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On the eigenvalue problem for the Hardy-Sobolev operator with indefinite weights *

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Abstract

In this paper we study the eigenvalue problem

$$-\Delta_p u - a(x)|u|^{p-2}u = \lambda |u|^{p-2}u, \quad u \in W_0^{1,p}(\Omega),$$

where $1 < p \leq N$, Ω is a bounded domain containing 0 in $\mathbb{R}^N$, Δ_p is the p-Laplacean, and $a(x)$ is a function related to Hardy-Sobolev inequality. The weight function $V(x) \in L^s(\Omega)$ may change sign and has nontrivial positive part. We study the simplicity, isolatedness of the first eigenvalue, nodal domain properties. Furthermore we show the existence of a nontrivial curve in the Fučik spectrum.

1 Introduction

Let Ω be a bounded domain containing 0 in $\mathbb{R}^N$. Then the Hardy-Sobolev inequality for $1 < p < N$ states that

$$\int_\Omega |\nabla u|^p dx \geq \Big(\frac{N-p}{p}\Big)^p \int_\Omega \frac{|u|^p}{|x|^p} dx \tag{1.1}$$

for all $u \in W_0^{1,p}(\Omega)$. It is known that $(\frac{N-p}{p})^p$ is the best constant in (1.1). In a recent work Adimurthi, Choudhuri and Ramaswamy [2] improved the above inequality. In particular, when $p = N$ their inequality reads

$$\int_\Omega |\nabla u|^N dx \geq \Big(\frac{N-1}{N}\Big)^N \int_\Omega \frac{|u|^N}{|x|^N (\log \frac{R}{|x|})^N} dx, \quad \forall u \in W_0^{1,N}(\Omega), \tag{1.2}$$

where $R > e^{2/N} \sup_\Omega |x|$. Subsequently it was shown in [4] that $(\frac{N-1}{N})^N$ is the best constant in (1.2). In view of the above two inequalities we define the Hardy-Sobolev Operator L_μ on $W_0^{1,p}(\Omega)$ as

$$L_\mu u := -\Delta_p u - \mu a(x)|u|^{p-2}u$$

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where

$$a(x) = \begin{cases} 1/|x|^p & 1 < p < N \\ 1/(|x| \log \frac{R}{|x|})^N & p = N \end{cases}$$

and $0 \leq \mu < (\frac{n-p}{p})^p$ or $(\frac{N-1}{N})^N$ depends on the value of p. Here $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ denotes the p-Laplacean. In the present work we consider the following eigenvalue problem:

$$\begin{gathered} L_\mu u = \lambda V(x)|u|^{p-2}u \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.3}$$

We assume that $V \in L^1_{\rm loc}(\Omega)$, $V^+ = V_1 + V_2 \not\cong 0$ with $V_1 \in L^{\frac{N}{p}}(\Omega)$ and V_2 is such that

$$\lim_{x\to y,\ x\in\Omega} |x-y|^p V_2(x) = 0 \quad \forall y \in \overline{\Omega} \quad \text{for } p < N$$

$$\lim_{x\to y,\ x\in\Omega} |x-y|^p (\log \frac{R}{|x-y|})^p V_2(x) = 0 \quad \forall y \in \overline{\Omega} \quad \text{for } p = N. \tag{1.4}$$

where $V^+(x) = \max\{V(x), 0\}$. We also assume

(H) There exists $r > \frac{N}{p}$ and a closed subset S of measure zero in $\mathbb{R}^N$ such that $\Omega\backslash S$ is connected and $V \in L^r_{\rm loc}(\Omega\backslash S)$.

We define the functional J_μ on $W^{1,p}_0(\Omega)$ as

$$J_\mu(u) := \int_\Omega |\nabla u|^p - \mu \int_\Omega a(x)|u|^{p-2}u.$$

Then J_μ is C^1 on $W^{1,p}_0(\Omega)$. Our goal here is to study the eigenvalue problem and some main properties (simplicity, isolatedness) of

$$\lambda_1 := \inf \left\{ J_\mu(u); u \in W^{1,p}_0(\Omega) \quad \text{and} \quad \int_\Omega V|u|^p dx = 1 \right\}$$

We use the following results in Section 2.

