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# STABILITY ESTIMATE FOR STRONG SOLUTIONS OF THE NAVIER-STOKES SYSTEM AND ITS APPLICATIONS

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Abstract. We obtain a 'stability estimate' for strong solutions of the Navier–Stokes system, which is an $L^\alpha$-version, $1 < \alpha < \infty$, of the estimate that Serrin [Se] used in obtaining uniqueness of weak solutions to the Navier-Stokes system. By applying this estimate, we obtain new results in stability and uniqueness of solutions, and non-blowup conditions for strong solutions.

## 1. Introduction

We consider the Navier-Stokes system in $\mathbb{R}^N$ $(N \geq 2)$,

$$\begin{aligned} u_t - \Delta u + (u \cdot \nabla)u + \nabla \pi = 0 \quad \text{in } \mathbb{R}^N \times \mathbb{R}^+, \\ \nabla \cdot u = 0 \quad \text{in } \mathbb{R}^N \times \mathbb{R}^+, \\ u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^N, \end{aligned} \tag{NS}$$

where $u(x,t) = (u_1, \cdots, u_N)$ is the velocity field and $\pi(x,t)$ is the scalar pressure. Let $P$ be the Helmholtz projection, and let $\|\cdot\|_p$ denote the $L^p(\mathbb{R}^N)$ norm. Kato [K] showed that for any $u_0 \in PL^N$ the problem (NS) has a unique local mild solution

$$u(t\,;\,u_0) \in C([0,T)\,;\,PL^N) \cap L^r((0,T)\,;\,PL^q)\,,$$

where $q, r > N$ and $N/q + 2/r = 1$. He also proved in [K, the end note] that if $\|u_0\|_N$ is sufficiently small then

$$u(t;u_0) \in C_0([0,\infty)\,;\,PL^N) := \{u \in C([0,\infty)\,;\,PL^N)\ ;\ \lim_{t\to\infty} \|u(t)\|_N = 0\}\,,$$

(see Theorem 2.1 below). See Section 2 for the definition of mild solution. This unique (local) mild solution also has the smoothing effect and is regular (see Remark 2.3). Therefore, we call it *strong solution* of (NS) for the remaining of this paper.

Our main result is Theorem 3.1 which establishes the 'stability estimate' (which we call) for strong solutions just mentioned. This estimate leads us to corollaries on uniqueness and stability, and to a non-blowup condition for strong solutions. First, we state a new uniqueness result.

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**Corollary 1.1. (Uniqueness).** *Mild solutions of (NS) are unique in the space* $C([0,T)\,;\,PL^N)\cap L^r_{\rm loc}((0,T)\,;\,PL^q)$ *with a pair of numbers* $(q,r)$ *that satisfies*

$$N<q<\infty \quad \textit{and} \quad \frac{N}{q}+\frac{2}{r}=1. \tag{1.1}$$

See Theorem 2.1 below for previous uniqueness results. Our improvement consists of imposing fewer restriction on the behavior of solution near $t=0$. This uniqueness result and its proof were suggested in [B] and [K2, Introduction]. Our proof in Section 3 is, however, different from the one suggested in [B] and [K2]. Next, we give a stability result.

**Corollary 1.2. (Global Stability).** *Let* $u(t;u_0)\in C_0([0,\infty)\,;\,PL^N)$ *be a global strong solution of (NS). We have the following.*

*(i) There exist constants* $\delta_0=\delta_0(N,u_0)\in\mathbb{R}^+$ *and* $C_0=C_0(N)\in\mathbb{R}^+$ *such that if*

$$v_0\in PL^N \quad \textit{and} \quad \|v_0-u_0\|_N\le\delta_0\,,$$

*then we have* $u(t\,;\,v_0)\in C_0([0,\infty)\,;\,PL^N)$ *and*

$$\|u(t\,;v_0)-u(t;u_0)\|_N\le\|v_0-u_0\|_N\exp\big(C_0\int_0^t\|u(s\,;u_0)\|_{N+2}^{N+2}\,ds\big) \tag{1.2}$$

*for all* $t\in\mathbb{R}^+$.

*(ii) In addition, we assume that* $u_0\in PL^N\cap PL^\alpha$ *for a constant* $\alpha\in(1,\infty)-\{N\}$. *Then there exist constants* $\delta_1=\delta_1(N,\alpha,u_0)\in(0,\delta_0]$, $q=q(N,\alpha)\in(N,\infty)$ *and* $C_1=C_1(N,\alpha)\in\mathbb{R}^+$ *such that if*

$$v_0\in PL^N\cap PL^\alpha \quad \textit{and} \quad \|v_0-u_0\|_N\le\delta_1$$

*then for* $\alpha\in[2,\infty)$ *we have*

$$\|u(t\,;v_0)-u(t;u_0)\|_\alpha\le\|v_0-u_0\|_\alpha\exp\big(C_1\int_0^t\|u(s\,;u_0)\|_q^r\,ds\big) \tag{1.3}$$

*for all* $t\in\mathbb{R}^+$ *and for* $\alpha\in(1,2)$ *we have (1.3) with the norm*$\|\cdot\|_\alpha$ *replaced by* $|\cdot|_\alpha$ *for all* $t\in\mathbb{R}^+$. *Here,* $r$ *is the constant that satisfies (1.1).*

See Notation (just after this section) for the difference between $\|\cdot\|_\alpha$ and $|\cdot|_\alpha$. Note that $u(t;u_0)\in C_0([0,\infty)\,;\,PL^N)$ implies $u(t;u_0)\in L^r(\mathbb{R}^+\,;\,PL^q)$ for any $(q,r)$ satisfying (1.1) (see Proposition 2.2). Global strong solutions in the class $C_0([0,\infty)\,;\,PL^N)$ are important since all global strong solutions belong to this class provided that $2\le N\le 4$ and $u_0\in PL^2\cap PL^N$ (see Propositions 4.1 and 4.2). Corollary 1.2 gives the first global $L^N$-stability result. Related global $L^2\cap L^q$-stability results with $q>N$ were given in [VS, Theorem A], [Wi, Theorem 2]. Global $H^1$-stability results for $N=3$ can be found in [PRST]. We remark that our estimates (1.2) and (1.3) are simpler than those in the previous works, and we clarify that the $L^\alpha$-estimate holds if the $L^N$-norm of $v_0-u_0$ is small. In Section 4 we give another version of Corollary 1.2 for non-global solutions (see Corollary 4.1).

We also have the following result. Let $[0,t_{\max}(u_0))$ be the maximal interval in which the strong solution $u(t;u_0)$ exists.

**Corollary 1.3 (Non-blowup condition).** *Let the strong solution $u(t;u_0)$ exist on $[0,T)$ with $T<\infty$. Then, we have $t_{\max}(u_0)>T$ if and only if $u(t;u_0)\in L^r((0,T)\,;\,PL^q)$. Here, $(q,r)$ is a pair of numbers that satisfies (1.1).*

This result has a 'global'-version (see Proposition 2.2). For example, we can apply Corollary 1.3 to obtain that $u(t;u_0)\in C_0([0,\infty)\,;\,PL^2)$ for any $u_0\in PL^2$ when $N=2$ (see Proposition 4.1).

The contents of this paper is presented as follows. In Section 2, preliminary results; In Section 3, statement and proof of main result; In Section 4, applications of our stability estimate and proofs of Corollaries 1.1-1.3. Notice that part of the contents of this paper was announced in [Ka3] and [Ka4].

**Notation.**

1. $\mathbb{R}^+ := (0,\infty)$.
2. $L^q := L^q(\mathbb{R}^N\,;\,\mathbb{R})$ or $L^q(\mathbb{R}^N\,;\,\mathbb{R}^N)$.
3. $f\in L^r_{\mathrm{loc}}((0,T)\,;\,L^q)$ means $f\in L^r((\varepsilon,T)\,;\,L^q)$ for any $\varepsilon\in(0,T)$.
4. We often write $C=C(\alpha,\beta,\gamma,\cdots)$ to indicate that $C$ depends only on $\alpha,\beta,\gamma,\cdots$.
5. For a Banach space $V$ with the norm $\|\cdot\|$ we set
$$C_0([0,\infty)\,;\,V) := \{u\in C([0,\infty)\,;\,V)\,;\,\lim_{t\to\infty}\|u(t)\|=0\}.$$
6. $P$ is the Helmholtz projection, i.e. the continuous projection from $L^p$ onto $\{u=(u_1,\cdots,u_N)\in L^p\,;\,\nabla\cdot u=0\}$.
7. We denote by $u(t;u_0)$ the strong solution, i.e. the unique mild solution of (NS) whose existence is ensured by Theorem 2.1. See Definitions 2.2.
8. For $u=(u_1,\cdots,u_N)\in L^q(\mathbb{R}^N\,;\,\mathbb{R}^N)$ we write
$$|u| := \sqrt{|u_1|^2+\cdots+|u_N|^2},\quad |\nabla u| := (\sum_{i,j=1}^N|\partial u_i/\partial x_j|^2)^{1/2}$$
$$|u|_q := (\int|u|^q dx)^{1/q}\quad\text{and}\quad \|u(t)\|_q := (\sum_{j=1}^N\int|u_j|^q dx)^{1/q}.$$
Note that $|\cdot|_q$ and $\|\cdot\|_q$ are equivalent $L^q$-norms.
9. We often write $|u|^{q-1}u=u^q$ $(0<q<\infty)$ for vector (or scalar) $u$.
10. $\partial_j := \partial/\partial x_j$.

## 2. Preliminaries

**Definition 2.1.** A mild solution $u$ of (NS) on $[0,T)$ is a function $u\in C([0,T)\,;\,PL^N)$ satisfying the integral equation
$$\begin{aligned}u(t) &= e^{t\Delta}u_0-\int_0^t e^{(t-s)\Delta}P(u\cdot\nabla)u(s)ds \qquad (2.1)\\ &= e^{t\Delta}u_0-\int_0^t P\nabla\cdot e^{(t-s)\Delta}(u\cdot u(s))ds\end{aligned}$$
for $t\in(0,T)$. Here, we assume that there exists a constant $\alpha\in(1,\infty)$ such that
$$P\nabla\cdot e^{(t-\cdot)\Delta}(u\cdot u(\cdot))\in L^1((0,t)\,;\,PL^\alpha)\quad\text{for}\quad t\in(0,T). \tag{2.2}$$

**Remark 2.1.**

(i) The second equality in (2.1) holds since $(u\cdot\nabla)u_j = \nabla\cdot(u_j u)$.

