omega)$ with $1 < q < \infty$ and $q + r \leq qr$.

(K2) There exist two measurable positive maps $a(\cdot), \eta(\cdot)$ such that

$$a(x)a(y) \leq K(x,y);\qquad \int_\Omega K(x,y)\eta(x)dx \leq a(y);$$

$$\eta \in L^{q'}(\Omega),\quad q' := \frac{q}{q-1};\quad \frac{\phi_0}{a^{p^*-1}} \in L^1(\Omega),\quad p^* := \max\{p,1\}.$$

(K3) For every $n \in \mathbb{N}^*$, the operator K is compact from $L^1(\Omega_n)$ in itself, where

$$\Omega_n := \big\{x \in \Omega : \frac{1}{n} \leq a(x)\big\},\quad n \in \mathbb{N}^*.$$

Observe that all the assumptions, except for (K1), are weaker than the ones in [9], in particular on p we require only the positivity. In [9] it possible to find a long list of kernels K satisfying $(\mathcal{K}_2), (\mathcal{K}_3)$, and within those we have the Green's function $-\Delta$ with Dirichlet boundary conditions and the Green's functions associated to several one-dimensional boundary value problems.

Remark 2.1. Hypothesis (G1) implies $g^*(\cdot,s) \in L^r(\Omega)$, $s > 0$. Hypotheses (K1), (K2) imply $a \in L^q(\Omega)$.

Remark 2.2. The condition $a > 0$ a.e. in Ω is equivalent to the fact that $(\Omega_n)_{n\in\mathbb{N}^*}$ covers Ω. Indeed, assuming by contradiction that $|\Omega \setminus (\cup_{n=1}^{\infty}\Omega_n)| > 0$. Since $a > 0$ a.e. in Ω and $(\Omega_n)_{n\in\mathbb{N}^*}$ is increasing we have

$$0 < \int_{(\Omega\setminus(\cup_{n=1}^{\infty}\Omega_n))} a(x)dx = \lim_n \int_{(\Omega\setminus\Omega_n)} a(x)dx \le \lim_n \frac{|\Omega\setminus\Omega_n|}{n} = 0,$$

that is absurd. The other implication is trivial. Finally, due to the continuity of the Lebesgue measure we have also $\lim_n |\Omega \setminus \Omega_n| = 0$.

Regarding the constant α (see (1.3)) we consider only the case $\alpha = 10q(p+1)$. The main results of this paper are the following.

Theorem 2.3. *If*

$$\mu_0|a^2|_{1,\Omega_0} > 1, \tag{2.1}$$

then there exists a solution $u_0 \in L^1_+(\eta,\Omega)$ *to* (1.1) *such that* $|ag(\cdot,u_0)|_{1,\Omega} > 0$ *and*

$$|ag(\cdot,u_0)|_{1,\Omega}a(x) \le u_0(x), \quad x \in \Omega \quad a.e.$$

Moreover, there exists $(\varepsilon_k)_{k\in\mathbb{N}}, \varepsilon_k \to 0$, *such that*

$$\lim_k |\eta(u_0 - u_{\varepsilon_k})|_{1,\Omega} = 0, \quad u_{\varepsilon_k} \to u_0 \quad a.e.\ in\ \Omega,$$

where $u_{\varepsilon_k} \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ *solves* $(1.3)_{\varepsilon_k}$. *Finally, there exist* $0 < \bar\varepsilon \le \frac{1}{2}$ *and a constant* $\bar c > 0$, *independent on* ε, *such that*

$$\varepsilon^{\alpha} \sum_{i,j=1}^{N} |\partial^2_{i,j}u_\varepsilon|_{q,\Omega} + \varepsilon^{\frac{\alpha}{2}} \sum_{i=1}^{N} |\partial_i u_\varepsilon|_{q,\Omega} \le \bar c\varepsilon^{p+2}, \quad 0 < \varepsilon \le \bar\varepsilon.$$

If $K(x,y)$ is the Green's function of $-\Delta$ on Ω, we get an existence result for the Dirichlet problem

$$\begin{gathered} -\Delta u = g(x,u) \quad \text{in } \Omega; \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

Theorem 2.4. *Let* $N \ge 2$. *Assume that* $g(y,s)$ *satisfies* (G1), (G2) *with*

$$q < \frac{N}{N-1}; \quad q + r \le rq; \quad \frac{\phi_0}{\delta^{p^*-1}} \in L^1(\Omega), \quad p^* = \max\{p,1\},$$

where $\delta(x) = \operatorname{dist}(x,\partial\Omega), \quad x \in \mathbb{R}^N$.

If

$$\mu_0|\phi_1^2|_{1,\Omega_0} > 1, \tag{2.2}$$

where ϕ_1 *is a positive eigenfunction of the Dirichlet problem for* $-\Delta$ *in* Ω *such that*

$$\phi_1(x)\phi_1(y) \le G(x,y),$$

then there exist $u_0 \in W^{2,r}_{\rm loc}(\Omega) \cap C(\bar\Omega)$ *and* $c_2 > 0$ *such that* $c_2\delta(x) \le u_0(x)$ *and*

$$\begin{gathered} -\Delta u_0(x) = g\big(x,u_0(x)\big) \quad x \in \Omega, \\ u_0(x) > 0 \quad x \in \Omega, \\ u_0(x) = 0 \quad x \in \partial\Omega. \end{gathered} \tag{2.3}$$

Moreover, for every $\varepsilon > 0$ *there exists* $u_\varepsilon \in W^{4,r}(\Omega)$*, a solution of*

$$\begin{gathered}\varepsilon^\alpha \Delta^2 u_\varepsilon(x) - \Delta u_\varepsilon(x) = g\big(x, \varepsilon + u_\varepsilon(x)\big) \quad x \in \Omega,\\ u_\varepsilon(x) > 0 \quad x \in \Omega,\\ \Delta u_\varepsilon(x) = u_\varepsilon(x) = 0 \quad x \in \partial\Omega,\end{gathered} \tag{2.4}$$

and $(\varepsilon_k)_{k\in\mathbb{N}}, \varepsilon_k \to 0$*, such that* $u_{\varepsilon_k} \to u_0$ *in* $W^{2,r}_{\rm loc}(\Omega) \cap L^q(\Omega)$*.*

In light of Lemma 4.1 below, $G(x, y)$ satisfies (K1), (K2), (K3) with

$$q < \frac{N}{N-1}; \quad a(x) = \frac{1}{\sqrt{c_1}}\delta(x); \quad \eta(x) = \frac{1}{c_1\sqrt{c_1}},$$

hence the integral equation associated with (2.3) satisfies the same hypotheses of (1.1). Since δ is equivalent to each positive eigenfunction of the Dirichlet problem for $-\Delta$ in Ω (see [4]), (2.2) coincides with (2.1) when $K = G$.

3. Proof of Theorem 2.3

In the following statements and proofs we write "cost" for positive constants independent of ε.

The first step of our analysis consists in the existence of solutions for (1.3). Thanks to (G1), (K1),

$$K(g^*(\cdot, \varepsilon)) = \int_\Omega K(\cdot, y) g^*(y, \varepsilon) dy \in L^q(\Omega), \quad \varepsilon > 0;$$

therefore, $K(g(\cdot, \varepsilon + u)) \in L^q(\Omega)$, $\varepsilon > 0$, $u \in L^q_+(\Omega)$. Due to [13, Theorem 9.15], for each $\varepsilon > 0$ and $u \in L^q_+(\Omega)$, there exists a unique $\Phi_\varepsilon(u) \in W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega)$ such that

$$\begin{gathered}-\varepsilon^\alpha \Delta \Phi_\varepsilon(u) + \Phi_\varepsilon(u) = K(g(\cdot, \varepsilon + u)) \quad \text{in } \Omega,\\ \Phi_\varepsilon(u) \geq 0 \quad \text{in } \Omega,\\ \Phi_\varepsilon(u) = 0 \quad \text{on } \partial\Omega.\end{gathered}$$

Analogously, there exists a unique $U_\varepsilon \in W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega)$ such that

$$\begin{gathered}-\varepsilon^\alpha \Delta U_\varepsilon + U_\varepsilon = K(g^*(\cdot, \varepsilon)) \quad \text{in } \Omega,\\ U_\varepsilon \geq 0 \quad \text{in } \Omega,\\ U_\varepsilon = 0 \quad \text{on } \partial\Omega.\end{gathered}$$

Since $0 < K(g(\cdot, \varepsilon + u)) \leq K(g^*(\cdot, \varepsilon))$, a.e. in Ω, the Maximum Principle states that $0 \leq \Phi_\varepsilon(u) \leq U_\varepsilon$. Hence, we have that

$$\Phi_\varepsilon(L^q_+(\Omega)) \subset S_\varepsilon = \big\{\omega \in W^{2,q}(\Omega) \,\big|\, 0 \leq \omega \leq U_\varepsilon\big\}.$$

Moreover, we have the following result.

Lemma 3.1. *Let* $\varepsilon > 0$*.*

(i) Φ_ε *is continuous in the sense that for every* $(u_n)_{n\in\mathbb{N}}$ *and* $\bar{u}$ *in* $L^q_+(\Omega)$*,*

$$u_n \to \bar{u} \text{ in } L^q(\Omega) \;\Rightarrow\; \Phi_\varepsilon(u_n) \to \Phi_\varepsilon(\bar{u}) \quad \text{in } W^{2,q}(\Omega).$$

(ii) $\Phi_\varepsilon(L^q_+(\Omega))$ *is compact in* $L^q(\Omega)$*.*

Proof. (i) Let $(u_n)_{n\in\mathbb{N}}$ and $\bar{u}$ in $L^q_+(\Omega)$ be such that $u_n \to \bar{u}$ in $L^q(\Omega)$. By (G1),

$$0 \le g(\cdot, \varepsilon + u_n) \le g^*(\cdot, \varepsilon) \in L^r(\Omega) \subset L^{q'}(\Omega), \quad n \in \mathbb{N}.$$

Due to the continuity of the Nemytskii operator $u \in L^q(\Omega) \mapsto g(\cdot, \varepsilon + u) \in L^{q'}(\Omega)$:

$$g(\cdot, \varepsilon + u_n) \to g(\cdot, \varepsilon + \bar{u}) \quad \text{in } L^{q'}(\Omega).$$

Then

$$K(g(\cdot, \varepsilon + u_n)) \to K(g(\cdot, \varepsilon + \bar{u})) \quad \text{in } L^q(\Omega). \tag{3.1}$$

Since $\Phi_\varepsilon(u_n) - \Phi_\varepsilon(\bar{u}) \in W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega)$ and

$$-\varepsilon^\alpha \Delta\big[\Phi_\varepsilon(u_n) - \Phi_\varepsilon(\bar{u})\big] + \big[\Phi_\varepsilon(u_n) - \Phi_\varepsilon(\bar{u})\big] = K(g(\cdot, \varepsilon + u_n)) - K(g(\cdot, \varepsilon + \bar{u})),$$

employing [13, Lemma 9.17], there exists $c_\varepsilon > 0$ independent on u_n and $\bar{u}$, such that

$$\|\Phi_\varepsilon(u_n) - \Phi_\varepsilon(\bar{u})\|_{W^{2,q}(\Omega)} \le c_\varepsilon |K(g(\cdot, \varepsilon + u_n)) - K(g(\cdot, \varepsilon + \bar{u}))|_{q,\Omega}.$$

Hence (i) follows from (3.1).