Proposition 1.1 ([5]) *Let $\Omega \subset \mathbb{R}^n$ is bounded domain and suppose $(u_n) \in W^{1,p}(\Omega)$ such that $u_n \to u$ weakly in $W^{1,p}_0(\Omega)$satisfies*

$$-\Delta_p u_n = f_n + g_n \text{ in } \mathcal{D}'(\Omega)$$

where $f_n \to f$ in $W^{-1,p'}$ and g_n is a bounded sequence of Radon measures, i.e.,

$$\langle g_n, \phi \rangle \leq C_K \, \|\phi\|_\infty$$

for all ϕ in $C^\infty_c(\Omega)$ with support in K. Then there exists a subsequence (u_n) of (u_n) such that $\nabla u_n(x) \to \nabla u(x)$ a.e. in Ω.

Proposition 1.2 ((Brezis-Lieb[6])) *Suppose $f_n \to f$ a.e. and $\|f_n\|_p \leq C < \infty$ for all n and for some $0 < p < \infty$. Then*

$$\lim_{n\to\infty} \{\|f_n\|_p^P - \|f_n - f\|_p^p\} = \|f\|_p^p.$$

In section 2 we study the eigenvalue problem for L_μ and show that the first eigenvalue is simple and the eigenfunctions corresponding to other eigenvalues changes sign. In section 3 we study the existence of nontrivial curve in the Fučik spectrum of L_μ. Finally in the last section we study some nodal domain properties of L_μ with a stronger assumption on V that $V \in L^r(\Omega)$ for some $r > \frac{N}{p}$.

We now provide a brief account of what is known about the problems of type (1.3). In case of $\mu = 0$, the above properties are well known when V is bounded(see[1]). For indefinite weights with different integrability conditions see[3] and [14]. In [14] the problem of simplicity and sign changing nature of other eigen functions are left open. In Theorem 2.1 below we prove the above properties. In a recent work Cuesta [7] proved above properties with stronger assumption that $V \in L^s(\Omega)$ for some $s > \frac{N}{p}$. When $\mu \neq 0$ and $V = 1$ the above properties are studied in [11],[12].

2 Eigenvalue Problem

In this section we show that the first eigenvalue is simple and the eigenfunctions corresponding to other eigenvalues changes sign. We prove the following theorem.

Theorem 2.1 *The first eigenvalue, λ_1, is simple and the eigenfunctions corresponding to the other eigenvalues changes sign.*

The next theorem is proven with the help of a deformation lemma for C^1 manifolds.

Theorem 2.2 *There exists a sequence $\{\lambda_n\}$ of eigenvalues of L_μ such that $\lambda_n \to \infty$.*

Let us define the operators

$$L(u,v) := |\nabla u|^p - (p-1)\frac{u^p}{v^p}|\nabla v|^p - p\frac{u^{p-1}}{v^{p-1}}\nabla u|\nabla v|^{p-2}\nabla v$$
$$R(u,v) := |\nabla u|^p - |\nabla v|^{p-2}\nabla v.\nabla\big(\frac{u^p}{v^{p-1}}\big)$$

Then $R(u,v) = L(u,v) \geq 0$ for all $u, v \in C^1(\Omega\backslash\{0\}) \cap W^{1,p}(\Omega)$ with $u \geq 0, v > 0$ and equal to 0 if and only if $u = kv$ for some constant k [3, Theorem 1.1]. We need following lemmas to prove our results.

Lemma 2.3 *The mapping $u \longrightarrow \int_\Omega V^+|u|^p dx$ is weakly continuous.*

Proof: In case the $1 < p < N$, the proof follows as in [14]. Here we give the proof when $p = N$. Clearly $u \to \int_\Omega V_1|u|^p$ is weakly continuous. Since $\overline{\Omega}$ is compact, there is a finite covering of $\overline{\Omega}$ by closed balls $B(x_i, r_i)$ such that, for $1 \le i \le k$,

$$|x - x_i| \le r_i \implies |x - x_i|^N (\log \frac{R}{|x - x_i|})^N V_2(x) \le \epsilon. \tag{2.1}$$

There exists $r > 0$ such that, for $1 \le i \le k$,

$$|x - x_i| \le r \implies |x - x_j|^N (\log \frac{R}{|x - x_i|})^N V_2(x) \le \epsilon/k.$$