(ii) The value of integral in (2.1) is independent of the choice of $\alpha$, and is unique. We understand that for each $t \in [0,T)$ the equalities (2.1) hold for a.e. $x \in \mathbb{R}^N$. The assumption $u \in C([0,T)\,;\,PL^N)$ guarantees (2.2) for e.g. $\alpha = 2N/3$ in view of the next $L^p$-$L^q$ estimate

$$\begin{aligned}\|P\nabla\cdot e^{(t-s)\Delta}(u\cdot u(s))\|_\alpha \le& C(t-s)^{-3/2+N/2\alpha}\|u\cdot u(s)\|_{N/2}\\ \le& C(t-s)^{-3/2+N/2\alpha}\|u(s)\|_N^2.\end{aligned} \tag{2.3}$$

(iii) The next Theorem 2.1 ensures that for any $u_0 \in PL^N$ Problem (NS) has a local mild solution.

The following result is a small extension of the results in [K] and [G].

**Theorem 2.1.**

*(i) (Existence) Let $1<\alpha<\infty$. For any $u_0 \in PL^N \cap PL^\alpha$ there exists $T \in \mathbb{R}^+$ such that (NS) has a unique local mild solution*

$$u(t) \in C([0,T]\,;\,PL^N \cap PL^\alpha) \cap L^r((0,T)\,;\,PL^q) \tag{2.4}$$

*for any pair of numbers $(q,r)$ satisfying (1.1), and*

$$t^{(1-N/q)/2}u \in C([0,T]\,;\,PL^q) \quad \textit{with the value zero at} \quad t=0 \tag{2.5}$$

*for any $q \in (N,\infty]$.*

*(ii) (Uniqueness)*
*(ii-a) Mild solutions of (NS) are unique in $C([0,T]\,;\,PL^N) \cap L^r((0,T)\,;\,PL^q)$ with a pair of numbers $(q,r)$ satisfying (1.1).*
*(ii-b) Mild solutions $u(t) \in C([0,T]\,;\,PL^N)$ of (NS) satisfying (2.5) for a number $q \in (N,\infty)$ are unique.*

*(iii) (Existence of global solutions) There exists a constant $\varepsilon_* = \varepsilon_*(N) \in \mathbb{R}^+$ such that if $\|u_0\|_N \le \varepsilon_*$ then (NS) has a unique global mild solution*

$$u(t) \in C_0([0,\infty)\,;\,PL^N) \cap L^r(\mathbb{R}^+\,;\,PL^q).$$

*Here, $(q,r)$ is any pair of numbers which satisfies (1.1).*

**Remark 2.2.**

(i) More precisely, Giga [G] obtained the uniqueness of solutions of (NS) in the class $L^r((0,T)\,;\,PL^q)$ which satisfy the integral equation (2.1).

(ii) Kato [K] and Giga [G] obtained Theorem 2.1 with (1.1) replaced by

$$N < q < N^2/(N-2) \quad \textit{and} \quad N/q + 2/r = 1\,, \tag{1.1'}$$

which is more restrictive than (1.1). We obtain Theorem 2.1 from these previous results and Lemma 2.1 below.

**Definition 2.2.** For $u_0 \in PL^N$ we denote by $u(t;u_0)$ the unique mild solution of (NS) satisfying (2.4). We call $u(t;u_0)$ the *strong solution* of (NS). We set

$$t_{\max}(u_0) := \sup\{T \in \mathbb{R}^+\,;\,u(t;u_0) \text{ exists on the time interval } [0,T]\},$$

i.e., $[0,t_{\max}(u_0))$ be the maximal time-interval where the strong solution $u(t;u_0)$ exists.

**Remark 2.3.**
(i) The strong solution $u(t;u_0)$ is regular for $t \in (0, t_{\max}(u_0))$ (see e.g. [GM]).
(ii) The strong solution has the semigroup property, i.e. $u(t\,;\, u(s\,;\, u_0)\,) = u(s+t\,;\, u_0)$ for $s, t > 0$ and $s + t < t_{\max}(u_0)$.

**Lemma 2.1.** *Let $u(t)$ be a mild solution on $[0,T)$. Assume that $u \in L^r((0,T)\,;\, PL^q)$ with a pair of numbers $(q,r)$satisfying (1.1). Then we have $u \in L^r((0,T)\,;\, PL^q)$ for any pair of numbers $(q,r)$ satisfying (1.1).*

**Proof.** By (2.1), $u(t) = e^{t\Delta}u_0 - I(t)$, where

$$I(t) := \int_0^t P\nabla \cdot e^{(t-s)\Delta}(u \cdot u(s))\,ds.$$

In [G, Lemma (p.196)] we find that

$$e^{t\Delta}u_0 \in L^r((0,T)\,;\, PL^q)$$

for any pair of numbers $(q,r)$ which satisfy (1.1). We will estimate $I(t)$. Let $q' \geq q/2$. We apply the $L^p$-$L^q$ estimate and have

$$\|I(t)\|_{q'} \leq C\int_0^t (t-s)^{-1/2-(2/q-1/q')N/2}\|u(s)\|_q^2\,ds\,.$$

Here, $C = C(N,q,q') \in \mathbb{R}^+$ is a constant. By the generalized Young inequality (see [RS, p.31]) we have $(P) \Rightarrow (Q)$ and $(R) \Rightarrow (S)$, where $(P), (Q), (R), (S)$ are defined as follows:
(P) $u \in L^r((0,T)\,;\, PL^q)$ with a $(q,r)$ satisfying (1.1) and $2N \leq q$
(Q) $u \in L^{r'}((0,T)\,;\, PL^{q'})$ for any $(q',r')$ satisfying $q/2 \leq q'$ and

$$N < q' < \infty \quad \text{and} \quad \frac{N}{q'} + \frac{2}{r'} = 1 \tag{2.6}$$

(R) $u \in L^r((0,T)\,;\, PL^q)$ with a $(q,r)$ satisfying (1.1) and $N < q \leq 2N$
(S) $u \in L^{r'}((0,T)\,;\, PL^{q'})$ for any $(q',r')$ satisfying (2.6) and $q' < Nq/(2N-q)$.

It suffices to show that $u \in L^4((0,T)\,;\, PL^{2N})$ is necessary and sufficient condition for $u \in L^r((0,T)\,;\, PL^q)$ with a pair of numbers $(q,r)$ satisfying (1.1). The sufficiency is obvious by $(P) \Rightarrow (Q)$ with setting $q = 2N$. We obtain the necessity in the case $q > 2N$ (resp. $N < q < 2N$) by applying $(P) \Rightarrow (Q)$ (resp. $(R) \Rightarrow (S)$) finitely many times. $\square$

The next result shows how $t_{\max}(u_0)$ depend on $u_0$.

**Proposition 2.1.** *Let $u_0 \in PL^N$ and $q \in (N,\infty)$. Then there exists a number $S_q = S_q(N,q) \in \mathbb{R}^+$ such that for any $\tau \in \mathbb{R}^+$ if*

$$t^{(1-N/q)/2}\|e^{t\Delta}u_0\|_q \leq S_q \quad \textit{for} \quad t \in [0,\tau] \tag{2.7}$$

*then the strong solution $u(t;u_0)$ exists on the time interval $[0,\tau]$.*

**Sketch of Proof.** We can verify this result by observing carefully the method for the construction of strong solutions in the proof of [K, Theorem 1] and [G, Section 2]. For the convenience of the reader, we will sketch the proof.

By induction we define the sequence of functions $\{U_n(t)\}_{n=0}^{\infty}$:

$$U_0(t) := e^{t\Delta}u_0, \quad U_{n+1}(t) := U_0(t) - \int_0^t P\nabla \cdot e^{(t-s)\Delta}(U_n \cdot U_n(s))ds.$$

Set $\nu := (1 - N/q)/2$ and $K_n := \sup_{t\in[0,\tau]} t^{\nu}\|U_n(t)\|_q$. We denote by $C_j \in \mathbb{R}^+$ $(j = 1, 2, \cdots)$ constants depending only on $N$ and $q$. It follows from the usual $L^p$-$L^q$ estimates (see e.g. [K, (2.3) and (2.3)']) that

$$\begin{aligned} \|U_{n+1}(t)\|_q &\le \|U_0(t)\|_q + C_1 \int_0^t (t-s)^{-(1-\nu)} \|U_n(s)\|_q^2 ds \\ &\le \|U_0(t)\|_q + C_1 K_n^2 \int_0^t (t-s)^{-(1-\nu)} s^{-2\nu} ds \end{aligned} \tag{2.8}$$

for $t \ge 0$. It follows that

$$K_{n+1} \le K_0 + C_2 K_n^2.$$

If the algebraic equation $x = K_0 + C_2 x^2$ has real solutions, i.e. $1 - 4C_2K_0 \ge 0$ then we see by induction that $\{K_n\}_{n=1}^{\infty}$ is bounded and

$$K_n \le K_* := \frac{1 - \sqrt{1 - 4C_2K_0}}{2C_2} \quad \text{for} \quad n \ge 0\,.$$

Here, $K_*$ is the smaller solution of the algebraic equation. By an estimation similar to (2.8),

$$\begin{aligned} &\|U_{n+1}(t) - U_n(t)\|_q \\ &\le C_1 \int_0^t (t-s)^{-(1-\nu)} (\|U_{n+1}(s)\|_q + \|U_n(s)\|_q) \|U_{n+1}(s) - U_n(s)\|_q \, ds \\ &\le 2C_1K_*(\int_0^t (t-s)^{-(1-\nu)} s^{-2\nu} ds) \sup_{s\in[0,\tau]} \|U_{n+1}(s) - U_n(s)\|_q \end{aligned}$$

for $t \in [0, \tau]$, which leads to

$$\sup_{t\in[0,\tau]} t^{\nu}\|U_{n+1}(t) - U_n(t)\|_q \le 2C_2K_* \sup_{t\in[0,\tau]} t^{\nu}\|U_{n+1}(t) - U_n(t)\|_q\,.$$

Now, set $S_q := K_0 < 1/4C_2$. Then $2C_2K_* < 1$. Therefore, $\{t^{\nu}U_n(t)\}_{n=1}^{\infty}$ is a Cauchy sequence in $C([0, \tau]\,; PL^q)$ and $u(t) = \lim_{n\to\infty} U_n(t)$ exists on $[0, \tau]$. We verify that $u(t)$ is a strong solution on $[0, \tau]$, i.e. $u(t) = u(t; u_0)$ (see [G]). $\square$

**Remark 2.4.** Under the same assumption of Proposition 2.1 we have

$$\lim_{t\to+0} t^{(1-N/q)/2}\|e^{t\Delta}u_0\|_q = 0. \tag{2.9}$$

This well-known result follows from the density of $L^q \cap L^N$ in $L^N$ and the estimate (2.11) below.

