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DETERMINISTIC HOMOGENIZATION OF PARABOLIC MONOTONE OPERATORS WITH TIME DEPENDENT COEFFICIENTS

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Abstract. We study, beyond the classical periodic setting, the homogenization of linear and nonlinear parabolic differential equations associated with monotone operators. The usual periodicity hypothesis is here substituted by an abstract deterministic assumption characterized by a great relaxation of the time behaviour. Our main tool is the recent theory of homogenization structures by the first author, and our homogenization approach falls under the two-scale convergence method. Various concrete examples are worked out with a view to pointing out the wide scope of our approach and bringing the role of homogenization structures to light.

1. Introduction

Let $2 \leq p < \infty$. Let $(y, \tau, \lambda) \to a(y, \tau, \lambda)$ be a function from $\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$ to $\mathbb{R}^N$ ($N \geq 1$) with the properties:

For each fixed $\lambda \in \mathbb{R}^N$, the function $(y, \tau) \to a(y, \tau, \lambda)$ (denoted by $a(\cdot, \cdot, \lambda)$) from $\mathbb{R}^N \times \mathbb{R}$ to $\mathbb{R}^N$ is measurable (1.1)

$a(y, \tau, \omega) = \omega$ almost everywhere (a.e.) in $(y, \tau) \in \mathbb{R}^N \times \mathbb{R}$, where ω denotes the origin in $\mathbb{R}^N$ (1.2)

There are two constants $\alpha_0, \alpha_1 > 0$ such that, a.e. in $(y, \tau) \in \mathbb{R}^N \times \mathbb{R}$:
(i) $(a(y, \tau, \lambda) - a(y, \tau, \mu)) \cdot (\lambda - \mu) \geq \alpha_0 \mid \lambda - \mu \mid^p$
(ii) $|a(y, \tau, \lambda) - a(y, \tau, \mu)| \leq \alpha_1 (|\lambda| + |\mu|)^{p-2} |\lambda - \mu|$ for all $\lambda, \mu \in \mathbb{R}^N$, where the dot denotes the usual Euclidean inner product in $\mathbb{R}^N$, and $|\cdot|$ the associated norm. (1.3)

Let T be a positive real number, Ω a smooth bounded open set in $\mathbb{R}^N_x$ (the space $\mathbb{R}^N$ of variables $x = (x_1, \dots, x_N)$), and $f \in L^{p'}(0, T; W^{-1,p'}(\Omega; \mathbb{R}))$ with $p' = \frac{p}{p-1}$.

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For each given $\varepsilon > 0$, we consider the initial-boundary value problem

$$\begin{gathered} \frac{\partial u_\varepsilon}{\partial t} - \operatorname{div} a\big(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, Du_\varepsilon\big) = f \quad \text{in } Q = \Omega \times (0,T) \\ u_\varepsilon = 0 \quad \text{on } \partial\Omega \times (0,T) \\ u_\varepsilon(x,0) = 0 \quad \text{in } \Omega \end{gathered} \tag{1.4}$$

where D denotes the usual gradient, i.e., $D = (D_{x_i})_{1 \le i \le N}$ with $D_{x_i} = \frac{\partial}{\partial x_i}$, and div the divergence with respect to the variable x.

Provided the diffusion term of the differential operator in (1.4) is rigorously defined (see [18, Subsection 4.1]) and further an existence and uniqueness result for (1.4) is pointed out (all that will be accomplished in Section 2), our goal in this paper is to investigate the limiting behaviour, as $\varepsilon \to 0$, of u_ε (the solution of (1.4)). In all probability such an undertaking is hopeless without any further suitable assumption termed a *structure hypothesis* [18, 20], which specifies the behaviour of the function $(y, \tau) \to a(y, \tau, \lambda)$ (for fixed λ).

The common structure hypothesis is the so-called periodicity hypothesis. The latter states that there exist two networks $\mathcal{R} \subset \mathbb{R}^N_y$ and $\mathcal{S} \subset \mathbb{R}_\tau$, e.g., $\mathcal{R} = \mathbb{Z}^N$ and $\mathcal{S} = \mathbb{Z}$, such that for any given $k \in \mathcal{R}$ and $l \in \mathcal{S}$, we have $a(y + k, \tau + l, \lambda) = a(y, \tau, \lambda)$ a.e. in $(y, \tau) \in \mathbb{R}^N \times \mathbb{R}$, where λ is arbitrarily fixed. Under the periodicity hypothesis, homogenization results for problem (1.4) are available; see, e.g., [17, 22, 24] (see also [25] for specific corrector results). It should be mentioned in passing that the homogenization of linear parabolic operators in the periodic setting is now a classical theory (see, e.g., [2, 3, 8, 12, 13]) with, further, an extension to the almost periodic setting (see [27]).

However, much yet remains to be done in this area. To a large extent, non-stochastic homogenization theory seems to confine itself to the periodic setting, and that in spite of the gap to be filled between periodic and stochastic homogenization [23]. No doubt, to arrive −via homogenization− at a thorough understanding of physical problems we need to be released from the classical periodicity hypothesis, especially with regard to the behaviour in the time variable.

Specifically, we study here the homogenization of problem (1.4) in a very general setting characterized by an abstract assumption on $a(y, \tau, \lambda)$ (for fixed λ) covering a wide range of behaviours, especially with respect to the time variable $\tau = \frac{t}{\varepsilon}$. Broadly speaking, this abstract assumption is *proper* [21] with respect to the space variable $y = \frac{x}{\varepsilon}$ and hence covers a great variety of concrete behaviours in y (see [21, Section 5]) whereas, surprisingly enough, with respect to $\tau = \frac{t}{\varepsilon}$ it sets no further significant restriction on $a(y, \tau, \lambda)$ (fixed λ), which we express by referring to the *quasi-properness* introduced in Definition 3.1. This is a true advance in the homogenization of parabolic partial differential equations, and a great step towards a better understanding of evolution phenomena.

Our main tool is the recent theory of homogenization structures earlier developed in [18, 21] and our homogenization approach falls under the two-scale convergence method. For an obvious reason (see the diffusion term of the differential operator in (1.4)) the present study greatly leans on the elliptic case [21] of which it is a natural continuation.

The rest of the paper is organized as follows. In Section 2 we rigorously define the diffusion term of the differential operator in (1.4) and we point out those of its basic properties that ensure an existence and uniqueness result for the initial-boundary value problem under consideration. The homogenization of problem (1.4) proper begins with Section 3. Under an abstract deterministic hypothesis on $a(\cdot,\cdot,\lambda)$ (for fixed λ) we achieve fundamental homogenization results that prove quite similar to those obtained in the periodic setting. Finally, to illustrate the preceding abstract setting and point out its wide scope, we consider in Section 4 a few concrete homogenization problems for (1.4). In particular it is shown how such concrete problems reduce in a natural way to the abstract setting of Section 3.

In order that we may make use of basic tools provided by the classical Banach algebras theory, the vector spaces throughout are generally considered over $\mathbb{C}$ and the scalar functions are assumed to take complex values. If X and F denote a locally compact space and a Banach space, respectively, then we write $\mathcal{C}(X;F), \mathcal{B}(X;F)$ and $\mathcal{K}(X;F)$ for continuous mappings of X into F, bounded continuous mappings of X into F, and those mappings in $\mathcal{C}(X;F)$ having compact supports, respectively. We shall always assume that $\mathcal{B}(X;F)$ is equipped with the supremum norm $\|u\|_\infty = \sup_{x\in X}\|u(x)\|$ ($\|\cdot\|$ denotes the norm in F). For shortness we will write $\mathcal{C}(X) = \mathcal{C}(X;\mathbb{C})$, $\mathcal{B}(X) = \mathcal{B}(X;\mathbb{C})$ and $\mathcal{K}(X) = \mathcal{K}(X;\mathbb{C})$. Likewise the usual spaces $L^p(X;F)$ and $L^p_{\mathrm{loc}}(X;F)$ (X provided with a positive Radon measure) will be denoted by $L^p(X)$ and $L^p_{\mathrm{loc}}(X)$, respectively, in the case when $F = \mathbb{C}$. We refer to [6, 7, 9] for integration theory. On the other hand, for convenience we will most of the time put $\mathcal{C}_\mathbb{R}(X) = \mathcal{C}(X;\mathbb{R})$, $\mathcal{B}_\mathbb{R}(X) = \mathcal{B}(X;\mathbb{R})$, $\mathcal{K}_\mathbb{R}(X) = \mathcal{K}(X;\mathbb{R})$ and $L^p_\mathbb{R}(X) = L^p(X;\mathbb{R})$. Finally, the numerical space $\mathbb{R}^d$ ($d \geq 1$) and its open sets are each provided with Lebesgue measure denoted by $dx = dx_1 \dots dx_d$.

2. Preliminaries

Let $1 < p < \infty$. Let G be a function from $\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$ to $\mathbb{R}$ with the following properties:

For each $\lambda \in \mathbb{R}^N$, the function $(y,\tau) \to G(y,\tau,\lambda)$ from $\mathbb{R}^N \times \mathbb{R}$ to $\mathbb{R}$, denoted by $G(\cdot,\cdot,\lambda)$, is measurable (2.1)

$$G(y,\tau,\omega) = 0 \text{ a.e. in } (y,\tau) \in \mathbb{R}^N \times \mathbb{R} \tag{2.2}$$

There exists a positive constant α_1 such that $|G(y,\tau,\lambda) - G(y,\tau,\mu)| \leq \alpha_1(|\lambda| + |\mu|)^{p-2}|\lambda - \mu|$ for all $\lambda,\mu \in \mathbb{R}^N$ and for almost all $(y,\tau) \in \mathbb{R}^N \times \mathbb{R}$. (2.3)

With a view to giving a meaning to the diffusion term in (1.4), we wish to define, for each $\mathbf{u}$ in $L^p_\mathbb{R}(Q)^N = L^p_\mathbb{R}(Q) \times \cdots \times L^p_\mathbb{R}(Q)$ (N times), the function $(x,t) \to G(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, \mathbf{u}(x,t))$ from $Q = \Omega \times (0,T)$ to $\mathbb{R}$, where $\varepsilon > 0$ is freely fixed. As was pointed out in [18, Subsection 4.1], it is worth emphasizing that this is a delicate matter because the set $\mathcal{Q}_\varepsilon = \{(x,t,y,\tau) : y = \frac{x}{\varepsilon} \text{ and } \tau = \frac{t}{\varepsilon} \text{ for } (x,t) \in Q\}$ is negligible in $\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$.

For $u \in L^1_{\mathrm{loc}}(Q \times \mathbb{R}^N_y \times \mathbb{R}_\tau)$, we set

$$u^\varepsilon(x,t) = u\big(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon}\big) \quad (x \in \Omega,\ 0 < t < T) \tag{2.4}$$

whenever the right-hand side has meaning (see [18]). We will need the following two basic lemmas. For the proofs we refer to [21, Lemmas 2.1 and 2.2].

Lemma 2.1. *The transformation* $u \to u^\varepsilon$ *(*u^ε *given by (2.4)) considered as a mapping of* $\mathcal{C}(\overline{Q}) \otimes L^\infty(\mathbb{R}^{N+1})$ *into* $L^\infty(Q)$ *extends by continuity to a linear mapping, still denoted by* $u \to u^\varepsilon$*, of* $\mathcal{C}(\overline{Q}; L^\infty(\mathbb{R}^{N+1}))$ *into* $L^\infty(Q)$ *with* $\|u^\varepsilon\|_{L^\infty(Q)} \leq \sup_{(x,t)\in\overline{Q}} \|u(x,t)\|_{L^\infty(\mathbb{R}^{N+1})}$ *for* u *in* $\mathcal{C}(\overline{Q}; L^\infty(\mathbb{R}^{N+1}))$*.*

Lemma 2.2. *Let* $u \in \mathcal{C}(\overline{Q}; L^\infty(\mathbb{R}^{N+1}))$*. Suppose for each* $(x,t) \in \overline{Q}$ *we have* $u(x,t,y,\tau) \geq 0$ *a.e. in* $(y,\tau) \in \mathbb{R}^N \times \mathbb{R}$*. Then* $u^\varepsilon(x,t) \geq 0$ *a.e. in* Q*, where* u^ε *is considered in the meaning of Lemma 2.1.*

Now, given $\Phi \in \mathcal{C}_\mathbb{R}(\overline{Q})^N = \mathcal{C}_\mathbb{R}(\overline{Q}) \times \cdots \times \mathcal{C}_\mathbb{R}(\overline{Q})$ (N times), it is immediate by (2.1)-(2.3) that the function $(x,t,y,\tau) \to u(x,t,y,\tau) = G(y,\tau,\Phi(x,t))$ of $\overline{Q}\times\mathbb{R}^N\times\mathbb{R}$ into $\mathbb{R}$ lies in $\mathcal{C}(\overline{Q}; L^\infty(\mathbb{R}^{N+1}))$. Hence, the real function $(x,t) \to G(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, \Phi(x,t))$ on Q, denoted below by $G^\varepsilon(\cdot,\cdot,\Phi)$, is defined in the sense of Lemma 2.1 as a function in $L^\infty_\mathbb{R}(Q)$.

