Electronic Journal of Differential Equations, Vol. **2003**(2003), No. 01, pp. 1–7.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

An ϵ-regularity result for generalized harmonic maps into spheres *

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Abstract

For $m, n \geq 2$ and $1 < p < 2$, we prove that a map $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{S}^{n-1})$ from an open domain $\Omega \subset \mathbb{R}^m$ into the unit $(n-1)$-sphere, which solves a generalized version of the harmonic map equation, is smooth, provided that $2 - p$ and $[u]_{\mathrm{BMO}(\Omega)}$ are both sufficiently small. This extends a result of Almeida [1]. The proof is based on an inverse Hölder inequality technique.

1 Introduction

For integers $m, n \geq 2$, let $\Omega \subset \mathbb{R}^m$ be an open domain, and let $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ denote the $(n-1)$-dimensional unit sphere. Define the space

$$H^1(\Omega, \mathbb{S}^{n-1}) = \{v \in H^1(\Omega, \mathbb{R}^n) \colon \ |v| = 1 \text{ almost everywhere}\},$$

and consider the functional

$$E(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx, \quad u \in H^1(\Omega, \mathbb{S}^{n-1}).$$

A map $u \in H^1(\Omega, \mathbb{S}^{n-1})$ is called a weakly harmonic map, if it is a critical point of E, i. e.

$$\frac{d}{dt}\Big|_{t=0} E\big(\frac{u + t\phi}{|u + t\phi|}\big) = 0$$

for all $\phi \in C^\infty_0(\Omega, \mathbb{R}^n)$. The Euler-Lagrange equation for this variational problem is

$$\Delta u + |\nabla u|^2 u = 0 \quad \text{in } \Omega \tag{1.1}$$

(in the distributions sense). Denote by $\wedge$ the exterior product $\wedge : \mathbb{R}^n \times \mathbb{R}^n \to \Lambda_2 \mathbb{R}^n$, then (1.1) is equivalent to

$$\operatorname{div}(u \wedge \nabla u) = 0 \quad \text{in } \Omega. \tag{1.2}$$

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[*] *Mathematics Subject Classifications:* 58E20.
Key words: Generalized harmonic maps, regularity.

Submitted December 13, 2002. Published January 2, 2003.

This form of the equation provides a natural extension of the notion of weakly harmonic maps into spheres. Whereas we need a map in $H^1_{\rm loc}(\Omega, \mathbb{S}^{n-1})$ to make any sense of (1.1), the equation (1.2) only requires

$$u \in W^{1,1}_{\rm loc}(\Omega, \mathbb{S}^{n-1}) = \{v \in W^{1,1}_{\rm loc}(\Omega, \mathbb{R}^n) : |v| = 1 \text{ almost everywhere}\}.$$

A map in this space satisfying (1.2) is called a generalized harmonic map.

For $m = 2$, it was proven by Hélein [8, 9], that any weakly harmonic map is smooth (also for more general target manifolds than spheres). For higher dimensions, this is no longer true. Indeed Rivière [13] constructed a weakly harmonic map in three dimensions which is discontinuous everywhere. But there exists an ϵ-regularity result, due to Evans [4] (and to Bethuel [2] for more general targets), which can be stated as follows.

Theorem 1.1 *There exists a number $\epsilon > 0$, depending only on m and n, such that any weakly harmonic map $u \in H^1(\Omega, \mathbb{S}^{n-1})$ with the property $[u]_{\mathrm{BMO}(\Omega)} \le \epsilon$ is smooth in Ω.*

Here we use the notation

$$[u]_{\mathrm{BMO}(\Omega)} = \sup_{B_r(x_0) \subset \Omega} \fint_{B_r(x_0)} |u - \bar{u}_{B_r(x_0)}| \, dx, \tag{1.3}$$

where $B_r(x_0)$ denotes the ball in $\mathbb{R}^m$ with centre x_0 and radius r, and

$$\bar{u}_{B_r(x_0)} = \fint_{B_r(x_0)} u \, dx = \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} u \, dx.$$

Together with the well-known monotonicity formula for so-called stationary weakly harmonic maps, e. g. weakly harmonic maps which satisfy $\frac{d}{dt}|_{t=0} E(u(x + t\psi(x))) = 0$ for all $\psi \in C^\infty_0(\Omega, \mathbb{R}^m)$ (see Price [12]), one concludes that weakly harmonic maps with this property are smooth away from a closed singular set of vanishing $(m-2)$-dimensional Hausdorff measure.