From Proposition 2.1, we can immediately obtain

**Corollary 2.1.** *Let $u(t;u_0)$ be a strong solution on $[0,T)$ with $T<\infty$. Then statements (a) - (f) are equivalent.*

*(a)* $\lim_{t\to+0}\sup_{s\in(T-\varepsilon,\,T)} t^{(1-N/q)/2}\|e^{t\Delta}u(s\,;u_0)\|_q = 0$ *for some* $q\in(N,\infty)$ *and* $\varepsilon\in(0,T)$.

*(b)* $\lim_{t\to+0}\sup_{s\in(T-\varepsilon,\,T)} t^{(1-N/q)/2}\|e^{t\Delta}u(s\,;u_0)\|_q < S_q$ *for some* $q\in(N,\infty)$ *and* $\varepsilon\in(0,T)$*, where* $S_q$ *is the same constant in the statement of Proposition 2.1.*

*(c)* $t_{\max}(u_0) > T$*, i.e. the strong solution* $u(t;u_0)$ *exists on* $[0,\,T+\delta)$ *with a small constant* $\delta\in\mathbb{R}^+$.

*(d)* $\liminf_{t\to T-0}\|u(t;u_0)\|_q<\infty$ *for some* $q\in(N,\infty)$.

*(e)* $\limsup_{t\to T-0}\|u(t;u_0)\|_\infty<\infty$.

*(f)* $\lim_{t\to T-0}u(t;u_0)$ *exists in* $L^N$.

**Proof**. First we will prove the equivalence of (a), (b) and (c).

$(a)\Rightarrow(b)$: This is obvious.

$(b)\Rightarrow(c)$: We can choose $\delta>0$ satisfying

$$\sup_{s\in(T-\varepsilon,\,T)} t^{(1-N/q)/2}\|e^{t\Delta}u(s\,;u_0)\|_r \le S_q \quad \text{for} \quad t\in[0,\delta).$$

Thus, it follows from Proposition 2.1 and the semigroup property of the mild solution (see Remark 2.3 (ii) ) that $u(t;u_0)$ exists on $[0,\,T+\delta)$.

$(c)\Rightarrow(a)$: By (c) and the definition of the strong solution (see Definition 2.2) we have

$$\sup_{s\in(T-\varepsilon,T)}\|u(s\,;u_0)\|_q<\infty. \tag{2.10}$$

We obtain (a) from (2.10) and the basic inequality:

$$\|e^{t\Delta}f\|_q\le\|f\|_q. \tag{2.11}$$

Thus, (a), (b) and (c) are equivalent.

$(c)\Rightarrow(d)$ and $(e)$ and $(f)$: This is obvious from Theorem 2.1.

$(d)\Rightarrow(c)$: We fix $\tau\in\mathbb{R}^+$ so small that

$$\liminf_{t\to T-0}\|u(t;u_0)\|_q<\tau^{-(1-N/q)/2}S_q.$$

Then, we choose a constant $s$ such that

$$T-\frac{\tau}{2}<s<T \quad \text{and} \quad \|u(s\,;u_0)\|_q<\tau^{-(1-N/q)/2}S_q.$$

It follows that

$$t^{(1-N/q)/2}\|e^{t\Delta}u(s\,;u_0)\|_q\le t^{(1-N/q)/2}\|u(s\,;u_0)\|_q<S_q \quad \text{for} \quad t\in[0,\tau].$$

Thus, by Proposition 2.1 $u(t;u_0)$ exists on $[\,0,\,T+\tau/2\,]$.

$(e)\Rightarrow(d)$: Let $\varepsilon>0$ be a small number which will be determined later. We set $u(t):=u(t;u_0)$. We have

$$u(t+T-\varepsilon)=e^{t\Delta}u(T-\varepsilon)-\int_0^t P\nabla\cdot e^{(t-s)\Delta}(u\cdot u(s+T-\varepsilon))ds \tag{2.12}$$

for $t \in (0, \varepsilon)$. We fix a number $q \in (N, \infty)$. We have

$$\|u(t+T-\varepsilon)\|_q \le \|u(T-\varepsilon)\|_q + C_0 \int_0^t (t-s)^{-1/2} \|u(s+T-\varepsilon)\|_\infty \|u(s+T-\varepsilon)\|_q ds, \tag{2.13}$$

where $C_0 = C_0(N, q) \in \mathbb{R}^+$. We set

$$C_\varepsilon = 2C_0 \varepsilon^{1/2} \sup\{\|u(\tau)\|_\infty \,;\, \tau \in [T-\varepsilon, T)\},$$
$$M(t) = \sup\{\|u(\tau)\|_q \,;\, \tau \in [T-\varepsilon, T-\varepsilon+t)\}.$$

It follows from (2.13) that

$$M(t) \le \|u(T-\varepsilon)\|_q + C_\varepsilon M(t) \quad \text{for} \quad t \in [0, \varepsilon). \tag{2.14}$$

We fix $\varepsilon > 0$ so small that $C_\varepsilon < 1$. Then (2.14) implies $M(\varepsilon) < \infty$. Thus, we obtain (d).

$(f) \Rightarrow (b)$: We write $u(t) := u(t; u_0)$ and

$$u(T) := \lim_{t \to T-0} u(t) \in PL^N.$$

$$t^{(1-N/q)/2} \|e^{t\Delta} u(s\,; u_0)\|_q \le t^{(1-N/q)/2} \|e^{t\Delta}(u(s) - u(T))\|_q + t^{(1-N/q)/2} \|e^{t\Delta} u(T)\|_q.$$

It follows from the $L^p$-$L^q$ estimate and (2.9) that

$$t^{(1-N/q)/2} \|e^{t\Delta}(u(s) - u(T))\|_q \le C_1 \|u(s) - u(T)\|_N,$$
$$\lim_{t \to +0} t^{(1-N/q)/2} \|e^{t\Delta} u(T)\|_q = 0\,.$$

Thus, we have (b). The proof is complete. □

**Remark 2.5.** It seems to be an open problem whether (c) is equivalent to (d) with $q = n$.

We will characterize the strong solutions belonging to $C_0([0, \infty)\,; PL^N)$. The next result is a 'global'-version of Corollary 1.3.

**Proposition 2.2.** *Let $u(t) = u(t; u_0)$ be a global strong solution of (NS). Then, we have $u \in C_0([0, \infty)\,; PL^N)$ if and only if $u \in L^r(\mathbb{R}^+\,; PL^q)$ for a pair of (or equivalently for any pair of) numbers $(q, r)$ satisfying (1.1).*

**Remark 2.6.** When $N = 3$, Ponce et al [PRST] obtained a similar result under an assumption: $u_0 \in PL^2 \cap H^1$.

**Proof of Proposition 2.2.** Let $u \in C_0([0, \infty)\,; PL^N)$. Fix a constant $T \in \mathbb{R}^+$ such that $\|u(T)\|_N \le \varepsilon_*$, where $\varepsilon_*$ is the constant appeared in the statement of Theorem 2.1. Set $u(T)$ as the initial value and apply Theorem 2.1 (iii). Then we have $u \in L^r([T, \infty)\,; PL^q)$. Combining this with $u \in L^r([0, T)\,; PL^q)$, we conclude that $u \in L^r(\mathbb{R}^+\,; PL^q)$.

Next, we will prove the inverse. Let $u \in L^r(\mathbb{R}^+\,; PL^q)$. In view of Lemma 2.1, we can assume without loss of generality that $N < q \le 2N$. Although the essence of the proof below is given in the proof of [K, Theorem 2'], we will describe it for the convenience of the reader. Applying the $L^p$-$L^q$ estimate to (2.1), we have

$$\|u(t)\|_N \le \|e^{t\Delta} u_0\|_N + C_1 \int_0^t (t-s)^{-(2/q-1/N)N/2-1/2} \|u(s)^2\|_{q/2} ds := I_1(t) + I_2(t).$$

The function $I_1(t)$ is a decreasing function. Moreover, we can easily verify from the density of $L^1 \cap L^N$ in $L^N$ that

$$I_1(t) \to 0 \quad \text{as} \quad t \to \infty. \tag{2.15}$$

Now we estimate $I_2(t)$. By the Hölder's inequality, we obtain

$$\begin{aligned}\frac{1}{T}\int_0^T I_2(t)dt &= \frac{C_1}{(1-N/q)T}\int_0^T (T-s)^{1-N/q}\|u(s)\|_q^2 ds \\ &\le \frac{C_1}{1-N/q}T^{-N/q}\int_0^T \|u(s)\|_q^2 ds \\ &\le \frac{C_1}{1-N/q}\left(\int_0^T \|u(s)\|_q^r\, ds\right)^{2/r}.\end{aligned} \tag{2.16}$$

It follows from (2.15) and (2.16) that

$$\liminf_{t\to\infty} \|u(t)\|_N \le \frac{C_1}{1-N/r}\left(\int_0^\infty \|u(s)\|_q^r ds\right)^{2/r}. \tag{2.17}$$

If we choose $u(T)$ as the initial value then we obtain from the same argument as above that

$$\liminf_{t\to\infty} \|u(t)\|_N \le \frac{C_1}{1-N/q}\left(\int_T^\infty \|u(s)\|_q^r ds\right)^{2/r}. \tag{2.18}$$

Here, $T \in \mathbb{R}^+$ is any constant. Therefore, we have

$$\liminf_{t\to\infty} \|u(t)\|_N = 0.$$

It follows from Theorem 2.1 (iii) that $u(t;u_0) \in C_0([0,\infty)\,;\, PL^N)$. $\square$