(ii) Let $(u_n)_{n\in\mathbb{N}}$, $u_n \in L^q_+(\Omega)$ be bounded. We prove that $(\Phi_\varepsilon(u_n))_{n\in\mathbb{N}}$ has a converging subsequence in $L^q(\Omega)$. Due to (G1) and ((K1), $(K(g(\cdot, \varepsilon + u_n)))_{n\in\mathbb{N}}$ is bounded in $L^q(\Omega)$. Hence $(\Phi_\varepsilon(u_n))_{n\in\mathbb{N}}$ is bounded in $W^{1,q}_0(\Omega)$ (see [13, Lemma 9.17]). Using $W^{1,q}_0(\Omega) \hookrightarrow \hookrightarrow L^q(\Omega)$, $(\Phi_\varepsilon(u_n))_{n\in\mathbb{N}}$ has a converging subsequence in $L^q(\Omega)$. The lemma is proved. □

Corollary 3.2. *For each $\varepsilon > 0$ there exists $u_\varepsilon \in S_\varepsilon \subset W^{2,q}(\Omega)$, such that $u_\varepsilon = \Phi_\varepsilon(u_\varepsilon)$, namely*

$$\begin{gathered} -\varepsilon^\alpha \Delta u_\varepsilon + u_\varepsilon = K(g(\cdot, \varepsilon + u_\varepsilon)) \qquad \text{in } \Omega, \\ u_\varepsilon \ge 0 \quad \text{in } \Omega, \\ u_\varepsilon = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

The claim of the above corollary follows directly from the Schauder theorem. The following two lemmas play a key role in our argument.

Lemma 3.3 ([15, Proposition 2.1])**.** *Let $1 < \rho < \infty$ and λ_1 be the first eigenvalue of the Dirichlet problem for $-\Delta$ on Ω. For every $0 < \varepsilon < \frac{1}{\lambda_1}$ and $\omega \in W^{2,\rho}(\Omega) \cap W^{1,\rho}_0(\Omega)$ we have that*

$$\varepsilon^\alpha \|\omega\|_{W^{2,\rho}(\Omega)} + \varepsilon^{\frac{\alpha}{2}} \|\omega\|_{W^{1,\rho}(\Omega)} + |\omega|_{\rho,\Omega} \le \text{const}\, | - \varepsilon^\alpha \Delta\omega + \omega|_{\rho,\Omega}.$$

Moreover, if $\omega_\varepsilon \in W^{2,\rho}(\Omega)$ solves

$$\begin{gathered} -\varepsilon^\alpha \Delta\omega_\varepsilon + \omega_\varepsilon = h_\varepsilon \quad \text{in } \Omega, \\ \omega_\varepsilon = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

with $h_\varepsilon \in L^\rho(\Omega)$ converging as $\varepsilon \to 0$ in $L^\rho(\Omega)$, there results

$$\lim_{\varepsilon\to 0} |\omega_\varepsilon - h_\varepsilon|_{\rho,\Omega} = 0; \quad \lim_{\varepsilon\to 0} \varepsilon^\alpha \|\omega_\varepsilon\|_{W^{2,\rho}(\Omega)} = 0.$$

Lemma 3.4. *For each $\varepsilon > 0$ there exists a unique $K_\varepsilon \in W^{2,q}(\Omega\times\Omega) \cap W^{1,q}_0(\Omega\times\Omega)$ such that*

$$\begin{gathered} -\varepsilon^{6q(p+1)} \Delta K_\varepsilon + K_\varepsilon = K \quad \text{in } \Omega\times\Omega, \\ K_\varepsilon \ge 0 \quad \text{in } \Omega\times\Omega, \\ rK_\varepsilon = 0 \quad \text{on } \partial(\Omega\times\Omega). \end{gathered}$$

Moreover,

(i) $\varepsilon^{6q(p+1)}\|K_\varepsilon\|_{W^{2,q}(\Omega\times\Omega)} + \varepsilon^{3q(p+1)}\|K_\varepsilon\|_{W^{1,q}(\Omega\times\Omega)} + |K_\varepsilon|_{q,\Omega\times\Omega}$
$\le \mathrm{const}\,|K|_{q,\Omega\times\Omega}$.

(ii) *There exists* $\varepsilon_0 > 0$ *such that* $|K_\varepsilon - K|_{q,\Omega\times\Omega}$
$\le \mathrm{const}(\varepsilon^{6q(p+1)})^{\frac{1}{3q}}\|K\|_{W^{\frac{1}{q},q}(\Omega\times\Omega)}$, $0 < \varepsilon < \varepsilon_0$.

Proof. Part (i) follows from the previous lemma, and (ii) follows from [11, Theorem 1.2]. □

For short, introduce the notation

$$g_\varepsilon := g(\cdot, \varepsilon + u_\varepsilon); \quad \bar\varepsilon := \min\{\varepsilon_0, \frac{1}{2}\}.$$

For the rest of this article, we assume that $0 < \varepsilon \le \bar\varepsilon$. The second step of our argument consists in the following estimates.

Lemma 3.5. *The following results hold:*

(a) $|K(g_\varepsilon)|_{q,\Omega} \le \mathrm{const}\,\varepsilon^{-p}$. *(It suffices to assume* $K \in L^q(\Omega\times\Omega)$*.)*
(b) $|K_\varepsilon(g_\varepsilon)|_{q,\Omega} \le \mathrm{const}\,\varepsilon^{-p}$.
(c) $\|K_\varepsilon(g_\varepsilon)\|_{W^{1,q}(\Omega)} \le \mathrm{const}\,\varepsilon^{-p-3q(p+1)}$.
(d) $\|K_\varepsilon(g_\varepsilon)\|_{W^{2,q}(\Omega)} \le \mathrm{const}\,\varepsilon^{-p-6q(p+1)}$.
(e) $|K(g_\varepsilon) - K_\varepsilon(g_\varepsilon)|_{q,\Omega} \le \mathrm{const}\,\varepsilon^{p+2}$.
(f) $|u_\varepsilon|_{q,\Omega} \le \mathrm{const}\,\varepsilon^{-p}$.
(g) $\|u_\varepsilon\|_{W^{1,q}(\Omega)} \le \mathrm{const}\,\varepsilon^{-p-5q(p+1)}$.

Proof. We begin by proving

$$|K(g_\varepsilon)|_{q,\Omega} \le \mathrm{const}\,\varepsilon^{-p}|K|_{q,\Omega\times\Omega}, \tag{3.2}$$

which implies (a). Since $q + r \le qr$ is equivalent to $q' \le r$, thanks to (G1), $\phi_0, g^*(\cdot, \frac{1}{2}) \in L^{q'}(\Omega)$. Define $X = (u_\varepsilon \le \frac{1}{2})$,

$$\begin{aligned}
&|K(g_\varepsilon)|_{q,\Omega}\\
&\le |K(g_\varepsilon\chi_X)|_{q,\Omega} + |K(g_\varepsilon\chi_{\Omega\setminus X})|_{q,\Omega}\\
&\le \Big(\int_\Omega \big(\int_\Omega K(x,y)\frac{\phi_0(y)}{(\varepsilon + u_\varepsilon(y))^p}dy\big)^q dx\Big)^{\frac{1}{q}} + \Big(\int_\Omega \big(\int_\Omega K(x,y)g^*(y,\frac{1}{2})dy\big)^q dx\Big)^{\frac{1}{q}}\\
&\le \frac{1}{\varepsilon^p}\Big(\int_\Omega \big(\int_\Omega K(x,y)^q dy\big)|\phi_0|^q_{q',\Omega}dx\Big)^{\frac{1}{q}} + \Big(\int_\Omega \big(\int_\Omega K(x,y)^q dy\big)\Big|g^*(\cdot,\frac{1}{2})\Big|^q_{q',\Omega}dx\Big)^{\frac{1}{q}}\\
&\le \Big(\frac{|\phi_0|_{q',\Omega}}{\varepsilon^p} + \Big|g^*(\cdot,\frac{1}{2})\Big|_{q',\Omega}\Big)|K|_{q,\Omega\times\Omega}.
\end{aligned}$$

Hence (3.2) is proved. Analogously we can prove

$$|K_\varepsilon(g_\varepsilon)|_{q,\Omega} \le \mathrm{const}\,\varepsilon^{-p}|K_\varepsilon|_{q,\Omega\times\Omega}.$$

Then employing (3.4)(i), we get (b). Since

$$\partial_i K_\varepsilon(g_\varepsilon)(x) = \int_\Omega \partial_{x_i}K_\varepsilon(x,y)g_\varepsilon(y)dy; \quad \partial^2_{i,j}K_\varepsilon(g_\varepsilon)(x) = \int_\Omega \partial^2_{x_i,x_j}K_\varepsilon(x,y)g_\varepsilon(y)dy,$$

arguing in the same way we have

$$\begin{gathered}
|\partial_i K_\varepsilon(g_\varepsilon)|_{q,\Omega} \le \mathrm{const}\,\varepsilon^{-p}|\partial_{x_i}K_\varepsilon|_{q,\Omega\times\Omega};\\
|\partial^2_{i,j}K_\varepsilon(g_\varepsilon)|_{q,\Omega} \le \mathrm{const}\,\varepsilon^{-p}|\partial^2_{x_i,x_j}K_\varepsilon|_{q,\Omega\times\Omega}.
\end{gathered}$$

Employing again (3.4)(ii) we get (c), (d).

Using the same argument of (3.2),

$$|K(g_\varepsilon) - K_\varepsilon(g_\varepsilon)|_{q,\Omega} \le \operatorname{const} \varepsilon^{-p} |K - K_\varepsilon|_{q,\Omega\times\Omega},$$

by (3.4)(ii),

$$|K(g_\varepsilon) - K_\varepsilon(g_\varepsilon)|_{q,\Omega} \le \operatorname{const} \varepsilon^{p+2} \|K\|_{W^{\frac{1}{q},q}(\Omega\times\Omega)},$$

that is (e). Applying Lemma 3.3 to u_ε in light of (3.2)

$$\varepsilon^{5q(p+1)} \|u_\varepsilon\|_{W^{1,q}(\Omega)} + |u_\varepsilon|_{q,\Omega} \le \operatorname{const} |K(g_\varepsilon)|_{q,\Omega} \le \operatorname{const} \varepsilon^{-p},$$

from which (f) and (g) follow. □

Lemma 3.6. *The following estimate holds*

$$\varepsilon^\alpha \sum_{i,j=1}^N |\partial^2_{i,j} u_\varepsilon|_{q,\Omega} + \varepsilon^{\frac{\alpha}{2}} \sum_{i=1}^N |\partial_i u_\varepsilon|_{q,\Omega} + |u_\varepsilon - K(g_\varepsilon)|_{q,\Omega} \le \bar{c}\varepsilon^{p+2},$$

for some constant $\bar{c} > 0$ independent of ε.