Define $A := \cup_{j=1}^k B(x_j, r)$. Then by inequality (1.2)

$$\int_A V_2 |u_n|^N dx \le \epsilon c^N, \quad \int_A V_2 |u|^N dx \le \epsilon c^N \tag{2.2}$$

where $c = \frac{N}{N-1} \sup_n \|u_n\|$. It follows from (2.1) that $V_2 \in L^1(\Omega \backslash A)$ so that

$$\int_{\Omega \backslash A} V_2 |u_n|^N dx \longrightarrow \int_{\Omega \backslash A} V_2 |u|^N dx \tag{2.3}$$

Now the conclusion follows from (2.2) and (2.3). $\square$

Define $M := \left\{ u \in W_0^{1,p}(\Omega); \ \int_\Omega V|u|^p = 1 \right\}$

Lemma 2.4 *The eigenvalue λ_1 is attained.*

Proof: Let u_n be a sequence in M such that $J_\mu(u_n) \to \lambda_1$. Since $W_0^{1,p}(\Omega)$ is reflexive, there exists a subsequence $\{u_n\}$ of $\{u_n\}$ such that $u_n \to u$ weakly in $W_0^{1,p}$ and a.e. in Ω. Now for $n \in \mathbb{N}$ choose u_n such that $J_\mu(u_n) \le \inf_M J_\mu + \frac{1}{n^2}$. Now by The Ekeland Variational Principle, there exists a sequence $\{v_n\}$ such that

$$\begin{gathered} J_\mu(v_n) \le J_\mu(u_n) \\ \|u_n - v_n\| \le \frac{1}{n} \\ J_\mu(v_n) \le J_\mu(u) + \frac{1}{n}\|v_n - u\| \quad \forall u \in M \end{gathered}$$

Now standard calculations from above three equations, as in [10], gives

$$\left|J'_\mu(v_n)w - J_\mu(v_n) \int_\Omega V|v_n|^{p-2} v_n w\right| \le C\frac{1}{n}\|w\|. \tag{2.4}$$

By Proposition 1.1, there exists a subsequence of $\{v_n\}$, which we still denote by $\{v_n\}$ such that $v_n \to v$ weakly in $W_0^{1,p}(\Omega)$ and $\nabla v_n \to \nabla v$ a.e. in Ω. Since

$|\nabla v_n|^{p-2}\nabla v_n$ is bounded in $(L^{p'}(\Omega))^N, 1/p + 1/p' = 1$, and $\nabla v_n \to \nabla v$ a.e. in Ω, we have

$$|\nabla v_n|^{p-2}\nabla v_n \to |\nabla v|^{p-2}\nabla v \quad \text{a.e. in } \Omega$$
$$|\nabla v_n|^{p-2}\nabla v_n \to |\nabla v|^{p-2}\nabla v \quad \text{weakly in } (L^{p'}(\Omega))^N$$

which allows us to pass the limit as $n \to \infty$ in (2.4), obtaining

$$-\Delta_p v - a(x)|v|^{p-2}v - \lambda_1 |v|^{p-2}v = 0 \quad \text{in } \mathcal{D}'(\Omega).$$

Observe that

$$\int_\Omega V^- |v_n|^p dx = \int_\Omega V^+ |v_n|^p dx - 1 \to \int_\Omega V^+ |v|^p dx - 1$$

as $n \to \infty$. Now using Fatau's lemma we can conclude that $v \not\equiv 0$. □

Lemma 2.5 *The eigenvalue λ_1 is simple.*

Proof: This is an adaptation from a proof in [3]. Let $\{\psi_n\}$ be a sequence of functions such that $\psi_n \in C_c^\infty(\Omega), \psi_n \geq 0, \psi_n \to \phi_1$ in $W^{1,p}$, a.e. in Ω and $\nabla\psi_n \to \nabla\phi_1$ a.e. in Ω. Then we have

$$\begin{aligned} 0 &= \int_\Omega (|\nabla \phi_1|^p - (\mu a(x) + \lambda_1 V)\phi_1^p)\, dx \\ &= \lim_{n\to\infty} \int_\Omega (|\nabla \psi_n|^p - (\mu a(x) + V\lambda_1)\psi_n^p)\, dx. \end{aligned} \tag{2.5}$$

Consider the function $w_1 := \psi_n^p/(u_2 + \frac{1}{n})^{p-1}$. Then $w_1 \in W_0^{1,p}(\Omega)$. So testing the equation satisfied by u_2 with w_1 we get,

$$\int_\Omega (\lambda_1 V + \mu a(x))\psi_n^p \Big(\frac{u_2}{u_2 + \frac{1}{n}}\Big)^{p-1} = \int_\Omega |\nabla u_2|^{p-2}\nabla u_2 . \nabla\Big(\frac{\psi_n^p}{(u_2 + \frac{1}{n})^{p-1}}\Big) \tag{2.6}$$