## 3. Main result and its proof

Our main result is the following:

**Theorem 3.1. (Stability estimate).** *Let $\alpha \in (1,\infty)$ be a constant, and let $u_0$, $v_0$ be in $PL^N \cap PL^\alpha$. Let $q \in (N, q_*]$ be a constant, where $q_* = q_*(N,\alpha)$ is the number defined by*

$$q_* = \begin{cases} \dfrac{N\alpha}{(N-\alpha)(\alpha-1)} & \text{if } \ 1<\alpha<2 \text{ and } \alpha > N-2, \\ \dfrac{N\alpha}{2(\alpha-1)} & \text{if } \ 1<\alpha<2 \text{ and } \alpha \le N-2, \\ \infty & \text{if } \ \alpha = 2 \text{ or } [N \le 4 \text{ and } \alpha \ge N], \\ \dfrac{N\alpha}{\alpha-2} & \text{if } \ 2<\alpha<N-2, \\ \dfrac{N\alpha}{N-4} & \text{if } \ N-2 \le \alpha \text{ and } \alpha > 4 \text{ and } N \ge 5, \\ \dfrac{2N\alpha}{(\alpha-2)(N-\alpha)} & \text{if } \ N-2 \le \alpha < N \text{ and } 2 < \alpha \le 4. \end{cases} \tag{3.1}$$

*Then there exist constants $d_{\alpha q}$, $A_{\alpha q} \in \mathbb{R}^+$ depending only on $N$, $\alpha$ and $q$ such that if*

$$0 < T < \min(\, t_{\max}(u_0), t_{\max}(v_0)\,) \tag{3.2}$$

*and*

$$\|u(t\,; v_0) - u(t; u_0)\|_N \le d_{\alpha q} \quad \textit{for} \quad t \in [0, T] \tag{3.3}$$

*then we have the estimate*

$$\|u(t\,; v_0) - u(t; u_0)\|_\alpha \le \|v_0 - u_0\|_\alpha \exp\big(A_{\alpha q} \int_0^t \|u(s\,; u_0)\|_q^r\, ds\big) \quad \textit{for} \quad t \in [0, T] \tag{3.4}$$

*for the case $\alpha \in [2, \infty)$, and*

$$|u(t\,; v_0) - u(t; u_0)|_\alpha \le |v_0 - u_0|_\alpha \exp\big(A_{\alpha q} \int_0^t \|u(s\,; u_0)\|_q^r\, ds\big) \quad \textit{for} \quad t \in [0, T] \tag{3.5}$$

*for the case $\alpha \in (1, 2]$. Here, $r \in \mathbb{R}^+$ is the constant satisfying (1.1).*

*Moreover, for the special case $\alpha = 2$ we can take $d_{\alpha q} = \infty$.*

**Remark 3.1.**

(i) Theorem 3.1 applies to the $N$-$\alpha$ region:

$$\{\, (N, \alpha)\,;\; N \in \mathbf{N},\, N \ge 2 \text{ and } 1 < \alpha < \infty\}.$$

(ii) For the special case $\alpha = 2$, the estimate (3.4) was obtained by [Se, Theorem 6] for more general (what we call) weak solutions.

(iii) It seems difficult to state the contents of Theorem 3.1 by using only one norm $\|\cdot\|_q$ or $|\cdot|_q$. The main reason is that the estimate (3.9) [resp. (3.31)] below does *not* hold in the case $1 < \alpha < 2$ [ resp. in the case $\alpha > 2$].

(iv) It seems that $\alpha = 2$ is the exceptional case in which we can take $d_{\alpha q} = \infty$. Indeed, if $d_{\alpha q} = \infty$ then by setting $u(t; u_0) \equiv 0$ in (3.4) we have

$$\|u(t; v_0)\|_\alpha \le \|v_0\|_\alpha, \quad t \ge 0 \tag{3.6}$$

for any $v_0 \in PL^N \cap PL^\alpha$. This monotonicity is valid for the special case $\alpha = 2$. However, it does not seem to hold for any $\alpha \ne 2$. To confirm it for each case, we have only to find a initial value $v_0$ which does not satisfy (3.6). Combining the analytical method and the numerical method, we will show in [Ka5] that for the two cases $(N, \alpha) = (3, 4), (3, 3)$ there exist $v_0$ which do not satisfy (3.6). We remark that it is possible to apply the same arguments as in [Ka5] to the other cases $(N, \alpha)$ with $\alpha \ne 2$.

We obtain immediately the following.

**Corollary 3.1.** *Let $\alpha \in (1, \infty)$ and $q \in (N, q_*)$ be constants and $u_0, v_0 \in PL^N \cap PL^\alpha$. We set $d_q := d_{Nq}$ and $A_q := A_{Nq}$. Here, we use the notations in the statement of Theorem 3.1. Assume (3.2) and*

$$\|v_0 - u_0\|_N \exp\big(A_{N+2} \int_0^T \|u(s\,; u_0)\|_{N+2}^{N+2}\, ds\big) < \min\,(d_{N+2}, d_{\alpha q}). \tag{3.7}$$

*Then we have (3.4) [resp. (3.5)] for the case $\alpha \in [2,\infty)$ [resp. for the case $\alpha \in (1,2)$].*

**Proof of Corollary 3.1.** The proof is complete if we derive (3.3). By Theorem 3.1 and (3.7) it suffices to obtain (3.4) for $(\alpha, q) = (N, N+2)$. To this end we will prove $t_* = T$, where

$$t_* := \max\{\tau \in [0,T]\,;\ (3.4) \text{ with } (\alpha,q) = (N,N+2) \text{ holds for } t \in [0,\tau]\}.$$

We proceed by contradiction. Suppose $t_* < T$. Then, it follows from the continuity of $\|u(t;v_0) - u(t;u_0)\|_N$ that there exists a small constant $\varepsilon > 0$ such that $\|u(t;v_0)-u(t;u_0)\|_N < d_{N+2}$ for $t \in [\,0,\ t_*+\varepsilon\,]$. Therefore, by Theorem 3.1 we have (3.4) with $(\alpha,q) = (N,N+2)$ for $t \in [\,0,\ t_*+\varepsilon\,]$. This contradicts the definition of $t_*$. Thus we conclude $t_* = T$. The proof is complete. □

**Proof of Theorem 3.1.** Our method of proof is close to the argument in [N] and [Ka] for the porous media equations. Set $w(t) := u(t;v_0) - u(t;u_0)$ and $u(t) := u(t;u_0)$ for simplicity. The solutions $u(t;u_0)$ and $u(t;v_0)$ are regular for $t \in (0,T]$. In particular, $w, w_t \in C((0,T]\,;\,W^{2,p})$ for $p \ge \min(N,\alpha)$ (see e.g. [K] and [GM, Section 3]). We verify that $w = (w_1, \cdots, w_N)$ satisfies

$$\begin{gathered} w_t - \Delta w + (w\cdot\nabla)w + (u\cdot\nabla)w + (w\cdot\nabla)u + \nabla\pi = 0 \quad \text{in } \mathbb{R}^N \times \mathbb{R}^+, \\ \nabla\cdot w = 0 \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}^+. \end{gathered} \tag{3.8}$$

For short notation, we use $\partial_j := \partial/\partial x_j$.

**Case** $\alpha > 2$**.** We denote $w_j^p := |w_j|^{p-1}w_j$ $(0 < p < \infty)$. By the integration by parts, we have

$$\begin{aligned} \frac{1}{\alpha}\frac{d}{dt}\|w(t)\|_\alpha^\alpha &= \frac{1}{\alpha}\frac{d}{dt}\sum_{j=1}^N \int_{\mathbb{R}^N} |w_j(t)|^\alpha = \sum_{j=1}^N \int w_j^{\alpha-1}(w_j)_t\,dx \\ &= -\frac{4(\alpha-1)}{\alpha^2}J(w)^2 - I_1 - I_2 - I_3 - I_4, \end{aligned} \tag{3.9}$$

where we set

$$J(w) = (\sum_{j=1}^N \|\nabla w_j^{\alpha/2}\|_2^2)^{1/2}, \qquad I_1 = \sum_{j=1}^N \int w_j^{\alpha-1}(w\cdot\nabla)w_j,$$

$$I_2 = \sum_{j=1}^N \int w_j^{\alpha-1}(u\cdot\nabla)w_j, \quad I_3 = \sum_{j=1}^N \int w_j^{\alpha-1}(w\cdot\nabla)u_j, \quad I_4 = \sum_{j=1}^N \int w_j^{\alpha-1}\partial_j\pi\,.$$

We will estimate $I_j$ $(j = 1,2,3,4)$.

$$I_1 = \frac{1}{\alpha}\sum_{j=1}^N \int \nabla(|w_j|^\alpha)\cdot w = -\frac{1}{\alpha}\sum_{j=1}^N \int |w_j|^\alpha\,\nabla\cdot w = 0.$$

It follows from the integration by parts and the Hölder's inequality that

$$\begin{aligned} |I_2| + |I_3| &\le C\sum_{j=1}^N \int |u||w|^{\alpha/2}|\nabla w_j^{\alpha/2}| \\ &\le C(\int |u|^2|w|^\alpha)^{1/2}J(w) \le C|u|_q\,|w|_{\alpha q/(q-2)}^{\alpha/2}J(w). \end{aligned} \tag{3.10}$$

By the Hölder's inequality and the Sobolev inequality,

$$|w|_{\alpha q/(q-2)} \le |w|_{\alpha}^{1-N/q}|w|_{N\alpha/(N-2)}^{N/q}, \tag{3.11}$$

$$|w|_{N\alpha/(N-2)} = |\,|w|^{\alpha/2}|_{2N/(N-2)}^{2/\alpha} \le CJ(w)^{2/\alpha} \quad \text{for} \quad N \ge 3. \tag{3.12}$$

Combining (3.11) and (3.12), we have

$$|w|_{\alpha q/(q-2)} \le C\|w\|_{\alpha}^{1-N/q}J(w)^{2N/\alpha q}. \tag{3.13}$$