Proof. We begin by proving that $K_\varepsilon(g_\varepsilon) \in W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega)$. The fact $K_\varepsilon(g_\varepsilon) \in W^{2,q}(\Omega)$ follows from (3.5)(d). Due to Lemma 3.4, $K_\varepsilon \in W^{1,q}_0(\Omega\times\Omega)$. By definition of $W^{1,q}_0(\Omega \times \Omega)$ there exists $(\Phi_n)_n, \Phi_n \in C^\infty_0(\Omega \times \Omega)$ such that

$$\lim_n \|K_\varepsilon - \Phi_n\|_{W^{1,q}(\Omega\times\Omega)} = 0. \tag{3.3}$$

Denote

$$\Phi_n(g_\varepsilon)(x) = \int_\Omega \Phi_n(x, y) g_\varepsilon(y) dy, \quad x \in \Omega.$$

Since $\partial\Omega \times \Omega \subset \partial(\Omega \times \Omega)$ we have that $\Phi_n(g_\varepsilon) \in C^\infty_0(\Omega)$. Using the Hölder Inequality,

$$\begin{aligned}
\|K_\varepsilon(g_\varepsilon) - \Phi_n(g_\varepsilon)\|_{W^{1,q}(\Omega)} &= \sum_{i=1}^N |\partial_i K_\varepsilon(g_\varepsilon) - \partial_i \Phi_n(g_\varepsilon)|_{q,\Omega} + |K_\varepsilon(g_\varepsilon) - \Phi_n(g_\varepsilon)|_{q,\Omega} \\
&\le \Big(\sum_{i=1}^N |\partial_{x_i} K_\varepsilon - \partial_{x_i} \Phi_n|_{q,\Omega} + |K_\varepsilon - \Phi_n|_{q,\Omega} \Big) |g_\varepsilon|_{q',\Omega} \\
&= \|K_\varepsilon - \Phi_n\|_{W^{1,q}(\Omega)} |g_\varepsilon|_{q',\Omega}.
\end{aligned}$$

Employing (3.3),

$$\lim_n \|K_\varepsilon(g_\varepsilon) - \Phi_n(g_\varepsilon)\|_{W^{1,q}(\Omega)} = 0,$$

then by definition, $K_\varepsilon(g_\varepsilon) \in W^{1,q}_0(\Omega)$. We continue by observing that

$$-\varepsilon^\alpha \Delta(u_\varepsilon - K_\varepsilon(g_\varepsilon)) + (u_\varepsilon - K_\varepsilon(g_\varepsilon)) = K(g_\varepsilon) - K_\varepsilon(g_\varepsilon) + \varepsilon^\alpha \Delta K_\varepsilon(g_\varepsilon).$$

From Lemma 3.3,

$$\begin{aligned}
&\varepsilon^\alpha \|u_\varepsilon - K_\varepsilon(g_\varepsilon)\|_{W^{2,q}(\Omega)} + \varepsilon^{\frac{\alpha}{2}} \|u_\varepsilon - K_\varepsilon(g_\varepsilon)\|_{W^{1,q}(\Omega)} + |u_\varepsilon - K_\varepsilon(g_\varepsilon)|_{q,\Omega} \\
&\le \operatorname{const} |K(g_\varepsilon) - K_\varepsilon(g_\varepsilon) + \varepsilon^\alpha \Delta K_\varepsilon(g_\varepsilon)|_{q,\Omega};
\end{aligned}$$

hence

$$\begin{aligned}
L_\varepsilon &:= \varepsilon^\alpha \|u_\varepsilon\|_{W^{2,q}(\Omega)} + \varepsilon^{\frac{\alpha}{2}} \|u_\varepsilon\|_{W^{1,q}(\Omega)} + |u_\varepsilon - K(g_\varepsilon)|_{q,\Omega} \\
&\le \operatorname{const} \Big(|K(g_\varepsilon) - K_\varepsilon(g_\varepsilon)|_{q,\Omega} + \varepsilon^\alpha \|K_\varepsilon(g_\varepsilon)\|_{W^{2,q}(\Omega)} + \varepsilon^{\frac{\alpha}{2}} \|K_\varepsilon(g_\varepsilon)\|_{W^{1,q}(\Omega)} \Big).
\end{aligned}$$

Using Lemma 3.5,

$$L_\varepsilon \le \text{const}\left(\varepsilon^{p+2} + \varepsilon^{4q(p+1)-p} + \varepsilon^{2q(p+1)-p}\right).$$

Since $p+2 < 2q(p+1)-p < 4q(p+1)-p$, we have $L_\varepsilon \le \text{const}\,\varepsilon^{p+2}$. This gives the claim. $\square$

Lemma 3.7. *The sequence $(ag_\varepsilon)_{0<\varepsilon<\bar\varepsilon}$ is equiabsolutely continuous, more precisely for each $E \subset\subset \Omega$,*

$$|ag_\varepsilon|_{1,E} \le T(E), \quad 0 < \varepsilon \le \bar\varepsilon,$$

where

$$T(E) = A(E) + C(E) + \sqrt{B(E)+C(E)}; \quad A(E) = \Big|ag^*(\cdot,\frac{1}{2})\Big|_{1,E};$$

$$B(E) = \bar c|\phi_0|_{q',E}; \quad C(E) = \Big|\frac{\phi_0}{a^{p^*-1}}\Big|_{1,E}^{\frac{1}{p^*}}.$$

Proof. Define $X = E \cap (u_\varepsilon \le \frac{1}{2})$. Multiplying (1.3) by g_ε and integrating on X :

$$-\varepsilon^\alpha \int_X (\Delta u_\varepsilon) g_\varepsilon dx + \int_X u_\varepsilon g_\varepsilon dx = \int_X K(g_\varepsilon) g_\varepsilon dx. \tag{3.4}$$

We continue by estimating separately the three terms. By (K1), $q' \le r$, hence $\phi_0 \in L^{q'}(\Omega)$ (see (G1)) so

$$-\varepsilon^\alpha \int_X \Delta u_\varepsilon g_\varepsilon dx \le \varepsilon^{\alpha-p} \sum_{i,j=1}^N \int_X |\partial^2_{i,j} u_\varepsilon| \phi_0 dx \le \varepsilon^{\alpha-p} \sum_{i,j=1}^N |\partial^2_{i,j} u_\varepsilon|_{q,X} \cdot |\phi_0|_{q',X}.$$

Since $\varepsilon \le \frac{1}{2}$, from Lemma 3.6,

$$-\varepsilon^\alpha \int_X \Delta u_\varepsilon g_\varepsilon dx \le \bar c \varepsilon^2 |\phi_0|_{q',E} \le \frac{\bar c}{4} |\phi_0|_{q',E} \le B(E). \tag{3.5}$$

We distinguish two cases. If $p \le 1$,

$$\int_X u_\varepsilon g_\varepsilon dx \le \int_X u_\varepsilon^{1-p} \phi_0 dx \le \frac{1}{2^{1-p}} |\phi_0|_{1,X} \le |\phi_0|_{1,E}.$$

If $p > 1$,

$$\int_X u_\varepsilon g_\varepsilon dx = \int_X u_\varepsilon g_\varepsilon^{1/p} g_\varepsilon^{\frac{1}{p'}} dx \le \int_X \frac{\phi_0^{\frac{1}{p}}}{a^{\frac{1}{p'}}} (a g_\varepsilon)^{\frac{1}{p'}} dx \le \Big|\frac{\phi_0}{a^{p-1}}\Big|_{1,E}^{1/p} \cdot |ag_\varepsilon|_{1,X}^{\frac{1}{p'}},$$

where $p' = \frac{p}{p-1}$. Therefore,

$$\int_X u_\varepsilon g_\varepsilon dx \le \begin{cases} C(E) & \text{if } p \le 1, \\ C(E) \cdot |ag_\varepsilon|_{1,X}^{\frac{1}{p'}} & \text{if } p > 1. \end{cases} \tag{3.6}$$

Finally, from (K2),

$$\begin{aligned} \int_X K(g_\varepsilon) g_\varepsilon dx &\ge \int_{X\times\Omega} a(x) g_\varepsilon(x) a(y) g_\varepsilon(y) dx\, dy \\ &\ge |ag_\varepsilon|_{1,\Omega} \cdot |ag_\varepsilon|_{1,X} \\ &\ge |ag_\varepsilon|_{1,E}\big(|ag_\varepsilon|_{1,E} - |ag_\varepsilon|_{1,E\setminus X}\big) \\ &\ge |ag_\varepsilon|_{1,E}\big(|ag_\varepsilon|_{1,E} - \big|ag^*(\cdot,\frac{1}{2})\big|_{1,E}\big). \end{aligned} \tag{3.7}$$

Using (3.5), (3.6), (3.7) in (3.4), we obtain that $p \leq 1$ implies

$$\bar{c}|\phi_0|_{q',E} + |\phi_0|_{1,E} + |ag_\varepsilon|_{1,E}\Big|ag^*(\cdot,\frac{1}{2})\Big|_{1,E} \geq |ag_\varepsilon|^2_{1,E}$$

which in turn implies

$$|ag_\varepsilon|_{1,E} \leq A(E) + \sqrt{B(E)+C(E)}.$$

Also $p > 1$ implies

$$\bar{c}|\phi_0|_{q',E} + \Big|\frac{\phi_0}{a^{p-1}}\Big|^{1/p}_{1,E} \cdot |ag_\varepsilon|^{\frac{1}{p'}}_{1,E} + |ag_\varepsilon|_{1,E}\Big|ag^*(\cdot,\frac{1}{2})\Big|_{1,E} \geq |ag_\varepsilon|^2_{1,E}.$$

Denoting $\theta = |ag_\varepsilon|_{1,E}$, the previous estimate becomes

$$\theta^2 \leq A(E)\theta + C(E)\theta^{\frac{1}{p'}} + B(E).$$

Then $p > 1$ and $\theta \leq 1$ imply

$$\theta^2 \leq A(E)\theta + B(E) + C(E) \;\Rightarrow\; \theta \leq A(E) + \sqrt{B(E)+C(E)}.$$

Also $p > 1$ and $\theta > 1$ imply

$$\theta^2 \leq (A(E)+C(E))\theta + B(E) \;\Rightarrow\; \theta \leq A(E) + C(E) + \sqrt{B(E)}.$$

In conclusion, for every p, we have

$$\theta \leq A(E) + C(E) + \sqrt{B(E)+C(E)}.$$

The proof is complete. □

In light of the estimates of the previous lemmas we are now able to prove that the family $(u_\varepsilon)_{0<\varepsilon\leq\bar{\varepsilon}}$ is compact and has a subsequence that converges to a positive solution of (1.1).

Lemma 3.8. *There exists $(\varepsilon_k)_{k\in\mathbb{N}}$, $\varepsilon_k \to 0$, such that $(K(g_{\varepsilon_k}\chi_{\Omega_n}))_{k\in\mathbb{N}}$ converges in $L^1(\Omega_n)$, for each $n \in \mathbb{N}^*$.*

Proof. Due to the previous lemma, $(g_\varepsilon)_{0<\varepsilon\leq\bar{\varepsilon}}$ is bounded in $L^1(\Omega_1)$, and by $(\mathcal{K}_3)$ there exists $(\varepsilon_{1,k})_{k\in\mathbb{N}}, \varepsilon_{1,k} \to 0$, such that $(K(g_{\varepsilon_{1,k}}\chi_{\Omega_1}))_{k\in\mathbb{N}}$ is converging in $L^1(\Omega_1)$. Iterating this argument, for each $n \in \mathbb{N}$ there exist $(\varepsilon_{i,k})_{k\in\mathbb{N}}, 1 \leq i \leq n$, tending to 0 with $(\varepsilon_{j+1,k})_{k\in\mathbb{N}}$ subsequence of $(\varepsilon_{j,k})_{k\in\mathbb{N}}$ and $(K(g_{\varepsilon_{j,k}}\chi_{\Omega_i}))_{k\in\mathbb{N}}$ converging in $L^1(\Omega_j), 1 \leq j \leq n$. Hence, by induction there exists $(\varepsilon_{i,k})_{k\in\mathbb{N}}$ playing the same game. $(\varepsilon_{k,k})_{k\in\mathbb{N}^*}$ is a subsequence of every $(\varepsilon_{i,k})_{k\in\mathbb{N}}$, hence it fulfills the claim. □

Thanks to the previous lemma we can define

$$v_n := \begin{cases} \lim_k K(g_{\varepsilon_k}\chi_{\Omega_n}), & \text{in } \Omega_n,\\ 0, & \text{in } \Omega\setminus\Omega_n. \end{cases}$$

From Lemma 3.8, $v_n \in L^1(\Omega)$, and by construction $(v_n)_{n\in\mathbb{N}^*}$ is increasing, so

$$u_0 := \lim_n v_n = \sup_n v_n.$$

Lemma 3.9. *u_0 satisfies the following conditions:*

(a) *$u_0 \in L^1_+(\eta,\Omega)$ and $|\eta u_0|_{1,\Omega} \leq T(\Omega)$.*
(b) $\lim_n |\eta(u_0 - v_n)|_{1,\Omega} = 0$.
(c) *For all $n \in \mathbb{N}^*$, $\lim_k \Big|\frac{\eta}{1+\eta}(u_0 - K(g_{\varepsilon_k}))\Big|_{1,\Omega_n} = 0$.*

(d) *For all* $n \in \mathbb{N}^*$, $\lim_k \left|\frac{\eta}{1+\eta}(u_0 - u_{\varepsilon_k})\right|_{1,\Omega_n} = 0$.