Now from (2.5) and (2.6) we obtain

$$\begin{aligned} 0 &= \lim_{n\to\infty} \int_\Omega \Big(|\nabla \psi_n|^p - |\nabla u_2|^{p-2}\nabla u_2 . \nabla\Big(\frac{\psi_n^p}{(u_2 + \frac{1}{n})^{p-1}}\Big)\Big) \\ &= \lim_{n\to\infty} \int_\Omega L(\psi_n, u_2) \geq \int_\Omega L(\phi_1, u_2) \geq 0 \end{aligned}$$

by Fatau's lemma. Now by assumption (H), ϕ_1, u_2 are in $C^1(\Omega\backslash S \cup \{0\})$ [9, 15]. Therefore $\phi_1 = k u_2$ for some constant k. □

Proof of Theorem 2.1, completed: Let ϕ_1, u be the eigenfunctions corresponding to λ_1 and λ respectively. Then ϕ_1, u satisfies

$$-\Delta_p \phi_1 - \mu a(x)\phi_1^{p-1} = \lambda_1 V(x)\phi_1^{p-1} \quad \text{in } \mathcal{D}'(\Omega), \tag{2.7}$$

$$-\Delta_p u - \mu a(x)|u|^{p-2}u = \lambda V(x)|u|^{p-2}u \quad \text{in } \mathcal{D}'(\Omega) \tag{2.8}$$

respectively. Suppose u does not change sign. We may assume $u \geq 0$ in Ω. Let $\{\psi_n\}$ be a sequence in C_c^∞ such that $\psi_n \to \phi_1$ as $n \to \infty$. Now consider the test functions $w_1 = \phi_1, w_2 = \frac{\psi_n^p}{(u+\frac{1}{n})^{p-1}}$. Then $w_1, w_2 \in W_0^{1,p}(\Omega)$. Testing (2.7) with w_1 and (2.8) with w_2 we get

$$\int_\Omega |\nabla \phi_1|^p dx - \int_\Omega (\lambda_1 V(x) + \mu a(x))\phi_1^p dx = 0 \tag{2.9}$$

$$\int_\Omega |\nabla u|^{p-2}\nabla u.\nabla\Big(\frac{\psi_n^p}{(u+\frac{1}{n})^{p-1}}\Big)dx - \int_\Omega (\lambda V(x) + \mu a(x))\psi_n^p \big(\frac{u}{u+\frac{1}{n}}\big)^{p-1} dx = 0$$

Since $R(u, v) \geq 0$, we get

$$\int_\Omega |\nabla \psi_n|^p dx - \int_\Omega (\lambda V(x) + \mu a(x)\psi_n^p \Big(\frac{u}{u+\frac{1}{n}}\Big)^{p-1} dx \geq 0. \tag{2.10}$$

Subtracting (2.9) from (2.10) and taking the limit as $n \to \infty$ we get,

$$(\lambda - \lambda_1)\int_\Omega V(x)\phi_1^p \leq 0$$

This is a contradiction to the fact that $\lambda > \lambda_1$. □

Proof of Theorem 2.2: Let $\tilde{J}_\mu$ be the restriction of J_μ to the set M. Define

$$\lambda_k = \inf_{\gamma(A)\geq n} \sup_{u\in A} J_\mu(u)$$

where A is a closed subset of M such that $A = -A$, and $\gamma(A)$ is the *Krasnosel'skiĭ genus* of A. Now we show that $\tilde{J}_\mu$ satisfies (P.S.) condition at level λ_k. Let $\{u_n\}$ be a sequence in M such that $J_\mu(u_n) \to \lambda_k$ and

$$\langle J_\mu(u_n), \phi\rangle - J_\mu(u_n)\int_\Omega |u_n|^{p-2}u_n\phi V dx = o(1). \tag{2.11}$$

Since u_n is bounded, there exists a subsequence $\{u_n\}, u$ such that $u_n \to u$ weakly in $W_0^{1,p}(\Omega)$. Since $\lambda_k > 0$ we may assume that $J_\mu(u_n) \geq 0$. Using Lemma 2.3 and (2.11), we get

$$\langle J_\mu(u_n) - J_\mu(u), u_n - u\rangle + J_\mu(u_n)\int_\Omega \big[|u_n|^{p-2}u_n - |u|^{p-2}u\big](u_n - u)V^- dx = o(1).$$