Proposition 2.1. *The transformation* $\Phi \to G^\varepsilon(\cdot,\cdot,\Phi)$ *of* $\mathcal{C}_\mathbb{R}(\overline{Q})^N$ *into* $L^\infty(Q)$ *extends by continuity to a mapping, still denoted by* $\Phi \to G^\varepsilon(\cdot,\cdot,\Phi)$*, of* $L^p_\mathbb{R}(Q)^N$ *into* $L^{p'}(Q)$ *(*$p' = \frac{p}{p-1}$*) with the property*

$$\|G^\varepsilon(\cdot,\cdot,\Phi) - G^\varepsilon(\cdot,\cdot,\Psi)\|_{L^{p'}(Q)} \leq \alpha_1 \||\Phi| + |\Psi|\|^{p-2}_{L^p(Q)} \|\Phi - \Psi\|_{L^p(Q)^N} \tag{2.5}$$

for $\Phi, \Psi \in L^p_\mathbb{R}(Q)^N$*.*

Proof. Let $\Phi, \Psi \in \mathcal{C}_\mathbb{R}(\overline{Q})^N$. Thanks to (2.3), we may apply Lemma 2.2 with

$$\begin{aligned} u(x,t,y,\tau) = \alpha_1\big(&|\Phi(x,t)| + |\Psi(x,t)|\big)^{p-2}|\Phi(x,t) - \Psi(x,t)| \\ &- |G(y,\tau,\Phi(x,t)) - G(y,\tau,\Psi(x,t))|. \end{aligned}$$

This leads immediately to

$$|G^\varepsilon(\cdot,\cdot,\Phi) - G^\varepsilon(\cdot,\cdot,\Psi)|^{p'} \leq \alpha_1^{p'} (|\Phi| + |\Psi|)^{(p-2)p'} |\Phi - \Psi|^{p'}.$$

Considering the functions $|\Phi - \Psi|^{p'}$ and $(|\Phi| + |\Psi|)^{(p-2)p'}$ as belonging to $L^q(Q)$ and $L^{q'}(Q)$, respectively, where $q = \frac{p}{p'}$ and $q' = \frac{q}{q-1}$, and using Hölder's inequality, one easily arrives at (2.5) with $\mathcal{C}_\mathbb{R}(\overline{Q})^N$ in place of $L^p_\mathbb{R}(Q)^N$. With this in mind, let $B_r = \{\mathbf{v} \in L^p_\mathbb{R}(Q)^N : \|\mathbf{v}\|_{L^p_\mathbb{R}(Q)^N} \leq \frac{r}{2}\}$ with $r > 0$. Let g_r be the restriction to $B_r \cap \mathcal{C}_\mathbb{R}(\overline{Q})^N$ of the mapping $\mathbf{v} \to G^\varepsilon(\cdot,\cdot,\mathbf{v})$ (where $\varepsilon > 0$ is fixed, of course). Clearly

$$\|g_r(\Phi) - g_r(\Psi)\|_{L^{p'}(Q)} \leq \alpha_1 r^{p-2} \|\Phi - \Psi\|_{L^p(Q)^N} \quad \text{for all } \Phi, \Psi \in B_r \cap \mathcal{C}_\mathbb{R}(\overline{Q})^N. \tag{2.6}$$

Since $B_r \cap \mathcal{C}_\mathbb{R}(\overline{Q})^N$ is dense in B_r (the verification is an elementary exercise), it follows that g_r extends by continuity to a continuous mapping, still denoted by g_r, of B_r into $L^{p'}(Q)$ such that (2.6) holds with B_r in place of $B_r\cap\mathcal{C}_\mathbb{R}(\overline{Q})^N$. Whence we deduce a sequence $(g_n)_{n\geq 1}$ of mappings $g_n : B_n \to L^{p'}(Q)$ with $g_n(\Phi) = G^\varepsilon(\cdot,\cdot,\Phi)$ for $\Phi \in B_n \cap \mathcal{C}_\mathbb{R}(\overline{Q})^N$. Noticing that $L^p_\mathbb{R}(Q)^N$ is the union of the balls B_n ($n \geq 1$) and, on the other hand, $g_{n+1}(\Phi) = g_n(\Phi)$ for $\Phi \in B_n$,we are led to a uniquely defined continuous mapping $g : L^p_\mathbb{R}(Q)^N \to L^{p'}(Q)$ such that $g(\Phi) = G^\varepsilon(\cdot,\cdot,\Phi)$ for any $\Phi \in \mathcal{C}_\mathbb{R}(\overline{Q})^N$. Hence the proposition follows by the density of $\mathcal{C}_\mathbb{R}(\overline{Q})^N$ in $L^p_\mathbb{R}(Q)^N$. □

Corollary 2.1. *We have*

$$a^{\varepsilon}(\cdot,\cdot,\omega)=\omega \quad a.e.\ in\ Q, \tag{2.7}$$

$$\|a^{\varepsilon}(\cdot,\cdot,Du)-a^{\varepsilon}(\cdot,\cdot,Dv)\|_{L^{p'}(Q)^N}\leq \alpha_1 \left\||Du|+|Dv|\right\|^{p-2}_{L^p(Q)}\|Du-Dv\|_{L^p(Q)^N} \tag{2.8}$$

and

$$\begin{aligned}\Big[a\big(\frac{x}{\varepsilon},\frac{t}{\varepsilon},Du(x,t)\big)-a\big(\frac{x}{\varepsilon},\frac{t}{\varepsilon},Dv(x,t)\big)\Big]\cdot(Du(x,t)-Dv(x,t))\\ \geq \alpha_0\,|Du(x,t)-Dv(x,t)|^p \quad a.e.\ in\ (x,t)\in Q\end{aligned} \tag{2.9}$$

for all $u,v\in L^p\left(0,T;W^{1,p}(\Omega;\mathbb{R})\right)$, *where* $a^{\varepsilon}(\cdot,\cdot,Du)=\{a_i^{\varepsilon}(\cdot,\cdot,Du)\}_{1\leq i\leq N}$.

Due to (1.1)-(1.3) and Lemma 2.2, this corollary is a direct consequence of Proposition 2.1 with $G=a_i$ (the i$^{\text{th}}$ component of the function $(y,\tau,\lambda)\to a(y,\tau,\lambda)$).

Remark 2.1. Thanks to Proposition 2.1, the diffusion term in (1.4) can now be rigorously defined. Specifically, let $u\in L^p\left(0,T;W^{1,p}(\Omega;\mathbb{R})\right)$. Then $a^{\varepsilon}(\cdot,\cdot,Du)\in L^{p'}(Q)^N$, as pointed out above. But we may as well view $a^{\varepsilon}(\cdot,\cdot,Du)$ as a function in $L^{p'}(0,T;L^{p'}(\Omega)^N)$. Consequently, $\operatorname{div} a^{\varepsilon}(\cdot,\cdot,Du)$ turns out to precisely represent the function $t\to \operatorname{div} a^{\varepsilon}(\cdot,t,Du(\cdot,t))$ of $(0,T)$ into $W^{-1,p'}(\Omega;\mathbb{R})$, which lies in $L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}))$ (this is straightforward).

Corollary 2.2. *Let* $2\leq p<\infty$. *For each given real* $\varepsilon>0$, *there exists a unique* $u_{\varepsilon}\in L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}))$ *satisfying* (1.4).

The statement of this corollary is guaranteed by (2.7)-(2.9). For more details we refer to, e.g., [1, 16, 26].

Remark 2.2. More precisely, u_{ε} lies in

$$V^p=\Big\{v\in L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R})): v'=\frac{\partial v}{\partial t}\in L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}))\Big\}.$$

With the norm $\|v\|_{V^p}=\|v\|_{L^p\left(0,T;W_0^{1,p}(\Omega)\right)}+\|v'\|_{L^{p'}\left(0,T;W^{-1,p'}(\Omega)\right)}$, V^p is a Banach space. For further needs it is worth noting that, since $p\geq 2$, the space $W_0^{1,p}(\Omega;\mathbb{R})$ is continuously and densely embedded in $L^2_{\mathbb{R}}(\Omega)$. Hence, identifying $L^2_{\mathbb{R}}(\Omega)$ with its dual, it follows

$$W_0^{1,p}(\Omega;\mathbb{R})\subset L^2_{\mathbb{R}}(\Omega)\subset W^{-1,p'}(\Omega;\mathbb{R})$$

with continuous embeddings. This has two important consequences:
1) We will use the same symbol, to denote both the inner product in $L^2_{\mathbb{R}}(\Omega)$ and the duality between the spaces $W^{-1,p'}(\Omega;\mathbb{R})$ and $W_0^{1,p}(\Omega;\mathbb{R})$.
2) The space V^p is continuously embedded in $\mathcal{C}([0,T];L^2_{\mathbb{R}}(\Omega))$ (this is a classical result). Thus, we may define $v(t)$ for $v\in V^p$ and $0\leq t\leq T$, and further the mapping $v\to v(t)$ sends continuously V^p into $L^2_{\mathbb{R}}(\Omega)$. Hence, we may consider the space $V_0^p=\{v\in V^p: v(0)=0\}$, a Banach space with the V^p-norm, which turns out to contain the solution u_{ε} of (1.4).

3. The Abstract Homogenization problem

For any notation, notion and result concerning homogenization structures and homogenization algebras we refer the reader to [18, 21]. The letter E throughout will denote exclusively a family of positive real numbers admitting 0 as an accumulation point. In the particular case where $E=(\varepsilon_n)_{n\geq 0}$ with $0<\varepsilon_n\leq 1$ and $\varepsilon_n\to 0$ as $n\to\infty$, we will refer to E as a *fundamental sequence*.

3.1. **Fundamentals of homogenization structures.** To the benefit of the reader we summarize below a few basic notions and results about the homogenization structures. We refer to [18, 21] for further details.

We start with one underlying concept. We say that a set $\Gamma \subset \mathcal{B}(\mathbb{R}^N_y)$ is a *structural representation* on $\mathbb{R}^N$ if

(1) Γ is a group under multiplication in $\mathcal{B}(\mathbb{R}^N_y)$
(2) Γ is countable
(3) $\gamma \in \Gamma$ implies $\overline{\gamma} \in \Gamma$ ($\overline{\gamma}$ the complex conjugate of γ)
(4) $\Gamma \subset \Pi^\infty$.

Here, Π^∞ denotes the space of functions $u \in \mathcal{B}(\mathbb{R}^N_y)$ such that $u^\varepsilon \to M(u)$ in $L^\infty(\mathbb{R}^N_x)$-weak $*$ as $\varepsilon \to 0$ ($\varepsilon > 0$), where $u^\varepsilon(x) = u\left(\frac{x}{\varepsilon}\right)$ ($x \in \mathbb{R}^N$) and $M(u) \in \mathbb{C}$.

We recall in passing that the complex mapping $u \to M(u)$ on Π^∞ is a positive continuous linear form with $M(1) = 1$ and $M(\tau_h u) = M(u)$ (for $u \in \Pi^\infty$ and $h \in \mathbb{R}^N$) where $\tau_h u(y) = u(y-h)$ ($y \in \mathbb{R}^N$). Thus, M is a mean value (see [18, 19] for further details).

Now, in the collection of all structural representations on $\mathbb{R}^N$ we consider the equivalence relation $\sim$ defined as: $\Gamma \sim \Gamma'$ if and only if $CLS(\Gamma) = CLS(\Gamma')$, where $CLS(\Gamma)$ denotes the closed vector subspace of $\mathcal{B}(\mathbb{R}^N_y)$ spanned by Γ. By an *H-structure* on $\mathbb{R}^N_y$ (H stands for *homogenization*) is understood any equivalence class modulo $\sim$. An H-structure is fully determined by its image. Specifically, let Σ be an H-structure on $\mathbb{R}^N$. Put $A = CLS(\Gamma)$ where Γ is any equivalence class representative of Σ (such a Γ is termed a *representation* of Σ). The space A is a so-called H-algebra on $\mathbb{R}^N_y$, that is, a closed subalgebra of $\mathcal{B}(\mathbb{R}^N_y)$ with the properties:

(5) A with the supremum norm is separable
(6) A contains the constants
(7) If $u \in A$ then $\overline{u} \in A$
(8) $A \subset \Pi^\infty$.

Furthermore, A depends only on Σ and not on the chosen representation Γ of Σ. Thus, we may set $A = \mathcal{J}(\Sigma)$ (the *image* of Σ). This yields a mapping $\Sigma \to \mathcal{J}(\Sigma)$ that carries the collection of all H-structures bijectively over the collection of all H-algebras on $\mathbb{R}^N_y$ (see [18, Theorem 3.1]).

Let A be an H-algebra on $\mathbb{R}^N_y$. Clearly A (with the supremum norm) is a commutative $\mathcal{C}^*$-algebra with identity (the involution is here the usual one of complex conjugation). We denote by $\Delta(A)$ the spectrum of A and by $\mathcal{G}$ the Gelfand transformation on A. We recall that $\Delta(A)$ is the set of all nonzero multiplicative linear forms on A, and $\mathcal{G}$ is the mapping of A into $\mathcal{C}(\Delta(A))$ such that $\mathcal{G}(u)(s) = \langle s, u\rangle$ ($s \in \Delta(A)$), where $\langle , \rangle$ denotes the duality between A' (the topological dual of A) and A. The topology on $\Delta(A)$ is the relative weak $*$ topology on A'. So topologized, $\Delta(A)$ is a metrizable compact space, and the Gelfand transformation is an isometric isomorphism of the $\mathcal{C}^*$-algebra A onto the $\mathcal{C}^*$-algebra $\mathcal{C}(\Delta(A))$. For further details concerning the Banach algebras theory we refer to [15]. The basic measure on $\Delta(A)$ is the so-called M-measure for A, namely the positive Radon measure β (of total mass 1) on $\Delta(A)$ such that $M(u) = \int_{\Delta(A)} \mathcal{G}(u)d\beta$ for $u \in A$ (see [18, Proposition 2.1]).

The partial derivative of index i ($1 \leq i \leq N$) on $\Delta(A)$ is defined to be the mapping $\partial_i = \mathcal{G} \circ D_{y_i} \circ \mathcal{G}^{-1}$ (usual composition) of $\mathcal{D}^1(\Delta(A)) = \{\varphi \in \mathcal{C}(\Delta(A)) : \mathcal{G}^{-1}(\varphi) \in A^1\}$ into $\mathcal{C}(\Delta(A))$, where $A^1 = \{\psi \in \mathcal{C}^1(\mathbb{R}^N) : \psi, D_{y_i}\psi \in A\ (1 \leq i \leq N)\}$.