This estimate is what we call (a version of) the Gagliardo-Nirenberg inequality (see e.g. [N]). Note that (3.12) holds for $N \ge 3$, but *not* for $N = 2$. However, (3.13) does hold for $N \ge 2$. In what follows , we often use the Hölder's inequality and the Sobolev inequality in the same way as above in order to make clear the essence of the argument below. Although the case $N = 2$ is exceptional in such situation, we will not mention it. But it is easy to rewrite it rigorously by the same argument as above. It follows from (3.10) and (3.13) that

$$|I_2| + |I_3| \le C\|u\|_q\|w\|_{\alpha}^{\alpha(q-N)/2q}J(w)^{1+N/q} \le \varepsilon J(w)^2 + C_\varepsilon\|u\|_q^r\|w\|_{\alpha}^{\alpha}. \tag{3.14}$$

Next we will estimate $I_4$, which is the most difficult part. It follows from the integration by parts and the Hölder's inequality that

$$\begin{aligned} |I_4| &\le \frac{2(\alpha-1)}{\alpha}\sum_{j=1}^{N}\int |\,\pi\, w_j^{\alpha/2-1}\partial_j w_j^{\alpha/2}| \\ &\le \frac{2(\alpha-1)}{\alpha}(\sum_{j=1}^{N}\int \pi^2|w_j|^{\alpha-2})^{1/2}J(w) \\ &\le C\|\pi^2\|_a^{1/2}\,\|\,|w|^{\alpha-2}\,\|_b^{1/2}J(w) = C\|\pi\|_{2a}|w|_{b(\alpha-2)}^{\alpha/2-1}J(w). \end{aligned} \tag{3.15}$$

Here, $a$ and $b$ are positive constants which satisfy

$$\frac{1}{a} + \frac{1}{b} = 1 \quad \text{and} \quad 1 < b < \infty. \tag{3.16}$$

We will later determine $a$ and $b$. By (3.8) we have

$$-\Delta\pi = \sum_{i,j=1}^{N}\partial_j w_i \cdot \partial_i(2u_j + w_j) = \sum_{i,j=1}^{N}\partial_i\partial_j[\,w_i(2u_j + w_j)\,]. \tag{3.17}$$

With the aid of the Calderón - Zygmund inequality and the Hölder's inequality,

$$\|\pi\|_{2a} \le C\sum_{i,j}\|w_i(2u_j + w_j)\|_{2a} \le C\|w\|_{4a}^2 + C\|u\|_q\|w\|_{2aq/(q-2a)}. \tag{3.18}$$

Combining (3.15) and (3.18), we have

$$|I_4| \le C\|w\|_{4a}^2\|w\|_{b(\alpha-2)}^{\alpha/2-1}J(w) + C\|u\|_q\|w\|_{2aq/(q-2a)}\|w\|_{b(\alpha-2)}^{\alpha/2-1}J(w). \tag{3.19}$$

We choose $a$ and $b$ such that the following two conditions hold.

(P) Both $4a$ and $b(\alpha-2)$ are between $N$ and $N\alpha/(N-2)$

(Q) Both $2aq/(q-2a)$ and $b(\alpha-2)$ are between $\alpha$ and $N\alpha/(N-2)$

Assuming that (P) and (Q) hold, from the Hölder's inequality and (3.12) we obtain

$$\|w\|_{4a} \le \|w\|_N^{1-\theta_1}\|w\|_{N\alpha/(N-2)}^{\theta_1} \le C\|w\|_N^{1-\theta_1}J(w)^{2\theta_1/\alpha}, \tag{3.20}$$

$$\|w\|_{b(\alpha-2)} \le \|w\|_N^{1-\theta_2}\|w\|_{N\alpha/(N-2)}^{\theta_2} \le C\|w\|_N^{1-\theta_2}J(w)^{2\theta_2/\alpha}, \tag{3.21}$$

$$\|w\|_{2aq/(q-2a)} \le \|w\|_\alpha^{1-\theta_3}\|w\|_{N\alpha/(N-2)}^{\theta_3} \le C\|w\|_\alpha^{1-\theta_3}J(w)^{2\theta_3/\alpha}, \tag{3.22}$$

$$\|w\|_{b(\alpha-2)} \le \|w\|_\alpha^{1-\theta_4}\|w\|_{N\alpha/(N-2)}^{\theta_4} \le C\|w\|_\alpha^{1-\theta_4}J(w)^{2\theta_4/\alpha}. \tag{3.23}$$

Here, $\theta_j \in [0,1]$ $(1 \le j \le 4)$ are constants, and exactly $(\theta_1,\theta_2,\theta_3,\theta_4) =$

$$\Big(\frac{\alpha\,(N-4a)}{4a\,(N-2-\alpha)},\ \frac{\alpha\,(N+2b-\alpha b)}{\alpha bN-2bN+4b-\alpha^2 b},\ \frac{N}{2}-\frac{\alpha N}{4a}+\frac{\alpha N}{2q},\ \frac{N\,(\alpha b-\alpha-2b)}{2b(\alpha-2)}\Big). \tag{3.24}$$

It follows from (3.16), (3.19) and (3.20)-(3.23) that

$$\begin{aligned} |I_4| &\le C\|w\|_N^{2(1-\theta_1)+(\alpha/2-1)(1-\theta_2)}J(w)^{1+4\theta_1/\alpha+(1-2/\alpha)\theta_2} \\ &\quad + C\|u\|_q\|w\|_\alpha^{(1-\theta_3)+(\alpha/2-1)(1-\theta_4)}J(w)^{1+2\theta_3/\alpha+(1-2/\alpha)\theta_4} \\ &= C\|w\|_N J(w)^2 + C\|u\|_q\|w\|_\alpha^{\alpha(q-N)/2q}J(w)^{1+N/q}. \end{aligned} \tag{3.25}$$

Therefore, when $q > N$ we have

$$|I_4| \le C\|w\|_N J(w)^2 + \varepsilon J(w)^2 + C_\varepsilon\|u\|_q^r\|w\|_\alpha^\alpha. \tag{3.26}$$

Thus we obtain

$$\frac{1}{\alpha}\frac{d}{dt}\|w(t)\|_\alpha^\alpha \le -\Big(\frac{\alpha-1}{\alpha^2} - C_0\|w\|_N\Big)J(w)^2 + A_{\alpha q}\|u\|_q^r\|w\|_\alpha^\alpha.$$

Set $d_{\alpha q} := (\alpha-1)/\alpha^2 C_0$. Then, by (3.3) we have

$$\frac{1}{\alpha}\frac{d}{dt}\|w(t)\|_\alpha^\alpha \le A_{\alpha q}\|u\|_q^r\|w\|_\alpha^\alpha \quad \text{for} \quad t \in [0,T].$$

Therefore, we obtain (3.4) for $t \in [0,T]$.

Finally, we observe how the conditions (P) and (Q) ($\Longleftrightarrow 0 \le \theta_j \le 1$ for $1 \le j \le 4$) restrict the range of $q$. First, we study the case $2 < \alpha < N-2$. We have $\alpha < N\alpha/(N-2) < N$. In order to satisfy (P) and (Q), we need to choose $b(\alpha-2) = N\alpha/(N-2)$. Then we have

$$a = \frac{N\alpha}{2(N+\alpha-2)}, \qquad (\theta_1,\theta_2,\theta_3,\theta_4) = \Big(\frac{1}{2},\,1,\,1-\frac{\alpha(q-N)}{2q},\,1\Big).$$

The condition $0 \le \theta_3 \le 1$ is equivalent to $N \le q \le q_* := N\alpha/(\alpha-2)$. Next, we study the case $N-2 \le \alpha$. The condition $0 \le \theta_1 \le 1$ is equivalent to

$$(N-4)b \le N \quad \text{and} \quad (N\alpha-4N+8)b \ge N\alpha\,. \tag{3.27}$$

By (3.24) we have

$$\frac{1}{q} = \frac{1}{2} - \frac{1}{\alpha} + \frac{2\theta_3}{N\alpha} - \frac{1}{2b}. \tag{3.28}$$

Let $N-2 \le \alpha < N$ and $2 < \alpha \le 4$. Since $\alpha < N \le N\alpha/(N-2)$, the conditions (P) and (Q) implies $N \le b(\alpha-2) \le N\alpha/(N-2)$. Thus we have

$$\frac{N}{\alpha-2} \le b \le \frac{N\alpha}{(\alpha-2)(N-2)}. \tag{3.29}$$

Remark that (3.29) leads to $\theta_2, \theta_4 \in [0,1]$ and (3.27). We see that $q$ achieves the maximum at $(\theta_3, b) = (0, N/(\alpha-2))$ and the minimum at $(\theta_3, b) = (1, \frac{N\alpha}{(\alpha-2)(N-2)})$. Thus we have

$$N < q \le q_* := \frac{2N\alpha}{(N-\alpha)(\alpha-2)}.$$

Let $N-2 \le \alpha$ and $\alpha > 4$ and $N \ge 5$. It follows from (3.27) and the condition $\theta_2, \theta_4 \in [0,1]$ that

$$\frac{N\alpha}{N(\alpha-4)+8} \le b \le \frac{N}{N-4},$$
$$\frac{\max(N,\alpha)}{\alpha-2} \le b \le \frac{N\alpha}{(N-2)(\alpha-2)}.$$

Here, we see that

$$\frac{\max(N,\alpha)}{\alpha-2} \le \frac{N\alpha}{N(\alpha-4)+8} \quad \text{and} \quad \frac{N\alpha}{(N-2)(\alpha-2)} \le \frac{N}{N-4}.$$

It follows that

$$\frac{N\alpha}{N(\alpha-4)+8} \le b \le \frac{N\alpha}{(N-2)(\alpha-2)}.$$

In view of this inequality and (3.28), $q$ achieves the maximum at $(\theta_3, b) = (0, N\alpha/(N(\alpha-4)+8))$ and the minimum at $(\theta_3, b) = (1, N\alpha/(\alpha-2)(N-2))$. Therefore, we have

$$N < q \le q_* := \frac{N\alpha}{N-4}.$$

Finally, let $N \le 4$ and $\alpha \ge N$. Since $N \le \alpha$, the conditions (P) and (Q) lead to $\alpha \le b(\alpha-2) \le N\alpha/(N-2)$. Thus we have

$$\frac{\alpha}{\alpha-2} \le b \le \frac{N\alpha}{(N-2)(\alpha-2)},$$

which implies that $\theta_2, \theta_4 \in [0,1]$ and also that $\theta_1 \in [0,1]$. By this inequality and (3.28) we see that $q$ achieves the maximum at $(\theta_3, b) = (0, \alpha/(\alpha-2))$ and the minimum at $(\theta_3, b) = (1, N\alpha/(\alpha-2)(N-2))$. We conclude that $N < q \le q_* := \infty$.