But

$$\int_\Omega \big[|u_n|^{p-2}u_n - |u|^{p-2}u\big][u_n - u]V^- \geq 0.$$

By Propositions 1.1 and 1.2, we have

$$\|u_n - u\|_{1,p} = \|u_n\|_{1,p} - \|u\|_{1,p} + o(1)$$
$$\|\frac{u_n - u}{|x|}\|_{0,p} = \|\frac{u_n}{|x|}\|_{0,p} - \|\frac{u}{|x|}\|_{0,p} + o(1)$$

Therefore

$$\begin{aligned} o(1) =& \langle J_\mu(u_n) - J_\mu(u), (u_n - u)\rangle \\ & + J_\mu(u_n) \int_\Omega [|u_n|^{p-2}u_n - |u|^{p-2}u](u_n - u)V^- dx \\ \geq & \int_\Omega |\nabla u_n - \nabla u|^p - \int_\Omega \mu a(x)|u_n - u|^p + o(1) \\ \geq & C\|u_n - u\|_{1,p} + o(1). \end{aligned}$$

Now by the classical critical point theory for C^1 manifolds [13], it follows that λ_k's are critical points of J_μ on M. Since $\lambda_k \geq c\lambda_k^0$, where λ_k^0 are eigenvalues of L_0, we have $\lambda_k \to \infty$. □

3 Fučik Spectrum

In this section we study the existence of a non-trivial curve in the Fučik spectrum $\sum_{p,\mu}$ of L_μ. The Fučik spectrum of L_μ is defined as the set of $(\alpha, \beta) \in \mathbb{R}^2$ such that

$$\begin{gathered} L_\mu u = \alpha V(u^+)^{p-1} + \beta V(u^-)^{p-1} \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

has a nontrivial solution $u \in W_0^{1,p}(\Omega)$. The variational approach that we follow here is same as that of [8, 12]. We prove the following statement.

Theorem 3.1 *There exists a nontrivial curve $\mathcal{C}$ in $\sum_{p,\mu}$.*

Let us consider the functional

$$J_s(u) = \int_\Omega |\nabla u|^p - \int_\Omega \mu a(x)|u|^p - s\int_\Omega V{u^+}^p$$

J_s is a C^1functional on $W_0^{1,p}(\Omega)$. We are interested in the critical points of the restriction $\tilde{J}_s$ of J_s to M. By Lagrange multiplier rule, $u \in M$ is a critical point of $\tilde{J}_s$ if and only if there exist $t \in \mathbb{R}$ such that $J_s'(u) = t.I'(u)$, i.e., for all $v \in W_0^{1,p}$ we have

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v - \int_\Omega \mu a(x)|u|^{p-2}uv - s\int_\Omega V{u^+}^{p-1}v = t\int_\Omega V|u|^{p-2}uv\,.(\Omega) \tag{3.1}$$

This implies that

$$\begin{gathered} -\Delta_p u - \mu a(x)|u|^{p-2}u = (s+t)V(x)(u^+)^{p-1} - tV(x)(u^-)^{p-1} \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{gathered}$$

holds in the weak sense. i.e., $(s+t, t) \in \sum_{p,\mu}$, taking $v = u$ in (3.1), we get t as a critical value of $\tilde{J}_s$. Thus the points in $\sum_{p,\mu}$ on the parallel to the diagonal passing through (s,0) are exactly of the form $(s+\tilde{J}_s(u), \tilde{J}_s(u))$ with u a critical point of $\tilde{J}_s$.

A first critical point of $\tilde{J}_s$ comes from global minimization . Indeed

$$\tilde{J}_s(u) \geq \lambda_1 \int_\Omega |u|^p - s\int_\Omega u^{+^p} \geq \lambda_1 - s$$

for all u $\in M$, and $\tilde{J}_s(u) = \lambda_1 - s$ for $u = \phi_1$.