Higher order derivatives are defined analogously. At the present time, let A^∞ be the space of $\psi \in \mathcal{C}^\infty(\mathbb{R}^N_y)$ such that $D^\alpha_y \psi = \frac{\partial^{|\alpha|}\psi}{\partial y_1^{\alpha_1} \dots \partial y_N^{\alpha_N}} \in A$ for every multi-index $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$, and let $\mathcal{D}(\Delta(A)) = \{\varphi \in \mathcal{C}(\Delta(A)) : \mathcal{G}^{-1}(\varphi) \in A^\infty\}$. Endowed with a suitable locally convex topology (see [18]), A^∞ (resp. $\mathcal{D}(\Delta(A))$) is a Fréchet space and further, $\mathcal{G}$ viewed as defined on A^∞ is a topological isomorphism of A^∞ onto $\mathcal{D}(\Delta(A))$. Any continuous linear form on $\mathcal{D}(\Delta(A))$ is referred to as a distribution on $\Delta(A)$. The space of all distributions on $\Delta(A)$ is then the dual, $\mathcal{D}'(\Delta(A))$, of $\mathcal{D}(\Delta(A))$. We endow $\mathcal{D}'(\Delta(A))$ with the strong dual topology. If we assume that A^∞ is dense in A (this condition is always fulfilled in practice), which amounts to assuming that $\mathcal{D}(\Delta(A))$ is dense in $\mathcal{C}(\Delta(A))$, then $L^p(\Delta(A)) \subset \mathcal{D}'(\Delta(A))$ $(1 \le p \le \infty)$ with continuous embedding (see [18] for more details). Hence we may define

$$W^{1,p}(\Delta(A)) = \{u \in L^p(\Delta(A)) : \ \partial_i u \in L^p(\Delta(A)) \ (1 \le i \le N)\}$$

where the derivative $\partial_i u$ is taken in the distribution sense on $\Delta(A)$ (exactly as the Schwartz derivative is taken in the classical case). We equip $W^{1,p}(\Delta(A))$ with the norm

$$\|u\|_{W^{1,p}(\Delta(A))} = \|u\|_{L^p(\Delta(A))} + \sum_{i=1}^N \|\partial_i u\|_{L^p(\Delta(A))} \quad \big(u \in W^{1,p}(\Delta(A))\big),$$

which makes it a Banach space. However, we will be mostly concerned with the space

$$W^{1,p}(\Delta(A))/\mathbb{C} = \Big\{u \in W^{1,p}(\Delta(A)) : \int_{\Delta(A)} u(s)d\beta(s) = 0\Big\}$$

provided with the seminorm

$$\|u\|_{W^{1,p}(\Delta(A))/\mathbb{C}} = \sum_{i=1}^N \|\partial_i u\|_{L^p(\Delta(A))} \quad \big(u \in W^{1,p}(\Delta(A))/\mathbb{C}\big).$$

So topologized, $W^{1,p}(\Delta(A))/\mathbb{C}$ is in general nonseparated and noncomplete. We denote by $W^{1,p}_\#(\Delta(A))$ the separated completion of $W^{1,p}(\Delta(A))/\mathbb{C}$ and by J the canonical mapping of $W^{1,p}(\Delta(A))/\mathbb{C}$ into its separated completion (see, e.g., chapter II of [6] and page 29 of [9]). $W^{1,p}_\#(\Delta(A))$ is a Banach space and $W^{1,2}_\#(\Delta(A))$ is a Hilbert space. Furthermore, as pointed out in [18], the distribution derivative ∂_i viewed as a mapping of $W^{1,p}(\Delta(A))/\mathbb{C}$ into $L^p(\Delta(A))$ extends to a unique continuous linear mapping, still denoted by ∂_i, of $W^{1,p}_\#(\Delta(A))$ into $L^p(\Delta(A))$ such that $\partial_i J(v) = \partial_i v$ for $v \in W^{1,p}(\Delta(A))/\mathbb{C}$ and

$$\|u\|_{W^{1,p}_\#(\Delta(A))} = \sum_{i=1}^N \|\partial_i u\|_{L^p(\Delta(A))} \text{ for } u \in W^{1,p}_\#(\Delta(A)).$$

To an H-structure Σ on $\mathbb{R}^N$ there are attached the important concepts of weak and strong Σ-convergence in L^p $(1 \le p < \infty)$ for which we refer to [18].

3.2. **The abstract structure hypothesis.** Let Σ_y and Σ_τ be two H-structures of class $\mathcal{C}^\infty$ on $\mathbb{R}^N_y$ and $\mathbb{R}_\tau$, respectively, and let $\Sigma = \Sigma_y \times \Sigma_\tau$ be their product, which is an H-structure of class $\mathcal{C}^\infty$ on $\mathbb{R}^N \times \mathbb{R}$. We introduce their respective images (i.e., the associated H-algebras) : $A_y = \mathcal{J}(\Sigma_y)$, $A_\tau = \mathcal{J}(\Sigma_\tau)$ and $A = \mathcal{J}(\Sigma)$. The same letter, $\mathcal{G}$, will denote the Gelfand transformation on A_y, A_τ, and A, as well. Points in $\triangle(A_y)$ (resp. $\triangle(A_\tau)$) are denoted by s (resp. s_0). The compact space $\triangle(A_y)$ (resp. $\triangle(A_\tau)$) is equipped with the M -measure, β_y (resp. β_τ), for A_y (resp. A_τ). We have $\triangle(A) = \triangle(A_y) \times \triangle(A_\tau)$ (Cartesian product) and the M-measure for A, with which $\triangle(A)$ is equipped, is precisely the product $\beta = \beta_y \otimes \beta_\tau$.

Now, let $1 \leq p < \infty$. Let Ξ^p denote the space of all $u \in L^p_{\mathrm{loc}}\left(\mathbb{R}^N_y \times \mathbb{R}_\tau\right)$ for which the sequence $(u^\varepsilon)_{0<\varepsilon\leq 1}$ [where $u^\varepsilon(x,t) = u\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right)$ $\left(x \in \mathbb{R}^N, t \in \mathbb{R}\right)$] is bounded in $L^p_{\mathrm{loc}}\left(\mathbb{R}^N_x \times \mathbb{R}_t\right)$. This is a Banach space with norm

$$\|u\|_{\Xi^p} = \sup_{0<\varepsilon\leq 1} \Big(\int_{B_{N+1}} \Big| u\Big(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\Big) \Big|^p dx\, dt \Big)^{1/p}$$

where B_{N+1} is the open unit ball in $\mathbb{R}^{N+1}$. Next, we define $\mathfrak{X}^p_\Sigma$ to be the closure of A in Ξ^p. We equip $\mathfrak{X}^p_\Sigma$ with the Ξ^p-norm, which makes it a Banach space. It is worth recalling that the Gelfand transformation $\mathcal{G} : A \to \mathcal{C}\left(\triangle(A)\right)$ extends by continuity to a continuous linear mapping, still denoted by $\mathcal{G}$, of $\mathfrak{X}^p_\Sigma$ into $L^p\left(\triangle(A)\right)$. This is referred to as the canonical mapping of $\mathfrak{X}^p_\Sigma$ into $L^p\left(\triangle(A)\right)$. We are now in a position to state the so-called abstract homogenization problem for (1.4). Let

$$A_{\mathbb{R}} = A \cap \mathcal{C}_{\mathbb{R}}\left(\mathbb{R}^N \times \mathbb{R}\right).$$

The main purpose of the present section is to investigate the limiting behaviour, as $\varepsilon \to 0$, of u_ε (the solution of (1.4)) under the *abstract structure hypothesis*

$$a_i\left(\cdot,\cdot,\Psi\right) \in \mathfrak{X}^{p'}_\Sigma \quad \text{for all } \Psi \in \left(A_{\mathbb{R}}\right)^N \ \ (1 \leq i \leq N) \tag{3.1}$$

with $2 \leq p < \infty$ and $p' = \frac{p}{p-1}$, where $a_i(\cdot,\cdot,\Psi)$ denotes the function $(y,\tau) \to a_i(y,\tau,\Psi(y,\tau))$ from $\mathbb{R}^N \times \mathbb{R}$ to $\mathbb{R}$, which belongs to $L^\infty_{\mathbb{R}}\left(\mathbb{R}^N \times \mathbb{R}\right)$ (see point (4.1) of [21]). The problem thus stated is precisely what is referred to as the *abstract homogenization problem* for (1.4) in a deterministic setting.

However, as will be seen later, one further assumption on Σ, the *quasi-properness* hypothesis, will be necessary to the resolution of the preceding abstract homogenization problem. Meanwhile, let us prove a few basic results we will need. In the sequel we assume that (3.1) holds. Thus, if $\Psi \in \left(A_{\mathbb{R}}\right)^N$, then $a_i\left(\cdot,\cdot,\Psi\right)$ lies in $\mathfrak{X}^{p',\infty}_\Sigma = \mathfrak{X}^{p'}_\Sigma \cap L^\infty_{\mathbb{R}}\left(\mathbb{R}^N \times \mathbb{R}\right)$. Consequently, $\mathcal{G}\left(a_i\left(\cdot,\cdot,\Psi\right)\right) \in L^\infty\left(\triangle(A)\right)$ [18, corollary 2.2]. With this in mind, let the index $1 \leq i \leq N$ be arbitrarily fixed. For $\varphi = \left(\varphi_j\right)_{1\leq j\leq N}$ in $\mathcal{C}_{\mathbb{R}}\left(\triangle(A)\right)^N$, let

$$b_i\left(\varphi\right) = \mathcal{G}\left(a_i\left(\cdot,\cdot,\mathcal{G}^{-1}\varphi\right)\right)$$

where $\mathcal{G}^{-1}\varphi = \left(\mathcal{G}^{-1}\varphi_j\right)_{1\leq j\leq N}$. This defines a transformation b_i of $\mathcal{C}_{\mathbb{R}}\left(\triangle(A)\right)^N$ into $L^\infty\left(\triangle(A)\right)$.

Proposition 3.1. *Let $2 \leq p < \infty$. Suppose (3.1) holds. For $\Psi = \left(\psi_j\right)_{1\leq j\leq N}$ in $\mathcal{C}(\overline{Q}; (A_{\mathbb{R}})^N)$, let $b_i(\widehat{\Psi}(x,t)) = \mathcal{G}\left(a_i\left(\cdot,\cdot,\Psi\left(x,t\right)\right)\right)$ for $(x,t) \in \overline{Q}$, where $\widehat{\Psi} = (\widehat{\psi}_j)_{1\leq j\leq N}$ with $\widehat{\psi}_j = \mathcal{G} \circ \psi_j$. This defines a mapping $(x,t) \to b_i(\widehat{\Psi}(x,t))$, still denoted by $b_i(\widehat{\Psi})$, of $\overline{Q}$ into $L^\infty(\triangle(A))$. The following assertions are true:*

(i) *We have $b_i(\widehat{\Psi}) \in \mathcal{C}(\overline{Q}; L^\infty(\triangle(A)))$ and*

$$a^\varepsilon_i\left(\cdot,\cdot,\Psi^\varepsilon\right) \to b_i(\widehat{\Psi}) \quad \text{in } L^{p'}(Q)\text{-weak } \Sigma \text{ as } \varepsilon \to 0, \tag{3.2}$$

where $\Psi^\varepsilon = (\psi^\varepsilon_j)_{1\leq j\leq N}$, ψ^ε_j defined as in (2.4).

(ii) *The mapping* $\Phi \to b(\Phi) = (b_i(\Phi))_{1\leq i\leq N}$ *of* $\mathcal{C}(\overline{Q};\mathcal{C}_{\mathbb{R}}(\triangle(A))^N)$ *into* $L^{p'}(Q \times \triangle(A))^N$ *extends by continuity to a mapping, still denoted by* b*, of the space* $L^p(Q;L^p_{\mathbb{R}}(\triangle(A))^N)$ *into* $L^{p'}(Q\times\triangle(A))^N$ *such that*

$$\|b(\mathbf{u}) - b(\mathbf{v})\|_{L^{p'}(Q\times\triangle(A))^N} \leq \alpha_1 \||\mathbf{u}| + |\mathbf{v}|\|^{p-2}_{L^p(Q\times\triangle(A))} \|\mathbf{u}-\mathbf{v}\|_{L^p(Q;L^p_{\mathbb{R}}(\triangle(A))^N)} \tag{3.3}$$

and

$$(b(\mathbf{u}) - b(\mathbf{v}))\cdot(\mathbf{u}-\mathbf{v}) \geq \alpha_0 |\mathbf{u}-\mathbf{v}|^p \text{ a.e. in } Q\times\triangle(A) \tag{3.4}$$

for all $\mathbf{u},\mathbf{v} \in L^p(Q;L^p_{\mathbb{R}}(\triangle(A))^N)$.

The proof of [21, Proposition 4.1] carries over mutatis mutandis to the present setting.

Remark 3.1. We have in particular

(1) $b(\omega) = \omega$
(2) $|b(\lambda) - b(\mu)| \leq \alpha_1(|\lambda| + |\mu|)^{p-2} |\lambda-\mu|$ $(\lambda,\mu\in\mathbb{R}^N)$
(3) $(b(\lambda) - b(\mu))\cdot(\lambda-\mu) \geq \alpha_0 |\lambda-\mu|^p$ $(\lambda,\mu\in\mathbb{R}^N)$.

As a consequence of Proposition 3.1, there is the following important corollary.