**Case** $\alpha = 2$**.** The above estimates of $I_1$, $I_2$, $I_3$ holds in this case. Moreover, by the integration by parts we have

$$I_4 = \sum_{j=1}^{N} \int_{\mathbb{R}^N} w_j \partial_j \pi = -\int \pi \, \nabla \cdot w = 0.$$

Therefore, we have

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|_2^2 \le A_{2,q}\|u\|_q^r \|w\|_2^2. \tag{3.30}$$

Here, $q \in (N, \infty]$ is any number. The estimate (3.30) implies (3.4) with $\alpha = 2$.

**Case:** $1 < \alpha < 2$**.** We write $w^p := |w|^{p-1}w$ $(1 < p < \infty)$ for simplicity. By Lemma 3.2 (at the end of this section) and integration by parts, we have

$$\begin{aligned} \frac{1}{\alpha}\frac{d}{dt}|w(t)|_\alpha^\alpha &= \frac{1}{\alpha}\frac{d}{dt}\int_{\mathbb{R}^N}|w(t)|^\alpha = \int \frac{w\cdot w_t}{|w|^{2-\alpha}} \\ &= (\int w^{\alpha-1}\cdot \Delta w\,) - I_1 - I_2 - I_3 - I_4 \\ &\le -(\alpha-1)K(w)^2 - I_1 - I_2 - I_3 - I_4, \end{aligned} \tag{3.31}$$

where we set

$$K(w) = (\int_{\mathbb{R}^N}\frac{|\nabla w|^2}{|w|^{2-\alpha}})^{1/2}, \qquad I_1 = \int w^{\alpha-1}\cdot(w\cdot\nabla)w = \int w^\alpha\cdot\nabla|w|,$$

$$I_2 = \int w^{\alpha-1}\cdot(u\cdot\nabla)w, \qquad I_3 = \int w^{\alpha-1}\cdot(w\cdot\nabla)u, \qquad I_4 = \int w^{\alpha-1}\cdot\nabla\pi\,.$$

It follows from (3.12) and Lemma 3.2 that

$$|w|_{N\alpha/(N-2)} \le CK(w)^{2/\alpha} \quad \text{for} \quad N\ge 3\,. \tag{3.32}$$

We have

$$|I_1| \le \int |w|^\alpha|\nabla w| \le |w|_{\alpha+2}^{(\alpha+2)/2}K(w) \le |w|_N|w|_{N\alpha/(N-2)}^{\alpha/2}K(w) \le C\|w\|_N K(w)^2. \tag{3.33}$$

We note that (3.33) holds also for $N=2$ (see the argument just after (3.13) ). By the integration by parts,

$$|I_2| + |I_3| \le C\int |u||w|^{\alpha-1}|\nabla w| \le C(\int |u|^2|w|^\alpha)^{1/2}K(w).$$

This is the same estimate as (3.10). Therefore, the same argumentation as in the case $\alpha\in(2,\infty)$ leads to

$$|I_2| + |I_3| \le \varepsilon K(w)^2 + C\|u\|_q^r|w|_\alpha^\alpha. \tag{3.34}$$

Next, we will estimate $I_4$ in the similar way as in the case: $\alpha\in(2,\infty)$. We denote by $R_k$ the Riesz operator, i.e.

$$R_k := \mathcal{F}^{-1}\frac{\xi_k}{|\xi|}\mathcal{F}.$$

Here, $\mathcal{F}$ is the Fourier transform operator. It follows from (3.17) that

$$-\partial_k\pi = \sum_{i,j}R_kR_i\partial_j[w_i(2u_j+w_j)] = \sum_{i,j}R_kR_i[(\partial_jw_i)(2u_j+w_j)]. \tag{3.35}$$

Let $a\in(1,2)$ be a constant which we will determine later. By $L^p$-boundedness of the Riesz operator (see [St, Chapter 3]), we have

$$|\nabla\pi|_a \le C\sum_{i,j}\|\partial_jw_i(2u_j+w_j)\|_a.$$

Therefore, we obtain

$$
\begin{aligned}
|I_4| \le & |\nabla\pi|_a |w|^{\alpha-1}_{a(\alpha-1)/(a-1)} \\
\le & C\sum_{i,j} \|\partial_j w_i(2u_j + w_j)\|_a \, |w|^{\alpha-1}_{a(\alpha-1)/(a-1)} \\
\le & C|w|^{(4-\alpha)/2}_{a(4-\alpha)/(2-a)} |w|^{\alpha-1}_{a(\alpha-1)/(a-1)} K(w) \\
& + C|u|_q |w|^{(2-\alpha)/2}_{aq(2-\alpha)/(2q-aq-2a)} |w|^{\alpha-1}_{a(\alpha-1)/(a-1)} K(w).
\end{aligned} \tag{3.36}
$$

We estimate this in a similar way as in (3.25). Choose $a$ such that the following two conditions hold.

(R) Both $a(4-\alpha)/(2-a)$ and $a(\alpha-1)/(a-1)$ are between $N$ and $N\alpha/(N-2)$

(S) Both $aq(2-\alpha)/(2q-aq-2a)$ and $a(\alpha-1)/(a-1)$ are between $\alpha$ and $N\alpha/(N-2)$

Assuming that (R) and (S) hold, it follows from the Hölder's inequality and (3.32) that

$$
|w|_{a(4-\alpha)/(2-a)} \le |w|^{1-\theta_1}_N |w|^{\theta_1}_{N\alpha/(N-2)} \le C\|w\|^{1-\theta_1}_N K(w)^{2\theta_1/\alpha}, \tag{3.37}
$$

$$
|w|_{a(\alpha-1)/(a-1)} \le |w|^{1-\theta_2}_N |w|^{\theta_2}_{N\alpha/(N-2)} \le C\|w\|^{1-\theta_2}_N K(w)^{2\theta_2/\alpha}, \tag{3.38}
$$

$$
|w|_{aq(2-\alpha)/(2q-aq-2a)} \le |w|^{1-\theta_3}_\alpha |w|^{\theta_3}_{N\alpha/(N-2)} \le C|w|^{1-\theta_3}_\alpha K(w)^{2\theta_3/\alpha}, \tag{3.39}
$$

$$
|w|_{a(\alpha-1)/(a-1)} \le |w|^{1-\theta_4}_\alpha |w|^{\theta_4}_{N\alpha/(N-2)} \le C|w|^{1-\theta_4}_\alpha K(w)^{2\theta_4/\alpha}, \tag{3.40}
$$

where$(\theta_1,\theta_2,\theta_3,\theta_4) =$

$$
\Big(\frac{\alpha(2N-aN+a\alpha-4a)}{a(4-\alpha)(N-\alpha-2)},\ \frac{\alpha(aN+a-a\alpha-N)}{a(\alpha-1)(N-\alpha-2)},\ \frac{N(a\alpha+aq-\alpha q)}{aq(2-\alpha)},\ \frac{N(\alpha-a)}{2a(\alpha-1)}\Big). \tag{3.41}
$$

It follows from (3.36) and (3.37)-(3.41) that

$$
|I_4| \le C\|w\|_N K(w)^2 + C\|u\|_q |w|^{\alpha(q-N)/2q}_\alpha K(w)^{1+N/q}.
$$

Therefore, when $q > N$ we have

$$
|I_4| \le C\|w\|_N K(w)^2 + \varepsilon K(w)^2 + C_\varepsilon \|u\|^r_q \, |w|^\alpha_\alpha.
$$

Thus, we obtain

$$
\frac{1}{\alpha}\frac{d}{dt}|w(t)|^\alpha_\alpha \le -\Big(\frac{\alpha-1}{2} - C_0\|w\|_N\Big)K(w)^2 + A_{\alpha q}\|u\|^r_q \, |w|^\alpha_\alpha. \tag{3.42}
$$

Set $d_{\alpha q} := (\alpha-1)/2C_0$. Then, by (3.3) and (3.42) we have (3.5) for $t \in [0,T]$.

Finally, we determine the available range of $q$. First we consider the case: $1 < \alpha \le N-2$. We have $\alpha < N\alpha/(N-2) \le N$. By (R) and (S), we need to choose $a(\alpha-1)/(a-1) := N\alpha/(N-2)$, i.e.

$$
a := \frac{N\alpha}{N+2(\alpha-1)} \quad (<2)\,.
$$

It leads to
$$\theta_1 = \frac{2-\alpha}{4-\alpha} \in (0,1), \qquad \theta_3 = 1 - \frac{\alpha(q-N)}{q(2-\alpha)}.$$
By the conditions $\theta_3 \in [0,1]$ and $q > N$ we have
$$N < q \le \frac{N\alpha}{2(\alpha-1)} := q_*.$$

Next, let $N-2 < \alpha < 2$. Since $\alpha < N < N\alpha/(N-2)$, (R) and (S) implies $N \le a(\alpha-1)/(a-1) \le N\alpha/(N-2)$, i.e.
$$\frac{N\alpha}{N+2\alpha-2} \le a \le \frac{N}{N-\alpha+1}. \tag{3.43}$$
We remark that (3.43) is equivalent to $\theta_2$, $\theta_4 \in [0,1]$. We verify that (3.43) also implies $\theta_1 \in [0,1]$. By (3.41) we have
$$\frac{1}{q} = \frac{2-\alpha}{N\alpha}\,\theta_3 + \frac{1}{a} - \frac{1}{\alpha}.$$

Thus, $q$ achieves the minimum at $(\theta_3, a) = (\,1,\ N\alpha/(N+2\alpha-2)\,)$ and the maximum at $(\theta_3, a) = (\,0,\ N/(N-\alpha+1)\,)$. Combining this and the condition $q > N$, we have
$$N < q \le \frac{N\alpha}{(N-\alpha)(\alpha-1)} := q_*.$$

The proof is complete. $\square$

**Lemma 3.2.** . *Let $\alpha \in (1,2)$ be a number and $w = (w_1, \cdots, w_N) \in W^{2,\alpha}(\mathbb{R}^N\,;\mathbb{R}^N)$. Then we have $|w|^{\alpha/2} \in H^1 (:= W^{1,2})$ and*
$$\frac{4(\alpha-1)}{\alpha^2}\|\nabla|w|^{\alpha/2}\|_2^2 \le (\alpha-1)\int_{\mathbb{R}^N} \frac{|\nabla w|^2}{|w|^{2-\alpha}} \le -\int_{\mathbb{R}^N} w^{\alpha-1}\cdot\Delta w\,dx\ (<\infty). \tag{3.44}$$
*Here, we define $|\nabla w|^2/|w|^{2-\alpha} = 0$ when $|w(x)| = 0$.*

**Proof.** Since $|\nabla|w|\,| \le |\nabla w|$, we have
$$\int_{\mathbb{R}^N} \frac{|\nabla w|^2}{|w|^{2-\alpha}} \ge \int_{\mathbb{R}^N} |w|^{\alpha-2}|\nabla|w|\,|^2 = \frac{4}{\alpha^2}\int |\nabla|w|^{\alpha/2}\,|^2.$$
Thus we obtained the first inequality in (3.44).