Generalized harmonic maps on the other hand may have singularities even in two dimensions. A typical example is the map $u(x) = x/|x|$ in $\mathbb{R}^2$. For $m = 2$ and for any $p \in [1, 2)$, Almeida [1] even constructed generalized harmonic maps in $W^{1,p}(\Omega, \mathbb{S}^1)$ which are nowhere continuous. Nevertheless, there is an ϵ-regularity result for generalized harmonic maps in two dimensions, due to Almeida [1]. (Another proof was given by Ge [6].)

Theorem 1.2 *For $m = 2$, there exists $\epsilon > 0$, depending only on n, such that any weakly harmonic map $u \in W^{1,1}_{\rm loc}(\Omega, \mathbb{S}^{n-1})$ with the property $\|\nabla u\|_{L^{2,\infty}(\Omega)} \le \epsilon$ is smooth in Ω.*

Here $\|\cdot\|_{L^{2,\infty}(\Omega)}$ is the norm of the Lorentz space $L^{2,\infty}(\Omega, \mathbb{R}^{m \times n})$. (For a definition and properties of Lorentz spaces, see e. g. [14], Chapter V.)

2 Results

The aim of this note is to extend and improve this result. We replace the smallness in the $L^{2,\infty}$-norm by a weaker condition (reminding of Theorem 1.1), and we prove the result for all dimensions. More precisely, we have the following theorem.

Theorem 2.1 *There exist $p < 2$ and $\epsilon > 0$, depending only on m and n, such that any generalized harmonic map $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{S}^{n-1})$ with the property $[u]_{\mathrm{BMO}(\Omega)} \le \epsilon$ is in $C^\infty(\Omega, \mathbb{S}^{n-1})$.*

To prove this theorem, it suffices to show that under these conditions, the generalized harmonic map u is in $H^1_{\mathrm{loc}}(\Omega, \mathbb{S}^{n-1})$. Higher regularity is then implied by Theorem 1.1 (provided that ϵ is chosen accordingly). For this first step on the other hand, we can also admit a non-vanishing right hand side in (1.2).

Theorem 2.2 *For any $q > 2$, there exist $p < 2$ and $\epsilon > 0$, depending only on m, n, and q, with the following property. Suppose that $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{S}^{n-1})$ is a distributional solution of*

$$\operatorname{div}(u \wedge \nabla u) = F + \operatorname{div} G, \tag{2.1}$$

where $F \in L^{mq/(m+q)}_{\mathrm{loc}}(\Omega, \Lambda_2\mathbb{R}^n)$ and $G \in L^q_{\mathrm{loc}}(\Omega, \mathbb{R}^m \otimes \Lambda_2\mathbb{R}^n)$. If $[u]_{\mathrm{BMO}(\Omega)} \le \epsilon$, then $u \in W^{1,p/(p-1)}_{\mathrm{loc}}(\Omega, \mathbb{S}^{n-1})$.

As mentioned above, Theorem 2.1 is an immediate consequence of Theorem 1.1 and Theorem 2.2. The proof of the latter is inspired by the inverse Hölder inequality technique used by Iwaniec–Sbordone [11] to prove regularity for solutions of equations of the form

$$\operatorname{div} A(x, \nabla u) = F + \operatorname{div} G,$$

where $A(x,\xi) = \frac{\partial \mathcal{F}}{\partial \xi}(x,\xi)$ for a quasi-convex function $\mathcal{F}$ (satisfying certain conditions). We combine these methods with arguments from the regularity theory for weakly harmonic maps.