Corollary 3.1. *Let*

$$\Phi_\varepsilon = \psi_0 + \varepsilon\psi_1^\varepsilon, \tag{3.5}$$

i.e., $\Phi_\varepsilon(x,t) = \psi_0(x,t) + \varepsilon\psi_1(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon})$ *for* $(x,t)\in Q$*, where* $\psi_0 \in \mathcal{D}_{\mathbb{R}}(Q) = \mathcal{K}_{\mathbb{R}}(Q)\cap \mathcal{C}^\infty(Q)$ *and* $\psi_1 \in \mathcal{D}_{\mathbb{R}}(Q)\otimes A^\infty_{\mathbb{R}}$ *with* $A^\infty_{\mathbb{R}} = A^\infty \cap A_{\mathbb{R}}$*. Then, as* $\varepsilon\to 0$,

$$a_i^\varepsilon(\cdot,\cdot,D\Phi_\varepsilon) \to b_i(D\psi_0 + \partial\widehat{\psi}_1) \text{ in } L^{p'}(Q)\text{-weak } \Sigma \quad (1\leq i\leq N)$$

where ∂ *stands for the gradient operator on* $\Delta(A_y)$ *[specifically, we have here* $\partial\widehat{\psi}_1 = (\partial_j\widehat{\psi}_1)_{1\leq j\leq N}$ *with* $\partial_j\widehat{\psi}_1 = \partial_j\circ\widehat{\psi}_1$ *viewed as a function of* $\overline{Q}\times\Delta(A_\tau)$ *into* $\mathcal{D}(\Delta(A_y))$*, where* ∂_j *is the partial derivative of index* j *on* $\Delta(A_y)$*]. Furthermore, if* $(v_\varepsilon)_{\varepsilon\in E}$ *is a sequence in* $L^p(Q)$ *such that* $v_\varepsilon \to v_0$ *in* $L^p(Q)$*-weak* Σ *as* $E\ni\varepsilon\to 0$*, then, as* $E\ni\varepsilon\to 0$,

$$\int_Q a_i^\varepsilon(\cdot,\cdot,D\Phi_\varepsilon)\, v_\varepsilon dx dt \to \int\int_{Q\times\Delta(A)} b_i(D\psi_0 + \partial\widehat{\psi}_1) v_0 \; dx\, dt\, d\beta \quad (1\leq i\leq N).$$

The proof of this corollary is a simple adaptation of the proof of [21, Corollary 4.1].

3.3. **Quasi-proper H-structures.** The basic notation and hypotheses are as in the preceding subsection. Now, for $1\leq p<\infty$, we put

$$\mathcal{H} = L^p(\Delta(A_\tau); W^{1,p}_\#(\Delta(A_y);\mathbb{R})),$$

a Banach space with an obvious norm. The canonical mapping of $W^{1,p}(\Delta(A_y))/\mathbb{C}$ into its separated completion, $W^{1,p}_\#(\Delta(A_y))$, will be denoted by J_y.

Definition 3.1. The H-structure $\Sigma = \Sigma_y\times\Sigma_\tau$ is said to be quasi-proper for some real $p>1$ if the following two conditions are fulfilled:

(QP1) Σ_y is total for p, i.e., $\mathcal{D}(\Delta(A_y))$ is dense in $W^{1,p}(\Delta(A_y))$

(QP2) Given a bounded sequence $(u_\varepsilon)_{\varepsilon\in E}$ in V^p (see Remark 2.2), where E is a fundamental sequence, there exist a subsequence E' from E and some $\mathbf{u} = (u_0, u_1) \in V^p \times L^p(Q;\mathcal{H})$ such that, as $E' \ni \varepsilon \to 0$,

$$u_\varepsilon \to u_0 \quad \text{in } V^p\text{-weak} \tag{3.6}$$

$$\frac{\partial u_\varepsilon}{\partial x_j} \to \frac{\partial u_0}{\partial x_j} + \partial_j u_1 \ \ \textit{in } L^p(Q)\text{-weak } \Sigma \ \ (1 \le j \le N). \tag{3.7}$$

Remark 3.2. The partial derivative $\partial_j u_1$ in (3.7) needs an explanation. First, let us once for all keep in mind that for $1 \le j \le N$, the symbol ∂_j denotes the partial derivative of index j on $\Delta(A_y)$ whereas ∂_0 denotes the derivative on $\Delta(A_\tau)$. Now, let $1 \le j \le N$. It is to be noted that ∂_j yields a transformation, still denoted by ∂_j, that maps continuously and linearly $W^{1,p}_{\#}(\Delta(A_y))$ into $L^p(\Delta(A_y))$ and in particular $W^{1,p}_{\#}(\Delta(A_y);\mathbb{R})$ into $L^p_{\mathbb{R}}(\Delta(A_y))$ (see [21]). With this in mind, if $\Phi \in \mathcal{H}$, then $\partial_j\Phi$ is understood as $\partial_j \circ \Phi$ (usual composition). We have $\partial_j\Phi \in L^p_{\mathbb{R}}(\Delta(A))$, and the transformation $\Phi \to \partial_j\Phi$ maps continuously and linearly $\mathcal{H}$ into $L^p_{\mathbb{R}}(\Delta(A))$. Accordingly if $u_1 \in L^p(Q;\mathcal{H})$, then $\partial_j u_1$ is naturally defined as being the function $(x,t) \to \partial_j(u_1(x,t))$ from Q to $L^p_{\mathbb{R}}(\Delta(A))$. We have $\partial_j u_1 \in L^p_{\mathbb{R}}(Q\times\Delta(A))$, and the transformation $u_1 \to \partial_j u_1$ maps continuously and linearly $L^p(Q;\mathcal{H})$ into $L^p_{\mathbb{R}}(Q\times\Delta(A))$.

Remark 3.3. Let $E \ni \varepsilon \to 0$. In order that (3.6) hold, it is necessary and sufficient that we have $u_\varepsilon \to u_0$ in $L^p(0,T;W^{1,p}_0(\Omega))$-weak and $\frac{\partial u_\varepsilon}{\partial t} \to \frac{\partial u_0}{\partial t}$ in $L^{p'}(0,T;W^{-1,p'}(\Omega))$-weak.

3.4. **Homogenization results.** Throughout this subsection we assume that $2 \le p < \infty$ and the H-structure $\Sigma = \Sigma_y \times \Sigma_\tau$ is quasi-proper for p. In the sequel, the space $\mathbb{H} = L^p(0,T;W^{1,p}_0(\Omega;\mathbb{R})) \times L^p(Q;\mathcal{H})$ is equipped with the norm $\|\mathbf{v}\|_{\mathbb{H}} = \|v_0\|_{L^p(0,T;W^{1,p}_0(\Omega;\mathbb{R}))} + \|v_1\|_{L^p(Q;\mathcal{H})}$, $\mathbf{v} = (v_0, v_1) \in \mathbb{H}$, which makes it a Banach space. We will need the following lemma.

Lemma 3.1. $\mathfrak{F}^\infty_0 = \mathcal{D}_{\mathbb{R}}(Q) \times (\mathcal{D}_{\mathbb{R}}(Q) \otimes [\mathcal{D}_{\mathbb{R}}(\Delta(A_\tau)) \otimes J_y(\mathcal{D}_{\mathbb{R}}(\Delta(A_y))/\mathbb{C})])$ *is dense in* $L^p(0,T;W^{1,p}_0(\Omega;\mathbb{R})) \times L^p(Q;\mathcal{H})$.

Proof. In view of (QP1) (Definition 3.1), the space $\mathcal{D}_{\mathbb{R}}(\Delta(A_\tau)) \otimes J_y(\mathcal{D}_{\mathbb{R}}(\Delta(A_y))/\mathbb{C})$ is dense in $\mathcal{H}$ (use [21, Remark 3.5] and the fact that Σ_τ is of class $\mathcal{C}^\infty$). We deduce immediately that $\mathcal{D}_{\mathbb{R}}(Q) \otimes [\mathcal{D}_{\mathbb{R}}(\Delta(A_\tau)) \otimes J_y(\mathcal{D}_{\mathbb{R}}(\Delta(A_y))/\mathbb{C})]$ is dense in $L^p(Q;\mathcal{H})$. Hence, the lemma follows by the density of $\mathcal{D}_{\mathbb{R}}(Q)$ in $L^p(0,T;W^{1,p}_0(\Omega;\mathbb{R}))$. □

Remark 3.4. We have

$$\mathcal{D}_{\mathbb{R}}(\Delta(A_\tau)) \otimes [\mathcal{D}_{\mathbb{R}}(\Delta(A_y))/\mathbb{C}] = \mathcal{G}\left({}_{\mathbb{R}}A^\infty_\tau \otimes \left[{}_{\mathbb{R}}A^\infty_y/\mathbb{C}\right]\right)$$

where $\mathcal{G}$ is here the Gelfand transformation on A, and where ${}_{\mathbb{R}}A^\infty_\tau = A^\infty_\tau \cap \mathcal{C}_{\mathbb{R}}(\mathbb{R})$ and ${}_{\mathbb{R}}A^\infty_y/\mathbb{C} = \left\{\psi \in A^\infty_y \cap \mathcal{C}_{\mathbb{R}}(\mathbb{R}^N) : M(\psi) = 0\right\}$ (M denotes the mean value on $\mathbb{R}^N$ in the sense of [18, Subsection 2.1]).

Lemma 3.2. *The variational problem*

$$\mathbf{u} = (u_0, u_1) \in \mathbb{F}_0^{1,p} = V_0^p \times L^p(Q;\mathcal{H}) \quad :$$

$$\int_0^T (u_0'(t), v_0(t))dt + \int\int_{Q\times\Delta(A)} b(Du_0 + \partial u_1)\cdot(Dv_0 + \partial v_1)\,dx\,dt\,d\beta = \int_0^T (f(t), v_0(t))dt, \tag{3.8}$$

for all $\mathbf{v} = (v_0, v_1) \in \mathbb{F}_0^{1,p}$*, has at most one solution.*

The proof of this lemma follows in a quite classical way (use in particular (3.4) and $b(\omega) = \omega$). We are now in a position to state and prove the main result in the present section.

Theorem 3.1. *Let* $2 \le p < \infty$*. Suppose* (3.1) *holds and* $\Sigma = \Sigma_y \times \Sigma_\tau$ *is quasi-proper for p. For each fixed real number* $\varepsilon > 0$*, let* u_ε *be the solution of the initial-boundary value problem* (1.4)*. As* $\varepsilon \to 0$*, we have*

$$u_\varepsilon \to u_0 \quad \text{in } L^p(0,T;W_0^{1,p}(\Omega))\text{-weak} \tag{3.9}$$

$$\frac{\partial u_\varepsilon}{\partial t} \to \frac{\partial u_0}{\partial t} \quad \text{in } L^{p'}(0,T;W^{-1,p'}(\Omega))\text{-weak} \tag{3.10}$$

$$\frac{\partial u_\varepsilon}{\partial x_j} \to \frac{\partial u_0}{\partial x_j} + \partial_j u_1 \quad \text{in } L^p(Q)\text{-weak } \Sigma \quad (1 \le j \le N), \tag{3.11}$$

where $\mathbf{u} = (u_0, u_1)$ *is the unique solution of* (3.8)*.*

Proof. The first point is to check that the sequence $(u_\varepsilon)_{\varepsilon>0}$ is bounded in V^p. To this end, observe that $u_\varepsilon \in V_0^p$ (Remark 2.2) and

$$\int_0^T (u_\varepsilon'(t), v(t))dt + \int_Q a^\varepsilon(x,t,Du_\varepsilon(x,t))\cdot Dv(x,t)dx\,dt = \int_0^T (f(t), v(t))dt \tag{3.12}$$

for all $v \in V_0^p$, where $\varepsilon > 0$ is arbitrarily fixed. Taking in particular $v = u_\varepsilon$ and using

$$\int_0^T (u_\varepsilon'(t), u_\varepsilon(t))dt = \frac{1}{2}\,\|u_\varepsilon(T)\|^2_{L^2(\Omega)} \ge 0 \tag{3.13}$$

and (2.7)-(2.9), we obtain by mere routine

$$\sup_{\varepsilon>0} \|u_\varepsilon\|_{L^p(0,T;W_0^{1,p}(\Omega))} < \infty. \tag{3.14}$$

Using (2.7)-(2.8), once again, it follows

$$\sup_{\varepsilon>0} \|a^\varepsilon(\cdot,\cdot,Du_\varepsilon)\|_{L^{p'}(Q)^N} < \infty,$$

hence $\sup_{\varepsilon>0} \|\operatorname{div} a^\varepsilon(\cdot,\cdot,Du_\varepsilon)\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} < \infty$. We deduce by (1.4) that

$$\sup_{\varepsilon>0}\Big\|\frac{\partial u_\varepsilon}{\partial t}\Big\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} < \infty,$$

which combines with (3.14) to show that the sequence $(u_\varepsilon)_{\varepsilon>0}$ is bounded in V^p, hence also in V_0^p.

Thus, given an arbitrary fundamental sequence E, the quasi-properness of Σ (see especially (QP2)) guarantees the existence of a subsequence E' from E and of some $\mathbf{u} = (u_0, u_1) \in \mathbb{F}_0^{1,p} = V_0^p \times L^p(Q;\mathcal{H})$ such that as $E' \ni \varepsilon \to 0$, (3.9)-(3.11) hold true (see Remark 3.3). Therefore, thanks to Lemma 3.2, the theorem is proved once we have established that the vector function $\mathbf{u} = (u_0, u_1)$ satisfies the variational equation in (3.8) (the conclusive argument is classical; see, e.g., the proof of [21, Theorem 4.1]).