(e) *Passing to a subsequence* $K(g_{\varepsilon_k}) \to u_0$, $u_{\varepsilon_k} \to u_0$, *a.e. in* Ω.

Proof. (a) By Lemma 3.7 and (K2),

$$T(\Omega) \ge |ag_{\varepsilon_k}|_{1,\Omega} \ge \int_{\Omega_n} g_{\varepsilon_k}(y)dy \int_{\Omega} K(x,y)\eta(x)dx \ge |\eta K(g_{\varepsilon_k}\chi_{\Omega_n})|_{1,\Omega_n}.$$

Sending $n, k \to +\infty$,

$$T(\Omega) \ge \lim_n \lim_k |\eta K(g_{\varepsilon_k}\chi_{\Omega_n})|_{1,\Omega_n} = \lim_n |\eta v_n|_{1,\Omega_n} = \lim_n |\eta v_n|_{1,\Omega} = |\eta u_0|_{1,\Omega}.$$

Part (b) is a direct consequence of the definition of u_0, (a) and the Dominate Convergence Theorem.

(c) Let $m \ge n > 0$ be integer numbers. Observe that

$$\begin{aligned} &\left|\frac{\eta}{1+\eta}(u_0 - K(g_{\varepsilon_k}))\right|_{1,\Omega_n} \\ &\le |\eta(u_0 - v_m)|_{1,\Omega_n} + |v_m - K(g_{\varepsilon_k}\chi_{\Omega_m})|_{1,\Omega_n} + |\eta K(g_{\varepsilon_k}\chi_{\Omega\setminus\Omega_m})|_{1,\Omega} \end{aligned}$$

and $\lim_k |v_m - K(g_{\varepsilon_k}\chi_{\Omega_m})|_{1,\Omega_n} = 0$. By (K2) and Lemma 3.7,

$$|\eta K(g_{\varepsilon_k}\chi_{\Omega\setminus\Omega_m})|_{1,\Omega} \le |ag_{\varepsilon_k}|_{1,\Omega\setminus\Omega_m} \le T(\Omega \setminus \Omega_m).$$

Hence

$$\limsup_k \left|\frac{\eta}{1+\eta}(u_0 - K(g_{\varepsilon_k}))\right|_{1,\Omega_n} \le |\eta(u_0 - v_m)|_{1,\Omega_n} + T(\Omega \setminus \Omega_m).$$

Since $|\Omega \setminus \Omega_m| \to 0$, using the absolute continuity of the integrals in $T(\cdot)$ we have that $T(\Omega \setminus \Omega_m) \to 0$. Hence (b) implies (c).

(d) Due to Lemma 3.6 and (K2),

$$\begin{aligned} \left|\frac{\eta}{1+\eta}(u_0 - u_{\varepsilon_k})\right|_{1,\Omega_n} &\le \left|\frac{\eta}{1+\eta}(u_0 - K(g_{\varepsilon_k}))\right|_{1,\Omega_n} + |\eta(K(g_{\varepsilon_k}) - u_{\varepsilon_k})|_{1,\Omega_n} \\ &\le \left|\frac{\eta}{1+\eta}(u_0 - K(g_{\varepsilon_k}))\right|_{1,\Omega_n} + |\eta|_{q',\Omega}|K(g_{\varepsilon_k}) - u_{\varepsilon_k}|_{q,\Omega} \\ &\le \left|\frac{\eta}{1+\eta}(u_0 - K(g_{\varepsilon_k}))\right|_{1,\Omega_n} + |\eta|_{q',\Omega}\bar{c}{\varepsilon_k}^{p+2}, \end{aligned}$$

using (c) we have (d).

Part e) is a consequence of (c), (d) and of the positivity of the map η a.e. in Ω; see (K2). ☐

In the proof of Lemma 3.11 we will use the following convergence theorem that will be proved in the appendix.

Lemma 3.10. *Let* $f_k \in L^1(\Omega)$, $\phi_k \in L^\infty(\Omega)$, $k \in \mathbb{N}$, *and* $\Omega \subset \mathbb{R}^N$. *If* $|\Omega| < \infty$, $(f_k)_{k\in\mathbb{N}}$ *is bounded in* $L^1(\Omega)$ *and equiabsolutely continuous,* $(\phi_k)_{k\in\mathbb{N}}$ *is bounded in* $L^\infty(\Omega)$, *and converging in measure to* $\phi \in L^\infty(\Omega)$, *then*

(i) $\lim_k |f_k\phi_k - f_k\phi|_{1,\Omega} = 0$

(ii)
$$\limsup_k |f_k\phi_k|_{1,\Omega} = \limsup_k |f_k\phi|_{1,\Omega}; \quad \liminf_k |f_k\phi_k|_{1,\Omega} = \liminf_k |f_k\phi|_{1,\Omega}.$$

Lemma 3.11. *Let* $n \in \mathbb{N}^*$ *and* $L > 0$. *Then*

$$\limsup_k |ag_{\varepsilon_k}|_{1,\Omega_n\cap(u_0\le L)} \le nL(1+L); \quad |ag(\cdot,u_0)|_{1,\Omega_n\cap(u_0\le L)} \le nL(1+L).$$

Proof. Denote $X_L = (u_0 \le L)$. Multiplying (1.3) with ε replaced by ε_k, by $\frac{g_{\varepsilon_k}}{1+u_{\varepsilon_k}}$ and integrating on $\Omega_n \cap X_L$:

$$\begin{aligned}&\int_{(\Omega_n\cap X_L)\times\Omega} \frac{g_{\varepsilon_k}(x)}{1+u_{\varepsilon_k}(x)} K(x,y) g_{\varepsilon_k}(y) dx\, dy\\&= \int_{\Omega_n\cap X_L} \frac{g_{\varepsilon_k} u_{\varepsilon_k}}{1+u_{\varepsilon_k}} dx - {\varepsilon_k}^{\alpha} \int_{\Omega_n\cap X_L} \Delta u_{\varepsilon_k} \frac{g_{\varepsilon_k}}{1+u_{\varepsilon_k}} dx.\end{aligned} \tag{3.8}$$

Due to (K2),

$$\int_{(\Omega_n\cap X_L)\times\Omega} \frac{g_{\varepsilon_k}(x)}{1+u_{\varepsilon_k}(x)} K(x,y) g_{\varepsilon_k}(y) dx\, dy \ge \Big| \frac{a g_{\varepsilon_k}}{1+u_{\varepsilon_k}} \Big|_{1,\Omega_n\cap X_L} \cdot |a g_{\varepsilon_k}|_{1,\Omega};$$

hence from (3.8),

$$\begin{aligned}&\limsup_k \Big(\Big| \frac{a g_{\varepsilon_k}}{1+u_{\varepsilon_k}} \Big|_{1,\Omega_n\cap X_L} \cdot |a g_{\varepsilon_k}|_{1,\Omega} \Big)\\&\le \limsup_k \int_{\Omega_n\cap X_L} \frac{g_{\varepsilon_k} u_{\varepsilon_k}}{1+u_{\varepsilon_k}} dx + \limsup_k {\varepsilon_k}^{\alpha} \int_{\Omega} |\Delta u_{\varepsilon_k}| \frac{g_{\varepsilon_k}}{1+u_{\varepsilon_k}} dx.\end{aligned} \tag{3.9}$$

Moreover, by Lemmas 3.7 and 3.10,

$$\begin{aligned}\limsup_k \Big(\Big| \frac{a g_{\varepsilon_k}}{1+u_{\varepsilon_k}} \Big|_{1,\Omega_n\cap X_L} \cdot |a g_{\varepsilon_k}|_{1,\Omega} \Big) &\ge \limsup_k \Big| \frac{a g_{\varepsilon_k}}{1+u_{\varepsilon_k}} \Big|^2_{1,\Omega_n\cap X_L}\\&= \Big(\limsup_k \Big| \frac{a g_{\varepsilon_k}}{1+u_{\varepsilon_k}} \Big|_{1,\Omega_n\cap X_L} \Big)^2\\&= \Big(\limsup_k \Big| \frac{a g_{\varepsilon_k}}{1+u_0} \Big|_{1,\Omega_n\cap X_L} \Big)^2\\&\ge \frac{1}{(1+L)^2} \big(\limsup_k |a g_{\varepsilon_k}|_{1,\Omega_n\cap X_L} \big)^2,\end{aligned} \tag{3.10}$$

$$\begin{aligned}\limsup_k \int_{\Omega_n\cap X_L} \frac{g_{\varepsilon_k} u_{\varepsilon_k}}{1+u_{\varepsilon_k}} dx &\le n \limsup_k \int_{\Omega_n\cap X_L} \frac{a g_{\varepsilon_k} u_{\varepsilon_k}}{1+u_{\varepsilon_k}} dx\\&= n \limsup_k \int_{\Omega_n\cap X_L} a g_{\varepsilon_k} \frac{u_0}{1+u_0} dx\\&\le \frac{nL}{1+L} \limsup_k |a g_{\varepsilon_k}|_{1,\Omega_n\cap X_L}.\end{aligned} \tag{3.11}$$

Finally, since $\varepsilon_k \le \bar{\varepsilon} \le \frac{1}{2}$, we get

$$\begin{aligned}{\varepsilon_k}^{\alpha} \int_{\Omega} |\Delta u_{\varepsilon_k}| \frac{g_{\varepsilon_k}}{1+u_{\varepsilon_k}} dx &\le {\varepsilon_k}^{\alpha} \Big(\int_{(\varepsilon_k + u_{\varepsilon_k} \le 1)} + \int_{(\varepsilon_k + u_{\varepsilon_k} > 1)} \Big) |\Delta u_{\varepsilon_k}| g_{\varepsilon_k} dx\\&\le {\varepsilon_k}^{\alpha - p} \int_{\Omega} |\Delta u_{\varepsilon_k}| \phi_0 dx + {\varepsilon_k}^{\alpha} \int_{\Omega} |\Delta u_{\varepsilon_k}| g^*(\cdot,1) dx\\&\le {\varepsilon_k}^{\alpha - p} \sum_{i,j=1}^N \int_{\Omega} |\partial^2_{i,j} u_{\varepsilon_k}| (\phi_0 + g^*(\cdot,1)) dx\\&\le {\varepsilon_k}^{\alpha - p} \sum_{i,j=1}^N |\partial^2_{i,j} u_{\varepsilon_k}|_{q,\Omega} \cdot |\phi_0 + g^*(\cdot,1)|_{q',\Omega},\end{aligned}$$

so due to Lemma 3.6

$$\lim_k {\varepsilon_k}^\alpha \int_\Omega |\Delta u_{\varepsilon_k}| \frac{g_{\varepsilon_k}}{1+u_{\varepsilon_k}} dx = 0. \tag{3.12}$$