Proposition 3.2 *The function ϕ_1 is a global minimum of $\tilde{J}_s$ with $\tilde{J}_s(\phi_1) = \lambda_1 - s$, the corresponding point in $\sum_{p,\mu}$ is $(\lambda_1, \lambda_1 - s)$ which lies on the vertical line through (λ_1, λ_1).*

Lemma 3.3 *Let $0 \neq v_n \in W_0^{1,p}$ satisfy $v_n \geq 0$ a.e and $|v_n > 0| \to 0$, then $\int_\Omega [|\nabla v_n|^p - \mu a(x)|v_n|^p]dx / \int_\Omega V|v_n|^p \to +\infty$.*

Proof: Let $w_n = v_n/\|v_n\|_{V,p}$ and assume by contradiction that $\int_\Omega |\nabla w_n|^p - \int_\Omega \mu a(x)|w_n|^p$ has a bounded subsequence. By (1.1) or (1.2), we get w_n bounded in $W_0^{1,p}(\Omega)$. Then for a further subsequence, $w_n \to w$ in $L^p(\Omega, V^+)$. Now observe that

$$\int_\Omega V^-(x)|w|^p \leq \lim_{n\to\infty}\int_\Omega V^-|w_n|^p = \lim_{n\to\infty}\int_\Omega V^+|w_n|^p - 1 = \int_\Omega V^+|w|^p - 1.$$

Then $w \geq 0$ and $\int_\Omega V^+(x)w^p \geq 1$. So for some $\epsilon > 0, \delta = |w > \epsilon| > 0$, we deduce that $|w_n > \epsilon/2| > \frac{\delta}{2}$ for n sufficiently large, which contradicts the assumption $|v_n > 0| \to 0$. □

A second critical point of $\tilde{J}_s$ comes next.

Proposition 3.4 *$-\phi_1$ is a strict local minimum of $\tilde{J}_s$, and $\tilde{J}_s(-\phi_1) = \lambda_1$, the corresponding point in $\sum_p$ is $(\lambda_1 + s, \lambda_1)$.*

Proof: We follow the ideas in [8, Prop. 2.3]. Assume by contradiction that there exist a sequence $u_n \in M$ with $u_n \neq -\phi_1$, $u_n \to -\phi_1$ in $W_0^{1,p}(\Omega)$ and $\tilde{J}_s(u_n) \leq \lambda_1$.

Claim: u_n changes sign for n sufficiently large. Since $u_n \to -\phi_1, u_n$, it must follow that $u_n \leq 0$ some where. If $u_n \leq 0$ a.e., in Ω, then

$$\tilde{J}_s(u_n) = \int_\Omega |\nabla u_n|^p - \int_\Omega \mu a(x)|u_n|^p > \lambda_1$$

since $u_n \neq \pm\phi_1$, and this contradicts $\tilde{J}_s(u_n) \leq \lambda_1$. This completes the proof of claim. Let $r_n = [\int_\Omega |\nabla u_n^+|^p - \int_\Omega \mu a(x)|u_n^+|^p] / \int_\Omega V u_n^{+p}$, we have

$$\begin{aligned}\tilde{J}_s(u_n) = &\int_\Omega |\nabla u_n^+|^p + \int_\Omega |\nabla u_n^-|^p - \int_\Omega \mu a(x)|u_n^+|^p \\ &- \int_\Omega \mu a(x)|u_n^-|^p - s\int_\Omega V|u_n^+|^p \\ \geq &(r_n - s)\int_\Omega V u_n^{+^p} + \lambda_1 \int_\Omega V u_n^{-^p}\end{aligned}$$

on the other hand

$$\tilde{J}_s(u_n) \leq \lambda_1 = \lambda_1 \int_\Omega V u_n^{+^p} + \int_\Omega V u_n^{-^p}$$

combining the two inequalities, we get $r_n \leq \lambda_1 + s$. Now since, $u_n \to -\phi_1$ in $L^p(\Omega)$, $|u_n > 0| \to 0$. The Lemma 3.3 then implies $r_n \to +\infty$, which contradicts $r_n \leq \lambda_1 + s$. □

Now as in the proof of Theorem 2.2, one can show that $\tilde{J}_s$ satisfies the P.S. condition at any positive level.