To do this, let $\Phi \in \mathfrak{F}_0^\infty$ (see Lemma 3.1), i.e., $\Phi = (\psi_0, J_y(\widehat{\psi}_1))$ with $\psi_0 \in \mathcal{D}_{\mathbb{R}}(Q)$, $\psi_1 \in \mathcal{D}_{\mathbb{R}}(Q) \otimes [{}_{\mathbb{R}}A_\tau^\infty \otimes ({}_{\mathbb{R}}A_y^\infty/\mathbb{C})]$ (see Remark 3.4), $\widehat{\psi}_1 = \mathcal{G} \circ \psi_1$ and $J_y(\widehat{\psi}_1) = J_y \circ \widehat{\psi}_1$ ($\widehat{\psi}_1$ viewed as a function of $\overline{Q} \times \Delta(A_\tau)$ into $\mathcal{D}(\Delta(A_y))/\mathbb{C}$). Define Φ_ε as in (3.5). Clearly $\Phi_\varepsilon \in \mathcal{D}_{\mathbb{R}}(Q)$. In (3.12), take $v = \Phi_\varepsilon$ and then use (2.9) to get

$$0 \le \int_0^T (f(t) - u_\varepsilon'(t), u_\varepsilon(t) - \Phi_\varepsilon(t))dt - \int_Q a^\varepsilon(\cdot,\cdot,D\Phi_\varepsilon)\cdot(Du_\varepsilon - D\Phi_\varepsilon)dxdt$$

or, according to (3.13),

$$\begin{aligned}\frac{1}{2}\|u_\varepsilon(T)\|^2_{L^2(\Omega)} \le \int_0^T (f(t), u_\varepsilon(t) - \Phi_\varepsilon(t))dt + \int_0^T (u_\varepsilon'(t), \Phi_\varepsilon(t))dt \\ - \int_Q a^\varepsilon(\cdot,\cdot,D\Phi_\varepsilon)\cdot(Du_\varepsilon - D\Phi_\varepsilon)dxdt\end{aligned} \tag{3.15}$$

and that for any $\varepsilon > 0$. Our goal now is to pass to the limit when $E' \ni \varepsilon \to 0$.

First, as $\varepsilon \to 0$, we have

$$\frac{\partial \Phi_\varepsilon}{\partial x_j} \to \frac{\partial \psi_0}{\partial x_j} + \partial_j \widehat{\psi}_1 \quad \text{in } L^q(Q)\text{-weak } \Sigma \quad (1 \le j \le N) \tag{3.16}$$

$$\frac{\partial \Phi_\varepsilon}{\partial t} \to \frac{\partial \psi_0}{\partial t} + \partial_0 \widehat{\psi}_1 \quad \text{in } L^q(Q)\text{-weak } \Sigma, \tag{3.17}$$

and that for any given $1 \le q < \infty$. Choosing in particular $q = p$ and using [18, Propositions 2.5 and 4.4], it follows that $\Phi_\varepsilon \to \psi_0$ in $W_0^{1,p}(Q)$-weak. Hence $\Phi_\varepsilon \to \psi_0$ in $L^p(0,T;W_0^{1,p}(\Omega))$-weak as $\varepsilon \to 0$, since $W_0^{1,p}(Q)$ is continuously embedded in $L^p(0,T;W_0^{1,p}(\Omega))$. Recalling (3.9) (when $E' \ni \varepsilon \to 0$), we finally arrive at $\int_0^T (f(t), u_\varepsilon(t) - \Phi_\varepsilon(t))dt \to \int_0^T (f(t), u_0(t) - \psi_0(t))dt$ when $E' \ni \varepsilon \to 0$. Next, observe that

$$\int_0^T (u_\varepsilon'(t), \Phi_\varepsilon(t))dt = -\int_Q u_\varepsilon \frac{\partial \Phi_\varepsilon}{\partial t} dxdt.$$

Thanks to the fact that V^p (for $2 \le p < \infty$) is compactly embedded in the space $L^p(0,T;L^2(\Omega))$ (this is a classical property; use, e.g., [16, p.58, Theorem 5.1]) and that the latter is continuously embedded in $L^2(Q)$, we have (from (3.9)-(3.10)) $u_\varepsilon \to u_0$ in $L^2(Q)$ as $E' \ni \varepsilon \to 0$. Combining this with (3.17) (for $q = 2$), it follows that

$$\int_0^T (u_\varepsilon'(t), \Phi_\varepsilon(t))dt \to \int_0^T (u_0'(t), \psi_0(t))dt \quad \text{as } E' \ni \varepsilon \to 0.$$

Now, based on (3.11) (when $E' \ni \varepsilon \to 0$, of course) and (3.16) (with $q = p$), a quick application of Corollary 3.1 yields

$$\int_Q a^\varepsilon(\cdot,\cdot,D\Phi_\varepsilon)\cdot(Du_\varepsilon - D\Phi_\varepsilon)dxdt \to \int\int_{Q\times\Delta(A)} b(\mathbb{D}\Phi)\cdot\mathbb{D}(\mathbf{u} - \Phi)\,dx\,dt\,d\beta$$

as $E' \ni \varepsilon \to 0$, where, for $\mathbf{v} = (v_0, v_1) \in L^p(0,T;\ W_0^{1,p}(\Omega)) \times L^p(Q;\mathcal{H})$, we denote $\mathbb{D}\mathbf{v} = Dv_0 + \partial v_1$ with $D = (D_{x_i})_{1\le i\le N}$ and $\partial = (\partial_i)_{1\le i\le N}$. Finally, as pointed out in Remark 2.2, the transformation $v \to \|v(T)\|^2_{L^2(\Omega)}$ is continuous on V_0^p. On the other hand, according to (3.9)-(3.10), we have $u_\varepsilon \to u_0$ in V_0^p-*weak* as $E' \ni \varepsilon \to 0$. Hence, by a classical argument it follows that

$$\|u_0(T)\|^2_{L^2(\Omega)} \leq \liminf_{E'\ni\varepsilon\to 0} \|u_\varepsilon(T)\|^2_{L^2(\Omega)}.$$

Therefore, taking the $\liminf_{E'\ni\varepsilon\to 0}$ of both sides of (3.15) and using

$$\frac{1}{2}\|u_0(T)\|^2_{L^2(\Omega)} = \int_0^T (u_0'(t), u_0(t))dt,$$

one arrives at

$$0 \leq \int_0^T (f(t) - u_0'(t), u_0(t) - \psi_0(t))dt - \int\int_{Q\times\Delta(A)} b(\mathbb{D}\Phi)\cdot\mathbb{D}(\mathbf{u}-\Phi)dx\,dt\,d\beta$$

and that for any $\Phi \in \mathfrak{F}_0^\infty$. Thanks to Lemma 3.1, this still holds true for $\Phi \in L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}))\times L^p(Q;\mathcal{H})$, hence for $\Phi \in \mathbb{F}_0^{1,p}$. Therefore the theorem follows by a classical line of reasoning (proceed as in the proof of [21, Theorem 4.1]). □

The variational problem (3.8) is called the global homogenized problem for (1.4) under the abstract structure hypothesis (3.1) with Σ quasi-proper (for the given real $p \geq 2$). The term *global* is used here to lay emphasis on the fact that (3.8) includes both the local (or microscopic) equation for $u_1(x,t)$ (where (x,t) is fixed in Q) and the macroscopic homogenized equation for u_0. Specifically, by choosing in (3.8) the test function $\mathbf{v} = (v_0, v_1)$ such that $v_0 = 0$ and $v_1(x,t) = \varphi(x,t)w$ with $\varphi \in \mathcal{D}_\mathbb{R}(Q)$ and $w \in \mathcal{H}$, we obtain the so-called local equation at (fixed) $(x,t) \in Q$:

$$\int_{\Delta(A)} b(Du_0(x,t) + \partial u_1(x,t))\cdot\partial w \;\; d\beta = 0 \;\text{ for all } w \in \mathcal{H}. \tag{3.18}$$

As regards the derivation of the macroscopic homogenized equation, let $r \in \mathbb{R}^N$ be freely fixed. Consider the so-called cell problem

$$\pi(r) \in \mathcal{H} :$$
$$\int_{\Delta(A)} b(r + \partial\pi(r))\cdot\partial w\,d\beta = 0 \quad \text{for all } w \in \mathcal{H}$$

which uniquely determines $\pi(r)$, thanks to Remark 3.1 (see [14, Chap.3]). Then, taking in particular $r = Du_0(x,t)$ with (x,t) arbitrarily fixed in Q, and comparing with (3.18), it follows at once

$$u_1 = \pi(Du_0) \tag{3.19}$$

where the right-hand side stands for the function $(x,t) \to \pi(Du_0(x,t))$ from Q to $\mathcal{H}$. Hence, substituting (3.19) in (3.8) and choosing there the test functions $\mathbf{v} = (v_0, v_1)$ such that $v_1 = 0$, we are led to the so-called macroscopic homogenized problem for (1.4), viz.

$$\begin{gathered} \frac{\partial u_0}{\partial t} - \operatorname{div} q(Du_0) = f \quad \text{in } Q \\ u_0 = 0 \quad \text{on } \partial\Omega\times(0,T) \\ u_0(x,0) = 0 \quad \text{in } \Omega, \end{gathered} \tag{3.20}$$

where $q(r) = \int_{\Delta(A)} b(r + \partial\pi(r))d\beta$ $(r \in \mathbb{R}^N)$.

Remark 3.5. A vector function $\mathbf{u} = (u_0, u_1)$ satisfies (3.8) if and only if the macroscopic component u_0 solves (3.20) and the microscopic component, $u_1(x,t)$, at a given point $(x,t) \in Q$ solves (3.18). Thanks to Lemma 3.2, this guarantees the uniqueness in (3.20).

Remark 3.6. We have $q(\omega) = 0$ and further it can be shown that the function $r \to q(r)$ satisfies inequalities of the same type *mutatis mutandis* as in Remark 3.1.

3.5. **Study of a concrete case. Harmonic H-structures.** We start with the following definition.

Definition 3.2. The H-structure (of class $\mathcal{C}^\infty$) Σ_y on $\mathbb{R}^N$ is termed p-harmonic (for some given $1 < p < \infty$) if the following conditions are satisfied:

(H1) Σ_y is total for p

(H2) To any $\mathbf{f} = (f_j)_{1\leq j\leq N} \in L^p(\Delta(A_y)^N$ satisfying

$$\int_{\Delta(A_y)} \mathbf{f}\cdot\widehat{\Psi} d\beta_y \equiv \sum_{j=1}^N \int_{\Delta(A_y)} f_j \widehat{\psi}_j d\beta_y = 0 \;\; (\text{with } \widehat{\psi}_j = \mathcal{G}(\psi_j)) \tag{3.21}$$

for all $\Psi = (\psi_j) \in \mathcal{V}_{div} = \{\mathbf{u} \in (A_y^\infty)^N : \operatorname{div}_y \mathbf{u} = 0\}$, there is attached a unique $\chi \in W_\#^{1,p}(\Delta(A_y))$ such that $f_j = \partial_j \chi$ $(1 \leq j \leq N)$.

We turn now to the proof of the following statement.

Proposition 3.2. *Suppose Σ_y is p-harmonic (for some given real $p > 1$). Then $\Sigma = \Sigma_y \times \Sigma_\tau$ is quasi-proper for p.*

Proof. We need verify only (QP2). So let $(u_\varepsilon)_{\varepsilon\in E}$ be a bounded sequence in V^p, E being fundamental. Based on the reflexivity of V^p and on the Σ-reflexivity of $L^p(Q)$ [18, Theorem 4.1], we can find a subsequence E' from E, a function $u_0 \in V^p$ and a family $(z_j)_{1\leq j\leq N} \subset L^p_{\mathbb{R}}(Q \times \Delta(A))$ such that as $E' \ni \varepsilon \to 0$, we have $u_\varepsilon \to u_0$ in V^p-*weak* and $\frac{\partial u_\varepsilon}{\partial x_j} \to z_j$ in $L^p(Q)$-*weak* Σ $(1 \leq j \leq N)$. Thus, the proposition is proved if we can establish that there is some function $u_1 \in L^p(Q; \mathcal{H})$ such that

$$z_j = \frac{\partial u_0}{\partial x_j} + \partial_j u_1 \quad (1 \leq j \leq N). \tag{3.22}$$

To do this, let $\Phi = (\phi_j)_{1\leq j\leq N}$, $\phi_j \in L^{p'}(Q; A)$, with

$$\phi_j(x,t,y,\tau) = \varphi(x,t)\psi_j(y)w(\tau) \quad ((x,t) \in Q,\ y \in \mathbb{R}^N,\ \tau \in \mathbb{R}),$$

where $\varphi \in \mathcal{D}(Q)$, $\Psi = (\psi_j) \in \mathcal{V}_{div}$ and $w \in A_\tau^\infty$. Clearly

$$\sum_{j=1}^N \int_Q \frac{\partial u_\varepsilon}{\partial x_j} \psi_j^\varepsilon w^\varepsilon \varphi dx\, dt = -\sum_{j=1}^N \int_Q u_\varepsilon \psi_j^\varepsilon w^\varepsilon \frac{\partial \varphi}{\partial x_j} dx\, dt.$$

Passing to the limit (as $E' \ni \varepsilon \to 0$) on both sides gives

$$\sum_{j=1}^N \int\int_{Q\times\Delta(A)} z_j \widehat{\psi}_j \widehat{w} \varphi\, dx\, dt\, d\beta = \sum_{j=1}^N \int\int_{Q\times\Delta(A)} \frac{\partial u_0}{\partial x_j} \widehat{\psi}_j \widehat{w} \varphi\, dx\, dt\, d\beta$$

where, regarding the right-hand side, we have used the facts that $u_\varepsilon \to u_0$ in $L^2(Q)$ as $E' \ni \varepsilon \to 0$ (see the proof of Theorem 3.1) and $\psi_j^\varepsilon w^\varepsilon \to \int_{\Delta(A)} \widehat{\psi}_j \widehat{w} d\beta$ in $L^2(Q)$-weak as $\varepsilon \to 0$. Using first the arbitrariness of φ and then that of w, we quickly arrive at (3.21) for all $\Psi \in \mathcal{V}_{div}$, where

$$f_j(s) = z_j(x,t,s,s_0) - \frac{\partial u_0}{\partial x_j}(x,t) \quad (s \in \Delta(A_y)),$$

$(x,t) \in Q$ and $s_0 \in \Delta(A_\tau)$ being fixed. Thanks to the p-harmonicity of Σ_y, this yields a function $u_1 \in L^p(Q;\mathcal{H})$ such that (3.22) holds (to show this is an easy matter), as claimed. □

This is worth illustrating the results above.

Example 3.1. Suppose Σ_y is an almost periodic H-structure on $\mathbb{R}^N$ [18, Example 3.3]. Then $\Sigma = \Sigma_y \times \Sigma_\tau$ (where Σ_τ is any H-structure of class $\mathcal{C}^\infty$ on $\mathbb{R}$) is quasi-proper for $p = 2$. Indeed, Σ_y is 2-harmonic (this is established in a preprint by the first author) and so the claimed property follows by Proposition 3.2.