Using (3.10), (3.11), (3.12) in (3.9)

$$\frac{1}{(1+L)^2}\big(\limsup_k |ag_{\varepsilon_k}|_{1,\Omega_n\cap X_L}\big)^2 \le \frac{nL}{1+L}\limsup_k |ag_{\varepsilon_k}|_{1,\Omega_n\cap X_L},$$

that is the first estimate of the claim. The second one follows from the first one, (3.9)and the Fatou's Lemma. □

Proof of Theorem 2.3. Fix $n, l \in \mathbb{N}^*$, and introduced the notation $Y_l = (u_0 \le \frac{1}{l})$. From (K2),

$$\begin{aligned}
\Big|\frac{\eta}{1+\eta}(K(g_{\varepsilon_k}\chi_{\Omega_n\setminus Y_l}) - v_n)\Big|_{1,\Omega_n} &= \Big|\frac{\eta}{1+\eta}(K(g_{\varepsilon_k}\chi_{\Omega_n}) - K(g_{\varepsilon_k}\chi_{\Omega_n\cap Y_l}) - v_n)\Big|_{1,\Omega_n}\\
&\le |K(g_{\varepsilon_k}\chi_{\Omega_n}) - v_n|_{1,\Omega_n} + |\eta K(g_{\varepsilon_k}\chi_{\Omega_n\cap Y_l})|_{1,\Omega_n}\\
&\le |K(g_{\varepsilon_k}\chi_{\Omega_n}) - v_n|_{1,\Omega_n} + |ag_{\varepsilon_k}|_{1,\Omega_n\cap Y_l}.
\end{aligned}$$

Due to the definition of v_n and Lemma 3.11, sending $k \to +\infty$, we obtain

$$\limsup_k \Big|\frac{\eta}{1+\eta}(K(g_{\varepsilon_k}\chi_{\Omega_n\setminus Y_l}) - v_n)\Big|_{1,\Omega_n} \le \frac{n}{l}(1+\frac{1}{l}) \le \frac{2n}{l}.$$

Using Fatou's Lemma and (3.9),

$$\Big|\frac{\eta}{1+\eta}(K(g(\cdot,u_0)\chi_{\Omega_n\setminus Y_l}) - v_n)\Big|_{1,\Omega_n} \le \frac{2n}{l}.$$

Passing to the limit as $l \to +\infty$,

$$v_n(x) = \lim_l \int_{\Omega_n\setminus Y_l} K(x,y)g(y,u_0(y))dy = \int_{\Omega_n\cap(0<u_0)} K(x,y)g(y,u_0(y))dy,$$

$x \in \Omega_n$. Then sending $n \to +\infty$,

$$u_0(x) = \int_{(0<u_0)} K(x,y)g(y,u_0(y))dy, \quad x \in \Omega. \tag{3.13}$$

We claim that $|(0 < u_0)| = |\Omega|$, namely $|ag(\cdot,u_0)|_{1,\Omega} > 0$. Assume, by contradiction, that $\mathcal{N} = (u_0 = 0)$ has positive measure. We have that

$$0 = \int_{\Omega\setminus\mathcal{N}} K(x,y)g(y,u_0(y))dy, \quad x \in \mathcal{N},$$

and using (K2)

$$0 = \int_{\Omega\setminus\mathcal{N}} a(y)g(y,u_0(y))dy.$$

Since $a(y) > 0$ a.e. in Ω, we have that $g(y,u_0(y)) = 0$ in $\Omega\setminus\mathcal{N}$. From (3.13) $|\mathcal{N}| = |\Omega|$.

Due to (3.9) we know $u_{\varepsilon_k} \to 0$ a.e. in Ω and in particular in Ω_0. By fixed $0 < \sigma < \mu_0$, in light of (G3) there exists $s_0 > 0$ such that

$$y \in \Omega_0, \quad 0 < s < s_0 \ \Rightarrow\ g(y,s) > (\mu_0 - \sigma)s.$$

For the reason that $|\Omega_0| < \infty$, there exists $\Omega_\sigma \subset \Omega_0$ such that $|\Omega_\sigma| < \sigma$ and $u_{\varepsilon_k} \to 0$ uniformly in $\Omega_0 \setminus \Omega_\sigma$ (Egorov-Severini Theorem). Then

$$k > k_0,\ y \in \Omega_0 \setminus \Omega_\sigma \ \Rightarrow\ g_{\varepsilon_k}(y) > (\mu_0 - \sigma)(\varepsilon_k + u_{\varepsilon_k}(y)),$$

for some $k_0 \in \mathbb{N}$. Multiplying (1.3) with ε replaced by ${\varepsilon_k}$, by $a_\lambda(x) := \frac{a(x)}{1+\lambda a(x)}$ and integrating on $\Omega_0 \setminus \Omega_\sigma$

$$-{\varepsilon_k}^\alpha \int_{\Omega_0\setminus\Omega_\sigma} a_\lambda \Delta u_{\varepsilon_k} dx + \int_{\Omega_0\setminus\Omega_\sigma} a_\lambda u_{\varepsilon_k} dx = \int_\Omega g_{\varepsilon_k}(y)dy \int_{\Omega_0\setminus\Omega_\sigma} K(x,y)a_\lambda(x)dx. \tag{3.14}$$

Since $a_\lambda \in L^\infty(\Omega)$, Lemma 3.6 implies

$$\begin{aligned} -{\varepsilon_k}^\alpha \int_{\Omega_0\setminus\Omega_\sigma} a_\lambda \Delta u_{\varepsilon_k} dx &\le {\varepsilon_k}^\alpha \sum_{i,j=1}^N |\partial^2_{i,j} u_{\varepsilon_k}|_{q,\Omega_0\setminus\Omega_\sigma} |a_\lambda|_{q',\Omega_0\setminus\Omega_\sigma} \\ &\le \bar{c}{\varepsilon_k}^{p+2} |a_\lambda|_{q',\Omega_0\setminus\Omega_\sigma} \\ &= \frac{\bar{c}{\varepsilon_k}^{p+1}|a_\lambda|_{q',\Omega_0\setminus\Omega_\sigma}}{|a_\lambda|_{1,\Omega_0\setminus\Omega_\sigma}} \int_{\Omega_0\setminus\Omega_\sigma} a_\lambda \varepsilon_k dx \\ &\le \frac{\bar{c}|a_\lambda|_{q',\Omega}}{|a_\lambda|_{1,\Omega_0\setminus\Omega_\sigma}} {\varepsilon_k}^{p+1} \int_{\Omega_0\setminus\Omega_\sigma} a_\lambda(\varepsilon_k + u_{\varepsilon_k}) dx. \end{aligned} \tag{3.15}$$

Using now (K2) and (3), for every $k > k_0$, we have that

$$\int_\Omega g_{\varepsilon_k}(y)dy \int_{\Omega_0\setminus\Omega_\sigma} K(x,y)a_\lambda(x)dx \ge (\mu_0 - \sigma) \int_{\Omega_0\setminus\Omega_\sigma} a_\lambda(\varepsilon_k + u_{\varepsilon_k})dx \int_{\Omega_0\setminus\Omega_\sigma} a a_\lambda dx. \tag{3.16}$$

Substituting (3.15), (3.16) in (3.14),

$$\begin{aligned} &\frac{\bar{c}|a_\lambda|_{q',\Omega}}{|a_\lambda|_{1,\Omega\setminus\Omega_\sigma}} {\varepsilon_k}^{p+1} \int_{\Omega_0\setminus\Omega_\sigma} a_\lambda(\varepsilon_k + u_{\varepsilon_k})dx + \int_{\Omega_0\setminus\Omega_\sigma} a_\lambda(\varepsilon_k + u_{\varepsilon_k})dx \\ &\ge (\mu_0 - \sigma)|a a_\lambda|_{1,\Omega_0\setminus\Omega_\sigma} \int_{\Omega_0\setminus\Omega_\sigma} a_\lambda(\varepsilon_k + u_{\varepsilon_k})dx, \end{aligned}$$

that gives

$$\frac{\bar{c}|a_\lambda|_{q',\Omega}}{|a_\lambda|_{1,\Omega\setminus\Omega_\sigma}} {\varepsilon_k}^{p+1} + 1 \ge (\mu_0 - \sigma)|a a_\lambda|_{1,\Omega_0\setminus\Omega_\sigma}.$$

Sending first $k \to +\infty$, then $\sigma \to 0$ and finally $\lambda \to 0$, we get

$$1 \ge \mu_0 |a^2|_{1,\Omega_0}.$$

That contradicts (2.1), hence $u_0 > 0$ a.e. in Ω.

We continue by proving that $u_{\varepsilon_k} \to u_0$ in $L^1_+(\eta,\Omega)$. Reminding that u_0, u_{ε_k} are solutions of (1.1) and (1.3) we get

$$\begin{aligned} I_k &:= |\eta(u_0 - u_{\varepsilon_k})|_{1,\Omega} \\ &\le |\eta(K(g(\cdot,u_0)) - K(g_{\varepsilon_k}))|_{1,\Omega} + {\varepsilon_k}^\alpha |\eta \Delta u_{\varepsilon_k}|_{1,\Omega} \\ &\le |a(g(\cdot,u_0) - g_{\varepsilon_k})|_{1,\Omega} + {\varepsilon_k}^\alpha |\eta|_{q',\Omega} \sum_{i,j=1}^N |\partial^2_{i,j} u_{\varepsilon_k}|_{q,\Omega}. \end{aligned}$$

In light of Lemma 3.6,

$$I_k \le |a(g(\cdot,u_0) - g_{\varepsilon_k})|_{1,\Omega} + \bar{c}|\eta|_{q',\Omega} {\varepsilon_k}^{p+2}.$$

Due to the positivity of u_0, $g_{\varepsilon_k} \to g(\cdot,u_0)$ a.e. in Ω; hence the equiabsolute continuity of the integrals in $a g_{\varepsilon_k}$ (see Lemma 3.7) and Vitali's Theorem say

$$|a(g(\cdot,u_0) - g_{\varepsilon_k})|_{1,\Omega} \to 0. \tag{3.17}$$

Therefore, $I_k \to 0$, so the claim is proved.

We conclude by proving that $|ag(\cdot,u_0)|_{1,\Omega}a(x) \le u_0(x)$, $x \in \Omega$ a.e. Due to (K2),

$$\begin{aligned}-{\varepsilon_k}^{\alpha}\Delta u_{\varepsilon_k}(x) + u_{\varepsilon_k}(x) &= K(g_{\varepsilon_k})(x)\\ &\ge a(x)|ag_{\varepsilon_k}|_{1,\Omega}\\ &\ge a(x)\big(|ag(\cdot,u_0)|_{1,\Omega} - |a\big(g_{\varepsilon_k} - g(\cdot,u_0)\big)|_{1,\Omega}\big).\end{aligned}$$

Since $u_{\varepsilon_k} \to u_0$ a.e. in Ω in light of Lemma 3.6 that must be ${\varepsilon_k}^{\alpha}\Delta u_{\varepsilon_k} \to 0$ in $L^q(\Omega)$, and passing to a subsequence ${\varepsilon_k}^{\alpha}\Delta u_{\varepsilon_k} \to 0$ a.e. in Ω. Hence the claim follows from (3.17).

The last part of the statement was proved in Lemma 3.6. □

4. Proof of Theorem 2.4

Let us list some of the properties of the Green's function $G(x,y)$ of the Dirichlet problem for $-\Delta$ in Ω.

Lemma 4.1 ([3, Lemma 3.1])**.** *There exists a constant $c_1 > 0$ such that*

(i) $\frac{\delta(x)\delta(y)}{c_1} \le G(x,y)$; $\int_\Omega G(x,y)dx \le c_1\delta(y)$.