To show the second inequality in (3.44), it suffices to derive
$$-\int_{\mathbb{R}^N} |w|^{\alpha-2}w\cdot\partial_j^2 w \ge (\alpha-1)\int_{\mathbb{R}^N}\frac{|\partial_j w|^2}{|w|^{2-\alpha}}. \tag{3.45}$$
Let $j=1$. (We omit the other cases: $j \ne 1$ since they are same.) With the aid of the Fubini Theorem we have
$$-\int_{\mathbb{R}^N} |w|^{\alpha-2}w\cdot\partial_1^2 w = -\int_{\mathbb{R}^{N-1}} dx' \int_{\mathbb{R}} |w|^{\alpha-2}w\cdot\partial_1^2 w\,dx_1. \tag{3.46}$$

Here, $x' := (x_2, \cdots, x_N)$. By the assumption: $w \in W^{2,\alpha}$ we have

$$w(\cdot, x') \in C^1(\mathbb{R}\,;\,\mathbb{R}^N) \tag{3.47}$$

for a.e. $x' \in \mathbb{R}^{N-1}$. We fix any $x'$ such that (3.47) holds and set

$$W(\cdot) := w(\cdot, x') \in C^1(\mathbb{R}\,;\,\mathbb{R}^N).$$

Then, there exist countable numbers of open intervals $\{I_k\}_{k=1}^{\infty}$ such that $W(x) \neq 0$ on each $I_k$. Actually, $W(x)$ is positive definite or negative definite on each interval $I_k$. Each $\partial I_k$ (:= the boundary of $I_k$ in $\mathbb{R}$) consists of at most two points. Therefore, $\bigcup_{k=1}^{\infty} \partial I_k$ is a set whose measure is zero. It follows from the integration by parts that

$$\begin{aligned} -\int_{\mathbb{R}} |W|^{\alpha-2} W \cdot W_{xx}\, dx &= -\sum_{k=1}^{\infty} \int_{I_k} |W|^{\alpha-2} W \cdot W_{xx} \\ &= \sum_{k=1}^{\infty} \int_{I_k} [\, -(2-\alpha)|W|^{\alpha-2}|W|_x W_x + |W|^{\alpha-2}(W_x)^2 \,] \\ &\geq \sum_{k=1}^{\infty} \int_{I_k} [\, -(2-\alpha)|W|^{\alpha-2}(W_x)^2 + |W|^{\alpha-2}(W_x)^2 \,] \\ &= (\alpha-1) \int_{\mathbb{R}} |W|^{\alpha-2}(W_x)^2. \end{aligned} \tag{3.48}$$

Combining (3.46) and (3.48), we obtain (3.45). $\square$

## 4. Applications

In this section we give the proofs of Corollaries 1.1-1.3 and some other applications of our Theorem 3.1.

**Proof of Corollary 1.1.** Let $v$ be any mild solution of (NS) which satisfies $v \in C([0,T)\,;\,PL^N) \cap L^r_{\mathrm{loc}}((0,T)\,;\,PL^q)$. It suffices to show that

$$v(t) = u(t\,;\,v(0)) \qquad \text{on} \qquad [0,T).$$

Here, $u(t\,;\,v(0))$ is a strong solution, which satisfies

$$u(t\,;\,v(0)) \in C([0,T)\,;\,PL^N) \cap L^{N+2}((0,T)\,;\,PL^{N+2})$$

(see Definition 2.2). We set $u(t) := u(t\,;\,v(0)\,)$ and $w(t) := v(t) - u(t)$. Let $d_{N+2}$ and $A_{N+2}$ be the constants defined in Corollary 3.1. We choose a small constant $\tau \in (0,T)$ such that

$$\|w(t)\|_N \leq d_{N+2} \quad \text{for} \quad t \in [0,\tau].$$

Let $\varepsilon \in (0,\tau)$. By the semigroup property of the mild solution, $v(t)$ is a strong solution on $[\varepsilon, T)$ with the initial value $v(\varepsilon)$. We apply Theorem 3.1 and obtain

$$\|w(t)\|_N \leq \|w(\varepsilon)\|_N \exp\big(A_{N+2} \int_{\varepsilon}^{\tau} \|u(s)\|_{N+2}^{N+2}\, ds\big) \quad \text{for} \quad t \in [\varepsilon, \tau].$$

Let $\varepsilon \to +0$. Then we have $w(t) = 0$ for $t \in [0,\tau]$. Now, we easily verify from the continuity of $w(t)$ that $T = \sup\{\tau\,;\, w(t) = 0 \quad \text{for} \quad t \in [0,\tau]\}$. $\square$

The next result is a 'local'-version of Corollary 1.2.

**Corollary 4.1. (Local Stability).** *Let $u_0 \in PL^N \cap PL^\alpha$ with a constant $\alpha \in (1,\infty)$ and $u(t;u_0)$ be a strong solution. Then, for any $T < t_{\max}(u_0)$ there exist constants $\delta_0 = \delta_0(N,u_0,T) \in \mathbb{R}^+$ and $C_0 = C_0(N) \in \mathbb{R}^+$ such that if*

$$v_0 \in PL^N \quad \textit{and} \quad \|v_0 - u_0\|_N \le \delta_0$$

*then we have $t_{\max}(v_0) > T$ and*

$$\|u(t\,;v_0) - u(t;u_0)\|_N \le \|v_0 - u_0\|_N \exp\big(C_0 \int_0^t \|u(s\,;u_0)\|_{N+2}^{N+2}\, ds\big) \tag{4.1}$$

*for $t \in [0,T]$.*

*Moreover, when $\alpha \ne N$ there exist constants $\delta_1 = \delta_1(N,\alpha,u_0,T) \in (0,\delta_0]$, $q = q(N,\alpha) \in (N,\infty)$ and $C_1 = C_1(N,\alpha) \in \mathbb{R}^+$ such that if $v_0 \in PL^N \cap PL^\alpha$ and $\|v_0 - u_0\|_N \le \delta_1$ then we have for $\alpha \in [2,\infty)$*

$$\|u(t\,;v_0) - u(t;u_0)\|_\alpha \le \|v_0 - u_0\|_\alpha \exp\big(C_1 \int_0^t \|u(s\,;u_0)\|_q^r\, ds\big) \tag{4.2}$$

*for $t \in [0,T]$, and for $\alpha \in (1,2)$ the estimate (4.2) with the norm $\|\cdot\|_\alpha$ replaced by $|\cdot|_\alpha$. Here, $r$ is the constant which satisfies (1.1).*

**Proof.** We use the notation in the statement of Theorem 3.1 and Corollary 3.1. We denote $u(t) := u(t;u_0)$ and $v(t) := u(t;v_0)$. We fix constants $\alpha_0 \in (N,\infty)$ and $q_0 \in (N, q_*(N,\alpha_0))$. For example, set $(\alpha_0,q_0) = (2N,2N)$. We choose $\delta_0 \in \mathbb{R}^+$ such that

$$\delta_0 < \min\big(d_{N+2}, d_{\alpha_0,q_0}\big) \exp\big(-A_{N+2} \int_0^T \|u(s)\|_{N+2}^{N+2}\, ds\big)$$

and assume $\|v_0 - u_0\|_N \le \delta_0$. Let $C_0 := A_{N+2}$. Then, by Corollary 3.1 we have (4.1) for $t \in [0,T_0)$ and

$$\|v(t) - u(t)\|_N < \min(d_{N+2}, d_{\alpha_0,q_0}) \quad \text{for} \quad t \in [0,T_0). \tag{4.3}$$

Here, $T_0 := \min(T, t_{\max}(v_0))$. We complete the proof if we prove $t_{\max}(v_0) > T$. We fix a constant $t_0 \in (0,T_0)$. Then, by (4.3) and Theorem 3.1 we have

$$\|v(t) - u(t)\|_{\alpha_0} \le \|v(t_0) - u(t_0)\|_{\alpha_0} \exp\big(A_{\alpha_0,q_0} \int_0^T \|u(s)\|_{q_0}^{r_0}\, ds\big) < \infty$$

for $t \in (0,T_0)$, where $r_0$ is the constant which satisfies $N/q_0 + 2/r_0 = 1$. Thus, $\|v(t)\|_{\alpha_0}$ is bounded on $[0,T_0)$. In view of Corollary 2.1 we have $T_0 = T < t_{\max}(v_0)$.

When $\alpha \ne N$, we choose $\delta_1 \in \mathbb{R}^+$ such that

$$\delta_1 < \min\big(d_{N+2}, d_{\alpha_0,q_0}, d_{\alpha q}\big) \exp\big(-A_{N+2} \int_0^T \|u(s)\|_{N+2}^{N+2}\, ds\big).$$

Here, $q \in (N, q_*(N,\alpha))$ is a constant. Then we can verify (4.2) in the same argument as above. □

The following lemma will be used in the proof of Corollary 1.2.