We will use the following well-known results. The first one is due to Giaquinta–Modica [7].

Proposition 2.3 *For $1 < a < b$, and for some ball $B_R(x_0) \subset \mathbb{R}^m$, suppose that $g \in L^a(B_R(x_0))$ and $f \in L^b(B_R(x_0))$ are non-negative functions which satisfy*

$$\fint_{B_{r/2}(x_1)} g^a\,dx \le A\Big[\Big(\fint_{B_r(x_1)} g\,dx\Big)^a + \fint_{B_r(x_1)} f^a\,dx\Big] + \theta \fint_{B_r(x_1)} g^a\,dx$$

for every ball $B_r(x_1) \subset B_R(x_0)$ and for certain constants $A, \theta > 0$. There exists a constant $\theta_0 = \theta_0(m,a,b) > 0$, such that whenever $\theta < \theta_0$, then $g \in L^c(B_{R/2}(x_0))$ with

$$\Big(\fint_{B_{R/2}(x_0)} g^c\,dx\Big)^{1/c} \le B\Big[\Big(\fint_{B_R(x_0)} g^a\,dx\Big)^{1/a} + \Big(\fint_{B_R(x_0)} f^c\,dx\Big)^{1/c}\Big]$$

for certain numbers $c > a$ and $B > 0$, both depending only on m, A, θ, a, and b.

The following is a combination of the compensated compactness results of Coifman–Lions–Meyer–Semmes [3], and the duality of the space $\mathrm{BMO}(\mathbb{R}^m) = \{f \in L^1_{\mathrm{loc}}(\mathbb{R}^m)\colon [f]_{\mathrm{BMO}(\mathbb{R}^m)} < \infty\}$ with the Hardy space $\mathcal{H}^1(\mathbb{R}^m)$. The latter is due to Fefferman–Stein [5].

Proposition 2.4 *For $1 < p < \infty$, suppose that a function $f \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^m)$ with $\|\nabla f\|_{L^p(\mathbb{R}^m)} < \infty$, a vector field $g \in L^{p/(p-1)}(\mathbb{R}^m, \mathbb{R}^m)$ with $\operatorname{div} g = 0$ in the distribution sense, and a function $h \in \mathrm{BMO}(\mathbb{R}^m)$ are given. Then*

$$\Big| \int_{\mathbb{R}^m} \nabla f \cdot g\, h\, dx \Big| \le C \|\nabla f\|_{L^p(\mathbb{R}^m)} \|g\|_{L^{p/(p-1)}(\mathbb{R}^m)} [h]_{\mathrm{BMO}(\mathbb{R}^m)}$$

for a constant C which depends only on m and p.

Having the ingredients ready, we can now prove Theorem 2.2.

Proof of Theorem 2.2. Suppose $q > 2$, $F \in L^{mq/(m+q)}_{\mathrm{loc}}(\Omega, \Lambda_2\mathbb{R}^n)$, and $G \in L^q_{\mathrm{loc}}(\Omega, \mathbb{R}^m \otimes \Lambda_2\mathbb{R}^n)$. Let for the moment p be any number in $(1, 2)$, and suppose that $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{S}^{n-1})$ is a solution of (2.1).

Let $\psi \in W^{2,mq/(m+q)}_{\mathrm{loc}}(\Omega, \Lambda_2\mathbb{R}^n)$ be a solution of

$$\Delta\psi = F \quad \text{in } \Omega.$$

Then $\nabla\psi \in W^{1,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^m \otimes \Lambda_2\mathbb{R}^n)$, and u satisfies

$$\operatorname{div}(u \wedge \nabla u) = \operatorname{div}(G + \nabla\psi).$$

Hence we may assume without loss of generality that $F = 0$. Choose a ball $B_r(x_0) \subset \Omega$ and a cut-off function $\zeta \in C^\infty_0(B_r(x_0))$ with $\zeta \equiv 1$ in $B_{r/2}(x_0)$, such that $|\nabla\zeta| \le 4r^{-1}$. Consider the Hodge decomposition