(ii)
$$\Big(\int_\Omega G(x,y)^\sigma dy\Big)^{\frac{1}{\sigma}} \le c_1 \int_\Omega G(x,y)dy, \quad 1 \le \sigma < \frac{N}{N-1}.$$

(iii) $|\nabla_x G(x,y)| \le \frac{c_1}{|x-y|^{N-1}}$, $x \ne y$.

Lemma 4.2. *Let $\psi \in L^r(\Omega)$, $1 < r < \infty$. The maps*

$$G(\psi)(x) := \int_\Omega G(x,y)\psi(y)dy; \quad \widetilde{G}(\psi)(x) := \int_\Omega \nabla_x G(x,y)\psi(y)dy, \quad x \in \Omega$$

satisfy the following conditions:

(i) $G(\psi) \in W^{2,r}(\Omega)$; $\widetilde{G}(\psi) \in W^{1,r}(\Omega)$; $\nabla G(\psi) = \widetilde{G}(\psi)$.

(ii) $-\Delta G(\psi) = \psi$ *in* Ω; $G(\psi) = 0$ *on* $\partial\Omega$.

In particular, if $r > N$, then $G(\psi) \in W^{2,r}(\Omega) \subset C^1(\bar{\Omega})$.

We will use a simplified version of the following Agmon's interior regularity result.

Lemma 4.3 ([1, Theorem 7.1])**.** *Let $u \in L^{\alpha}_{\rm loc}(\Omega)$, $1 < \alpha$, be such that $\Delta u \in L^{\beta}_{\rm loc}(\Omega)$, $1 < \beta$, where Δu is defined by*

$$\int_\Omega \Delta u \cdot \phi dx = \int_\Omega u \cdot \Delta\phi dx, \quad \forall \phi \in C_0^\infty(\Omega).$$

Then $u \in W^{2,\beta}_{\rm loc}(\Omega)$ and for every $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ there exists $c > 0$ such that

$$\|u\|_{W^{2,\beta}(\Omega')} \le c\big(|\Delta u|_{\beta,\Omega''} + |u|_{\beta,\Omega''}\big).$$

First of all we prove that the solutions of (1.3) are also solutions of (2.4).

Lemma 4.4. *For each $\varepsilon > 0$, the solutions to* (1.3) *in $W^{2,q}(\Omega)$ belong to $W^{4,r}(\Omega)$ and solves* (2.4).

Proof. Let $u_\varepsilon \in W^{2,q}(\Omega)$ be solution to (1.3) with $G(x,y)$ instead of $K(x,y)$ (see Corollary 3.2). Since $g_\varepsilon \in L^r(\Omega)$ (see (G1) and Remark 2.1), due to Lemma 4.2, we have that $G(g_\varepsilon) \in W^{2,r}(\Omega)$, hence $u_\varepsilon \in W^{2,r}(\Omega)$. From (1.3)

$$\Delta u_\varepsilon = \varepsilon^{-\alpha}(u_\varepsilon - G(g_\varepsilon)) \in W^{2,r}(\Omega); \quad \Delta u_\varepsilon = 0 \text{ on } \partial\Omega.$$

The lemma is established. □

Due to Theorem 2.3 there exist $u_0 \in L^1_+(\Omega)$ and $(\varepsilon_k)_{k\in\mathbb{N}}$, $0 < \varepsilon_k < \bar{\varepsilon} \leq \frac{1}{2}$, tending to 0, such that

$$\begin{gathered} u_0(x) = \int_\Omega G(x,y) g\big(y, u_0(y)\big) dy, \quad x \in \Omega, \\ \lim_k |u_0 - u_{\varepsilon_k}|_{1,\Omega} = 0; \quad u_{\varepsilon_k} \to u_0 \quad q.o \text{ in } \Omega; \quad u_{\varepsilon_k} \in W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega), \end{gathered} \tag{4.1}$$

and

$$-{\varepsilon_k}^\alpha \Delta u_{\varepsilon_k}(x) + u_{\varepsilon_k}(x) = \int_\Omega G(x,y) g\big(y, \varepsilon_k + u_{\varepsilon_k}(y)\big) dy, \quad x \in \Omega.$$

Lemma 4.5. *There exist $k_0 \in \mathbb{N}$ and $c_2 > 0$ such that*

$$k_0 < k \Rightarrow c_2 \delta(x) \leq u_{\varepsilon_k}(x); \quad c_2\delta(x) \leq u_0(x).$$

Proof. The second estimate comes from Theorem 2.3. We have to prove the first estimate. From (4.1),

$$G(g_{\varepsilon_k})(x) \geq \frac{\delta(x)}{c_1} |\delta g_{\varepsilon_k}|_{1,\Omega}, \quad x \in \bar{\Omega}, \quad k \in \mathbb{N}.$$

Since

$$\liminf_k |\delta g_{\varepsilon_k}|_{1,\Omega} \geq |\liminf_k \delta g_{\varepsilon_k}|_{1,\Omega} = |\delta g(\cdot, u_0)|_{1,\Omega} > 0,$$

there exists $k_0 \in \mathbb{N}$ such that

$$k > k_0 \;\Rightarrow\; |\delta g_{\varepsilon_k}|_{1,\Omega} > \frac{1}{2}|\delta g(\cdot, u_0)|_{1,\Omega} \;\Rightarrow\; G(g_{\varepsilon_k})(x) \geq \frac{\delta(x)}{2c_1}|\delta g(\cdot, u_0)|_{1,\Omega}.$$

Let ϕ_1 be a positive eigenfunction and λ_1 be the first eigenvalue of the Dirichlet problem for $-\Delta$ in Ω. Since ϕ_1 and δ are equivalent in the sense that

$$0 < \inf_{x\in\Omega} \frac{\delta(x)}{\phi_1(x)} < \sup_{x\in\Omega} \frac{\delta(x)}{\phi_1(x)} < \infty,$$

there exists an eigenfunction ϕ_1 relatively to λ_1 such that

$$k > k_0 \;\Rightarrow\; G(g_{\varepsilon_k})(x) \geq (\lambda_1 + 1)\phi_1(x) > ({\varepsilon_k}^\alpha \lambda_1 + 1)\phi_1(x) = -{\varepsilon_k}^\alpha \Delta\phi_1(x) + \phi_1(x).$$

From (1.3) with ε replaced by ε_k,

$$k > k_0 \;\Rightarrow\; -{\varepsilon_k}^\alpha \Delta u_{\varepsilon_k} + u_{\varepsilon_k} \geq -{\varepsilon_k}^\alpha \Delta \phi_1 + \phi_1.$$

Due to the Maximum Principle $u_{\varepsilon_k} \geq \phi_1, k > k_0$. Finally, since ϕ_1 and δ are equivalent we get the first estimate of the claim. □

Lemma 4.6. $u_0 \in W^{2,r}_{\rm loc}(\Omega) \cap L^q(\Omega)$ *and* $-\Delta u_0(x) = g(x, u_0(x))$.

Proof. Introduce the notation $v_{\varepsilon_k} := -\Delta u_{\varepsilon_k}$. From Lemma 4.4,

$$\begin{gathered} -{\varepsilon_k}^{\alpha}\Delta v_{\varepsilon_k} + v_{\varepsilon_k} = g_{\varepsilon_k} \quad \text{in } \Omega, \\ v_{\varepsilon_k} = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{4.2}$$

Hence for a fixed $\phi \in C_0^2(\Omega)$,

$$-{\varepsilon_k}^{\alpha}\Delta(\phi v_{\varepsilon_k}) + (\phi v_{\varepsilon_k}) = f_{\varepsilon_k}, \tag{4.3}$$

where

$$f_{\varepsilon_k} := \phi g_{\varepsilon_k} - {\varepsilon_k}^{\alpha}(2\nabla\phi\nabla v_{\varepsilon_k} + v_{\varepsilon_k}\Delta\phi).$$

From Lemma 4.5 and (G1), $g_{\varepsilon_k} \in L^r(\Omega)$, $g(\cdot, u_0) \in L^r_{\rm loc}(\Omega)$. We claim that

$$\lim_k |f_{\varepsilon_k} - \phi g(\cdot, u_0)|_{r,\Omega} = 0. \tag{4.4}$$

Observe that

$$\begin{aligned} |f_{\varepsilon_k} - \phi g(\cdot, u_0)|_{r,\Omega} &\le |\phi(g_{\varepsilon_k} - g(\cdot, u_0))|_{r,\Omega} + {\varepsilon_k}^{\alpha}|2\nabla\phi\nabla v_{\varepsilon_k} + v_{\varepsilon_k}\Delta\phi|_{r,\Omega} \\ &\le |\phi(g_{\varepsilon_k} - g(\cdot, u_0))|_{r,\Omega} + \text{const}\,{\varepsilon_k}^{\alpha}\|v_{\varepsilon_k}\|_{W^{1,r}(\Omega)}. \end{aligned}$$

Applying Lemma 3.3 to (4.2),

$$\begin{aligned} &|f_{\varepsilon_k} - \phi g(\cdot, u_0)|_{r,\Omega} \\ &\le |\phi(g_{\varepsilon_k} - g(\cdot, u_0))|_{r,\Omega} + \text{const}\,{\varepsilon_k}^{\frac{\alpha}{2}}|g_{\varepsilon_k}|_{r,\Omega} \\ &\le |\phi(g_{\varepsilon_k} - g(\cdot, u_0))|_{r,\Omega} + \text{const}\,{\varepsilon_k}^{\frac{\alpha}{2}}\Big(\big(\int_{\varepsilon_k + u_{\varepsilon_k} \le 1} + \int_{\varepsilon_k + u_{\varepsilon_k} \ge 1}\big) g_{\varepsilon_k}^r dx\Big)^{\frac{1}{r}} \\ &\le |\phi(g_{\varepsilon_k} - g(\cdot, u_0))|_{r,\Omega} + \text{const}\Big({\varepsilon_k}^{r(\frac{\alpha}{2} - p)}\int_\Omega \phi_0^r dx + {\varepsilon_k}^{\frac{r\alpha}{2}}\int_\Omega (g^*(x,1))^r dx\Big)^{\frac{1}{r}}. \end{aligned}$$

Recalling that $\frac{\alpha}{2} - p = 5q(p+1) - p > 0$ we have

$$\limsup_k |f_{\varepsilon_k} - \phi g(\cdot, u_0)|_{r,\Omega} \le \limsup_k |\phi(g_{\varepsilon_k} - g(\cdot, u_0))|_{r,\Omega}. \tag{4.5}$$

Since $u_{\varepsilon_k} \to u_0$ a.e. in Ω implies $g_{\varepsilon_k} \to g(\cdot, u_0)$ a.e. in Ω and $\operatorname{dist}(\operatorname{supp}\phi, \partial\Omega) > 0$, from Lemma 4.5, (4.1) and the Dominate Convergence Theorem,

$$\lim_k |\phi(g_{\varepsilon_k} - g(\cdot, u_0))|_{r,\Omega} = 0.$$

Hence (4.4) follows from (4.5). Applying Lemma 3.3 to (4.3), by (4.4),

$$\forall \phi \in C_0^2(\Omega): \quad \lim_k |\phi(v_{\varepsilon_k} - g(\cdot, u_0))|_{r,\Omega} = 0. \tag{4.6}$$