Example 3.2. Suppose Σ_y is the periodic H-structure on $\mathbb{R}^N$ represented by a network $\mathcal{R} \subset \mathbb{R}^N$, say $\mathcal{R} = \mathbb{Z}^N$ (see [18, Example 3.2]). Then Σ_y *is* p-harmonic for any real $p > 1$ (see [21, Subsection 3.3]). Consequently, according to Proposition 3.2, the H-structure $\Sigma = \Sigma_y \times \Sigma_\tau$ (where Σ_τ is an arbitrary H-structure of class $\mathcal{C}^\infty$ on $\mathbb{R}$) is quasi-proper for any real $p > 1$.

4. Concrete homogenization problems for (1.4)

This section provides concrete examples of homogenization problems for (1.4). More precisely, we study here the limiting behaviour, as $\varepsilon \to 0$, of u_ε (the solution of (1.4)) under various *concrete* structure hypotheses. It should be noted that in practice the statement of a homogenization problem makes no mention of the concept of a homogenization structure, still less of that of a quasi-proper H-structure. The term *concrete* used above is precisely intended to stress this fact, as opposed to the abstract nature of (3.1).

In fact, in view of the fundamental results achieved in the preceding section, our only concern in each example under consideration below will be to show that the concrete structure hypothesis supplementing (1.4) (so as to yield a solvable homogenization problem) can be reduced to (3.1) for a suitable quasi-proper H-structure Σ. This is the general point of view. We will see that the particular case where the diffusion term in (1.4) is linear entails considerable simplifications with regard to practice.

4.1. **General case.**

Just as in the preceding subsections, it is not specified here whether the diffusion term in (1.4) is linear or nonlinear.

Problem I. (Periodic setting) As we mentioned in Section 1, the homogenization of (1.4) under the periodicity hypothesis has been sufficiently investigated. We will only draw attention to the fact that the present study includes the periodic setting. Indeed, suppose for each fixed $\lambda \in \mathbb{R}^N$, the function $(y,\tau) \to a(y,\tau,\lambda)$ is Y-periodic in $y \in \mathbb{R}^N$ and Z-periodic in $\tau \in \mathbb{R}$ with, e.g., $Y = (0,1)^N$ and $Z = (0,1)$. It amounts to saying that for any $k \in \mathcal{R} = \mathbb{Z}^N$ and any $l \in \mathcal{S} = \mathbb{Z}$, we have $a(y+k,\tau+l,\lambda) = a(y,\tau,\lambda)$ a.e. in $(y,\tau) \in \mathbb{R}^N \times \mathbb{R}$. Immediately we see that the appropriate homogenization structures are the periodic H-structures $\Sigma_y = \Sigma_{\mathcal{R}}$ and $\Sigma_\tau = \Sigma_{\mathcal{S}}$ represented by $\mathcal{R} = \mathbb{Z}^N$ and $\mathcal{S} = \mathbb{Z}$, respectively (see [18, Example 3.2]). In other words, in the present case we have $A_y = \mathcal{C}_{\rm per}(Y)$, $A_\tau = \mathcal{C}_{\rm per}(Z)$, and hence $A = \mathcal{C}_{\rm per}(Y \times Z)$. Then, as pointed out in Example 3.2, the product homogenization structure $\Sigma = \Sigma_y \times \Sigma_\tau$ is quasi-proper for any $1 < p < \infty$. Hence, the results of Subsection 3.4 (and especially the conclusion of Theorem 3.1) are valid, as claimed. Finally, for the sake of exactness, we should change there $\Delta(A_y)$ (resp. $\Delta(A_\tau)$, $\Delta(A)$) in Y, (resp. Z, $Y \times Z$), ∂_j in D_{y_j} $(1 \leq j \leq N)$, ∂_0 in $\frac{d}{d\tau}$, and β_y (resp. β_τ, β) in dy (resp. $d\tau, dyd\tau$).

Problem II. (Almost periodic setting) For the benefit of the reader we begin by recalling the notion of an almost periodic function [4, 11, 15] we will be dealing with. By an almost periodic continuous complex function on $\mathbb{R}^d$ $(d \geq 1)$ is meant a function $u \in \mathcal{B}(\mathbb{R}^d)$ whose translates $\{\tau_h u\}_{h\in\mathbb{R}^d}$ (with $\tau_h u(y) = u(y-h)$, $y \in \mathbb{R}^d$) form a relatively compact set in $\mathcal{B}(\mathbb{R}^d)$. Such functions form a closed subalgebra of $\mathcal{B}(\mathbb{R}^d)$ denoted by $AP(\mathbb{R}^d)$. This fundamental notion, which is due to Bohr [5], has been generalized to L^p_{loc} spaces. A function $u \in L^p_{\text{loc}}(\mathbb{R}^d)$ $(1 \leq p < \infty)$ is said to be almost periodic in Stepanoff sense if u lies in the amalgam space $(L^p, l^\infty)(\mathbb{R}^d)$ [10, 20] and further the translates $\{\tau_h u\}_{h\in\mathbb{R}^d}$ form a relatively compact set in $(L^p, l^\infty)(\mathbb{R}^d)$. Such functions form a closed vector subspace of $(L^p, l^\infty)(\mathbb{R}^d)$ denoted by $L^p_{AP}(\mathbb{R}^d)$. It seems useful to recall that $(L^p, l^\infty)(\mathbb{R}^d)$ is the space of functions $u \in L^p_{\text{loc}}(\mathbb{R}^d)$ such that

$$\|u\|_{p,\infty} = \sup_{k\in\mathbb{Z}^d} \Big(\int_{k+(0,1)^d} |u(y)|^p \, dy \Big)^{1/p} < \infty.$$

Equipped with the norm $\|\cdot\|_{p,\infty}$, $(L^p, l^\infty)(\mathbb{R}^d)$ is a Banach space. The appropriate norm on $L^p_{AP}(\mathbb{R}^d)$ is the $(L^p, l^\infty)(\mathbb{R}^d)$-norm. It is also worth noting that $AP(\mathbb{R}^d)$ is a dense vector subspace of $L^p_{AP}(\mathbb{R}^d)$.

Now, let M denote the mean value on $\mathbb{R}^d$ as stated in Subsection 3.1. Considered as defined on $AP(\mathbb{R}^d)$, the mapping M extends to a continuous linear form on $L^p_{AP}(\mathbb{R}^d)$ still denoted by M. This follows immediately by the inequality

$$\Big(\int_B \Big| u\Big(\frac{x}{\varepsilon}\Big) \Big|^p dx \Big)^{1/p} \leq c \|u\|_{p,\infty} \quad (0 < \varepsilon \leq 1) \tag{4.1}$$

for all $u \in (L^p, l^\infty)(\mathbb{R}^d)$, where B is a fixed bounded open set in $\mathbb{R}^d$ and c is a positive constant independent of both ε and u.

Now, given a countable subgroup $\mathcal{R}$ of $\mathbb{R}^d$, we will put

$$AP_{\mathcal{R}}(\mathbb{R}^d) = \{u \in AP(\mathbb{R}^d) : Sp(u) \subset \mathcal{R}\}$$
$$L^p_{AP,\mathcal{R}}(\mathbb{R}^d) = \{u \in L^p_{AP}(\mathbb{R}^d) : Sp(u) \subset \mathcal{R}\}$$

where $Sp(u)$ stands for the spectrum of u, i.e., $Sp(u) = \{k \in \mathbb{R}^d : M(u\overline{\gamma}_k) \neq 0\}$ with $\gamma_k(y) = \exp(2i\pi k\cdot y)$ $(y \in \mathbb{R}^d)$. Equipped with the (L^p, l^∞)-norm, $L^p_{AP,\mathcal{R}}(\mathbb{R}^d)$ is a Banach space. As regards $AP_{\mathcal{R}}(\mathbb{R}^d)$, this is a homogenization algebra on $\mathbb{R}^d$, the associated H-structure being the so-called almost periodic H-structure represented by $\mathcal{R}$ [18, Examples 2.2 and 3.3]. It is worth knowing that $AP_{\mathcal{R}}(\mathbb{R}^d)$ is dense in $L^p_{AP,\mathcal{R}}(\mathbb{R}^d)$. Indeed, given $u \in L^p_{AP,\mathcal{R}}(\mathbb{R}^d)$ $(1 \leq p < \infty)$, the same procedure as followed in the proof of [21, Proposition 3.2] leads to

$$\|u * \theta_n - u\|^p_{p,\infty} \leq \int \theta_n(x) \|u - \tau_x u\|^p_{p,\infty} \, dx \quad (\text{integers } n \geq 1),$$

where $\theta_n \in \mathcal{D}(\mathbb{R}^d)$ is a mollifier. By using the fact that $\tau_x u \to u$ in $(L^p, l^\infty)(\mathbb{R}^d)$ as $|x| \to 0$ (there is no serious difficulty in verifying this), we deduce that $u * \theta_n \to u$ in $(L^p, l^\infty)(\mathbb{R}^d)$ as $n \to \infty$. But $u * \theta_n \in AP(\mathbb{R}^d)$ and $M([u * \theta_n]\overline{\gamma}_k) =$

$M(u\overline{\gamma}_k)\mathcal{F}\theta_n(-k)$, where $\mathcal{F}$ denotes the Fourier transformation on $\mathbb{R}^d$. Hence $u*\theta_n \in AP_{\mathcal{R}}(\mathbb{R}^d)$ and so the claimed result follows.

Having made this point, let us turn now to one fundamental result.

Proposition 4.1. *Let $(f_i)_{i\in I}$ be a countable family in $L^p_{AP}(\mathbb{R}^d)$. There exists a countable subgroup $\mathcal{R}$ of $\mathbb{R}^d$ such that $f_i \in L^p_{AP,\mathcal{R}}(\mathbb{R}^d)$ for all $i \in I$.*

Proof. Indeed, the set $\mathcal{U} = \cup_{i\in I} Sp(f_i)$ is countable, since I and $Sp(f_i)$ (for each fixed $i \in I$) are countable. Therefore, the set $\mathcal{R}$ of finite combinations $\sum_{\text{finite}} t_i k_i$ ($t_i \in \mathbb{Z}$, $k_i \in \mathcal{U}$) is countable. But $\mathcal{R}$ is a subgroup of $\mathbb{R}^d$ and $Sp(f_i) \subset \mathcal{R}$ for all $i \in I$. Hence the proposition follows. □

As a consequence of this, we have the following result.

Corollary 4.1. *Let X be a separable metric space and let $(\varphi_i)_{i\in I} \subset \mathcal{C}(X; L^p_{AP}(\mathbb{R}^d))$, where the index set I is countable. Then, there is some countable subgroup $\mathcal{R}$ of $\mathbb{R}^d$ such that $\varphi_i \in \mathcal{C}(X; L^p_{AP,\mathcal{R}}(\mathbb{R}^d))$ for every $i \in I$.*

Proof. Let D be a dense countable set in X. Thanks to Proposition 4.1, there is a countable subgroup $\mathcal{R}$ of $\mathbb{R}^d$ such that $\varphi_i(\zeta) \in L^p_{AP,\mathcal{R}}(\mathbb{R}^d)$ for all $i \in I$ and all $\zeta \in D$. Now let $i \in I$ be fixed. Fix also some $x \in X$. Let $\eta > 0$. By the continuity of φ and the density of D in X, we may consider some $\zeta \in D$ such that $\|\varphi_i(x) - \varphi_i(\zeta)\|_{p,\infty} \leq \frac{\eta}{c}$ where c is a positive constant such that $|M(u)| \leq c\,\|u\|_{p,\infty}$ ($u \in L^p_{AP}(\mathbb{R}^d)$). It follows that $|M(\varphi_i(x)\overline{\gamma}_k) - M(\varphi_i(\zeta)\overline{\gamma}_k)| \leq \eta$ for all $k \in \mathbb{R}^d$. But $M(\varphi_i(\zeta)\overline{\gamma}_k) = 0$ for all $k \in \mathbb{R}^d\backslash\mathcal{R}$. By the arbitrariness of η we deduce that $M(\varphi_i(x)\overline{\gamma}_k) = 0$ for all $k \in \mathbb{R}^d\backslash\mathcal{R}$. Hence $\varphi_i(x) \in L^p_{AP,\mathcal{R}}(\mathbb{R}^d)$. This completes the proof. □

We are now in a position to study the almost periodic homogenization of (1.4).

Example 4.1. Our goal here is to investigate the limiting behaviour, as $\varepsilon \to 0$, of u_ε, the solution of (1.4) for $p = 2$, under the structure hypothesis

$$a_i(\cdot,\cdot,\lambda) \in L^2_{AP}(\mathbb{R}^{N+1}) \quad \text{for fixed } \lambda \in \mathbb{R}^N \quad (1 \leq i \leq N). \tag{4.2}$$

According to Theorem 3.1, this homogenization problem is quite solvable and the results are available in Subsection 3.4 if we can find a suitable quasi-proper H-structure $\Sigma = \Sigma_y \times \Sigma_\tau$ for $p = 2$ such that (3.1) holds for $p = 2$. To achieve this, we shall require the following property: For $\Psi \in AP(\mathbb{R}^{N+1};\mathbb{R})^N$, we have

$$\sup_{k\in\mathbb{Z}^{N+1}} \int_{k+Z} |a(y-r,\tau-\sigma,\Psi(y,\tau)) - a(y,\tau,\Psi(y,\tau))|^2\, dyd\tau \to 0 \tag{4.3}$$

as $|r| \to 0$ and $\sigma \to 0$, where $Z = (0,1)^{N+1}$.