$$|\nabla(\zeta(u - \bar{u}_{B_r(x_0)}))|^{p-2}\, u \wedge \nabla(\zeta(u - \bar{u}_{B_r(x_0)})) = \nabla\phi + \Phi,$$

where $\phi \in W^{1,p/(p-1)}_{\mathrm{loc}}(\mathbb{R}^m, \Lambda_2\mathbb{R}^n)$ and $\Phi \in L^{p/(p-1)}(\mathbb{R}^m, \mathbb{R}^m \otimes \Lambda_2\mathbb{R}^n)$ have the properties $\operatorname{div}\Phi = 0$ and

$$\|\nabla\phi\|_{L^s(\mathbb{R}^m)} + \|\Phi\|_{L^s(\mathbb{R}^m)} \le C_1 \|\nabla(\zeta(u - \bar{u}_{B_r(x_0)}))\|^{p-1}_{L^{(p-1)s}(B_r(x_0))}$$

for any $s \in (\frac{1}{p-1}, \frac{p}{p-1}]$ and for a constant $C_1 = C_1(m, n, s)$. The existence of such a decomposition is due to Iwaniec–Martin [10]. In particular, we have

$$\fint_{B_r(x_0)} |\nabla\phi|^s\, dx \le C_2 \left(\fint_{B_r(x_0)} |\nabla u|^s\, dx \right)^{p-1} \tag{2.2}$$

for a constant $C_2 = C_2(m,n,s)$, owing to the Poincaré and the Hölder inequality. Observe that

$$\begin{aligned} 2^{-m}\fint_{B_{r/2}(x_0)} |\nabla u|^p \, dx &\le \fint_{B_r(x_0)} \left\langle u \wedge \nabla(\zeta(u - \bar{u}_{B_r(x_0)})), \nabla\phi + \Phi \right\rangle dx \\ &= \fint_{B_r(x_0)} \left\langle u \wedge \nabla(\zeta(u - \bar{u}_{B_r(x_0)})), \Phi \right\rangle dx \\ &\quad + \fint_{B_r(x_0)} \left\langle \nabla\zeta, (u \wedge (u - \bar{u}_{B_r(x_0)})) \cdot \nabla\phi \right\rangle dx \\ &\quad - \fint_{B_r(x_0)} \left\langle \nabla\zeta, (u \wedge \nabla u) \cdot (\phi - \bar{\phi}_{B_r(x_0)}) \right\rangle dx \\ &\quad + \fint_{B_r(x_0)} \left\langle G, \nabla(\zeta(\phi - \bar{\phi}_{B_r(x_0)})) \right\rangle dx, \end{aligned}$$

where we denote the standard scalar product in $\mathbb{R}^m$ and in $\mathbb{R}^m \otimes \Lambda_2\mathbb{R}^n$ by $\langle\cdot,\cdot\rangle$, whereas we use a dot in $\mathbb{R}^n$ to avoid confusion. We have the estimates

$$\begin{aligned} &\fint_{B_r(x_0)} \left\langle \nabla\zeta, (u \wedge (u - \bar{u}_{B_r(x_0)})) \cdot \nabla\phi \right\rangle dx \\ &\le \frac{4}{r}\Big(\fint_{B_r(x_0)} |\nabla\phi|^{\frac{2m}{m+1}} \, dx\Big)^{\frac{m+1}{2m}} \Big(\fint_{B_r(x_0)} |u - \bar{u}_{B_r(x_0)}|^{\frac{2m}{m-1}} \, dx\Big)^{\frac{m-1}{2m}} \\ &\le C_3\Big(\fint_{B_r(x_0)} |\nabla u|^{\frac{2m}{m+1}} \, dx\Big)^{\frac{p(m+1)}{2m}}, \end{aligned}$$

by (2.2) and the Sobolev inequality, and similarly

$$-\fint_{B_r(x_0)} \left\langle \nabla\zeta, (u \wedge \nabla u) \cdot (\phi - \bar{\phi}_{B_r(x_0)}) \right\rangle dx \le C_4 \left(\fint_{B_r(x_0)} |\nabla u|^{\frac{2m}{m+1}} \, dx\right)^{\frac{p(m+1)}{2m}},$$

for certain constants C_3, C_4 which depend only on m and n.