Observing that (4.1), (4.6) give

$$\begin{aligned} \int_\Omega \phi(x) g(x, u_0) dx &= \lim_k \int_\Omega \phi v_{\varepsilon_k} dx \\ &= \lim_k \int_\Omega \phi(-\Delta u_{\varepsilon_k}) dx \\ &= \lim_k \int_\Omega (-\Delta\phi) u_{\varepsilon_k} dx \\ &= \int_\Omega (-\Delta\phi) u_0 dx; \end{aligned}$$

therefore,

$$-\Delta u_0 = g(\cdot, u_0) \quad \text{in } \Omega \text{ in the sense of distributions.} \tag{4.7}$$

We claim that

$$u_0 \in L^q(\Omega). \tag{4.8}$$

Observe that

$$\begin{aligned} u_0(x) &= \int_\Omega G(x,y) g(y, u_0(y)) dy \\ &= \int_\Omega \frac{G(x,y)}{\delta(y)^{\frac{1}{q'}}} g(y,u_0(y))^{\frac{1}{q}} (\delta(y) g(y,u_0(y)))^{\frac{1}{q'}} dy \\ &\le \Big(\int_\Omega \frac{G(x,y)^q}{\delta(y)^{q-1}} g(y,u_0(y)) dy \Big)^{\frac{1}{q}} |\delta g(\cdot,u_0)|_{1,\Omega}^{\frac{1}{q'}}. \end{aligned}$$

Hence

$$\begin{aligned} |u_0|_{q,\Omega}^q &\le |\delta g(\cdot,u_0)|_{1,\Omega}^{q-1} \int_\Omega dx \int_\Omega \frac{G(x,y)^q}{\delta(y)^{q-1}} g(y,u_0(y)) dy \\ &= |\delta g(\cdot,u_0)|_{1,\Omega}^{q-1} \int_\Omega \frac{\delta(y) g(y,u_0(y))}{\delta^q(y)} dy \int_\Omega G(x,y)^q dx. \end{aligned}$$

From (4.1)(i) and (4.1)(ii),

$$|u_0|_{q,\Omega}^q \le |\delta g(\cdot,u_0)|_{1,\Omega}^{q-1} c_1^{2q} \int_\Omega \frac{\delta(y) g(y,u_0(y))}{\delta^q(y)} \delta^q(y) dy = c_1^{2q} |\delta g(\cdot,u_0)|_{1,\Omega}^q.$$

In light of Theorem 2.3, $g(\cdot,u_0) \in L^1_+(\delta,\Omega)$, hence (4.8) is true. We need to prove that

$$u_0 \in W^{2,r}_{\rm loc}(\Omega).$$

Since (4.7) holds in the sense of distributions we have simply to apply Lemma 4.3 with $\alpha = q$ and $\beta = r$. The lemma is proved. □

Lemma 4.7. *$u_0 \in C(\bar\Omega)$ and $u_0(x) = 0, x \in \partial\Omega$.*

Proof. Since $q < \frac{N}{N-1}$ and $q + r \le qr$ give $r > N$, due to Lemmas 4.5 and 4.6, $0 \le c_2\delta \le u_0 \in C(\Omega)$. Therefore, we have only to prove that

$$\lim_{x\to x_0} u_0(x) = 0, \quad x_0 \in \partial\Omega. \tag{4.9}$$

Define $\theta_1 : \mathbb{R}_+ \to \mathbb{R}$ by

$$\theta_1(t) = \begin{cases} ([p]+2)([p]+1)t^{[p]} & \text{if } 0 \le t \le 3, \\ 0 & \text{if } t > 3, \end{cases}$$

where $[p]$ is the integer part of p, and $\theta_2 \in C^\infty(\mathbb{R}_+)$ such that

$$0 \le \theta_2 \le 1, \quad 0 \le t \le 1 \Rightarrow \theta_2(t) = 1, \quad t \ge 2 \Rightarrow \theta_2(t) = 0.$$

Denoting

$$\theta(t) := \int_0^t dt_1 \int_0^{t_1} \theta_1(\tau)\theta_2(\tau) d\tau, \quad t \in \mathbb{R}_+,$$

observe that

$$\theta \ge 0; \quad \theta(t) = t^{[p]+2}, \quad 0 \le t \le 1;$$

$$\theta'(t) = \int_0^t \theta_1(\tau)\theta_2(\tau) d\tau \le \theta'(2); \quad \theta'' = \theta_1\theta_2 \ge 0; \quad \theta \in C^\infty(\mathbb{R}_+).$$

Since $\theta'' \geq 0$, $-\Delta(\theta(u_{\varepsilon_k})) \leq \theta'(u_{\varepsilon_k})(-\Delta u_{\varepsilon_k})$. As in the Proof of Lemma 4.6, denoting $-\Delta u_{\varepsilon_k} = v_{\varepsilon_k}$, from (2.4) with ε_k instead of ε, we get

$$-\Delta u_{\varepsilon_k} = g_{\varepsilon_k} - {\varepsilon_k}^{\alpha}\Delta^2 u_{\varepsilon_k} = g_{\varepsilon_k} + {\varepsilon_k}^{\alpha}\Delta v_{\varepsilon_k}.$$

Therefore,

$$-\Delta(\theta(u_{\varepsilon_k})) \leq \theta'(u_{\varepsilon_k})(g_{\varepsilon_k} + {\varepsilon_k}^{\alpha}\Delta v_{\varepsilon_k}). \tag{4.10}$$

Since $r > N$, $u_{\varepsilon_k} \in W^{4,r}(\Omega) \subset C^3(\bar{\Omega})$ and $u_{\varepsilon_k} = 0$ on $\partial\Omega$, from the properties of θ we deduce that $\theta(u_{\varepsilon_k}) \in W^{4,r}(\Omega)$ and $\theta(u_{\varepsilon_k}) = 0$ on $\partial\Omega$. Thanks to the positivity of the Green's function from (4.10), we get

$$\theta(u_{\varepsilon_k}) \leq G(\theta'(u_{\varepsilon_k})g_{\varepsilon_k}) + {\varepsilon_k}^{\alpha}G(\theta'(u_{\varepsilon_k})\Delta v_{\varepsilon_k}). \tag{4.11}$$

Integrating by parts

$$\begin{aligned} G(\theta'(u_{\varepsilon_k})\Delta v_{\varepsilon_k})(x) &= -\int_\Omega \nabla_y G(x,y)\theta'(u_{\varepsilon_k}(y))\nabla v_{\varepsilon_k}(y)dy \\ &\quad - \int_\Omega G(x,y)\nabla(\theta'(u_{\varepsilon_k}(y)))\nabla v_{\varepsilon_k}(y)dy, \end{aligned}$$

so

$$\begin{aligned} I_k &:= {\varepsilon_k}^{\alpha}|G(\theta'(u_{\varepsilon_k})\Delta v_{\varepsilon_k})|_{1,\Omega} \\ &\leq \int_{\Omega\times\Omega} |\nabla_y G(x,y)|\theta'(u_{\varepsilon_k}(y))|\nabla v_{\varepsilon_k}(y)|dx\,dy \\ &\quad + \int_{\Omega\times\Omega} G(x,y)|\nabla(\theta'(u_{\varepsilon_k}(y)))|\cdot|\nabla v_{\varepsilon_k}(y)|dx\,dy. \end{aligned}$$

Since

$$\sup_{y\in\Omega}\Big(\int_\Omega |\nabla_y G(x,y)|dx + \int_\Omega G(x,y)dx\Big) < \infty,$$

the boundedness of θ' and θ'' implies

$$I_k \leq \mathrm{const}\,{\varepsilon_k}^{\alpha}\int_\Omega (1+|\nabla u_{\varepsilon_k}|)|\nabla v_{\varepsilon_k}|dy \leq \mathrm{const}\,{\varepsilon_k}^{\alpha}(|\nabla v_{\varepsilon_k}|_{1,\Omega}+|\nabla u_{\varepsilon_k}|_{q,\Omega}|\nabla v_{\varepsilon_k}|_{q',\Omega}).$$

Since $q' \leq r$,

$$I_k \leq \mathrm{const}\,{\varepsilon_k}^{\alpha}(1+\sum_{i=1}^N |\partial_i u_{\varepsilon_k}|_{q,\Omega})\|v_{\varepsilon_k}\|_{W^{1,r}(\Omega)},$$

applying Theorem 2.3 and Lemma 3.3 to (2.4) with ε_k instead of ε,

$$I_k \leq \mathrm{const}({\varepsilon_k}^{\frac{\alpha}{2}} + {\varepsilon_k}^{p+2})|g_{\varepsilon_k}|_{r,\Omega} \leq \mathrm{const}({\varepsilon_k}^{\frac{\alpha}{2}} + {\varepsilon_k}^{p+2})\Big(\frac{|\phi_0|_{r,\Omega}}{{\varepsilon_k}^p} + |g^*(\cdot,1)|_{r,\Omega}\Big).$$

Hence

$$\lim_k I_k = 0. \tag{4.12}$$

Observe that $(\theta'(u_{\varepsilon_k})g_{\varepsilon_k})_{k\in\mathbb{N}}$ is bounded in $L^r(\Omega)$. Indeed from (G1) and the properties of θ',

$$\begin{aligned} |\theta'(u_{\varepsilon_k})g_{\varepsilon_k}|_{r,\Omega} &\leq |\theta'(u_{\varepsilon_k})g_{\varepsilon_k}\chi_{(\varepsilon_k+u_{\varepsilon_k}\leq 1)}|_{r,\Omega} + |\theta'(u_{\varepsilon_k})g_{\varepsilon_k}\chi_{(\varepsilon_k+u_{\varepsilon_k}\geq 1)}|_{r,\Omega} \\ &\leq \Big|\frac{([p]+2)u_{\varepsilon_k}^{[p]+1}\phi_0\chi_{(\varepsilon_k+u_{\varepsilon_k}\leq 1)}}{(\varepsilon_k+u_{\varepsilon_k})^p}\Big|_{r,\Omega} + |\theta'(2)g^*(\cdot,1)|_{r,\Omega} \\ &\leq \mathrm{const}(|\phi_0|_{r,\Omega} + |g^*(\cdot,1)|_{r,\Omega}). \end{aligned}$$

Fatou's Lemma and Vitali's Theorem give

$$\theta'(u_0)g(\cdot,u_0)\in L^r(\Omega);\quad \lim_k G(\theta'(u_{\varepsilon_k})g_{\varepsilon_k})=G(\theta'(u_0)g(\cdot,u_0)).\tag{4.13}$$

Hence, from (4.11) and (4.12) we get

$$0\le\theta(u_0)\le G(\theta'(u_0)g(\cdot,u_0)).$$

Since $r>N$, by (4.13) $G(\theta'(u_0)g(\cdot,u_0))\in C^1(\bar\Omega)$ and $G(\theta'(u_0)g(\cdot,u_0))=0$ on $\partial\Omega$, it follows that

$$\lim_{x\to x_0}\theta(u_0(x))=0,\quad x_0\in\partial\Omega.$$

Then the monotonicity of θ implies (4.9). □

Proof of Theorem 2.4. In light of previous lemmas we have to prove that $u_{\varepsilon_k}\to u_0$ in $W^{2,r}_{\rm loc}(\Omega)\cap L^q(\Omega)$. We begin by proving

$$u_{\varepsilon_k}\to u_0 \text{ in } L^q(\Omega).\tag{4.14}$$

Observing that

$$-\varepsilon_k{}^\alpha\Delta u_{\varepsilon_k}+(u_{\varepsilon_k}-u_0)=G(g_{\varepsilon_k}-g(\cdot,u_0)),$$

and using the same argument of the proof of (4.8), from Lemma 3.6,

$$|u_{\varepsilon_k}-u_0|_{q,\Omega}\le|G(g_{\varepsilon_k}-g(\cdot,u_0))|_{q,\Omega}+\varepsilon_k{}^\alpha|\Delta u_{\varepsilon_k}|_{q,\Omega}\le c_1^2|\delta(g_{\varepsilon_k}-g(\cdot,u_0))|_{1,\Omega}+\bar c\varepsilon_k{}^{p+2}.$$

Since the integrals that define δg_{ε_k} are equiabsolutely continuous (see Lemma 3.7) and $g_{\varepsilon_k}\to g(\cdot,u_0)$ a.e. in Ω, Vitali's Theorem gives (4.14).