Remark 4.1. Condition (4.3) is satisfied if the following condition holds: For each bounded set $\Lambda \subset \mathbb{R}^N$ and each real $\eta > 0$, there exists a real $\rho > 0$ such that

$$|a(y-r,\tau-\sigma,\lambda) - a(y,\tau,\lambda)| \leq \eta \tag{4.4}$$

for all $\lambda \in \Lambda$ and for almost all $(y,\tau) \in \mathbb{R}^{N+1}$ provided $|r| + |\sigma| \leq \rho$.

Indeed, if (4.4) holds and if Ψ is given in $AP(\mathbb{R}^{N+1};\mathbb{R})^N$, then by choosing $\Lambda = \Psi(\mathbb{R}^{N+1})$ (range of Ψ) we get at once (4.3).

This being so, let $(\theta_n)_{n\geq 1}$ be a sequence with $\theta_n \in \mathcal{D}_{\mathbb{R}}(\mathbb{R}^{N+1}), \theta_n \geq 0$, $\operatorname{Supp}\theta_n \subset \frac{1}{n}\overline{B}_{N+1}$ (B_{N+1} the open unit ball of $\mathbb{R}^{N+1}$, $\overline{B}_{N+1}$ its closure) and $\int \theta_n(y,\tau)dyd\tau = 1$. Let

$$\zeta_n^i(y,\tau,\lambda) = \int \theta_n(r,\sigma)a_i(y-r,\tau-\sigma,\lambda)drd\sigma \quad (1 \leq i \leq N)$$

for $\lambda, y \in \mathbb{R}^N$ and $\tau \in \mathbb{R}$, which defines a function $(y,\tau,\lambda) \to \zeta_n^i(y,\tau,\lambda)$ of $\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$ into $\mathbb{R}$. Clearly $\zeta_n^i(\cdot,\cdot,\lambda) \in AP(\mathbb{R}^{N+1})$ for each $\lambda \in \mathbb{R}^N$, and further $|\zeta_n(y,\tau,\lambda) - \zeta_n(y,\tau,\mu)| \leq \alpha_1 |\lambda - \mu|$ for all $\lambda, \mu, y \in \mathbb{R}^N$ and all $\tau \in \mathbb{R}$, where $\zeta_n = (\zeta_n^i)_{1\leq i\leq N}$. Now, thanks to Corollary 4.1, there exists a countable subgroup R of $\mathbb{R}^{N+1}$ such that $\zeta_n^i(\cdot,\cdot,\lambda) \in AP_R(\mathbb{R}^{N+1})$ $(1 \leq i \leq N)$ for all $\lambda \in \mathbb{R}^N$ and all integers $n \geq 1$. Let $\mathcal{R}_y = pr_y(R)$ and $\mathcal{R}_\tau = pr_\tau(R)$, where pr_y (resp. pr_τ) stands for the natural projection of $\mathbb{R}^{N+1} = \mathbb{R}_y^N \times \mathbb{R}_\tau$ onto $\mathbb{R}_y^N$ (resp. $\mathbb{R}_\tau$). The set $\mathcal{R}_y$ (resp. $\mathcal{R}_\tau$) is a countable subgroup of $\mathbb{R}^N$ (resp. $\mathbb{R}$). Therefore $\mathcal{R} = \mathcal{R}_y \times \mathcal{R}_\tau$ is a countable subgroup of $\mathbb{R}^{N+1}$ with moreover $R \subset \mathcal{R}$. Hence

$$\zeta_n^i(\cdot,\cdot,\lambda) \in A = AP_{\mathcal{R}}(\mathbb{R}^{N+1}) \quad (\lambda \in \mathbb{R}^N,\ n \in \mathbb{N}^*,\ 1 \leq i \leq N) \tag{4.5}$$

and

$$\Sigma_{\mathcal{R}} = \Sigma_{\mathcal{R}_y} \times \Sigma_{\mathcal{R}_\tau} \quad \text{(see [18, Example 3.6])} \tag{4.6}$$

where $\Sigma_{\mathcal{R}}$ (resp. $\Sigma_{\mathcal{R}_y}, \Sigma_{\mathcal{R}_\tau}$) is the almost periodic H-structure on $\mathbb{R}^{N+1}$ (resp. $\mathbb{R}^N$, $\mathbb{R}$) represented by $\mathcal{R}$ (resp. $\mathcal{R}_y$, $\mathcal{R}_\tau$). Recalling that $\Sigma_{\mathcal{R}}$ is quasi-proper for $p = 2$ (see Example 3.1), we see that the problem under consideration is completely solved if we show that (3.1) holds with $\Sigma = \Sigma_{\mathcal{R}}$ and $p = 2$. To this end, starting from (4.5) and following the same line of reasoning as in [21, Subsection 5.5] leads to $\zeta_n^i(\cdot,\cdot,\Psi) \in A$ for all $\Psi \in (A_{\mathbb{R}})^N$ $(n \in \mathbb{N}^*,\ 1 \leq i \leq N)$. On the other hand, by an obvious adaptation of the procedure in [21, Subsection 5.6] one quickly arrives at the following result :

> Given $\Psi \in (A_{\mathbb{R}})^N$ and $1 \leq i \leq N$, to each $\eta > 0$ there is assigned some integer $\nu \geq 1$ such that $\left\|\zeta_n^i(\cdot,\cdot,\Psi) - a_i(\cdot,\cdot,\Psi)\right\|_{2,\infty} \leq \eta$ for all $n \geq \nu$.

Since $(L^2, l^\infty)(\mathbb{R}^{N+1})$ is continuously embedded in $\Xi^2(\mathbb{R}^{N+1})$ (this follows immediately by (4.1)), the desired result follows from all that.

Remark 4.2. If instead of (4.2) we consider the structure hypothesis:

$$a_i(\cdot,\cdot,\lambda) \in AP(\mathbb{R}^{N+1}) \quad \text{for fixed } \lambda \in \mathbb{R}^N \quad (1 \leq i \leq N),$$

then (4.3) may be disregarded. Indeed, proceeding directly as in [21, Subsection 5.5] we arrive at $a_i(\cdot,\cdot,\Psi) \in A$ for all $\Psi \in (A_{\mathbb{R}})^N$ $(1 \leq i \leq N)$, which leads at once to (3.1) with $\Sigma_{\mathcal{R}}$ as in (4.6), and with $p = 2$, of course.

Problem III. The present problem deals with two closely connected examples.

Example 4.2. We assume here that the family $\{a(\cdot,\cdot,\lambda)\}_{\lambda\in\mathbb{R}^N}$ satisfies the condition

(BUE) For each bounded set $\Lambda \subset \mathbb{R}^N$ and each real $\eta > 0$, there exists a real $\rho > 0$ such that $|a(y-r,\tau-\sigma,\lambda) - a(y,\tau,\lambda)| \leq \eta$ for all $\lambda \in \Lambda$ and all $(y,\tau) \in \mathbb{R}^N \times \mathbb{R}$ provided $|r| + |\sigma| \leq \rho$.

Remark 4.3. Condition (BUE) is more practical than its analog (UE) in [21, Subsection 5.4]. In fact, [21, Proposition 5.2] and its proof remain unchanged if the sole points λ considered in (UE) are those lying in an arbitrarily fixed bounded set $\Lambda \subset \mathbb{R}^N$. This remark carries over *mutatis mutandis* to [21, Subsection 5.7].

Assuming (BUE), we want to study the homogenization of (1.4) (for any given real $p \geq 2$) under the structure hypothesis

$$a_i(\cdot,\cdot,\lambda) \in \mathcal{B}_\infty(\mathbb{R};\mathcal{C}_{\mathrm{per}}(Y)) \text{ for any } \lambda \in \mathbb{R}^N \quad (1 \leq i \leq N) \tag{4.7}$$

where $Y = (0,1)^N$. We recall that $\mathcal{C}_{\mathrm{per}}(Y)$ denotes the space of continuous complex functions on $\mathbb{R}^N$ that are Y-periodic (i.e., that satisfy $f(y+k) = f(y)$ for all $y \in \mathbb{R}^N$and all $k \in \mathbb{Z}^N$), and $\mathcal{B}_\infty(\mathbb{R};\mathcal{C}_{\mathrm{per}}(Y))$ denotes the space of those $f \in \mathcal{C}(\mathbb{R};\mathcal{C}_{\mathrm{per}}(Y))$ such that $f(\tau)$ has a limit in $\mathcal{B}(\mathbb{R}^N)$ when $|\tau| \to \infty$. Now, let $\Sigma_{\mathbb{Z}^N}$ be the periodic H-structure on $\mathbb{R}^N$ represented by the network $\mathbb{Z}^N$, and Σ_∞ be the H-structure on $\mathbb{R}$ of which $\mathcal{B}_\infty(\mathbb{R})$ is the image (see [18, Example 3.4]). The product H-structure $\Sigma = \Sigma_{\mathbb{Z}^N} \times \Sigma_\infty$ on $\mathbb{R}^N \times \mathbb{R}$ is quasi-proper for any $1 < p < \infty$ (see Example 3.2) and its image is precisely $A = \mathcal{B}_\infty(\mathbb{R};\mathcal{C}_{\mathrm{per}}(Y))$ (see [18, Proposition 3.3]). Thus, the present study falls under the framework of Section 3 provided it is shown that (3.1) holds true for the above H-structure. Clearly it suffices to check that $a_i(\cdot,\cdot,\Psi) \in A$ for all $\Psi \in (A_{\mathbb{R}})^N$ $(1 \leq i \leq N)$. But this follows by proceeding exactly as in the proof of [21, Proposition 5.2].

Example 4.3. Assuming here that (4.4) holds true, let us consider the homogenization problem for (1.4) (for any real $p \geq 2$) under the structure hypothesis

$$a_i(\cdot,\cdot,\lambda) \in \mathcal{B}_\infty(\mathbb{R};L^\infty_{\mathrm{per}}(Y)) \textit{ for any } \lambda \in \mathbb{R}^N \quad (1 \leq i \leq N). \tag{4.8}$$

By a simple adaptation of [21, Subsection 5.7] one is led to (3.1) with $\Sigma = \Sigma_{\mathbb{Z}^N} \times \Sigma_\infty$ as above. Hence, thanks to Theorem 3.1, the same conclusion as above follows.

4.2. **The linear case.** In this subsection we assume that the function $\lambda \to a(\cdot,\cdot,\lambda)$ of $\mathbb{R}^N$ into itself is linear. Then, there is a family $\{a_{ij}\}_{1\leq i,j\leq N}$, $a_{ij} \in L^\infty_{\mathbb{R}}(\mathbb{R}^N \times \mathbb{R})$ (thanks to (1.1)-(1.2) and part (ii) of (1.3)), such that

$$a_i(y,\tau,\lambda) = \sum_{j=1}^N a_{ij}(y,\tau)\lambda_j \quad \text{for all } \lambda \in \mathbb{R}^N \quad (1 \leq i \leq N).$$

In the sequel we suppose $p = 2$. Now, it is clear that the results obtained in Subsection 4.1 remain valid in the present case. In addition, by turning the linearity to good account we can get round technical difficulties and thus, with the help of further concrete examples, point out the wide scope of Theorem 3.1.

This being so, it is immediate that (3.1) holds true if and only if

$$a_{ij} \in \mathfrak{X}^2_\Sigma \quad (1 \leq i,j \leq N). \tag{4.9}$$

Thus, in the sequel, the aim will be to reduce to (4.9) the concrete examples under consideration.

Problem IV. Our purpose here is to study the homogenization of (1.4) under the structure hypothesis

$$a_{ij} \in F_{\infty,AP} \quad (1 \leq i,j \leq N) \tag{4.10}$$

where $F_{\infty,AP}$ denotes the closure of $\mathcal{B}_\infty(\mathbb{R};AP(\mathbb{R}^N))$ in $(L^2,l^\infty)(\mathbb{R}^{N+1})$. To this end, let us consider

$$\zeta_{nij} \in \mathcal{B}_\infty(\mathbb{R};AP(\mathbb{R}^N)) \quad (n \in \mathbb{N},\ 1 \leq i,j \leq N)$$

such that as $n \to \infty$, $\zeta_{nij} \to a_{ij}$ in $(L^2,l^\infty)(\mathbb{R}^{N+1})$ for $1 \leq i,j \leq N$. According to Corollary 4.1, there exists a countable subgroup $\mathcal{R}$ of $\mathbb{R}^N$ such that $\zeta_{nij} \in$

$\mathcal{B}_\infty(\mathbb{R}; AP_\mathcal{R}(\mathbb{R}^N))$ for all integers $n \in \mathbb{N}$ and all indices $1 \leq i, j \leq N$. Let $\Sigma = \Sigma_\mathcal{R} \times \Sigma_\infty$, where $\Sigma_\mathcal{R}$ is the almost periodic H -structure on $\mathbb{R}^N$ represented by $\mathcal{R}$ and Σ_∞ is the H-structure on $\mathbb{R}$ introduced in Example 4.2. The H-structure Σ on $\mathbb{R}^N \times \mathbb{R}$ is quasi-proper for $p = 2$ (Example 3.1) and its image is precisely the H-algebra $A = \mathcal{B}_\infty(\mathbb{R}; AP_\mathcal{R}(\mathbb{R}^N))$ [18, Proposition 3.3]. Therefore, the homogenization problem before us is solved through Theorem 3.1 if we can check that (4.9) holds for the preceding H-structure. But this follows immediately by the fact (already pointed out before) that $(L^2, l^\infty)(\mathbb{R}^{N+1})$ is continuously embedded in $\Xi^2(\mathbb{R}^{N+1})$. Let us illustrate this.

Example 4.4. The structure hypothesis (generalizing (4.7)) $a_{ij} \in \mathcal{B}_\infty(\mathbb{R}; AP(\mathbb{R}^N))$ ($n \in \mathbb{N}$, $1 \leq i, j \leq N$) reduces to (4.10). The same is true of the structure hypothesis (generalizing (4.8)) $a_{ij} \in \mathcal{B}_\infty(\mathbb{R}; L^2_{AP}(\mathbb{R}^N))$ ($1 \leq i, j \leq N$). Indeed,the first assertion is evident. Regarding the next one, observe that $\mathcal{B}_\infty(\mathbb{R}; AP(\mathbb{R}^N))$ is dense in $\mathcal{B}_\infty(\mathbb{R}; L^2_{AP}(\mathbb{R}^N))$ provided with the $\mathcal{B}(\mathbb{R}; (L^2, l^\infty)(\mathbb{R}^N))$-norm, and the latter is continuously embedded in $(L^2, l^\infty)(\mathbb{R}^{N+1})$.