**Lemma 4.2.** *Let $v(t) = u(t; v_0) \in C_0([0, \infty)\,;\, PL^N)$ be a global strong solution. And let $\alpha \in [N, \infty)$ be a constant. Then there exists a constant $T_* = T_*(N, \alpha, v_0) \in \mathbb{R}^+$ such that $\|u(t; v_0)\|_\alpha$ is non-increasing on $[T_*, \infty)$.*

**Proof.** We use the notation in the statement of Theorem3.1 and Corollary 3.1. We choose $T_*$ such that $\|v(T_*)\|_N < \min(d_{N+2}, d_{\alpha\alpha})$. Let $u_0 := 0$. Then $u(t; u_0)$ is a trivial solution. It follows from Corollary 3.1 that $\|v(t)\|_\alpha \le \|v(T_*)\|_\alpha < \min(d_{N+2}, d_{\alpha\alpha})$ for $t \ge T_*$. Applying Corollary 3.1 again, we see that $\|u(t; v_0)\|_\alpha$ is non-increasing for $t \ge T_*$. $\square$

**Remark 4.1.** Actually, we can verify the decay estimate

$$\|v(t)\|_\alpha \le Ct^{-(1-N/\alpha)/2} \quad \text{for} \quad t \ge 1. \tag{4.4}$$

See [K, Theorem 4]. We can prove (4.4) using the same argument as in the proof of [Ka, Theorem 4.1]. We omit it since we do not use (4.4) in this paper.

Proof of Corollary 1.2. We use the notation in the statement of Theorem 3.1 and Corollary 3.1. We fix constants $\alpha_0 \in (N, \infty)$ and $q_0 \in (N, q_*(N, \alpha_0))$. For example, set $(\alpha_0, q_0) = (2N, 2N)$.

(i) We write $u(t) := u(t; u_0)$ and $v(t) := u(t; v_0)$. We choose $\delta_0 \in \mathbb{R}^+$ such that

$$\delta_0 \exp\big(A_{N+2} \int_0^\infty \|u(s)\|_{N+2}^{N+2}\, ds\big) < \min\big(d_{N+2}, d_{\alpha_0, q_0}, \frac{1}{2}\varepsilon_*\big).$$

Here, $\varepsilon_*$ is the constant in the statement of Theorem 2.1 (iii). We set $C_0 := A_{N+2}$. Let $\|v_0 - u_0\|_N \le \delta_0$. Then, by Corollary 3.1 we have (1.2) and

$$\|v(t) - u(t)\|_N < \min(d_{N+2}, d_{\alpha_0, q_0}, \frac{1}{2}\varepsilon_*) \tag{4.5}$$

for $t \in (t_0, t_{\max}(v_0))$. We will show $t_{\max}(v_0) = \infty$. We fix a constant $t_0 \in (0, t_{\max}(v_0))$. By (4.5) and Theorem 3.1 we have

$$\|v(t) - u(t)\|_{\alpha_0} \le \|v(t_0) - u(t_0)\|_{\alpha_0} \exp\big(A_{\alpha_0, q_0} \int_{t_0}^\infty \|u(s)\|_{q_0}^{r_0}\, ds\big) < \infty \tag{4.6}$$

for $t \in (t_0, t_{\max}(v_0))$, where $r_0$ is the constant which satisfies $N/q_0 + 2/r_0 = 1$. Since $\|u(t)\|_{\alpha_0}$ is bounded on $[t_0, \infty)$ (see Lemma 4.2), $\|v(t)\|_{\alpha_0}$ is bounded on $[t_0, t_{\max}(v_0))$. With the aid of Corollary 2.1, we have $t_{\max}(v_0) = \infty$. Thus, $v(t)$ is a global solution. We choose a constant $T_1$ such that $\|u(T_1)\|_N < \varepsilon_*/2$. Then, by (4.5) we have $\|v(T_1)\|_N < \varepsilon_*$. Therefore, we obtain from Theorem 2.1 that $v(t) \in C_0([0, \infty)\,;\, PL^N)$.

(ii) We choose a constant $q$ such that $N < q < q_*(N, \alpha)$. We choose $\delta_1 \in \mathbb{R}^+$ such that

$$\delta_1 < \min\big(d_{N+2}, d_{\alpha_0, q_0}, \frac{1}{2}\varepsilon_*, d_{\alpha q}\big) \exp\big(-A_{N+2} \int_0^\infty \|u(s)\|_{N+2}^{N+2}\, ds\big).$$

Then we can easily verify (1.3) for $t \in \mathbb{R}^+$ by the same argument as in (i). $\square$

**Proof of Corollary 1.3.** We denote $u(t) := u(t; u_0)$. It follows from Theorem 2.1 that $t_{\max}(u_0) > T$ implies $u(t) \in L^r((0, T)\,;\, PL^q)$. Next, we will prove the inverse. Let $u(t) \in L^r((0,T)\,;\,PL^q)$. In view of Corollary 2.1, it suffices to prove that $\lim_{t\to T-0} u(t)$ exists in $PL^N$. By Lemma 2.1 we have $u(t;u_0) \in L^{N+2}((0,T)\,;\,PL^{N+2})$. Let $\{t_j\}_{j=1}^{\infty}$ be any monotone increasing sequence such that $t_1 \geq T/2$ and $t_j \to T$ $(j \to \infty)$. Setting $u_0 := u(t_k - T/2)$ and $v_0 := u(t_l - T/2)$ $(k < l)$, we apply Corollary 3.1. We have

$$
\begin{aligned}
&\|u(t_l) - u(t_k)\|_N \\
&\leq \|u(t_l - T/2) - u(t_k - T/2)\|_N \exp(A_{N+2}\int_0^{T/2} \|u(s + t_k - T/2)\|_{N+2}^{N+2}\,ds) \\
&\to 0 \quad (k \to \infty).
\end{aligned}
$$

Thus, $\lim_{j\to\infty} u(t_j)$ exists in $PL^N$. □

Finally we mention the topological structure of the space of strong solutions. We set

$$A := \{\, u_0 \in PL^N\,;\, u(t\,;\,u_0) \in C_0([0,\infty)\,;\,PL^N)\,\}. \tag{4.7}$$

Then $A$ is open in $PL^N$ by Corollary 1.2. When $N = 3$ the open set $A$ is unbounded in $PL^3$ since $u_0 \in A$ if $u_0(x)$ is axially symmetric and $u_0 \in PL^2 \cap PL^3$ (which was shown in [UI]). We also set

$$
\begin{aligned}
B :=& \{u_0 \in PL^N\,;\, t_{\max}(u_0) < \infty\} \qquad (4.8)\\
=& \{u_0 \in PL^N\,;\, \|u(t;u_0)\|_\infty \text{ blows up in finite time}\,\}.
\end{aligned}
$$

The second equality holds by the equivalence of (e) and (c) in Corollary 2.1.

**Proposition 4.1.** *When $N = 2$, we have $A = PL^2$.*

This result was proved in [KM], [M] and [Wi]. However, we give a simple and different proof by using Corollary 1.3 and Proposition 2.2.

**Proof.** Let fix any $u_0 \in PL^2$. We write $u(t) := u(t;u_0)$. We have the energy equality

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u(s)\|_2^2 ds = \|u_0\|_2^2 \quad \text{for} \quad t \in [0, t_{\max}(u_0)). \tag{4.9}$$

By the Gagliardo-Nirenberg inequality

$$\|u(t)\|_4 \leq C\|u(t)\|_2^{1/2}\|\nabla u(t)\|_2^{1/2}. \tag{4.10}$$

In view of (4.9) and (4.10) we have $u(t;u_0) \in L^4((0,T)\,;\,PL^4)$. Here $T \in \mathbb{R}^+$ is a constant such that $T \leq t_{\max}(u_0)$. By Corollary 1.3 we have $T < t_{\max}(u_0)$. Thus, we obtain $t_{\max}(u_0) = \infty$, i.e. $u(t;u_0)$ is a global solution. Now we have $u(t;u_0) \in L^4(\mathbb{R}^+\,;\,PL^4)$. Therefore, we apply Proposition 2.2 to conclude $u(t;u_0) \in C_0([0,\infty)\,;\,PL^N)$. □

**Proposition 4.2.** *Let $N = 3, 4$. Then we have the following results:*

*(i) For $u_0 \in PL^2 \cap PL^N$ the solution $u(t;u_0)$ blows up in finite time or $u(t;u_0) \in C_0([0,\infty)\,;\,PL^N)$, i.e. we have*

$$(A \cup B) \cap PL^2 = PL^2 \cap PL^N. \tag{4.11}$$

*(ii) the set $B \cap PL^2$ is closed in $PL^2 \cap PL^N$.*

*(iii) the set $B$ is empty or $B$ is not open in $PL^N$.*

**Proof.** (i) Let $u_0 \in PL^2 \cap PL^N$ and $t_{\max}(u_0) = \infty$. In view of the energy equality (4.9), $\|u(t)\|_2$ is bounded for $t \geq 0$ and $\liminf_{t\to\infty} \|\nabla u(t)\|_2 = 0$. Thus, by the Gagliardo - Nirenberg inequality, we have $\liminf_{t\to\infty} \|u(t)\|_N = 0$. It follows from Theorem 2.1 that $u \in C_0([0,\infty)\,; PL^N)$. Hence, we have (4.11).

(ii) By Corollary 1.2, the set $A \cap PL^2$ is open in $PL^2 \cap PL^N$. Therefore, $B \cap PL^2 = (PL^2 \cap PL^N) - (A \cap PL^2)$ is closed in $PL^2 \cap PL^N$.

(iii) We proceed by contradiction. Suppose that $B$ is non-empty open set in $PL^N$. Since $PL^2 \cap PL^N$ is dense in $PL^N$, there exists $u_0 \in (PL^2 \cap B) - \{0\}$. We set

$$C := \{\tau \in \mathbb{R}\,; \tau u_0 \in A\}.$$

In view of Corollary 1.2 and Theorem 2.1 (iii), the set $C$ is non-empty open set in $\mathbb{R}$. Since $C \neq \mathbb{R}$, we have $\partial C \neq \phi$. Set $B_1 := \{\tau u_0\,; \tau \in \partial C\}$. Then, we obtain from (4.11) that $B_1 \subset \partial A \cap B$. This implies $A \cap B \neq \emptyset$, which is a contradiction. Thus, $B$ is empty or $B$ is not open in $PL^N$. $\square$

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