Lemma 3.5 *Let $\epsilon_0 > 0$ be such that*

$$\tilde{J}_s(u) > \tilde{J}_s(-\phi_1) \quad \forall u \in B(-\phi_1, \epsilon_0) \cap M \tag{3.2}$$

with $u \neq -\phi_1, B \subset W_0^{1,p}$. Then for any $0 < \epsilon < \epsilon_0$

$$\inf\{\tilde{J}_s(u); u \in M \quad \textit{and} \quad \|u - (-\phi_1)\|_{1,p} = \epsilon\} > \tilde{J}_s(-\phi_1). \tag{3.3}$$

The proof of this lemma follows from the Ekeland variational principle. Therefore, we omit it. For details we refer the reader to [8]. Let

$$\Gamma = \{\gamma \in C([-1,1]; M) : \gamma(-1) = -\phi_1, \gamma(1) = \phi_1\} \neq \varnothing$$

and the geometric assumptions of Mountain-pass Lemma are satisfied by previous Lemma. Therefore, there exists $u \in W_0^{1,p}$ such that $\tilde{J}_s'(u) = 0$ and $J_s(u) = c$, where c is given by

$$c(s) = \inf_\Gamma \sup_\gamma J_s(u). \tag{3.4}$$

Proceeding in this manner for each $s \geq 0$ we get a non-trivial curve $\mathcal{C}$: $s \in \mathbb{R}^+ \to (s + c(s), c(s)) \in \mathbb{R}^2$ in $\sum_{p,\mu}$, which completes the proof of Theorem 3.1.

4 Nodal Domain Properties

In this section we show that λ_1 is isolated in the spectrum under the assumption on V that $V \in L^s(\Omega)$ for some $s > \frac{N}{p}$. By the regularity results in [15, 9] the solutions of (1.3) are $C^1(\Omega\backslash\{0\})$. In [11] it is shown that the positive solutions of (1.3) when $V = 1$ tends to $+\infty$ as $|x| \to 0$. We prove the following theorem.

Theorem 4.1 *The eigenvalue λ_1 is isolated in the spectrum provided that $V \in L^s(\Omega)$ for some $s > \frac{N}{p}$. Moreover, for v an eigenfunction corresponding to an eigenvalue $\lambda \neq \lambda_1$ and O be a nodal domain of v, then*

$$|O| \geq (C\lambda\|V\|_s)^{-\gamma} \tag{4.1}$$

where $\gamma = \frac{sN}{sp-N}$ and C is a constant depending only on N and p.

Lemma 4.2 *Let $u \in C(\Omega\backslash\{0\}) \cap W_0^{1,p}(\Omega)$ and let O be a component of $\{x \in \Omega; u(x) > 0\}$. Then $u|_O \in W_0^{1,p}(O)$*

Proof: case (i): $1 < p < N$.
Let $u_n \in C_c(\Omega) \cap W_0^{1,p}(\Omega)$ such that $u_n \to u$ in $W_0^{1,p}(\Omega)$. Then $u_n^+ \to u^+$ in $W_0^{1,p}(\Omega)$. Let $v_n = \min(u_n, u)$ and let $\psi_r \in C(\Omega)$ be a cutoff function such that

$$\psi_r(x) = \begin{cases} 0 & \text{if } |x| \le r/2 \\ 1 & \text{if } |x| \ge r \end{cases}$$

and $|\nabla \psi_r(x)| \le \frac{C}{r}$ for some constant C. Now consider the sequence $w_{n,r}(x) = \psi_r v_n(x)\big|_O$. Since $\psi_r v_n \in C(\overline{\Omega})$, we have $w_{n,r} \in C(\overline{O})$ and vanishes on the boundary ∂O. Indeed for $x \in \partial O$ and $x = 0$ then $\psi_r = 0$ and so $w_{n,r} = 0$. If $x \in \partial O \cap \Omega$ and $x \neq 0$ then $u(x) = 0$(since u is continuous except at 0) and so $v_n(x) = 0$. If $x \in \partial\Omega$ then $u_n(x) = 0$ and hence $v_n(x) = 0$. So in all the cases $w_{n,r}(x) = 0$ for $x \in \partial O$. Therefore $w_{n,r} \in W_0^{1,p}(O)$ and

$$\begin{aligned}\int_\Omega |\nabla(w_{n,r}) - \nabla(\psi_r u)|^p &= \int_O |(\nabla\psi_r)v_n + \psi_r\nabla v_n - (\nabla\psi_r)u - \psi_r\nabla u|^p dx \\ &\le \|\nabla\psi_r v_n - \nabla\psi_r u\|^p_{L^p(O)} + \|\psi_r\nabla v_n - \psi_r\nabla u\|^p_{L^p(O)}\end{aligned}$$

which approaches 0 as $n \to \infty$. i.e., $w_{n,r} \to \psi_r u\big|_O$ in $W_0^{1,p}(O)$. Now