Example 4.5. The structure hypothesis $a_{ij} \in L^2(\mathbb{R}; L^2_{AP}(\mathbb{R}^N))$ ($1 \leq i, j \leq N$) reduces to (4.10). Indeed, $\mathcal{K}(\mathbb{R}; AP(\mathbb{R}^N))$ (a subspace of $\mathcal{B}_\infty(\mathbb{R}; AP(\mathbb{R}^N))$) is dense in $L^2(\mathbb{R}; L^2_{AP}(\mathbb{R}^N))$ and the latter is continuously embedded in $(L^2, l^\infty)(\mathbb{R}^{N+1})$.

Example 4.6. Suppose our goal is to study the homogenization of (1.4) under the following structure hypothesis, where the two indices $1 \leq i, j \leq N$ are arbitrarily fixed:

(1) The function $\tau \to a_{ij}(\cdot, \tau)$ maps continuously $\mathbb{R}$ into $(L^2, l^\infty)(\mathbb{R}^N)$
(2) As $|\tau| \to \infty$, $a_{ij}(\cdot, \tau)$ has a limit in $(L^2, l^\infty)(\mathbb{R}^N)$
(3) For each fixed $\tau \in \mathbb{R}$, the function $a_{ij}(\cdot, \tau)$ is Y_τ-periodic, where $Y_\tau = (0, c_\tau)^N$ with $c_\tau > 0$.

Then this leads us to Problem IV. Indeed, it is not hard to check that the preceding structure hypothesis implies that a_{ij} belongs to $\mathcal{B}_\infty(\mathbb{R}; L^2_{AP}(\mathbb{R}^N))$ (Example 4.4).

Our last problem states as follows.

Problem V. Let A_τ be an H-algebra on $\mathbb{R}$ with the property that A_τ^∞ is dense in A_τ. The matter in hand here is to study the homogenization of (1.4) under the hypothesis that

$$a_{ij} \text{ lies in the closure of } AP(\mathbb{R}^N) \otimes A_\tau \text{ in } (L^2, l^\infty)(\mathbb{R}^{N+1}) \ (1 \leq i, j \leq N). \quad (4.11)$$

To begin with, let $\zeta_{nij} \in AP(\mathbb{R}^N) \otimes A_\tau$ ($n \in \mathbb{N}$, $1 \leq i, j \leq N$) be such that $\zeta_{nij} \to a_{ij}$ in $(L^2, l^\infty)(\mathbb{R}^{N+1})$ ($1 \leq i, j \leq N$) as $n \to \infty$. By Proposition 4.1 one is easily led to some countable subgroup $\mathcal{R}$ of $\mathbb{R}^N$ such that $\zeta_{nij} \in AP_\mathcal{R}(\mathbb{R}^N) \otimes A_\tau$ for all $n \in \mathbb{N}$ and all indices $1 \leq i, j \leq N$. Let $\Sigma = \Sigma_\mathcal{R} \times \Sigma_\tau$, where $\Sigma_\mathcal{R}$ is as in Problem IV and Σ_τ is the H-structure of class $\mathcal{C}^\infty$ on $\mathbb{R}$ of which A_τ is the image. The H-structure Σ on $\mathbb{R}^N \times \mathbb{R}$ is quasi-proper for $p = 2$ and its image is the closure, A, of $AP_\mathcal{R}(\mathbb{R}^N) \otimes A_\tau$ in $\mathcal{B}(\mathbb{R}^N \times \mathbb{R})$ (see [18, Proposition 3.2]). Thus, we will be through if we have shown that (4.9) holds. But this is a direct consequence of the fact that $(L^2, l^\infty)(\mathbb{R}^{N+1})$ is continuously embedded in $\Xi^2(\mathbb{R}^{N+1})$. Therefore, the homogenization problem under consideration lies within the scope of Theorem 3.1 and so we are led to the results of Subsection 3.4.

Remark 4.4. According to (4.11), the function $(y,\tau) \to a_{ij}(y,\tau)$ is almost periodic in $y \in \mathbb{R}^N$ whereas in the variable $\tau \in \mathbb{R}$ it admits a great variety of behaviours. This is illustrated below.

Example 4.7. Property (4.11) includes (4.10) as a particular case. Indeed, this follows by choosing $A_\tau = \mathcal{B}_\infty(\mathbb{R})$ in (4.11) and observing that $AP(\mathbb{R}^N) \otimes \mathcal{B}_\infty(\mathbb{R})$ is a dense subspace of $\mathcal{B}_\infty(\mathbb{R}; AP(\mathbb{R}^N))$.

Example 4.8. Our purpose in the present example is to study the homogenization of (1.4) under the following assumptions, where the pair of indices $1 \le i,j \le N$ is arbitrarily fixed:

(SH1) $a_{ij}(\cdot,\tau) \in L^2_{AP}(\mathbb{R}^N)$ a.e. in $\tau \in \mathbb{R}$

(SH2) The function $\tau \to a_{ij}(\cdot,\tau)$ from $\mathbb{R}$ to $L^2_{AP}(\mathbb{R}^N)$ is piecewise constant in the sense that there exists a mapping $q_{ij} : \mathbb{Z} \to L^2_{AP}(\mathbb{R}^N)$ such that

$$a_{ij}(\cdot,\tau) = q_{ij}(k) \quad \text{a.e. in } k \le \tau < k+1 \ \ (k \in \mathbb{Z}). \tag{4.12}$$

However, in order to have a "well posed" homogenization problem, we need to be informed about the behaviour of the coefficient q_{ij}. We assume here that

(SH3) $q_{ij} \in \mathcal{B}_\infty(\mathbb{Z}; L^2_{AP}(\mathbb{R}^N)$ where $\mathcal{B}_\infty(\mathbb{Z}; L^2_{AP}(\mathbb{R}^N))$ denotes the space of mappings $q : \mathbb{Z} \to L^2_{AP}(\mathbb{R}^N)$ that converge at infinity, i.e., such that $q(k)$ has a limit in $L^2_{AP}(\mathbb{R}^N)$ when $|k| \to \infty$.

It is well to note in passing that such a q is necessarily bounded.

Let us show that the structure hypothesis made up of (SH1)–(SH3) reduces to (4.11), so that the problem under consideration is quite solvable.

Proposition 4.2. *Let $\mathcal{F}$ be the set of functions $f : \mathbb{R} \to \mathbb{C}$ of the form*

$$f = \sum_{k\in\mathbb{Z}} r(k)\tau_k\varphi \quad (r \in \mathcal{B}_\infty(\mathbb{Z}),\ \varphi \in \mathcal{K}(Z)) \tag{4.13}$$

with $Z = (0,1)$, where $\mathcal{K}(Z)$ is identified with the space of functions in $\mathcal{K}(\mathbb{R})$ with supports contained in Z. Let A_τ be the closure in $\mathcal{B}(\mathbb{R})$ of the space of functions of the form

$$\psi = c + \sum_{finite} f_i \quad (c \in \mathbb{C},\ f_i \in \mathcal{F}).$$

Then the following assertions are true.

(i) *A_τ is an H-algebra on $\mathbb{R}$ with the further property that A_τ^∞ is dense in A_τ.*

(ii) *The family $\{a_{ij}\}_{1\le i,j\le N}$ satisfies (4.11).*

Proof. Part (i) is proved in [20]. Thus, we need only show (ii). Let a pair of indices $1 \le i,j \le N$ be freely fixed. Based on the density of $AP(\mathbb{R}^N) \otimes \mathcal{B}_\infty(\mathbb{Z})$ in $\mathcal{B}_\infty(\mathbb{Z}; AP(\mathbb{R}^N))$ (this follows by [6, page 46, Proposition 5]) and recalling that $AP(\mathbb{R}^N)$ is a dense subspace of $L^2_{AP}(\mathbb{R}^N)$, we see immediately that $AP(\mathbb{R}^N) \otimes \mathcal{B}_\infty(\mathbb{Z})$ is dense in $\mathcal{B}_\infty(\mathbb{Z}; L^2_{AP}(\mathbb{R}^N))$. Hence, in view of (SH3), we may consider a sequence $(q_{nij})_{n\in\mathbb{N}}$ in $AP(\mathbb{R}^N) \otimes \mathcal{B}_\infty(\mathbb{Z})$ such that $q_{nij} \to q_{ij}$ in $\mathcal{B}_\infty(\mathbb{Z}; L^2_{AP}(\mathbb{R}^N))$ as $n \to \infty$. It is useful to specify that q_{nij} writes as

$$q_{nij}(y,k) = \sum_{l\in I} \zeta^l_{nij}(k) u^l_{nij}(y) \quad (y \in \mathbb{R}^N,\ k \in \mathbb{Z}) \tag{4.14}$$

where $\zeta_{nij}^l \in \mathcal{B}_\infty(\mathbb{Z})$, $u_{nij}^l \in AP(\mathbb{R}^N)$, and I is a finite set (depending on q_{nij}). On the other hand, it is worth bearing in mind that (4.12) is equivalent to

$$a_{ij}(y,\tau) = \sum_{k\in\mathbb{Z}} q_{ij}(y,k)\chi_{k+Z}(\tau) \quad \text{a.e. in } (y,\tau) \in \mathbb{R}^N \times \mathbb{R}$$

where χ_{k+Z} denotes the characteristic function of $k+Z=(k,k+1)$ in $\mathbb{R}$, and where the sum on the right is locally finite. Now, let

$$a_{nij}(y,\tau) = \sum_{k\in\mathbb{Z}} q_{nij}(y,k)\chi_{k+Z}(\tau) \quad (y \in \mathbb{R}^N,\ \text{a.e. in } \tau \in \mathbb{R}). \tag{4.15}$$

It is clear that a_{nij} lies in $L^\infty(\mathbb{R}; AP(\mathbb{R}^N)) \subset (L^2, l^\infty)(\mathbb{R}^{N+1})$ and further

$$\|a_{nij} - a_{ij}\|_{2,\infty} \le \sup_{k\in\mathbb{Z}} \|q_{nij}(k) - q_{ij}(k)\|_{2,\infty}.$$

Hence $a_{nij} \to a_{ij}$ in $(L^2, l^\infty)(\mathbb{R}^{N+1})$ as $n \to \infty$.

On the other hand, substituting (4.14) in (4.15) yields

$$a_{nij}(y,\tau) = \sum_{l\in I} u_{nij}^l(y) f_{nij}^l(\tau) \quad (y \in \mathbb{R}^N,\ \text{a.e. in } \tau \in \mathbb{R})$$

where $f_{nij}^l \in L^\infty(\mathbb{R})$ with

$$f_{nij}^l(\tau) = \sum_{k\in\mathbb{Z}} \zeta_{nij}^l(k)\chi_{k+Z}(\tau) \quad (\text{a.e. in } \tau \in \mathbb{R}).$$

But if $\eta > 0$ is arbitrarily given and if $\varphi \in \mathcal{K}(Z)$ is such that $\|\chi_Z - \varphi\|_{L^2(\mathbb{R})} = \|1-\varphi\|_{L^2(Z)} \le \frac{\eta}{c}$, where $c > 0$ with $|\zeta_{nij}^l(k)| \le c$ $(k \in \mathbb{Z})$, then $\|f_{nij}^l - \psi_{nij}^l\|_{2,\infty} \le \eta$ with $\psi_{nij}^l = \sum_{k\in\mathbb{Z}} \zeta_{nij}^l(k)\tau_k\varphi$ (see (4.13)).

Finally, let

$$\Phi_{nij}(y,\tau) = \sum_{l\in I} u_{nij}^l(y)\psi_{nij}^l(\tau) \quad (y \in \mathbb{R}^N,\ \tau \in \mathbb{R}),$$

which defines a function in $AP(\mathbb{R}^N) \otimes A_\tau$. It is an elementary exercise to deduce from the preceding development that for any $\eta > 0$, there is some integer $n \in \mathbb{N}$ such that $\|a_{ij} - \Phi_{nij}\|_{2,\infty} \le \eta$. This completes the proof. □

Example 4.9. The case to be examined here states as in Example 4.8 except that in (SH3), $\mathcal{B}_\infty(\mathbb{Z}; L^2_{AP}(\mathbb{R}^N))$ is substituted by the space $\ell^1(\mathbb{Z}; L^2_{AP}(\mathbb{R}^N))$ of mappings $q : \mathbb{Z} \to L^2_{AP}(\mathbb{R}^N)$ such that $\sum_{k\in\mathbb{Z}} \|q(k)\|_{2,\infty} < \infty$. Without going too deeply into details let us verify that the present case leads to the same conclusion as in the preceding example. First, let $\ell_0^1(\mathbb{Z})$ denote the closure in $\ell^\infty(\mathbb{Z})$ of the set of functions $r \in \ell^\infty(\mathbb{Z})$ of the form $r = c + r_0$ with $c \in \mathbb{C}$ and $r_0 \in \ell^1(\mathbb{Z})$. We claim that the statement of Proposition 4.2 is still valid when $\mathcal{B}_\infty(\mathbb{Z})$, in (4.13), is replaced by $\ell_0^1(\mathbb{Z})$. Indeed, there is no real difficulty in verifying that the proof of the said proposition holds when the symbol $\mathcal{B}_\infty$ is replaced by ℓ^1 (not ℓ_0^1!). The details are left to the reader.

Remark 4.5. The coefficient q_{ij} in Example 4.8 is $q_{ij}(k) = \int_k^{k+1} a_{ij}(\cdot,\tau)d\tau$ $(k \in \mathbb{Z})$.

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