Note that $[\zeta(u - \bar{u}_{B_r(x_0)})]_{\mathrm{BMO}(\mathbb{R}^m)} \le C_5[u]_{\mathrm{BMO}(B_r(x_0))}$ for a constant $C_5 = C_5(m,n)$. (This is proven in [4].) Extending ∇u to $\mathbb{R}^m$ and applying Proposition 2.4, we thus find

$$\begin{aligned} &\fint_{B_r(x_0)} \left\langle u \wedge \nabla(\zeta(u - \bar{u}_{B_r(x_0)})), \Phi \right\rangle dx \\ &= -\fint_{B_r(x_0)} \zeta \left\langle \nabla u \wedge (u - \bar{u}_{B_r(x_0)}), \Phi \right\rangle dx \\ &\le C_6[u]_{\mathrm{BMO}(\Omega)} \fint_{B_r(x_0)} |\nabla u|^p \, dx \end{aligned}$$

for a constant $C_6 = C_6(m,n,p)$.

Finally, choose a number $\sigma \in (2, q)$. We have

$$\begin{aligned} &\fint_{B_r(x_0)} \langle G, \nabla(\zeta(\phi - \bar{\phi}_{B_r(x_0)}))\rangle \, dx \\ &\le C_7 \Big(\fint_{B_r(x_0)} |G|^\sigma \, dx\Big)^{1/\sigma} \Big(\fint_{B_r(x_0)} |\nabla\phi|^{\sigma/(\sigma-1)} \, dx\Big)^{\frac{\sigma-1}{\sigma}} \\ &\le C_8 \Big(\fint_{B_r(x_0)} |G|^\sigma \, dx\Big)^{1/\sigma} \Big(\fint_{B_r(x_0)} |\nabla u|^{\sigma/(\sigma-1)} \, dx\Big)^{\frac{(p-1)(\sigma-1)}{\sigma}} \\ &\le C_8 \Big[\fint_{B_r(x_0)} |G|^\sigma \, dx + \Big(\fint_{B_r(x_0)} |\nabla u|^{\sigma/(\sigma-1)} \, dx\Big)^{\frac{p(\sigma-1)}{\sigma}} + 1\Big] \end{aligned}$$

(for constants C_7, C_8 which depend on m, n, and σ) by the Hölder inequality, the Poincaré inequality, the estimate (2.2), and Young's inequality.

Now choose $a \in (1, \min\{\frac{m+1}{m}, \frac{2(\sigma-1)}{\sigma}\})$, and set $b = \frac{qa}{\sigma}$. Let θ_0 be the constant from Proposition 2.3 (belonging to a and b), and choose a number $\theta \in (0, \theta_0)$. Then the conditions of Proposition 2.3 are satisfied for any ball $B_R(x_0) \subset\subset \Omega$, for the functions

$$g = |\nabla u|^{p/a}, \quad f = |G|^{\sigma/a} + 1,$$

and for a constant A which depends only on m, n, and σ, provided that $p \ge a \max\{\frac{2m}{m+1}, \frac{\sigma}{\sigma-1}\}$ (which is strictly less than 2) and $[u]_{\mathrm{BMO}(\Omega)} \le C_6^{-1}\theta$. Hence under these conditions, there exists a number $c > a$, not depending on p, such that $|\nabla u| \in L^{pc/a}_{\mathrm{loc}}(\Omega)$. If $2-p$ is sufficiently small, then $\frac{pc}{a} \ge \frac{p}{p-1}$, and therefore $u \in W^{1,p/(p-1)}_{\mathrm{loc}}(\Omega, \mathbb{S}^{n-1})$. This concludes the proof. □

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