$$\int_O |\nabla\psi_r u + \psi_r\nabla u - u|^p \le \int_O |\psi_r\nabla u - \nabla u|^p + \int_{O\cap\{r/2<|x|<r\}} |\nabla\psi_r|^p u$$

which approaches 0 as $r \to 0$ (by (1.1). Therefore, $u\big|_O \in W_0^{1,p}(O)$.
case(ii): $p = N$. In this case we use the following cut-off function which are introduced in [11]

$$\psi_r(x) = \begin{cases} 0 & \text{if } |x| \le r \\ 2\log\left(\frac{r}{|x|}\right)/\log(r) & \text{if } r \le |x| \le r^{1/2} \\ 1 & \text{if } |x| \ge r^{1/2}. \end{cases}$$

and we can proceed as in the previous case. ☐

Proof of Theorem 4.1: The proof follows as in [1, 7]. Let μ_n be a sequence of eigenvalues such that $\mu_n > \lambda_1$ and $\mu_n \to \lambda_1$. Let the corresponding eigenfunctions u_n converge to ϕ_1. such that $\|u_n\|_{L^p(V)} = 1$. i.e., u_n satisfies

$$-\Delta_p u_n - \mu a(x)|u_n|^{p-2}u_n = \lambda_n V(x)|u_n|^{p-2}u_n. \tag{4.2}$$

Testing (4.2) with u_n and applying weighted Hardy-Sobolev inequallity we get u_n to be bounded. Therefore by Proposition 1.1, there exists a subsequence (u_n) of (u_n) such that $u_n \to u$ weakly in $W_0^{1,p}(\Omega)$, strongly in $L^p(\Omega)$ and $\nabla u_n \to \nabla u$ a.e in Ω. Taking limit $n \to \infty$ in (4.2) we get

$$-\Delta_p u - \mu a(x)|u|^{p-2}u = \lambda_1 V(x)|u|^{p-2}u \quad \text{in } \mathcal{D}'(\Omega).$$

Therefore $u = \pm\phi_1$. By Theorem 2.1, u_n changes sign. Without loss of generality, we can assume that $u = +\phi_1$, then

$$|\{x; u_n < 0\}| \to 0. \tag{4.3}$$

Testing (4.2) with u_n^-, we get

$$\int_\Omega |\nabla u_n^-|^p - \int_\Omega \mu a(x) {u_n^-}^p = \int_\Omega \lambda_n V(x) {u_n^-}^p$$

By Hardy-Sobolev and Sobolev inequalities, we get

$$C_1\|u_n\|_{1,p}^p \le C\int_{\Omega^-} V(x)|u_n|^p \le C\|V\|_s\|u_n\|_{p^*}^p|\Omega_n^-|^\gamma \le C_3\|u_n\|_{1,p}^p|\Omega_n^-|^\gamma\|V\|_s,$$

for some positive $\gamma > 0$. This implies that

$$|\Omega_n^-| \ge C_4^{1/\gamma}, \quad \Omega_n^- = \{x \in \Omega; u_n < 0\}.$$

This contradicts (4.3).

Next we prove the estimate (4.1). Assume that $v > 0$ in O, the case $v < 0$ being treated similarly. We observe by Lemma 4.2, that $v|_O \in W_0^{1,p}(O)$. Hence the function defined as $w(x) = v(x)$ if $x \in O$ and $w(x) = 0$ if $x \in \Omega\backslash O$ belongs to $W_0^{1,p}(\Omega)$. Using w as test function in the equation satisfied by v, we find

$$\int_O |\nabla v|^p dx - \int_\Omega \mu a(x)|v|^p dx = \lambda\int_O V|v|^p dx \le \lambda\|V\|_s\|v\|_{p^*,O}|O|^{\frac{p^*-s'p}{s'p^*}}$$

by Holder inequality. On the other hand by Sobolev and Hardy-Sobolev inequalities we have that $\int_O |\nabla v|^p dx \ge C\|v\|_{p^*,O}^p$ for some constant $C = C(N,p)$. Hence

$$C \le \lambda\|V\|_s|O|^{\frac{p^*-s'p}{s'p^*}}$$

□

Corollary 4.3 *Each eigenfunction has a finite number of nodal domains.*

Proof: Let O_j be a nodal domain of an eigenfunction associated to some positive eigenvalue λ. It follows from (4.1) that

$$|\Omega| \ge \sum_j |O_j| \ge (C\lambda\|V\|_s)^{-\gamma}\sum_j 1$$

and the proof follows.

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