We continue by proving that

$$u_{\varepsilon_k}\to u_0 \text{ in } W^{2,r}_{\rm loc}(\Omega).\tag{4.15}$$

Let Ω',Ω'' be two open subsets of Ω such that $\Omega'\subset\subset\Omega''\subset\subset\Omega$. Denoting $v_{\varepsilon_k}=-\Delta u_{\varepsilon_k}$ from (2.3), (2.4), we get

$$-\Delta(u_{\varepsilon_k}-u_0)=g_{\varepsilon_k}-g(\cdot,u_0)+\varepsilon_k{}^\alpha\Delta v_{\varepsilon_k}.\tag{4.16}$$

Introducing the notation

$$J_k(\Omega^*)=|g_{\varepsilon_k}-g(\cdot,u_0)|_{r,\Omega^*}+\varepsilon_k{}^\alpha|\Delta v_{\varepsilon_k}|_{r,\Omega^*};\quad \delta_k=u_{\varepsilon_k}-u_0;\quad \Omega^*\subset\subset\Omega,$$

applying Lemma 4.3 with $\alpha=\beta=q$,

$$\begin{aligned}\|u_{\varepsilon_k}-u_0\|_{W^{2,q}(\Omega')}&\le\text{const}\left(|g_{\varepsilon_k}-g(\cdot,u_0)|_{q,\Omega''}+\varepsilon_k{}^\alpha|\Delta v_{\varepsilon_k}|_{q,\Omega''}+|u_{\varepsilon_k}-u_0|_{q,\Omega''}\right)\\&\le\text{const}(J_k(\Omega'')+|\delta_k|_{q,\Omega}).\end{aligned}\tag{4.17}$$

If $N=2$ we have $\frac{N}{2}=1<q$, hence $W^{2,q}_{\rm loc}(\Omega)\hookrightarrow C(\Omega)$. Therefore, (4.17) gives

$$|\delta_k|_{\infty,\Omega'}\le\text{const}(J_k(\Omega'')+|\delta_k|_{q,\Omega}).\tag{4.18}$$

If $N\ge3$ there results $q<\frac{N}{N-1}\le\frac{N}{2}$. Let $l\in\mathbb{N}^*$ be such that

$$l\le\frac{N}{2q}<l+1.$$

Denoting

$$q_i=\frac{Nq}{N-2iq},\quad i\in\mathbb{N},\quad 2iq<N,$$

we have

$$q_{i+1}=\frac{Nq_i}{N-2q_i},\quad 2(i+1)q<N;\quad q<q_i<\dots<q_{l-1}\le\frac{N}{2}.$$

Since $W^{2,q}_{\rm loc}(\Omega) \hookrightarrow L^{q_1}_{\rm loc}(\Omega)$, by (4.17),

$$|\delta_k|_{q_1,\Omega'} \le \mathrm{const}(J_k(\Omega'') + |\delta_k|_{q,\Omega}).$$

Applying Lemma 4.3 to (4.16) with $\alpha = q$ and $\beta = q_1$,

$$\|\delta_k\|_{W^{2,q_1}(\Omega')} \le \mathrm{const}(J_k(\Omega'') + |\delta_k|_{q,\Omega}).$$

Iterating this argument we get

$$\|\delta_k\|_{W^{2,q_{l-1}}(\Omega')} \le \mathrm{const}(J_k(\Omega'') + |\delta_k|_{q,\Omega}). \tag{4.19}$$

If $l = \frac{N}{2q}$ we have $q_{l-1} = \frac{N}{2}$, hence $W^{2,q_{l-1}}_{\rm loc}(\Omega) \hookrightarrow L^{\sigma}_{\rm loc}(\Omega)$, $1 \le \sigma < \infty$. From (4.19),

$$|\delta_k|_{\sigma,\Omega'} \le \mathrm{const}(J_k(\Omega'') + |\delta_k|_{q,\Omega}). \tag{4.20}$$

In the case $l < \frac{N}{2q} < l+1$, we have $q_{l-1} < \frac{N}{2} < q_l$. Therefore, $W^{2,q_{l-1}}_{\rm loc}(\Omega) \hookrightarrow L^{q_l}_{\rm loc}(\Omega)$ and $W^{2,q_l}_{\rm loc}(\Omega) \hookrightarrow C(\Omega)$. From (4.19),

$$|\delta_k|_{q_l,\Omega'} \le \mathrm{const}(J_k(\Omega'') + |\delta_k|_{q,\Omega}).$$

Hence if $r \le q_l$,

$$|\delta_k|_{r,\Omega'} \le \mathrm{const}(J_k(\Omega'') + |\delta_k|_{q,\Omega}) \tag{4.21}$$

and if $r > q_l$,

$$\|\delta_k\|_{W^{2,q_l}(\Omega')} \le \mathrm{const}(J_k(\Omega'') + |\delta_k|_{q,\Omega}),$$

which gives (4.18). In conclusion both (4.18) and (4.20) imply (4.21). Applying Lemma 4.3 to (4.16) with $\alpha = q$ and $\beta = r$,

$$\|\delta_k\|_{W^{2,r}(\Omega')} \le \mathrm{const}(J_k(\Omega'') + |\delta_k|_{q,\Omega}). \tag{4.22}$$

Since $g_{\varepsilon_k} \to g(\cdot, u_0)$ a.e. in Ω, from Lemma 4.5 and (G1),

$$\lim_k |g_{\varepsilon_k} - g(\cdot, u_0)|_{r,\Omega''} = 0.$$

Thanks to (4.4), applying Lemma 3.3 to (4.3) for each $\phi \in C^\infty_0(\mathbb{R}^N)$, with $\phi(x) = 1$ for $x \in \Omega''$, we have

$$\lim_k {\varepsilon_k}^\alpha |\Delta u_{\varepsilon_k}|_{r,\Omega''} \le \lim_k {\varepsilon_k}^\alpha |\Delta(\phi u_{\varepsilon_k})|_{r,\Omega} = 0.$$

Hence $\lim_k J_k(\Omega'') = 0$. Finally, in light of (4.14), (4.15) follows from (4.22). □

5. Appendix

In this appendix for the sake of completeness, we prove the following result from which Lemma 3.10 follows.

Lemma 5.1. *Let $f_k \in L^p(\Omega)$, $\phi_k \in L^q(\Omega)$, $k \in \mathbb{N}$, $1 \le p < \infty$, $1 < q \le \infty$, and $|\Omega| < \infty$. If $(f_k)_{k\in\mathbb{N}}$ is bounded in $L^p(\Omega)$, $(\phi_k)_{k\in\mathbb{N}}$ is bounded in $L^q(\Omega)$, is converging in measure to $\phi \in L^q(\Omega)$ and*

$$q' < p; \quad q' = \frac{q}{q-1}, \tag{5.1}$$

or

$$q' = p \text{ and } (|f_k|^p)_{k\in\mathbb{N}} \text{is equiabsolutely continuous }, \tag{5.2}$$

then (3.10)*(i) and* (3.10)*(ii) hold.*

Proof. Since $\phi_k \to \phi$ in measure, by fixing $\sigma > 0$, it follows that

$$\lim_k |\Omega_{\sigma,k}| = 0 \tag{5.3}$$

and

$$|f_k(\phi_k - \phi)|_{1,\Omega} = |f_k(\phi_k - \phi)|_{1,\Omega_{\sigma,k}} + |f_k(\phi_k - \phi)|_{1,\Omega\setminus\Omega_{\sigma,k}}, \tag{5.4}$$

where $\Omega_{\sigma,k} := \big\{x \in \Omega : |\phi_k(x) - \phi(x)| > \sigma\big\}$. We begin by considering the case in which (5.1) holds. Since $1 < p < \infty$, $q' < \infty$, and $(f_k)_k, (\phi_k)_k$ are bounded in $L^p(\Omega), L^q(\Omega)$, respectively, using (5.1) and the Hölder Inequality

$$\begin{aligned}
|f_k(\phi_k - \phi)|_{1,\Omega_{\sigma,k}} &\le |\phi_k - \phi|_{q,\Omega} |f_k|_{q',\Omega_{\sigma,k}} \\
&= |\phi_k - \phi|_{q,\Omega} \Big(\int_{\Omega_{\sigma,k}} |f_k|^{q'} dx\Big)^{\frac{1}{q'}} \\
&\le |\phi_k - \phi|_{q,\Omega} |\Omega_{\sigma,k}|^{\frac{1}{q'} - \frac{1}{p}} |f_k|_{p,\Omega} \\
&\le |\Omega_{\sigma,k}|^{\frac{1}{q'} - \frac{1}{p}} \sup_k \big(|\phi_k - \phi|_{q,\Omega} |f_k|_{p,\Omega}\big).
\end{aligned}$$

Hence, due to (5.3), there exists $k_0 \in \mathbb{N}$ such that

$$k \ge k_0 \Rightarrow |f_k(\phi_k - \phi)|_{1,\Omega_{\sigma,k}} \le \sigma. \tag{5.5}$$

Using again the boundedness of (f_k) in $L^p(\Omega)$, the definition of $\Omega_{\sigma,k}$ and the Hölder Inequality,

$$|f_k(\phi_k - \phi)|_{1,\Omega\setminus\Omega_{\sigma,k}} \le \sigma |f_k|_{1,\Omega} \le \sigma |\Omega|^{\frac{p-1}{p}} \sup_k |f_k|_{p,\Omega}. \tag{5.6}$$

Therefore, (5.4), (5.5), (5.6) imply

$$k \ge k_0 \Rightarrow |f_k(\phi_k - \phi)|_{1,\Omega} \le \sigma\big(1 + |\Omega|^{\frac{p-1}{p}} \sup_k |f_k|_{p,\Omega}\big),$$

from which (3.10)(i) and (3.10)(ii) follow. When (5.2) holds, observe that

$$\begin{aligned}
|f_k(\phi_k - \phi)|_{1,\Omega} &= \Big(\int_{\Omega_{\sigma,k}} + \int_{\Omega\setminus\Omega_{\sigma,k}}\Big) |f_k(\phi_k - \phi)| dx \\
&\le \sup_k |\phi_k - \phi|_{q,\Omega} \Big(\int_{\Omega_{\sigma,k}} |f_k|^p dx\Big)^{1/p} + \sigma \sup_k |f_k|_{1,\Omega}.
\end{aligned}$$

Thanks to the equiabsolute continuity of $(|f_k|^p)_k$ there exists $\delta > 0$ such that

$$|E| < \delta \;\Rightarrow\; \int_E |f_k|^p dx < \sigma^p, \quad k \in \mathbb{N}.$$

Moreover, due to (5.3), there is $k_0 \in \mathbb{N}$ such that $|\Omega_{\sigma,k}| < \delta$, $k > k_0$. Therefore,

$$k_0 < k \;\Rightarrow\; \Big(\int_{\Omega_{\sigma,k}} |f_k|^p dx\Big)^{1/p} < \sigma,$$

and then

$$k_0 < k \;\Rightarrow\; |f_k(\phi_k - \phi)|_{1,\Omega} \le \sigma\big(\sup_k |\phi_k - \phi|_{q,\Omega} + \sup_k |f_k|_{1,\Omega}\big),$$

that give (3.10)(i) and (3.10)(ii). $\square$

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